Research Article

The Solvability of Fractional Elliptic Equation with the Hardy Potential

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In this paper, we study the existence and nonexistence of solutions to fractional elliptic equations with the Hardy potential

$$\begin{cases} (-\Delta)^s u - \lambda (u/|x|^{2s}) = u^{r-1} + \delta g(u), & \text{in } \Omega, \\ u(x) > 0, & \text{in } \Omega, \\ u(x) = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad \text{where } \Omega \subset \mathbb{R}^N \text{ is a bounded Lipschitz domain with } 0 \in \Omega, (-\Delta)^s \text{ is a fractional}$$

Laplace operator, $s \in (0, 1)$, $N > 2s$, δ is a positive number, $2 < r < r(\lambda, s) \equiv (N + 2s - 2\alpha_\lambda/N - 2s - 2\alpha_\lambda) + 1$, $\alpha_\lambda \in (0, ((N - 2s)/2))$ is a parameter depending on λ , $0 < \lambda < \Lambda_{N,s}$, and $\Lambda_{N,s} = 2^{2s} (\Gamma^2((N + 2s)/4)) / (\Gamma^2((N - 2s)/4))$ is the sharp constant of the Hardy-Sobolev inequality.

1. Introduction

In this paper, we consider the solvability of the following fractional elliptic problem:

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = u^{r-1} + \delta g(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain with $0 \in \Omega$, $s \in (0, 1)$, $N > 2s$, $2 < r < r(\lambda, s) \equiv ((N + 2s - 2\alpha_\lambda)/(N - 2s - 2\alpha_\lambda)) + 1$, $0 < \lambda < \Lambda_{N,s}$, and $\Lambda_{N,s} = 2^{2s} (\Gamma^2((N + 2s)/4)) / (\Gamma^2((N - 2s)/4))$ is the sharp constant of the Hardy-Sobolev inequality; the fractional Laplace operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u = C(N, s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad (2)$$

where P.V. stands for the Cauchy principal value and constant $C(N, s)$ is a constant.

Recently, much attention has been devoted to the study of fractional Laplacian equations. One of the reasons comes from the fact that the fractional Laplacian arises in various areas and different applications, such as phase transitions, finance, stratified materials, flame propagation, ultra-relativistic limits of quantum mechanics, and water waves. For more details, see [1–6] and references therein.

For fractional elliptic problems with the Hardy potential, Abdellaoui et al. [7] obtained the existence and summability of solutions to a class of nonlocal elliptic problem:

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = f(x, u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (3)$$

with $f \in L^m(\Omega)$ and $0 < \lambda < \Lambda_{N,s}$. They mainly considered the summability of solutions to (3) with $f(x, u) = f(x)$ and the existence and regularity of solutions to (3) with $f(x, u) = h(x)/u^\sigma$. Mi et al. [8] obtained the combined influence of the Hardy potential and lower order terms on the existence and regularity of solutions to the problem:

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} + u^p = f(x), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (4)$$

Barrios et al. [9] discussed the existence and multiplicity of solutions to the following fractional elliptic equation:

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} = u^p + \mu u^q, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (5)$$

where $0 < \lambda < \Lambda_{N,s}$, $0 < q < 1$,

$$1 < p < p(\lambda, s) = \frac{N + 2s - 2\alpha_\lambda}{N - 2s - 2\alpha_\lambda}, \quad (6)$$

and $\alpha_\lambda \in (0, (N - 2s)/2)$ is a parameter depending on λ . They shown that problem (5) has at least one solution if $1 < p < p(\lambda, s)$ and problem (5) has no solution if $p > p(\lambda, s)$.

Recently, Shang et al. [10] studied the existence and multiplicity of positive solutions to the following problem:

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \lambda g(x)u^p + K(x)u^{2^*_s-1}, \quad \text{in } \mathbb{R}^N, \quad (7)$$

where $s \in (0, 1)$, $N > 2s$, $0 < p < 2^*_s - 1$, and $0 < \mu < \Lambda_{N,s}$. Some other results of fractional elliptic equations with the Hardy potential, see [7, 9, 11–14] and references therein.

The local version of quasilinear problem related to problem (8) has been considered by Boccardo et al. [15]. They analyzed the existence of nontrivial solutions to the following problem:

$$\begin{cases} -\Delta_p u = |u|^{r-2}u + \lambda g(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (8)$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $1 < p < N$, $r > p$, $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, and there exist constants $c_1 > 0$ and $q \in (1, p)$ such that $g(s) \leq c_1 s^{q-1}$ for any $s > 0$.

Motivated by the above works, the aim of this paper is to study the existence of solutions to problem (1) by the method of subsuper solutions and taking into advantage the combined effect of concave and convex nonlinearity.

We make the following assumptions:

(F1)

$$2 < r < r(\lambda, s) \equiv \frac{N + 2s - 2\alpha_\lambda}{N - 2s - 2\alpha_\lambda} + 1, \quad (9)$$

where $\alpha_\lambda \in (0, ((N - 2s)/2))$ is a parameter depending on λ .

(F2) $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is Carathéodory function, and there exist constants $c_1 > 0$ and $q \in (1, 2)$, such that, for any $\sigma > 0$,

$$g(\sigma) \leq c_1 \sigma^{q-1}. \quad (10)$$

(F3) The function

$$u \rightarrow u^{r-1} + \delta g(u) \text{ is nondecreasing.} \quad (11)$$

(F4) Define

$$M_0 = \lim_{z \rightarrow 0^+} \frac{g(z)}{z}, \quad (12)$$

for all $\tau > 0$,

$$M_1(\tau) = \inf_{0 \leq z \leq (\tau\lambda_1)^{1/(r-2)}} \frac{g(z)}{z}, \quad (13)$$

where λ_1 is the first eigenvalue of $(-\Delta)^s$ in Ω .

Now, we state our main result.

Theorem 1. Suppose (F1) – (F4) hold. Then, there exists a positive constant δ_0 , such that, for all $\delta \in (\lambda_1/M_0, \delta_0]$, problem (1) has at least a nonnegative solution if $M_0 > (\lambda_1/\delta_0)$, where M_0 is defined by (12).

Remark 1. In order to prove the above theorem, we study directly to the pseudodifferential operator, without the harmonic extension to an extra dimension by transforming the nonlocal problem into a local problem due to Caffarelli and Silvestre [16].

Remark 2. To establish the upper bound for r (see (9)), we consider a radial solution $w = A|x|^{((2s-N)/2)+\beta}$ with constant $A > 0$ to the problem:

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} = w^{r-1}. \quad (14)$$

We obtain

$$\begin{aligned} A\gamma_\beta |x|^{-2s+((2s-N)/2)+\beta} - \lambda A |x|^{-2s+((2s-N)/2)+\beta} \\ = A^{r-1} |x|^{((2s-N)/2)+\beta(r-1)}, \end{aligned} \quad (15)$$

where

$$\gamma_\beta := \frac{\pi^{2s}\Gamma((N+2s+2\beta)/4)\Gamma((N+2s-2\beta)/4)}{\Gamma((N-2s-2\beta)/4)\Gamma((N-2s+2\beta)/4)}. \quad (16)$$

In order to have homogeneity, we have

$$\frac{2s-N}{2} + \beta = \frac{-2s}{r-2}. \quad (17)$$

Thus, we deduce that $\gamma_\beta - \lambda = A^{r-2}$. Since $A > 0$, we conclude that $\gamma_\beta - \lambda > 0$. Note that the map $\gamma: [0, ((N-2s)/2)) \mapsto (0, \Lambda_{N,s}]$ is decreasing about β , see [17, 18]. Hence, there is a unique element α_λ such that $\gamma_{\alpha_\lambda} = \lambda$. Thus, we have $\alpha_\lambda > \beta$, that is,

$$\alpha_\lambda > \frac{-2s}{r-2} + \frac{N-2s}{2}, \quad (18)$$

which implies that

$$r < \frac{N+2s-2\alpha_\lambda}{N-2s-2\alpha_\lambda} + 1 := r(\lambda, s). \quad (19)$$

Therefore, we can construct a supersolution to problem (1) for $r < r(\lambda, s)$, just modifying the w found above. Thus, $r(\lambda, s)$ is the threshold for the existence to problem (1).

Now, we consider the nonexistence of solution to problem (1).

Theorem 2. Suppose (F1) – (F4) hold. Then, problem (1) has no solution in H_0^s if for some $\tau > 1$, $M_1(\tau) > 0$, and $\delta > \lambda_1/M_1(\tau)$, where M_0 and M_1 are defined by (12) and (13), respectively.

The following two examples also appeared in [15].

Remark 3. An example of function $g(\sigma) \equiv \sigma^q$ with $0 < q < 1$, which satisfies conditions (10) and (11) for any $\delta \in (0, \delta_0]$, such that problem (1) has at least one positive solution. In this condition $M_0 = \infty$, by (52), we have $\eta^{r-q}c_1^{-1}C_1^{2-q}(C_1 - C_1^r) \geq \delta$. Define

$$\Phi(C_1) = \eta^{r-q}c_1^{-1}C_1^{2-q}(C_1 - C_1^r). \quad (20)$$

It is easy to see that

$$\frac{d}{dC_1}\Phi(C_1) = 0 \iff C_1 = C_{1,0} := \left(\frac{3-q}{2-q+r}\right)^{1/(r-1)}. \quad (21)$$

We have to prove that δ is smaller than the minimum of $\Phi(C_1)$. Therefore, we have

$$\delta_0 = \Phi(C_{1,0}) = \eta^{r-q}c_1^{-1}\left(\frac{3-q}{2-q+r}\right)^{1/(r-1)}\left[\left(\frac{3-q}{2-q+r}\right)^{(2-q)/(r-1)} - \left(\frac{3-q}{2-q+r}\right)^{(1-q+r)/(r-1)}\right]. \quad (22)$$

Moreover, by (10) and (13), we have for any $\tau > 1$,

$$M_1(\tau)\delta_0 \leq M_1(1)\delta_0 \leq c_1\delta_0 \inf_{0 < z \leq \lambda_1^{1/(r-2)}} |z|^{q-2} \equiv c_1\delta_0\lambda_1^{(q-2)/(r-2)}. \quad (23)$$

Thus, for any $\tau > 1$, $M_1(\tau) \leq c_1\lambda_1^{(q-2)/(r-2)}$. Hence, problem (1) has no solution at least $\delta > c_1\lambda_1^{(q-2)/(r-2)}$. Therefore, the result of the above theorem is more general than [9].

Remark 4. We consider the function

$$g(\sigma) = \frac{\sigma|\sigma|^{\theta-1}}{1 + \sigma|\sigma|^{\alpha-1}}, \quad (24)$$

for $0 < \alpha < \theta < 1$. We easily deduce that conditions (10) and (11) are fulfilled and $M_0 = \infty$. On the contrary,

$$M_1(\tau) = \inf_{0 \leq z \leq (\tau\lambda_1)^{1/(r-2)}} \frac{z^{\theta-1}}{1 + z^\alpha}. \quad (25)$$

If $\alpha > \theta - 1$, the function $z^{\theta-1}/(1 + z^\alpha)$ is monotone decreasing for $z \geq 0$. Then,

$$M_1(\tau) = \frac{(\tau\lambda_1)^{(\theta-1)/(r-2)}}{1 + (\tau\lambda_1)^{\alpha/(r-2)}}. \quad (26)$$

Similarly, in this case, problem (1) has no solution provided

$$\delta > \lambda_1^{(r-\theta-1)/(r-2)}(1 + \lambda_1^{\alpha/(r-2)}). \quad (27)$$

Remark 5. The function $\tau \mapsto M_1(\tau)$ is nonincreasing. Hence, if $\delta > \lambda_1/M_1(\tau_0)$, for some $\tau_0 > 1$, the results of Theorem 1 will be true for any $\tau > \tau_0$.

The paper is organized as follows. In Section 2, we present some definitions and preliminary tools, which will be used in the Proof of Theorems 1 and 2. The Proof of Theorems 1 and 2 are given in Section 3 and Section 4, respectively.

2. Preliminaries and Function Setting

In this section, we recall some known results for reader's convenience.

Denote the space

$$\mathcal{L}^s := \left\{ u: \mathbb{R}^N \longrightarrow \mathbb{R} \text{ is measurable: } \int_{\mathbb{R}^N} \frac{|u(x)|}{(1 + |x|^{N+2s})} dx < \infty \right\}, \quad (28)$$

equipped with the norm

$$\|u\|_{\mathcal{L}^s} := \int_{\mathbb{R}^N} \frac{|u(x)|}{(1 + |x|^{N+2s})} dx. \quad (29)$$

Let Ω be an open subset of $\mathbb{R}^N$. Given $u \in \mathcal{L}^s$ and φ in the Schwartz class, the distribution $(-\Delta)^s u$ in $\mathcal{D}'(\Omega)$ is defined as

$$\langle (-\Delta)^s u, \varphi \rangle = \int_{\mathbb{R}^N} u (-\Delta)^s \varphi dx, \quad \text{for any } \varphi \in C_c^\infty(\Omega). \quad (30)$$

We give some useful facts for the fractional Sobolev space.

Definition 1. Let $s \in (0, 1)$, and define the fractional Sobolev space:

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\xi|^s \hat{u} \in L^2(\mathbb{R}^N)\}. \quad (31)$$

We need to consider the space $X_0^s(\Omega)$, which is defined as

$$X_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N), u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}, \quad (32)$$

with the norm

$$\|u\|_{X_0^s(\Omega)} = \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{1/2}, \quad (33)$$

where $Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$. The pair $(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)})$ yields a Hilbert space (see Lemma 7 in [19]).

We have to use the classical Sobolev theorem.

Theorem 3 (see [20], Theorem 6.5). *Let $s \in (0, 1)$, then there exists a positive constant $S = S(N, s)$, such that, for any measurable and compactly supported function $u: \mathbb{R}^N \rightarrow \mathbb{R}$, we have*

$$\|u\|_{L^{2_s^*}(\mathbb{R}^N)}^2 \leq S \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad (34)$$

where 2_s^* is the so-called Sobolev critical exponent.

In this paper, we consider the existences of energy solution to problem (1) with the critical and subcritical cases.

Definition 2. We say that $u \in X_0^s(\Omega)$ is an energy solution to problem (1), if for any $\varphi \in X_0^s(\Omega)$,

$$\begin{aligned} & \int_{\Omega} |u|^{r-2} u \varphi dx < \infty, \\ & \int_{\Omega} \frac{u \varphi}{|x|^{2s}} dx < \infty, \\ & \frac{C(N, s)}{2} \int_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \\ & - \lambda \int_{\Omega} \frac{u \varphi}{|x|^{2s}} dx \\ & = \int_{\Omega} u^{r-1} \varphi dx + \delta \int_{\Omega} g(u) \varphi dx. \end{aligned} \quad (35)$$

We also need to consider the weak solution to problem (1).

Definition 3. We say that $u \in L^1(\Omega)$ is a weak solution to problem (1), if $u \geq 0$ a.e. in Ω , $u = 0$ in $\mathbb{R}^N \setminus \Omega$,

$$\int_{\Omega} \left(\lambda \frac{u}{|x|^{2s}} + u^{r-1} + \delta g(u) \right) \delta^s dx < \infty, \quad (36)$$

and for all $\varphi \in C^{2s+\beta}(\Omega) \cap C^s(\overline{\Omega})$, $\beta > 0$,

$$\int_{\Omega} u (-\Delta)^s \varphi dx = \int_{\Omega} \left(\lambda \frac{u}{|x|^{2s}} + u^{r-1} + \delta g(u) \right) \varphi dx, \quad (37)$$

where $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$ and $\delta(x) := \text{dist}(x, \partial\Omega)$.

Definition 4. If u satisfies

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} \geq u^{r-1} + \delta g(u), & \text{in } \Omega, \\ u \geq 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (38)$$

in the weak sense, we say that u is a supersolution to problem (1).

If u satisfies

$$\begin{cases} (-\Delta)^s u - \lambda \frac{u}{|x|^{2s}} \leq u^{r-1} + \delta g(u), & \text{in } \Omega, \\ u \leq 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (39)$$

in the weak sense, we say that u is a subsolution to problem (1).

Now, we recall the comparison lemma.

Lemma 1 (see [9]). *Let $u \in H^s(\mathbb{R}^N)$ and $v \in H^s(\mathbb{R}^N)$ be solutions, respectively, to*

$$\begin{cases} (-\Delta)^s u = f_1, & \text{in } \Omega, \\ u = g_1, & \text{in } \mathbb{R}^N \setminus \Omega, \\ (-\Delta)^s v = f_2, & \text{in } \Omega, \\ v = g_2, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (40)$$

Then, $u(x) \leq v(x)$ for all $x \in \mathbb{R}^N$ if $f_1 \leq f_2$ and $g_1 \leq g_2$.

For the supercritical case, we need a prior regularity result, see [9], Lemma 2.2.

Lemma 2. *Given $f \in L^1(\Omega, \delta^s(x) dx)$, where $\delta(x) = \text{dist}(x, \partial\Omega)$. There exists a unique weak solution $v(x) \in L^1(\Omega)$ to*

$$\begin{cases} (-\Delta)^s v = f, & \text{in } \Omega, \\ v = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (41)$$

in the sense that

$$\int_{\Omega} v (-\Delta)^s \vartheta = \int_{\Omega} f \vartheta, \quad (42)$$

for all $\vartheta \in C^2(\overline{\Omega})$ with $\vartheta = 0$ in $\mathbb{R}^N \setminus \Omega$.

Moreover, $\|v\|_{L^1(\Omega)} \leq C \|f\|_{L^1(\Omega, \delta^s(x) dx)}$, for some constant C independent of f . In addition, if $f \geq 0$

$$\begin{cases} (-\Delta)^s v = f, & \text{in } \Omega, \\ v \geq 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (43)$$

Then, $v \geq 0$ a.e. in Ω .

3. The Existence Result

We are now ready to prove Theorem 1 by employing the idea contained in [9, 15], whose proof will be split into several steps.

Proof of Theorem 1.

Step 1: subsolution to problem (1). We first consider the eigenvalue problem:

$$\begin{cases} (-\Delta)^s \varphi_1 = \lambda_1 \varphi_1, & \text{in } \Omega, \\ \varphi_1 = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (44)$$

Note that the eigenfunction $\varphi_1 \geq 0$ belongs to $X_0^s \cap L^\infty(\Omega)$.

Suppose $\delta M_0 > \lambda_1$, where M_0 is given in (12), by (F4), for all $\delta \in (\lambda_1/M_0, \delta_0]$, taking t small enough, we have

$$\frac{g(t\varphi_1)}{t\varphi_1} > \frac{\lambda_1}{\delta}. \quad (45)$$

Therefore, for $x \in \Omega$,

$$(-\Delta)^s(t\varphi_1) = \lambda_1 t\varphi_1 < \delta g(t\varphi_1) \leq \delta g(t\varphi_1) + (t\varphi_1)^{r-1} + \lambda \frac{t\varphi_1}{|x|^{2s}}. \quad (46)$$

Therefore,

$$\begin{cases} (-\Delta)^s(t\varphi_1) \leq \delta g(t\varphi_1) + (t\varphi_1)^{r-1} + \lambda \frac{t\varphi_1}{|x|^{2s}}, & \text{in } \Omega, \\ t\varphi_1 = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (47)$$

Thus, $\underline{u} := t\varphi_1$ is a subsolution to problem (1).

Next, we consider supersolution to problem (1) in the subcritical and supercritical case, respectively.

Step 2: supersolution for subcritical and critical case: $2 < r \leq 2_s^*$. We look for a supersolution of the form $w(x) := A|x|^{-\beta}$ with $A \geq 0$ and $\beta > 0$ as real parameters and verify

$$\beta < \frac{N-2s}{2}. \quad (48)$$

Since $r \leq 2_s^*$, we obtain

$$(r-1)\beta < \beta + 2s, \quad (49)$$

$$\beta r < N. \quad (50)$$

By (49), we deduce that

$$(-\Delta)^s w - \lambda \frac{w}{|x|^{2s}} \geq w^{r-1}, \quad \text{in } \Omega, \quad (51)$$

for the appropriate choice of A .

Let $\eta := \inf_{\Omega} w > 0$. Taking $\bar{u} = C_1 w$ with $0 < C_1 < 1$, which is a suitable constant such that, for δ small enough, and by (10) we have

$$\eta^{r-p} \geq \frac{\delta c_1}{C_1^{1-q}(C_1 - C_1^r)}, \quad (52)$$

where δ appears in (1).

By (52), we obtain

$$\begin{cases} (-\Delta)^s \bar{u} - \lambda \frac{\bar{u}}{|x|^{2s}} \geq \bar{u}^{r-1} + \delta g(\bar{u}), & \text{in } \Omega, \\ \bar{u} \geq 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (53)$$

Thus, we have concluded that $C_1 w$ is a supersolution to (1) for $2 < r \leq 2_s^*$. Moreover, by (48) and (50), we obtain that

$$\begin{aligned} \bar{u} &\in L^r(\Omega), \\ \frac{\bar{u}^2}{|x|^{2s}} &\in L^1(\Omega). \end{aligned} \quad (54)$$

Define $\{w_j\}$ in $L^1(\mathbb{R}^N)$ is the weak solution to

$$\begin{cases} (-\Delta)^s w_{j+1} = w_j^{r-1} + \lambda \frac{w_j}{|x|^{2s}} + \delta g(w_j), & \text{in } \Omega, \\ w_{j+1} = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (55)$$

for $j \geq 1$ and $w_0 := \underline{u}$. We now check that this definition makes sense and $\{w_j\}$ are monotone and satisfy

$$0 \leq \underline{u} \leq w_1 \leq w_2 \leq \dots \leq w_j \leq w_{j+1} \leq \dots \leq \bar{u} \text{ a.e. } \Omega. \quad (56)$$

For $\underline{u}$, there is nothing to prove. Suppose the result is true up to order j . Then,

$$\begin{cases} (-\Delta)^s w_{j+1} = w_j^{r-1} + \lambda \frac{w_j}{|x|^{2s}} + \delta g(w_j), \\ \leq \bar{u}^{r-1} + \lambda \frac{\bar{u}}{|x|^{2s}} + \delta g(\bar{u}), & \text{in } \Omega, \\ w_{j+1} = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (57)$$

So $\{w_j\}$ is well-defined by (54) and Lemma 2. By the induction hypothesis, for $x \in \Omega$,

$$\begin{aligned} & (-\Delta)^s(w_{j+1} - w_j) \\ &= (w_j^{r-1} - w_{j-1}^{r-1}) + \lambda \frac{(w_j - w_{j-1})}{|x|^{2s}} + \delta(g(w_j) - g(w_{j-1})) \\ &\geq 0, \end{aligned} \quad (58)$$

and $w_{j+1} - w_j = 0$, in $\mathbb{R}^N \setminus \Omega$. Then, by Lemma 1, we obtain $w_{j+1} \geq w_j$ in Ω .

Similarly, for any $x \in \Omega$,

$$\begin{aligned} & (-\Delta)^s(\bar{u} - w_j) \geq (\bar{u}^{r-1} - w_j^{r-1}) + \lambda \frac{(\bar{u} - w_j)}{|x|^{2s}} + \delta(g(\bar{u}) \\ & \quad - g(w_j)) \geq 0, \end{aligned} \quad (59)$$

and $\bar{u} - w_{j+1} \geq 0$ in $\mathbb{R}^N \setminus \Omega$. Then, $w_{j+1} \leq \bar{u}$ a.e. in Ω . We conclude that (56) holds.

We can define $u_\delta := \lim_{j \rightarrow \infty} w_j$ in $L^1(\Omega)$. Moreover, by (9), (54), and (56),

$$\begin{aligned} \|(-\Delta)^{s/2} w_j\|_{L^2(\mathbb{R}^N)} &= \lambda \int_{\Omega} \frac{w_j w_{j-1}}{|x|^{2s}} dx + \int_{\Omega} w_j w_{j-1}^{r-1} dx \\ &\quad + \delta \int_{\Omega} w_j g(w_{j-1}) dx \\ &\leq \lambda \int_{\Omega} \frac{\bar{u}^2}{|x|^{2s}} dx + \int_{\Omega} \bar{u}^r dx \\ &\quad + \delta c_1 \int_{\Omega} \bar{u}^q dx \leq C. \end{aligned} \quad (60)$$

Hence, up to a subsequence, we know that $w_j \rightarrow u_\delta$ in $X_0^s(\Omega)$. By monotony, the whole sequence weakly converges. Therefore, we can pass to the limit in (55) and conclude that $u_\delta \geq 0$ is a minimal energy solution of (1).

Step 3: supersolution for supercritical case: $2_s^* < r < r(\lambda, s)$. If $r < r(\lambda, s)$, where $r(\lambda, s)$ is given in

(9). For constant $A \geq 0$, there exists a radial function $v(x) := A|x|^{-2s/(r-2)}$ such that

$$(-\Delta)^s v - \lambda \frac{v}{|x|^{2s}} = v^{r-1}, \quad \text{in } \mathbb{R}^N. \quad (61)$$

Since $r > 2_s^* > ((2N - 2s)/(N - 2s))$, then

$$\begin{aligned} v &\in L_{\text{loc}}^{r-1}(\mathbb{R}^N), \\ \frac{v}{|x|^{2s}} &\in L_{\text{loc}}^1(\mathbb{R}^N). \end{aligned} \quad (62)$$

Taking $\bar{u} = C_1 v$, where the constant $C_1 > 0$ is given by (54), we obtain

$$\begin{cases} (-\Delta)^s \bar{u} - \lambda \frac{\bar{u}}{|x|^{2s}} \geq \bar{u}^{r-1} + \delta g(\bar{u}), & \text{in } \Omega, \\ \bar{u} > 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (63)$$

Moreover by (62), $\bar{u} \geq 0$ satisfies (36). By Lemma 2, define $\{w_j\}$ to be the weak solutions of (55). Moreover, similarly, by the induction hypothesis, we can conclude that

$$0 \leq \underline{u} \leq w_1 \leq w_2 \leq \dots \leq w_j \leq w_{j+1} \leq \dots \leq \bar{u} \text{ a.e. } \Omega. \quad (64)$$

Note that (F2) and (F4) hold, for all $\delta \in (\lambda_1/M_0, \delta_0]$, define w_1 as the solution of

$$\begin{cases} (-\Delta)^s w_1 = \underline{u}^{r-1} + \lambda \frac{\underline{u}}{|x|^{2s}} + \delta g(\underline{u}), & \text{in } \Omega, \\ w_1 > 0, & \text{in } \Omega, \\ w_1 = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (65)$$

Set

$$F(\underline{u}) = \underline{u}^{r-1} + \lambda \frac{\underline{u}}{|x|^{2s}} + \delta g(\underline{u}) = (t\varphi_1)^{r-1} + \lambda \frac{t\varphi_1}{|x|^{2s}} + \delta g(t\varphi_1). \quad (66)$$

Then, $w_1 \in W_0^{1,p} \cap L^\infty(\Omega)$. By (44), we obtain that

$$\begin{cases} (-\Delta)^s w_1 = F(\underline{u}) \geq (-\Delta)^s \underline{u}, & \text{in } \Omega, \\ w_1 = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (67)$$

We deduce from the comparison principle that $\underline{u} \leq w_1$ in Ω .

On the contrary, by (11), the function F is nondecreasing. Therefore,

$$\begin{cases} (-\Delta)^s \bar{u} \geq F(\bar{u}) \geq F(\underline{u}) = (-\Delta)^s w_1, & \text{in } \Omega, \\ \bar{u} \geq w_1, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (68)$$

By the comparison principle we deduce that $w_1 \leq \bar{u}$ in Ω .

In particular, for all $x \in \Omega$, $\{w_j\}$ is a nondecreasing sequence which is bounded. Therefore, $\{w_j\}$ monotone converges in $L^1(\mathbb{R}^N)$ to a weak nonnegative solution u_δ to (1) for $2_s^* < r < r(\lambda, s)$.

Therefore, for δ small enough, we have built a minimal solution in both subcritical and supercritical case. Let

$$M = \sup\{\delta > 0: \text{problem (1) has a solution}\}, \quad (69)$$

that is, we show that $M > 0$.

Step 4: $M < \infty$, for $2 < r < r(\lambda, s)$. We consider the following eigenvalue problem with the Hardy potential:

$$\begin{cases} (-\Delta)^s \psi_1 - \lambda \frac{\psi_1}{|x|^{2s}} = \lambda_1 \psi_1, & \text{in } \Omega, \\ \psi_1 = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (70)$$

Since $0 < \lambda < \Lambda_{N,s}$, problem (70) is well defined. Taking ψ_1 as a test function in problem (1), we obtain that

$$\begin{aligned} & \frac{C(N, s)}{2} \int_{\Omega} \frac{(\psi_1(x) - \psi_1(y))(u(x) - u(y))}{|x - y|^{N+2s}} dx dy - \lambda \int_{\Omega} \frac{\psi_1 u}{|x|^{2s}} dx \\ &= \int_{\Omega} u^{r-1} \psi_1 dx + \delta \int_{\Omega} g(u) \psi_1 dx. \end{aligned} \quad (71)$$

Since ψ_1 is a solution to (70), it follows that

$$\int_{\Omega} (u^{r-1} + \delta g(u)) \psi_1 dx = \lambda_1 \int_{\Omega} u \psi_1 dx. \quad (72)$$

If $2_s^* < r < r(\lambda, s)$. Taking φ_1 as a test function in (1), where $\varphi_1 \geq 0$ is solution to problem (44), we have

$$\begin{aligned} \int_{\Omega} u (-\Delta)^s \varphi_1 dx &= \int_{\Omega} \left(\lambda \frac{u}{|x|^{2s}} + u^{r-1} + \delta g(u) \right) \varphi_1 dx \\ &\geq \int_{\Omega} (u^{r-1} + \delta g(u)) \varphi_1 dx. \end{aligned} \quad (73)$$

Moreover, φ_1 is also a classical solution (see Remark 2.1 in [21]). From (72), we immediately deduce that

$$\lambda_1 \int_{\Omega} u \varphi_1 dx \geq \int_{\Omega} (u^{r-1} + \delta g(u)) \varphi_1 dx. \quad (74)$$

Since there exist structural positive constants b_0 and b_1 such that $|t|^{r-2}t + \delta g(t) > b_0 \delta^{b_1} t$, for any $t > 0$. From (70) and (73), we obtain that $b_0 \delta^{b_1} < \lambda_1$. This implies that $M < \infty$ for $r < r(\lambda, s)$.

We complete the Proof of Theorem 1. $\square$

4. Nonexistence Result

In this section, we consider the nonexistence of solution to problem (1) in H_0^s .

Proof of Theorem 2. Suppose that problem (1) has a solution $u \in H_0^s$ under the conditions of Theorem 2. Then, there exists a constant $\delta > 0$ such that $\delta \varphi_1 \leq u$ in Ω , where φ_1 is the first eigenfunction of $(-\Delta)^s$, that is, φ_1 satisfies (44).

Let $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$, where $\varepsilon > 0$ is a small constant. Denote $\psi = \delta \varphi_1$. Then, we have

$$\begin{cases} (-\Delta)^s \psi = \lambda_1 \psi \leq \mu \psi < (\lambda_1 + \varepsilon) u, & \text{in } \Omega, \\ \psi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (75)$$

Furthermore, for any $\tau > 1$, $\delta < \lambda_1 / M_1(\tau)$; then, for ε small enough, we deduce that

$$\begin{cases} (\lambda_1 + \varepsilon) u \leq u^{r-1} + \delta g(u) + \lambda \frac{u}{|x|^{2s}} \equiv (-\Delta)^s u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (76)$$

Thus, according to (75) and (76), we have

$$\begin{cases} (-\Delta)^s \psi \leq \mu \psi, & \text{in } \Omega, \\ (-\Delta)^s u \geq \mu u, & \text{in } \Omega, \\ u \geq \psi, & \text{in } \Omega, \\ u = \psi = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (77)$$

Hence, it is possible to construct the subsolution and supersolution to the problem:

$$\begin{cases} (-\Delta)^s u = \mu u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (78)$$

with $\mu \in (\lambda_1, \lambda_1 + \varepsilon)$. However, this is impossible. $\square$

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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