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Research Article

Dynamics of Duopoly Models with Undecided Clients under Decentralized Affine Feedback Advertising Policies

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This paper extends the Deal-Vidal-Wolfe and Lanchester models of duopoly dynamics, which involve two populations, by explicitly introducing a third population of undecided users. An analysis of these extended models establishes conditions for the existence of equilibria, as well as their stability properties under different classes of advertising policies. This analysis also leads to the surprising result that the extended Vidale–Wolfe and Lanchester models, despite having different dynamics, under the general class of decentralized affine feedback advertising policies have equilibria in identical locations, with the same stability properties.

1. Introduction

A predominant form of market structure is the oligopoly which represents a market with few enterprises and a large number of buyers from the demand side [1]. The simplest type of oligopoly is the duopoly, where the market consists of two companies offering similar or identical products [2]. Markets are dynamic so that the strategies of competition existing in the firms which conform the market such as price, quality, and publicity are also influenced by market changes [3]. On the contrary, the consumer buying process is strongly influenced by cultural, social, personal, and psychological factors [4]. In order to influence the buying behavior of people [5], firms use advertising to promote the sale of their products [6]. Advertising can be understood as a form of communication used to induce consumers to take a particular action with respect to products or services [7].

Advertising results can be assessed by communication impacts, and the effects on sales and profits [4] and, under different hypotheses, corresponding classes of models have been developed and are surveyed in [7]. Studies of duopoly models based on differential games are carried out by Fruchter [8], Wang and Wu [7], Fletcher and Howell [9], Wang et al. [10], and Jørgensen and Sigué [11]. These references are mainly focused on the solution of the Hamilton–Jacobi–Bellman equations arising from the use of differential game theory, in contrast with this paper, which aims to provide a new perspective on the relations between the classical Vidale–Wolfe and Lanchester models and their proposed variants.

Vidale and Wolfe [12] proposed the following model for sales response to advertising:

$$\dot{S} = rA(t)\left(1 - \frac{S}{M}\right) - \lambda S, \tag{1}$$

where S represents the rate of sales at time t, $A(t)$ is the advertising expenditure (to be designed), r is referred to as the response constant, M the market saturation level, and λ the exponential sales decay constant. The interpretation is that the increase in the rate of sales, $\dot{S}$, is proportional to the intensity of the advertising policy (or control) $A(t)$ reaching the fraction of potential clients $(1 - S/M)$, less the number of defecting clients (λS). Rewriting the model in terms of the fraction of clients S/M, also called the market share $x = S/M$, gives

$$\dot{x} = \frac{rA(t)}{M}(1 - x) - \lambda x = u(1 - x) - \lambda x, \tag{2}$$

where $u = rA/M$. Deal [13] generalized the Vidale–Wolfe model to a duopoly as follows:

$$\dot{x}_1 = u_1 (1 - x_1 - x_2) - \lambda_1 x_1, \quad (3)$$

$$\dot{x}_2 = u_2 (1 - x_1 - x_2) - \lambda_2 x_2, \quad (4)$$

where for $i = 1, 2$, x_i (resp u_i) denotes the market share (resp. advertising policy) of firm i and λ_i are the respective sales decay constants. As is usual in the literature, Deal's generalization will also be referred to as a Vidale–Wolfe model [14].

Kimball [15] cited the Lanchester model as a saturated (i.e., the market shares sum to unity) market model, written as follows:

$$\dot{x}_1 = -\rho_2 u_2 x_1 + \rho_1 u_1 x_2, \quad (5)$$

$$\dot{x}_2 = \rho_2 u_2 x_1 - \rho_1 u_1 x_2. \quad (6)$$

Little [16] made the substitutions $x_2 = 1 - x_1$, $\rho_2 u_2 = \lambda$, and $\rho_1 = 1$, in the Lanchester model [17] and noted that it became identical to the Vidale–Wolfe model, thus generalizing the latter by considering competition. The decay term $(-\lambda x)$ referred to in [12] as a forgetting effect thus also models the effect of competition in reducing market share.

Clients in a duopolistic market model are restricted to choosing either a firm or its competitor. However, clients may also remain neutral or undecided in relation to both firms, and allowing this possibility leads to a more realistic description of market dynamics. Studies show advertising affects all market clients [7], which implies that the inclusion of a third population could be important in modeling and analysis. Proposals for the introduction of an undecided population have also been made in other areas such as opinion dynamics, social dynamics, religious affiliation, and political dynamics [18–21]. Finally, in a recent paper [22], the authors note that "the most popular models are still those proposed by Nerlove and Arrow [23] for monopolies, Vidale and Wolfe [12] and Lanchester [17] for duopolies. However, each model has limitations: the Vidale–Wolfe model does not consider competition explicitly, whereas the Lanchester model does not consider behavior of the firms before market saturation is reached." They then propose a three-population model for a monopoly, and discuss its dynamics, based on projected dynamical systems, in addition to considering control of market share under a class of nonlinear switching controls.

One important contribution of this paper is to introduce a class of models that extends and unifies the Lanchester and Vidale–Wolfe models so as to remove their limitations, by explicitly introducing a third class of undecided clients. The interactions between firms and clients (advertising), as well as the client-client interactions (leading to transitions between firm allegiances) are shown schematically in Figure 1(a). We also introduce a natural class of advertising policies, composed of the sum of two terms, one of which is a constant effort and the other proportional to the firm's market share. Such policies are referred to as decentralized affine controls, in the technical literature on control. We study the dynamics of market share in the extended class of models that we introduce and derive the new and, in our view, interesting result that, although the extended Lanchester and Vidale–Wolfe models have mathematically different dynamics, they share identical equilibrium points (in location and stability properties, even though the trajectories themselves are, of course, different), under the proposed class of affine policies or controls. We now emphasize the differences between the approach in [22] and that of this paper. This paper proposes new Vidale–Wolfe and Lanchester type duopoly advertising models that explicitly consider a third population of undecided clients. In [22], models with three populations are also considered, but only for monopolies. This paper carries out a complete analysis of equilibria of proposed duopoly models under affine feedback advertising policies. In contrast, the equilibrium point analysis in [22] is carried out for a monopoly model, under a nonlinear switching policy. One of the main results of this paper is showing equivalence, in terms of equilibrium market shares, of proposed Vidale–Wolfe and Lanchester type model under affine feedback advertising policies. There is no such analysis in [22] because it considers a different class of monopoly models, arising from projected dynamical systems.

2. Models of Duopolies with Undecided Clients

2.1. Vidale–Wolfe Model. Deal's extension of the Vidale–Wolfe model to the case of a duopoly assumes that the effects of advertising act only on the unconquered part of the market, thus discarding the influence of advertising on the market shares conquered by the competing firms. We now rewrite Deal's models (3) and (4) with the third class of undecided clients made explicit (see Figure 1(b)). Denoting the unconquered part of the market by x_3, which corresponds to the population of undecided clients, Deal's version of the Vidale–Wolfe model can be expressed by the following equations:

$$\begin{aligned} \dot{x}_1 &= x_3 u_1 - \lambda_1 x_1, \\ \dot{x}_2 &= x_3 u_2 - \lambda_2 x_2, \\ \dot{x}_3 &= -x_3 u_1 - x_3 u_2 + \lambda_1 x_1 + \lambda_2 x_2. \end{aligned} \quad (7)$$

Assuming that the total population size is constant and normalized to unity [5], i.e., $x_1 + x_2 + x_3 = 1$, model (7) can also be expressed as follows:

$$\begin{aligned} \dot{x}_1 &= u_1 - u_1 x_1 - u_1 x_2 - \lambda_1 x_1, \\ \dot{x}_2 &= u_2 - u_2 x_1 - u_2 x_2 - \lambda_2 x_2. \end{aligned} \quad (8)$$

Under constant advertising policies u_1 and u_2, the equilibrium point of system (8) is calculated to be

$$(x_1^*, x_2^*) = \left(\frac{\lambda_2 u_1}{\lambda_1 \lambda_2 + \lambda_1 u_2 + \lambda_2 u_1}, \frac{\lambda_1 u_2}{\lambda_1 \lambda_2 + \lambda_1 u_2 + \lambda_2 u_1} \right), \quad (9)$$

where x_1, x_2, and x_3 are the state variables representing the market shares of firm 1 and firm 2 and undecided users, respectively, u_1 and u_2 are the policies, assumed constant, of firm 1 and firm 2 representing advertising, and λ_1 and λ_2 are the decay terms of firm 1 and firm 2.

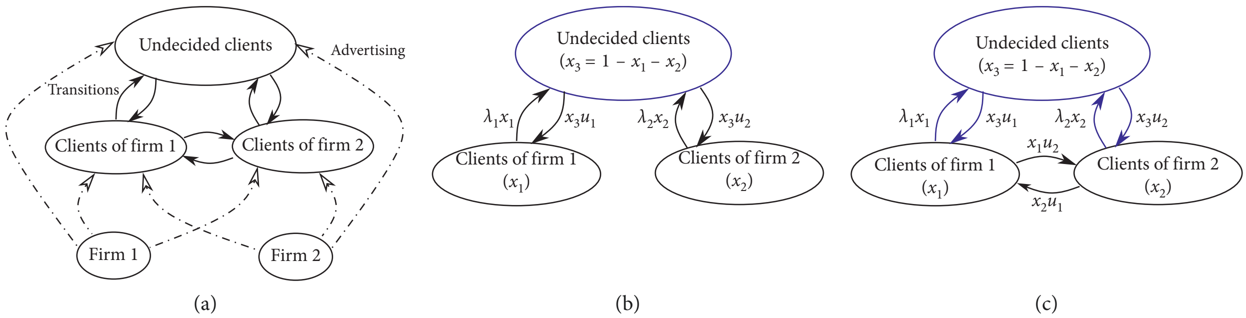


Figure 1: A graph representation of client and firm interactions in a duopoly with advertising. (a) The nodes represent clients and firms and the dotted arrows (edges) represent advertising which cause transitions (solid arrows) between them and (b) Vidale–Wolfe model and (c) extended Lanchester model, showing only transitions between client sets (refer to equations (7) and (11)). The extensions made to the original models are highlighted in blue.

2.2. Extended Lanchester Model. The Lanchester model [17] can be understood as an extension of the Vidale–Wolfe model [12] within a duopoly with competition in advertising [16]. The Lanchester model different from the Deal model [13] represents the dynamics of competition in advertising, considering advertising as the sole cause of variation of the market share of firms. For this reason, the Lanchester model does not consider the decay term contemplated in the Vidale–Wolfe model, which is used to represent the loss of market share produced by factors such as quality of the product or service, as well as competition in advertising with other firms not modeled in the duopoly. The classical Lanchester model is determined by the following expression:

$$\begin{aligned}\dot{x}_1 &= x_2 u_1 - x_1 u_2,\\ \dot{x}_2 &= x_1 u_2 - x_2 u_1.\end{aligned} \tag{10}$$

In order to extend the Lanchester model, we argue that the firm *i*'s advertising acts on the undecided consumers in a positive sense (i.e., to increase x_i), while it acts negatively on firm *j* (its competitor). In mathematical terms, the proposed extension of the Lanchester model (see Figure 1(c)) is as follows:

$$\begin{aligned}\dot{x}_1 &= -x_1 u_2 + (x_3 + x_2) u_1 - \lambda_1 x_1,\\ \dot{x}_2 &= -x_2 u_1 + (x_3 + x_1) u_2 - \lambda_2 x_2,\\ \dot{x}_3 &= -x_3 u_1 - x_3 u_2 + \lambda_1 x_1 + \lambda_2 x_2.\end{aligned} \tag{11}$$

Note that x_3 has the same dynamics as in the Vidale–Wolfe model because of the fact that the Lanchester model just adds and subtracts the terms $x_i u_j$ from the corresponding equations so that the sum of the first two equations in the Vidale–Wolfe model is the same as the corresponding sum in the Lanchester model. Since the total population size is constant and normalized to 1, model (11) can be expressed as follows:

$$\begin{aligned}\dot{x}_1 &= u_1 - x_1 (u_1 + u_2 + \lambda_1),\\ \dot{x}_2 &= u_2 - x_2 (u_1 + u_2 + \lambda_2).\end{aligned} \tag{12}$$

Thus, the fixed points of system (12), under constant advertising policies u_1 and u_2 are determined as follows:

$$(x_1^*, x_2^*) = \left(\frac{u_1}{\lambda_1 + u_1 + u_2}, \frac{u_2}{\lambda_2 + u_1 + u_2}\right), \tag{13}$$

where x_1, x_2, and x_3 are the state variables representing the market shares of firm 1, firm 2, and undecided users, respectively, u_1 and u_2 are the policies of firm 1 and firm 2 representing positive advertising, and λ_1 and λ_2 are the decay terms of firm 1 and firm 2, respectively.

2.3. Nonequivalence of Classical and Proposed Models. The introduction (see discussion below equation (6)) mentioned the manipulation made by Little [16] in the case of a monopoly which shows that the Lanchester model generalizes the Vidale–Wolfe model for a saturated market. This section examines whether such a manipulation is possible for the extended Lanchester model to be seen as a generalization of Deal's version of the Vidale–Wolfe model. Comparing (12), in which we set $\lambda_1 = \lambda_2 = 0$ with Deal's equations (3) and (4), it is clear that they will only have identical dynamics , for all possible choices of x_1 and x_2:

$$\begin{aligned}u_1 x_2 - u_2 x_1 &= -\lambda_1 x_1,\\ -u_1 x_2 + u_2 x_1 &= -\lambda_2 x_2,\end{aligned} \tag{14}$$

which, in matrix form, becomes

$$\begin{bmatrix}\lambda_1 - u_2 & u_1\\ u_2 & \lambda_2 - u_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}, \tag{15}$$

which can only be satisfied for all x_1 and x_2 if the coefficient matrix is the null matrix (all entries zero), which yields the trivial solution $u_1 = u_2 = \lambda_1 = \lambda_2 = 0$. Thus, differently from the case of a monopoly, it is not possible to directly rewrite the generalized Lanchester duopoly equations as the Deal (generalized Vidale–Wolfe) duopoly equations. In a similar manner, if the extended Lanchester duopoly equations with nonzero decay coefficients (12) are compared with Deal's duopoly equations (3) and (4), the two sets of equations are identical, for all x_1 and x_2

$$\begin{bmatrix}-u_2 & u_1\\ u_2 & -u_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}, \tag{16}$$

which, once again, leads to the trivial solution $u_1 = u_2 = 0$. In other words, we conclude that the proposed generalization of Lanchester's model to a duopoly does not lead to Deal's generalization of the Vidale–Wolfe model to a duopoly through a simple algebraic manipulation and renaming of parameters, which was possible for the corresponding models for monopolies. Despite this, we shall see that, under certain conditions, both the Deal model as well as the proposed generalized Lanchester model share important properties such as the location and nature of their singularities, under the general class of affine advertising policies.

3. Equilibria and Stability Analysis of Duopolies Models considering Undecided Users under Decentralized Affine Advertising Policy

In this section, we analyze the existence and stability of the equilibrium points of the duopoly models proposed in the previous section. We first make the following standard assumptions.

Assumption 1. The market shares of x_i of firm i, $i = 1, 2$ are assumed to be nonnegative values in the interval $[0, 1]$.

Assumption 2. The advertising policies of the firms are assumed as decentralized affine feedback, that is, $u_i = k_i x_i + c_i$, $i = 1, 2$.

Assumption 3. The decay coefficients λ_1 and λ_2 and advertising policies u_1 and u_2 and their coefficients k_1, k_2, c_1, and c_2 are assumed to be positive values.

Assumption 4. The decay rates of both firms are assumed to be equal, i.e., $\lambda_1 = \lambda_2 = \lambda$.

Remark. Note that, the advertising policy of the firms is expressed by two terms, where the first being proportional to market share ($k_i x_i$) and the second having a constant value (c_i). The advertising policy is also decentralized in the sense that the firm i bases its policy based only on information about its own market share x_i. Affine control has been proposed in different contexts, for example, in magnetic and electronic control [24, 25] and automation and robotic systems [26, 27]. It has also been proposed, using full state feedback, for predator-prey models in the textbook [28]. Prior to this, decentralized affine control, also in the context of predator-prey models, was used in [29, 30] and subsequently in [31]. Taking inspiration from these applications, it was then proposed to use decentralized affine advertising (DAA) policies in models of duopolies in [32, 33]. The main motivations for the use of DAA policies in this paper are summarized below:

(i) DAA policies have a natural interpretation as proportional plus constant control and are easy to implement, in contrast to policies derived from optimal control theory, which are usually very hard to calculate and also to implement.

(ii) The simple mathematical form of DAA policies also permits analytical derivations of stability and bifurcation results for the models proposed in this paper.

Given the fact that, in this work, all advertising policies or controls are affine and decentralized and determined by the choice of the parameters $(k_i, c_i, i = 1, 2)$, the results presented are analytical, allowing a policy designer to predict what happens under different scenarios for different choices of the parameter values. This is in contrast with the approach of optimal control, which determines a (usually an open loop and not necessarily decentralized) policy that takes the system state from an initial set of market shares to a desired final set of market shares, minimizing some cost function. In the proposed approach, costs can be evaluated by substituting the proposed controls into a specified cost function and using the results in a "flight simulator" mode [34]. It is also possible, for example, to draw isocost contours that connect reachable states with the same terminal cost.

3.1. Decentralized Affine Advertising Policy Applied to the Vidale–Wolfe Model.

Considering the Assumption 2, the advertising policies of the firms are assumed to be affine [32]:

$$u_1 = k_1 x_1 + c_1, \tag{17}$$

$$u_2 = k_2 x_2 + c_2. \tag{18}$$

Substituting the affine advertising policies in model (8) yields,

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1^2 - k_1 x_1 x_2 + k_1 x_1 - \lambda x_1 + c_1 - c_1 x_1 - c_1 x_2, \\ \dot{x}_2 &= -k_2 x_2^2 - k_2 x_1 x_2 + k_2 x_2 - \lambda x_2 + c_2 - c_2 x_1 - c_2 x_2. \end{aligned} \tag{19}$$

Reordering terms, we have

$$\begin{aligned} \dot{x}_1 &= -k_1 x_1^2 - k_1 x_1 x_2 - a x_1 - c_1 x_2 + c_1, \\ \dot{x}_2 &= -k_2 x_2^2 - k_2 x_1 x_2 - b x_2 - c_2 x_1 + c_2, \end{aligned} \tag{20}$$

where

$$\begin{aligned} a &= (c_1 + \lambda - k_1), \\ b &= (c_2 + \lambda - k_2). \end{aligned} \tag{21}$$

The dynamics and corresponding equilibrium points for model (20) are shown in Table 1. Note that for general cases of affine advertising policies, special cases of parameter values given in Table 2 are used in order to establish analytical solutions for equilibrium points.

Thus, for policy 4, $k_1 = c_1 = c_2 = c$. For policy 5, $k_1 = c_1 = k$ and for policy 6, $k_1 = k_2 = k$ and $c_1 = c_2 = c$. In addition, in Table 1, we have that for policy 4, $p = 3c + \lambda + g$ and $q = 3c + \lambda - g$, where $g = (5c^2 + 2c\lambda + \lambda^2)^{1/2}$, for policy 5, $f = \sqrt{\lambda^2 + 4k^2}$, and finally for policy 6, $f = \sqrt{e^2 + 2ec + c^2 + 8ck/4}$, where $e = c + \lambda - k$.

Table 1: Dynamic equations and equilibrium points for Vidale–Wolfe model (8) under different feedback advertising policies.

Policy	Control parameters	Vidale–Wolfe model under feedback	Equilibrium points
P_1	$u_1 = c_1$ $u_2 = c_2$	$\dot{x}_1 = -c_1x_1 - c_1x_2 - \lambda x_1 + c_1$ $\dot{x}_2 = -c_2x_1 - c_2x_2 - \lambda x_2 + c_2$	$(c_1/c_1 + c_2 + \lambda, c_2/c_1 + c_2 + \lambda)$
P_2	$u_1 = k_1x_1$ $u_2 = c_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -c_2x_1 - c_2x_2 - \lambda x_2 + c_2$	$(0, c_2/c_2 + \lambda)$ $(k_1 - c_2 - \lambda/k_1, c_2/k_1)$
P_3	$u_1 = k_1x_1$ $u_2 = k_2x_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -k_2x_2^2 - k_2x_1x_2 + k_2x_2 - \lambda x_2$	$(0, 0)$ $(0, k_2 - \lambda/k_2)$ $(k_1 - \lambda/k_1, 0)$
P_3^*	$u_1 = kx_1$ $u_2 = kx_2$	$\dot{x}_1 = -kx_1^2 - kx_1x_2 + kx_1 - \lambda x_1$ $\dot{x}_2 = -kx_2^2 - kx_1x_2 + kx_2 - \lambda x_2$	$(-kx_2 - k + \lambda/k, x_2)$ $(0, 0)$
P_4	$u_1 = k_1x_1 + c_1$ $u_2 = c_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1$ $\dot{x}_2 = -c_2x_1 - c_2x_2 - \lambda x_2 + c_2$	$(2c - p/2c, p/2(c + \lambda))$ $(2c - q/2c, q/2(c + \lambda))$
P_5	$u_1 = k_1x_1 + c_1$ $u_2 = k_2x_2$	$\dot{x}_1 = -k_1x_1^2 - k_2x_1x_2 - c_1x_1 + k_1x_1 - \lambda x_1 + c_1$ $\dot{x}_2 = -k_1x_1x_2 - k_2x_2^2 - c_1x_2 + k_2x_2 - \lambda x_2$	$(-\lambda - f/2k, 0)$ $(-\lambda + f/2k, 0)$ $(k/k_2 - k, 2kk_2 - k\lambda + k_2\lambda - k_2^2/kk_2 - k_2^2)$
P_6	$u_1 = k_1x_1 + c_1$ $u_2 = k_2x_2 + c_2$	$\dot{x}_1 = -k_1x_1^2 - k_1x_1x_2 + k_1x_1 - \lambda x_1 + c_1 - c_1x_1 - c_1x_2$ $\dot{x}_2 = -k_2x_2^2 - k_2x_1x_2 + k_2x_2 - \lambda x_2 + c_2 - c_2x_1 - c_2x_2$	$(-e - c - 2f/4k, -e - c - 2f/4k)$ $(-e - c + 2f/4k, -e - c + 2f/4k)$

Table 2: Advertising policy parameters and expressions of variables in special cases of affine policy.

Policy	Parameters	Expressions of variables
P_4	$k_1 = c_1 = c_2 = c$	$p = 3c + \lambda + g$, $q = 3c + \lambda - g$ $g = \sqrt{5c^2 + 2c\lambda + \lambda^2}$
P_5	$k_1 = c_1 = k$	$f = \sqrt{\lambda^2 + 4k^2}$
P_6	$k_1 = k_2 = k$ $c_1 = c_2 = c$	$f = \sqrt{e^2 + 2ec + c^2 + 8kc/4}$ $e = c + \lambda - k$

Stability is determined by the signs of the determinant and trace of the Jacobian matrix evaluated at the corresponding equilibrium points. The Jacobian matrix for model (20) with respect to x_1 and x_2 is given by

$$J_{VW} = \begin{bmatrix} -2k_1x_1 - k_1x_2 - a & -k_1x_1 - c_1 \\ -k_2x_2 - c_2 & -2k_2x_2 - k_2x_1 - b \end{bmatrix}. \quad (22)$$

The determinant and trace for the Jacobian matrix are

$$\begin{aligned} \det(J_{VW}) &= 2k_1k_2x_1^2 + 4k_1k_2x_1x_2 + 2k_1k_2x_2^2 + ak_2x_1 + 2ak_2x_2 \\ &\quad + 2bk1x_1 + bk_1x_1 - c_1k_2x_2 - c_2k_1x_1 + ab - c_1c_2, \\ \operatorname{tr}(J_{VW}) &= -2k_1x_1 - k_1x_2 - k_2x_1 - 2k_2x_2 - a - b. \end{aligned} \quad (23)$$

The conditions that ensure stability of the equilibrium points of the Vidale–Wolfe model are shown in Table 3. The expressions for the determinants and traces of the Jacobian matrix of the Vidale–Wolfe model, subject to different choices of affine advertising, are displayed in Table 4.

3.2. Decentralized Affine Advertising Policy Applied to the Extended Lanchester Model. Similar to the previous section, we consider the Assumption 2 for the affine advertising policy of the firms. Therefore, based on equations (17) and (18) we have $u_1 = k_1x_1 + c_1$ and $u_2 = k_2x_2 + c_2$.

Substituting the advertising policies in model (12) yields,

$$\begin{aligned} \dot{x}_1 &= -k_1x_1^2 - k_2x_1x_2 - c_1x_1 - c_2x_1 + k_1x_1 - \lambda x_1 + c_1, \\ \dot{x}_2 &= -k_1x_1x_2 - k_2x_2^2 - c_1x_2 - c_2x_2 + k_2x_2 - \lambda x_2 + c_2. \end{aligned} \quad (24)$$

Rearranging terms,

$$\begin{aligned} \dot{x}_1 &= -k_1x_1^2 - k_2x_1x_2 - wx_1 + c_1, \\ \dot{x}_2 &= -k_2x_2^2 - k_1x_1x_2 - zx_2 + c_2, \end{aligned} \quad (25)$$

where

$$\begin{aligned} w &= (c_1 + c_2 + \lambda - k_1), \\ z &= (c_1 + c_2 + \lambda - k_2). \end{aligned} \quad (26)$$

Table 5 shows the equations of dynamics and the equilibrium points of the extended Lanchester model for special cases of the affine policy. Note that, similar to the Vidale–Wolfe model, particular cases of advertising policy parameters given in Table 2 are considered in order to establish analytical solutions for equilibrium points. For policy 4, $k_1 = c_1 = c_2 = c$. For policy 5, $k_1 = c_1 = k$ and for policy 6, $k_1 = k_2 = k$ and $c_1 = c_2 = c$. In addition, in Table 5, for policy 4, $p = 3c + \lambda + g$ and $q = 3c + \lambda - g$, where $g = (5c^2 + 2c\lambda + \lambda^2)^{1/2}$, for policy 5, $f = \sqrt{\lambda^2 + 4k^2}$, and finally for policy 6, $f = \sqrt{e^2 + 2ec + c^2 + 8ck/4}$, where $e = c + \lambda - k$.

The Jacobian matrix for model (25) with respect to x_1 and x_2 is given by

$$J_L = \begin{bmatrix} -2k_1x_1 - k_2x_2 - w & -k_2x_1 \\ -k_1x_2 & -2k_2x_2 - k_1x_1 - z \end{bmatrix}, \quad (27)$$

when the determinant and trace for the Jacobian matrix are

$$\begin{aligned} \det(J_L) &= 2k_1^2x_1^2 + 4k_1k_2x_1x_2 + 2k_2^2x_2^2 + k_1wx_1 + 2k_1zx_1 \\ &\quad + 2k_2wx_2 + k_2zx_2 + zw, \\ \operatorname{tr}(J_L) &= -3k_1x_1 - 3k_2x_2 - z - w. \end{aligned} \quad (28)$$

Table 3: Conditions on the advertising policy parameters to guarantee stability of the equilibrium points. The expression "*None*" indicates that there is no restriction on the values of advertising policies parameters to guarantee the condition of stability of the equilibria of both models (cf. Theorem 1).

Policy	Equilibrium points	Stability conditions for VW	Stability conditions for extended Lanchester
P_1	$(c_1/c_1+c_2+\lambda, c_2/c_1+c_2+\lambda)$	*None*	*None*
P_2	$(0, c_2/c_2+\lambda)$ $(k_1-c_2-\lambda/k_1, c_2/k_1)$	$k_1 < c_2+\lambda$ $k_1 > c_2+\lambda$	$k_1 < c_2+\lambda$ $k_1 > c_2+\lambda$
P_3	$(0, k_2-\lambda/k_2)$ $(k_1-\lambda/k_1, 0)$	$k_1 < k_2$ $k_2 < k_1$	$k_1 < k_2$ $k_2 < k_1$
P_3^*	$(-kx_2-k+\lambda/k, x_2)$	$\lambda < k$	$\lambda < k$
P_4	$(2c-q/2c, q/2(c+\lambda))$	*None*	*None*
P_5	$(-\lambda+f/2k, 0)$ $(k/k_2-k, 2kk_2-k\lambda+k_2\lambda-k_2^2/kk_2-k_2^2)$	$k > k_2(\lambda-f)/2(\lambda+f-k_2)$ $k_2(\lambda-k_2)/\lambda-2k_2 < k < k_2^2/\lambda$	$k > 2k_2-f-\lambda/2$ $k_2(\lambda-k_2)/\lambda-2k_2 < k < 2k_2-\lambda/2$
P_6	$(-e-c+2f/4k, -e-c+2f/4k)$	$k < 2f+\lambda-2c$	$k < 2f+\lambda-2c$

Table 4: Expressions for determinants and traces in Vidale–Wolfe model (8) under special cases of affine feedback advertising policy.

Policy	Determinant for Vidale–Wolfe model	Trace for Vidale–Wolfe model
P_1	$\lambda(c_1+c_2+\lambda)$	$-c_1-c_2-2\lambda$
P_2	$c_2k_1x_1+c_2k_1x_2+2\lambda k_1x_1+\lambda k_1x_2+c_2\lambda-c_2k_1+\lambda^2-\lambda k_1$	$-2k_1x_1-k_1x_2-c_2-2\lambda+k_1$
P_3	$2k_1k_2x_1^2+4k_1k_2x_1x_2+2k_1k_2x_2^2+2\lambda k_1x_1+\lambda k_1x_2+\lambda k_2x_1+2\lambda k_2x_2-3k_1k_2x_1-3k_1k_2x_2+\lambda^2-\lambda k_1-\lambda k_2+k_1k_2$	$-2k_1x_1-k_1x_2-k_2x_1-2k_2x_2-2\lambda+k_1+k_2$
P_4	$c^2x_1+c^2x_2+2c\lambda x_1+c\lambda x_2-c^2+c\lambda+\lambda^2$	$-2cx_1-cx_2-c-2\lambda$
P_5	$2kk_2x_1^2+4kk_2x_1x_2+2kk_2x_2^2+2\lambda kx_1+\lambda kx_2+\lambda k_2x_1++2\lambda k_2x_2-2kk_2x_1-2kk_2x_2+\lambda^2-\lambda k_2$	$-2kx_1-kx_2-k_2x_1-2k_2x_2-2\lambda+k_2$
P_6	$(kx_1+kx_2+\lambda-k)(2kx_1+2kx_2+2c+\lambda-k)$	$-3kx_1-3kx_2-2c-2\lambda+2k$

Table 5: Dynamic equations and equilibrium points for extended Lanchester model (12) under different feedback advertising policies.

Policy	Control parameters	Extended Lanchester model under feedback	Equilibrium points
P_1	$u_1=c_1$ $u_2=c_2$	$\dot{x}_1=-c_1x_1-c_2x_1-\lambda x_1+c_1$ $\dot{x}_2=-c_1x_2-c_2x_2-\lambda x_2+c_2$	$(c_1/c_1+c_2+\lambda, c_2/c_1+c_2+\lambda)$
P_2	$u_1=k_1x_1$ $u_2=c_2$	$\dot{x}_1=-k_1x_1^2-c_2x_1+k_1x_1-\lambda x_1$ $\dot{x}_2=-k_1x_1x_2-c_2x_2-\lambda x_2+c_2$	$(0, c_2/c_2+\lambda)$ $(k_1-c_2-\lambda/k_1, c_2/k_1)$
P_3	$u_1=k_1x_1$ $u_2=k_2x_2$	$\dot{x}_1=-k_1x_1^2-k_2x_1x_2+k_1x_1-\lambda x_1$ $\dot{x}_2=-k_2x_2^2-k_1x_1x_2+k_2x_2-\lambda x_2$	$(0,0)$ $(0, k_2-\lambda/k_2)$ $(k_1-\lambda/k_1, 0)$
P_3^*	$u_1=kx_1$ $u_2=kx_2$	$\dot{x}_1=-kx_1^2-kx_1x_2+kx_1-\lambda x_1$ $\dot{x}_2=-kx_2^2-kx_1x_2+kx_2-\lambda x_2$	$(-kx_2-k+\lambda/k, x_2)$ $(0,0)$
P_4	$u_1=k_1x_1+c_1$ $u_2=c_2$	$\dot{x}_1=-k_1x_1^2-c_1x_1-c_2x_1+k_1x_1-\lambda x_1+c_1$ $\dot{x}_2=-k_1x_1x_2-c_1x_2-c_2x_2-\lambda$	$(2c-p/2c, p/2(c+\lambda))$ $(2c-q/2c, q/2(c+\lambda))$
P_5	$u_1=k_1x_1+c_1$ $u_2=k_2x_2$	$\dot{x}_1=-k_1x_1^2-k_2x_1x_2-c_1x_1+k_1x_1-\lambda x_1+c_1$ $\dot{x}_2=-k_1x_1x_2-k_2x_2^2-c_1x_2+k_2x_2-\lambda x_2$	$(-\lambda-f/2k, 0)$ $(-\lambda+f/2k, 0)$ $(k/k_2-k, 2kk_2-k\lambda+k_2\lambda-k_2^2/kk_2-k_2^2)$
P_6	$u_1=k_1x_1+c_1$ $u_2=k_2x_2+c_2$	$\dot{x}_1=-k_1x_1^2-k_2x_1x_2-c_1x_1-c_2x_1+k_1x_1-\lambda x_1+c_1$ $\dot{x}_2=-k_1x_1x_2-k_2x_2^2-c_1x_2-c_2x_2+k_2x_2-\lambda x_2+c_2$	$(-e-c-2f/4k, -e-c-2f/4k)$ $(-e-c+2f/4k, -e-c+2f/4k)$

The conditions that ensure stability of the equilibrium points of the extended Lanchester model are shown in Table 3. Conditions for the existence of equilibria for both models are given in Table 6. Expressions for the determinants and traces of the extended Lanchester model for special cases of affine policy are displayed in Table 7.

From the results presented in the Tables 1, 3, and 5, we can formulate the main result.

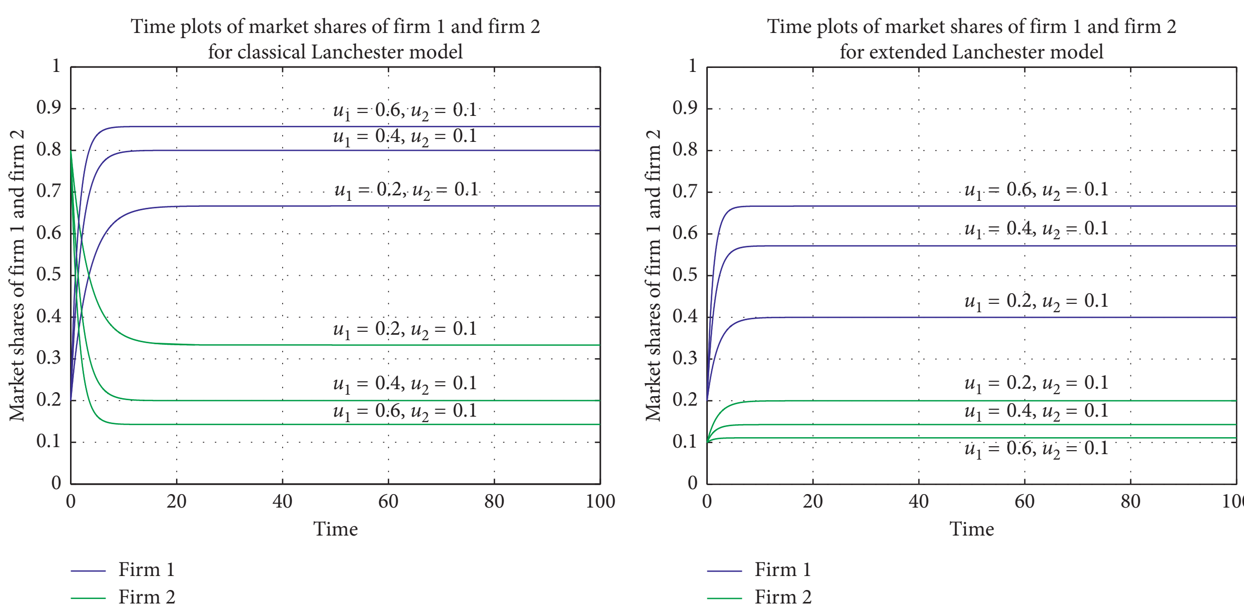
Theorem 1. *The extended Lanchester model (12) and the Vidale–Wolfe model (8), both under decentralized affine advertising policy $u_i = k_ix_i + c_i, i = 1, 2$, are equivalent for the same choices of affine policy parameters (k_i, c_i) in the*

Table 6: Equilibrium points and conditions of the advertising policy parameters for existence of the equilibrium points under special cases of affine policy, for both the Vidale–Wolfe and extended Lanchester models (cf. Theorem 1).

Policy	Control parameters	Equilibrium points	Conditions of existence
P_1	$u_1 = c_1,\ u_2 = c_2$	$(c_1/c_1 + c_2 + \lambda, c_2/c_1 + c_2 + \lambda)$	*Always exists*
P_2	$u_1 = k_1x_1,\ u_2 = c_2$	$(0, c_2/c_2 + \lambda)$ $(k_1 - c_2 - \lambda/k_1, c_2/k_1)$	*Always exists* $k_1 > c_2 + \lambda$
P_3	$u_1 = k_1x_1,\ u_2 = k_2x_2$	$(0, 0)$ $(0, k_2 - \lambda/k_2)$ $(k_1 - \lambda/k_1, 0)$	*Always exists* $k_2 > \lambda$ $k_1 > \lambda$
P_4	$u_1 = k_1x_1 + c_1,\ u_2 = c_2$	$(2c - p/2c, p/2(c + \lambda))$ $(2c - q/2c, q/2(c + \lambda))$	*Negative*$\Longrightarrow$*Never exists* $c + \lambda < g < 3c + \lambda$
P_5	$u_1 = k_1x_1 + c_1,\ u_2 = k_2x_2$	$(-\lambda - f/2k, 0)$ $(-\lambda + f/2k, 0)$ $(k/k_2 - k, 2kk_2 - k\lambda + k_2\lambda - k_2^2/kk_2 - k_2^2)$	*Negative*$\Longrightarrow$*Never exists* $\lambda < f < 2k + \lambda$ $k < k_2\lambda/\lambda - k_2 \wedge k2 < \lambda/2$
P_6	$u_1 = k_1x_1 + c_1,\ u_2 = k_2x_2 + c_2$	$(-e - c - 2f/4k, -e - c - 2f/4k)$ $(-e - c + 2f/4k, -e - c + 2f/4k)$	*Negative*$\Rightarrow$*Never exists* $2c + \lambda - k/2 < f < 3k + \lambda + 2c/2$

Table 7: Expressions for determinants and traces in extended Lanchester model (12) under special cases of affine feedback advertising policy.

Policy	Determinant for extended Lanchester model	Trace for extended Lanchester model
P_1	$(c_1 + c_2 + \lambda)^2$	$-2c_1 - 2c_2 - 2\lambda$
P_2	$(k_1x_1 + c_2 + \lambda)(2k_1x_1 + c_2 + \lambda - k_1)$	$-3k_1x_1 - 2c_2 - 2\lambda + k_1$
P_3	$2k_1^2x_1^2 + 4k_1k_2x_1x_2 + 2k_1^2x_2^2 + 3\lambda k_1x_1 + 3\lambda k_2x_2 -$ $k_1^2x_1 - 2k_1k_2x_1 - 2k_1k_2x_2 - k_2^2x_2 + \lambda^2 - \lambda k_1 - \lambda k_2 +$ k_1k_2	$-3k_1x_1 - 3k_2x_2 - 2\lambda + k_1 + k_2$
P_4	$(2cx_1 + c + \lambda)(cx_1 + 2c + \lambda)$	$-3cx_1 - 3c - 2\lambda$
P_5	$2k^2x_1^2 + 4kk_2x_1x_2 + 2k_2^2x_2^2 + 3\lambda kx_1 + 3\lambda k_2x_2 +$ $2k^2x_1 - 2kk_2x_1 + kk_2x_2 - k_2^2x_2 + \lambda^2 + \lambda k - \lambda k_2$	$-3kx_1 - 3k_2x_2 - 2\lambda - k + k_2$
P_6	$(2kx_1 + 2kx_2 + 2c + \lambda - k)(kx_1 + kx_2 + 2c + \lambda - k)$	$-3kx_1 - 3kx_2 - 4c - 2\lambda + 2k$



(a) (b)

Figure 2: Evolution of market shares of firms x_1 and x_2 under affine feedback advertising policies $P_1\,(u_i = c_i)$ with parameters $c_1 = 0.2,\ 0.4,\ 0.6$, for firm 1 and $c_2 = 0.1$ for firm 2 for (a) classical Lanchester model for $x_1(0) = 0.2$ and (b) extended Lanchester model with undecided users for $x_1(0) = 0.2$, $x_2(0) = 0.1$ and $\lambda = 0.2$. Note that in the classical Lanchester model the market shares are always on the line $x_1 + x_2 = 1$.

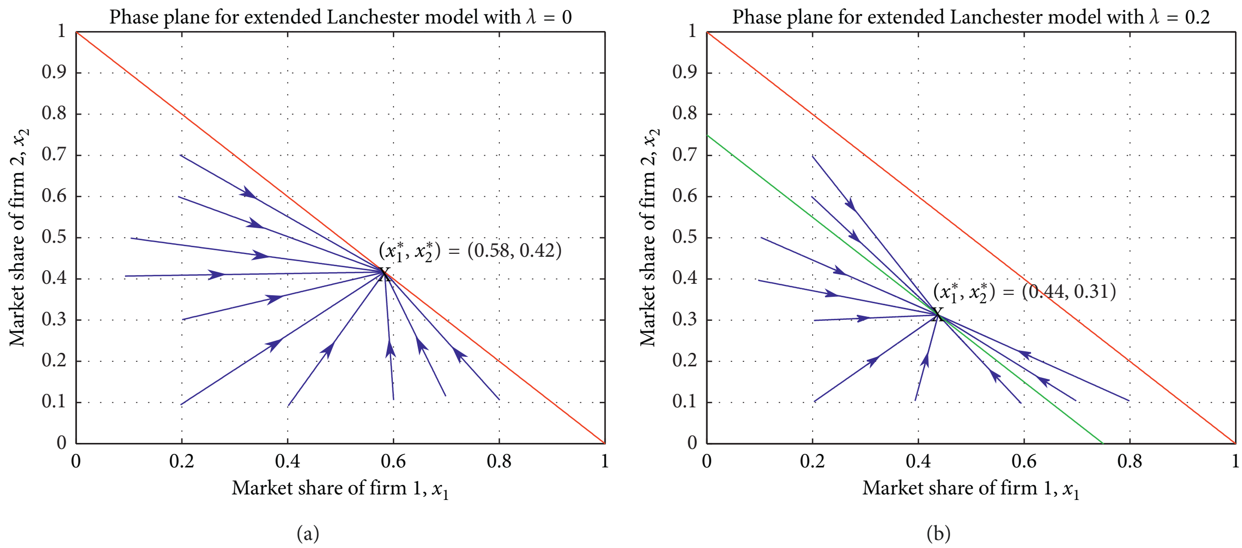


Figure 3: Phase plane of the market shares of firms x_1 and x_2 under affine feedback advertising policies $P_1\,(u_i = c_i)$ with parameters $c_1 = 0.35$ for firm 1 and $c_2 = 0.25$ for firm 2 for (a) extended Lanchester model with $\lambda = 0$ and (b) extended Lanchester model with $\lambda = 0.2$. Note the change in the line containing the equilibrium points. In this case, the new equation of line is $x_1 + x_2 = 0.75$.

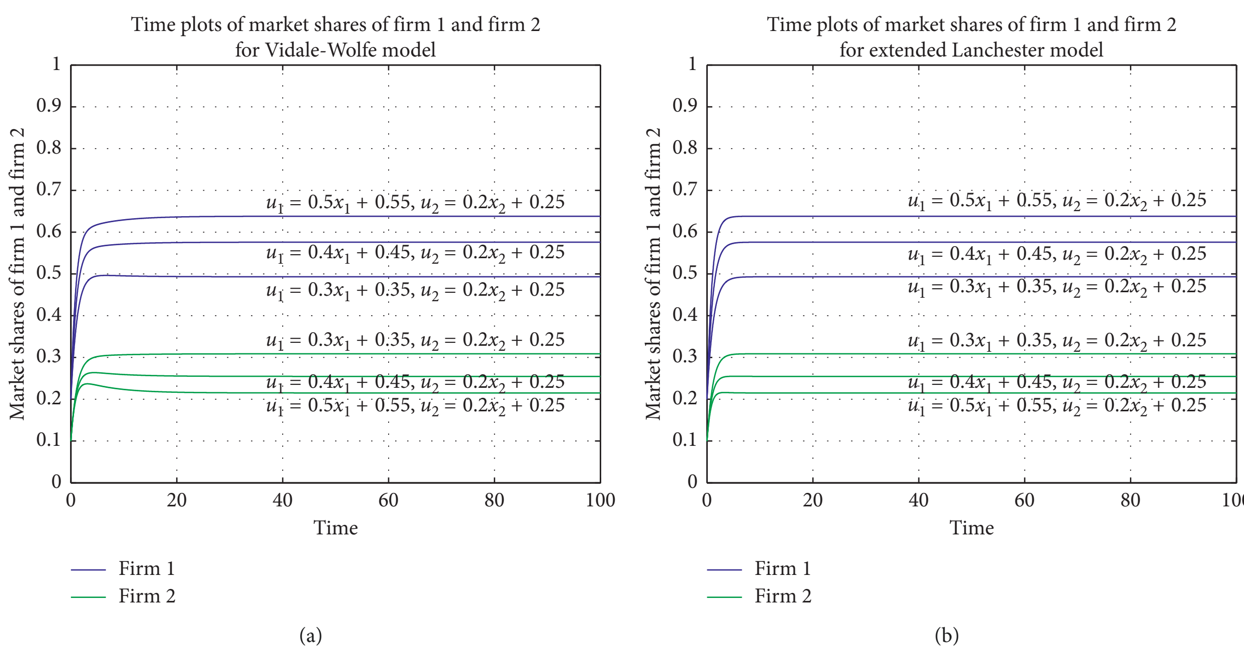


Figure 4: Evolution of market shares of firms x_1 and x_2 under affine advertising policies for (a) Vidale–Wolfe model expressed in equation (20) and (b) extended Lanchester model expressed in equation (25).

sense that both models have the same equilibrium points with the same stability properties (even though stability conditions, eigenvalues, and dynamics differ).

Remark. Note that Theorem 1 does not assert that the dynamics are identical but only that the equilibria and their stability properties are the same. However, for the special choice $u_i = k_i x_i, i = 1, 2$, the dynamics of the extended Lanchester model (12) and the Vidale–Wolfe model (7) are also identical.

Remark. We conjecture that Theorem 1 can be strengthened to assert that the flows of the Vidale–Wolfe and extended Lanchester models under decentralized affine feedback are, in fact, topologically equivalent [35].

4. Numerical Simulations

In this section, we present some numerical simulations to verify the results of the previous section. Firstly, Figure 2 allows the comparison of the classical Lanchester model with the extended Lanchester model, thus Figure 2(a) for the

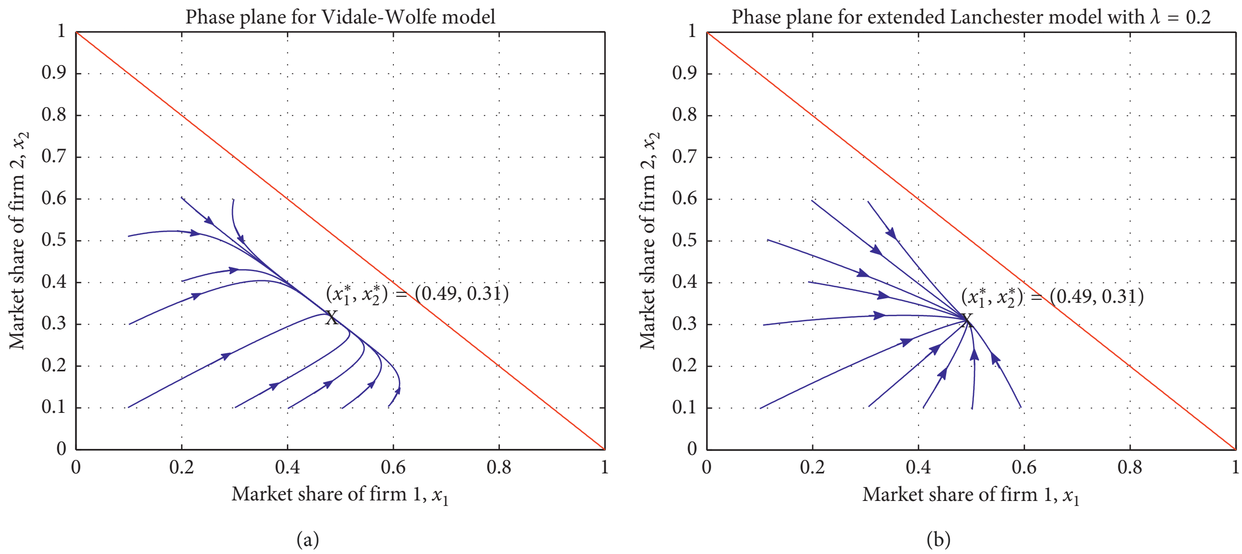


Figure 5: Phase plane of the market shares of firms x_1 and x_2 under affine advertising policies $P_6\,(u_i = k_i x_i + c_i)$ with parameters $k_1 = 0.3$ and $c_1 = 0.35$ for firm 1 and $k_2 = 0.2$ and $c_2 = 0.25$ for firm 2 for (a) Vidale–Wolfe model expressed in equation (20) and (b) extended Lanchester model expressed in equation (25). It is observed that the models have the same equilibrium point but different dynamics.

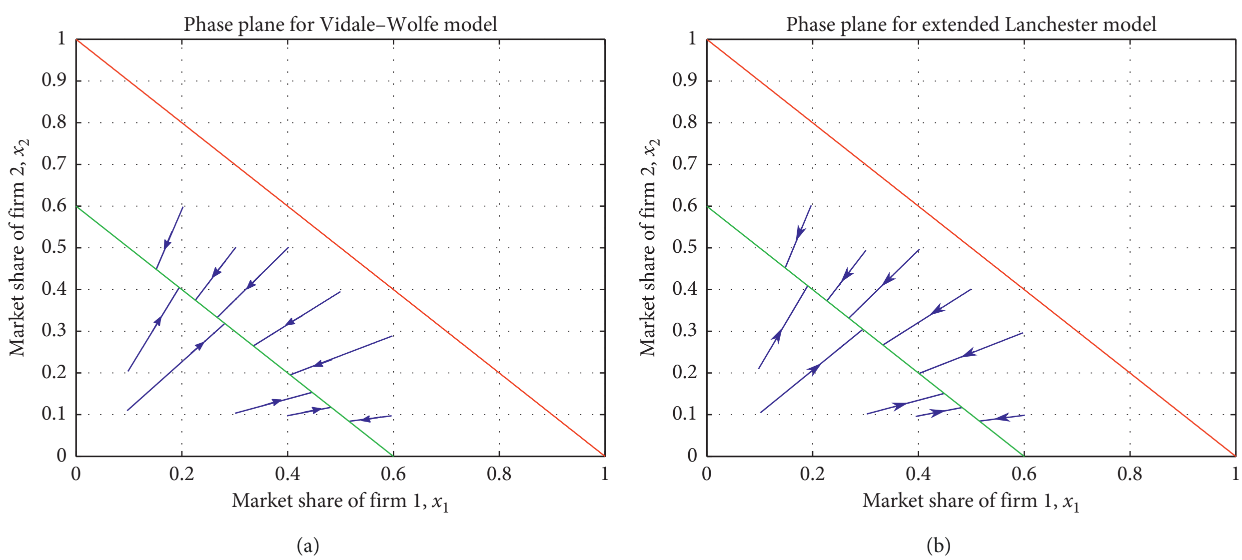


Figure 6: Phase plane of the market shares of firms x_1 and x_2 under affine advertising policies $P_3\,(u_i = k_i x_i)$ with parameters $k_1 = 0.5$ for firm 1 and $k_2 = 0.5$ for firm 2 for (a) Vidale–Wolfe model expressed in equation (20) and (b) extended Lanchester model expressed in equation (25).

classical Lanchester model shows that the equilibrium points are always on the saturated market line $x_1 + x_2 = 1$, while Figure 2(b) for the extended model shows the influence of considering an undecided population, namely, that the equilibrium points are no longer on the line $x_1 + x_2 = 1$, since x_3 may be positive at equilibrium.

Figure 3 shows the dynamics of the extended Lanchester model for different values of λ. Figure 3(a) (resp. Figure 3(b)) shows simulation of the extended Lanchester model for $\lambda = 0$ (resp. $\lambda = 0.2$), where the equilibrium point lies on the saturated market line $x_1 + x_2 = 1$ for $\lambda = 0$, shifting away from the saturated market line for positive λ.

Figure 4 shows the evolution of market shares of the firms under increase in advertising policy parameters for u_1 and u_2. Figure 4(a) shows the evolution of market shares for the Vidale–Wolfe model, while Figure 4(b) shows the evolution of market shares for the extended Lanchester model, for three different choices of affine advertising policy u_1. Note that the equilibrium points in both models under the same advertising policy are identical. The difference in the dynamics of models can be seen in the evolution of market share of firm 2 when the increment in advertising u_1 is greater. Next, Figures 5(a) and 5(b) show phase plane portraits for models (20) and (25) with the following choices of advertising policy parameters

$k_1 = 0.3, c_1 = 0.35$ and $k_2 = 0.2, c_2 = 0.25$. In accordance with Theorem 1, it can be observed that the models of Vidale–Wolfe and extended Lanchester have the same equilibrium points $(x_1 = 0.49, x_2 = 0.31)$ although they have different trajectories.

Finally, Figure 6 shows the phase plane portrait for the special case when $u_1 = k_1 x_1$ and $u_2 = k_2 x_2$ with $k_1 = k_2$, for which the models have an entire line of fixed points, i.e., an equilibrium set, which attracts all trajectories to itself.

Remark. The values of λ found in the literature related to empirical data [3, 7, 10, 36] are used for numerical simulations.

5. Concluding Remarks

This paper argues for the explicit introduction of a third class of undecided clients into Deal's version of the classical Vidale–Wolfe model and also into an extension of Lanchester's model, which includes both decay terms and the "spillover" effect of advertising on clients of the rival firm as well as on the undecided clients. The proposed modification of the Lanchester dynamics extends the classical model from the saturated market to the unsaturated market. A complete analysis of the location and stability properties of the equilibria of these two models under a general class of decentralized affine feedback advertising policies leads to the surprising conclusion that, even though the models are not equivalent, as shown by the qualitatively different trajectories in the phase plane, under identical advertising policies, the final outcome in terms of equilibrium market share is the same for both models. This is an indication of the fact that although Little's algebraic manipulation showed that the Lanchester model for two firms in a saturated market subsumes the single firm Vidale–Wolfe model, in the duopoly case, there is a deeper similarity between the two models. Further research will elucidate whether this similarity persists under the application of advertising policies derived from optimal control theory.

Data Availability

No real-world market share data were used in the theoretical study reported in this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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