Research Article

Range-Kernel Orthogonality and Elementary Operators: The Nonsmoothness Case

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Received 3 December 2019; Accepted 25 May 2020; Published 24 June 2020

Academic Editor: Eulalia Martínez

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The characterization of the points in $C_1(\mathcal{H})$, the trace class operators, that are orthogonal to the range of elementary operators has been carried out for certain kinds of elementary operators by many authors in the smooth case. In this note, we study that the characterization is a problem in nonsmoothness case for general elementary operators, and we give a counter example to S. Mecheri and M. Bounkhel results.

1. Introduction

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$ and $A \in B(\mathcal{H})$ be compact operator, and let $s_1(A) \geq s_2(A) \geq \dots \geq 0$ denote the eigenvalues of $|A| = (A^*A)^{1/2}$ arranged in their decreasing order. $A \in C_1(\mathcal{H})$, the class trace, if

$$\|A\|_1 = \sum_{i=1}^{\infty} s_i(A) = \text{tr}|A| < \infty, \quad (1)$$

where tr denotes the trace functional [1].

We recall the definition of Birkhoff–James’s orthogonality in Banach spaces [2, 3].

Definition 1. If $\mathcal{A}$ is a complex Banach space, then for any elements $a, b \in \mathcal{A}$, we say that a is orthogonal to b , noted by $a \perp b$, iff for all $\lambda, \beta \in \mathbb{C}$ there holds

$$\|\lambda b + \beta a\| \geq \|\beta a\|. \quad (2)$$

If M and N are linear subspaces in $\mathcal{A}$, we say that M is orthogonal to N , noted by $M \perp N$, if $\|a + b\| \geq \|a\|$ for all $a \in M$ and all $b \in N$. If $M = \text{span}\{a\}$, we simply write $a \perp N$.

- (i) The orthogonality in this sense is not symmetric
- (ii) If $\mathcal{A}$ is a Hilbert space with its inner product $\langle \cdot, \cdot \rangle$, then it follows from (2) that $\langle a, b \rangle = 0$ which means that Birkhoff–James’s orthogonality generalizes the usual sense in Hilbert space

We also recall the definition of the range-kernel orthogonality for a pair of operators (E, T) on Banach spaces introduced by R. Harte [4].

Definition 2. If $E: \mathcal{A} \longrightarrow Y$ and $T: \mathcal{B} \longrightarrow \mathcal{C}$ are bounded linear operators between Banach spaces. T is called range-kernel orthogonal to E provided $\ker T$ is orthogonal to $\text{ran} E$ in the sense of the Definition 1, i.e.,

$$s \in \ker T \implies \|s + E(x)\| \geq \|s\|, \quad \text{for all } x \in \mathcal{A}. \quad (3)$$

If $T = E$, we shall call that E is orthogonal.

The elementary operator is an operator E defined on Banach $(\mathcal{A}, \mathcal{B})$ -bimodule $\mathcal{M}$ with its representation $E(x) = \sum_{i=1}^n a_i x b_i$, where $a = (a_i)_i \in \mathcal{A}^n$ and $b = (b_i)_i \in \mathcal{B}^n$ are n -tuples of algebra elements. The length of E is defined to be the smallest number of multiplication terms required for any representation $\sum_j a_j x b_j$ for E . The case of the elementary operator is restricted to the operator $\delta_{A,B} \in B[B(\mathcal{H})]$, which is well known, as generalized derivation induced by fixed

operators A and B in $B(\mathcal{H})$ and defined by $\delta_{A,B}(X) = AX - XB$.

Anderson [5] proved that if A and B are normal operators then

$$\text{for all } X, S \in B(\mathcal{H}): S \in \ker \delta_{A,B} \implies \|\delta_{A,B}(X) + S\| \geq \|S\|, \quad (4)$$

where $\ker \delta_{A,B}$ denotes the kernel of $\delta_{A,B}$.

This means that the kernel of $\delta_{A,B}$ is orthogonal to its range.

This result has been generalized in different directions, to nonnormal operators (see [6, 7]), to $C_p(\mathcal{H})$ ($1 \leq p < \infty$), and to some elementary operators (see [8, 9]).

The main purpose of this paper is to characterize the elements that are orthogonal to the range of arbitrary elementary operators defined on $C_1(\mathcal{H})$ in nonsmoothness case and give a counter example to the results of S. Mecheri and M. Bounkhel [10].

In addition to the notations and the definitions already introduced, we set, if $\mathcal{X}$ is a normed linear space over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we denote by $B(\mathcal{X})$ the space of all linear bounded operator on $\mathcal{X}$, the closure of the range of an operator $T \in B(\mathcal{X})$ will be denoted by $\overline{\text{ran}}(T)$, the restriction of T to an invariant subspace M will be denoted by $T|_M$, and the commutator $AB - BA$ of the operators A, B will be denoted by $[A, B]$.

2. Main Results

Let $\mathcal{X}$ be a normed linear space over the field $\mathbb{K}$ and $\mathcal{X}^\dagger$ its topological dual. Let D be the (multivalued) mapping defined from $\mathcal{X}$ to $\mathcal{X}^\dagger$ as

$$\forall x \in \mathcal{X}: D(x) = \{\varphi \in \mathcal{X}^\dagger: \varphi(x) = \|x\|^2; \|\varphi\| = \|x\|\}, \quad (5)$$

called the normalized duality mapping. Hahn–Banach's theorem ensures that there always exists at least one support functional (a support functional ψ at $x \in \mathcal{X}$ is a norm-one linear functional in $\mathcal{X}^\dagger$ such that $\psi(x) = \|x\|$) at each vector $x \in \mathcal{X}$ and therefore $D(x)$ is nonempty for every $x \in \mathcal{X}$. Moreover, it is well known that $D(x)$ is convex and weak*-compact subset of $\mathcal{X}^\dagger$. D is not linear in general but it is homogeneous, i.e., $\forall \alpha \in \mathbb{R}: D(\alpha x) = \alpha D(x)$.

Theorem 1. *Let K be a closed linear subset of $\mathcal{X}$ and $x \notin K$; then,*

$$x \perp K \iff \exists \tilde{\varphi} \in D(x): K \subseteq \ker \tilde{\varphi}. \quad (6)$$

Proof. Let $\varphi \in D(x)$ such that $\langle \varphi, y \rangle = 0$, for all $y \in K$. Then,

$$\begin{aligned} \varphi(x + y) &= \varphi(x) = \|x\|^2, \\ \|x\|^2 &\leq \|\varphi\| \|x + y\| \leq \|x\| \|x + y\|, \end{aligned} \quad (7)$$

that is, $\|x\| \leq \|x + y\|$, for all $y \in K$. Hence, $x \perp K$.

Conversely, let $x \notin K$ such that $x \perp K$. Then, for all $y \in K$, x and y are linearly independent vectors. Let L be the closed subspace spanned by K and $\{x\}$, $L = [K, \{x\}]$, and define the

function φ on L by $\varphi(\alpha x + \beta y) = \alpha \|x\|^2$ for all $y \in K$ and all $\alpha, \beta \in \mathbb{C}$. Clearly φ is linear (by the assumption that K is a linear subset of $\mathcal{X}$). To prove the continuity of φ , let $z \in L$, then $z = \alpha x + \beta y$ and $\varphi(z) = \alpha \|x\|^2$. By the definition of $\perp$ and the assumption that $x \perp K$, we derive that $\|z\| \geq \|\alpha x\|$. If $\alpha \neq 0$, we have

$$|\varphi(z)| = |\alpha| \|x\|^2 \cdot \frac{\|z\|}{\|\alpha x\|} \leq \frac{|\alpha| \|x\|^2}{\|\alpha x\|} \|z\| = \|x\| \|z\|. \quad (8)$$

If $\alpha = 0$, then $|\varphi(z)| = |\varphi(\beta y)| = 0$. Hence,

$$|\varphi(z)| \leq \|x\| \|z\|, \quad \text{for all } z \in L. \quad (9)$$

Therefore, φ is continuous on L and $\|\varphi\| = \|x\|$. By Hahn–Banach theorem, there is a continuous linear functional $\tilde{\varphi}$ on $\mathcal{X}$ such that $\tilde{\varphi}|_L = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$, where $\tilde{\varphi}|_L$ is the restriction of $\tilde{\varphi}$ on L . It follows by the definition of φ and $\tilde{\varphi}|_L = \varphi$, that $K \subseteq \ker \tilde{\varphi}$ and $\tilde{\varphi} \in D(x)$. $\square$

Notation 1. Let K be a nonempty subset of $\mathcal{X}$ and $T \in B(\mathcal{X})$, and we denote the duality adjoint of T by $T^\dagger$ and set $K^\perp = \{x \in \mathcal{X}: x \perp y; \forall y \in K\}$.

Remark 1. It is clear that if $\{x_n\}_n$ is a sequence in a subset K converging to y and $x \perp x_n$, for all n , then $x \perp y$. Hence, $x \perp K \implies x \perp \overline{K}$.

Corollary 1

- (i) $\forall x, y \in \mathcal{X}, x \perp y \iff \exists \varphi \in D(x): \varphi(y) = 0$
- (ii) $\forall x \in \mathcal{X}, \forall \varphi \in D(x): x \perp \ker \varphi$
- (iii) $s \perp \text{ran } T \iff \exists \varphi \in D(s): \varphi \in \ker T^\dagger$

Proof (i) and (ii) follow immediately from the previous Theorem 1.

(iii) By Theorem 1 and Remark 1, $s \perp \text{ran } T$ is equivalent to the existence of $\varphi \in D(s)$ such that $\text{ran } T \subseteq \ker \varphi$, that is, $\varphi(Tx) = (T^\dagger \varphi)x = 0, \forall x \in \mathcal{X}$. Hence,

$$s \perp \text{ran } T \iff \exists \varphi \in D(s): \varphi \in \ker T^\dagger. \quad (10)$$

It is well known that $C_1(\mathcal{H})$ not reflexive, even not smooth and its dual $C_1(\mathcal{H})^\dagger$ is isometrically isomorphic to $B(\mathcal{H})$. This isomorphism is given by

$$\phi: B(\mathcal{H}) \longrightarrow C_1(\mathcal{H})^\dagger, R \mapsto \phi_R: \phi_R(X) = \text{tr}(XR). \quad (11)$$

Then,

$$\varphi \in C_1(\mathcal{H})^\dagger \iff \exists R \in B(\mathcal{H}): \phi_R = \varphi,$$

$$\|\varphi\| = \|R\| \text{ and } \varphi(X) = \text{tr}RX, \quad \forall X \in C_1(\mathcal{H}). \quad (12)$$

Let ξ be the elementary operator defined by $\xi(X) = \sum_{i=1, \dots, n} A_i X B_i$ on $C_1(\mathcal{H})$, where $A = (A_1, A_2, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ are n -tuples in $B(\mathcal{H})^n$. By isomorphism (11), we can assert that the duality of ξ on $C_1(\mathcal{H})$ has the form $\xi^\dagger(X) = \sum_{i=1, \dots, n} B_i X A_i$. Indeed, let $X \in C_1(\mathcal{H})$ and $R \in B(\mathcal{H})$; then,

$$\begin{aligned}
\phi_R(\xi(X)) &= \text{tr} \left(\sum_{i=1,2,\dots,n} A_i X B_i R \right) \\
&= \text{tr} \left(X \sum_{i=1,2,\dots,n} B_i R A_i \right) \\
&= \text{tr} (X \xi^\dagger(R)) \\
&= \phi_{\xi^\dagger(R)}(X).
\end{aligned} \tag{13}$$

Let K be a closed linear subspace of a hilbert space $\mathcal{H}$, and P_k be the orthogonal projection on K . For a bounded operator T on $\mathcal{H}$, the compression of T to K is the restriction K to the operator $P_k T$. We denote by $\sigma_p(T)$ for the point spectrum (set of eigenvalues of T) and by σ_{pr} for the reduced point spectrum of T , i.e.,

$$\sigma_{pr}(T) = \{ \lambda : Tx = \lambda x \text{ and } T^*x = \bar{\lambda}x; x \in \mathcal{H} \setminus \{0\} \}. \quad (14) \quad \square$$

Proposition 1. Let $T \in B(\mathcal{H})$ and $\mathcal{F}$ be a two-sided ideal of $B(\mathcal{H})$.

If $\ker \delta_T \cap \mathcal{F} \subseteq \ker \tilde{\delta} \cap \mathcal{F}$, then

$$\ker \delta_T \cap \mathcal{F} \neq \{0\} \iff \sigma_p(T) \cap \sigma_p(T^*)^* = \sigma_{pr}(T) \text{ and } \sigma_{pr}(T) \neq \emptyset, \quad (15)$$

where $\delta_T = \delta_{T,T}$.

Proof. ($\implies$)

(1) Suppose that $0 \neq S \in \ker \delta_T \cap \mathcal{F}$, then $TS = ST$ and thus $T^*S = ST^*$ or $TS^* = S^*T$. Hence, $T|S| = |S|T$, where $|S|$ is a normal compact operator. Since $\sigma(|S|) \neq \{0\}$, then there is $0 \neq \lambda \in \sigma(|S|)$ such that the Riesz projector P_λ is of finite rank and $P_\lambda \in T^0$ (the bicommutant of T). It follows from $S \in \ker \delta_T \subseteq \ker \tilde{\delta}_T$ that $P_\lambda \in \ker \delta_T \subseteq \ker \tilde{\delta}_T$ and therefore the subspace $\text{ran } P_\lambda$ reduces T and $m \text{ran } P_\lambda < \infty$. Let

$$\mathcal{F}_\lambda = \{ P_\lambda S|_{\text{ran } P_\lambda} : S \in \mathcal{F} \}, \quad (16)$$

where $P_\lambda S|_{\text{ran } P_\lambda}$ is the compression of S on $\text{ran } P_\lambda$ so that

$$\ker \delta_{T|_{\text{ran } P_\lambda}} \cap \mathcal{F}_\lambda \subseteq \ker \tilde{\delta}_{T|_{\text{ran } P_\lambda}} \cap \mathcal{F}_\lambda. \quad (17)$$

Indeed, let us consider the decomposition $\mathcal{H} = \text{ran } P_\lambda \oplus (\text{ran } P_\lambda)^\perp$, then T can be written $T = T_1 \oplus T_2$, where $T_1 = T|_{\text{ran } P_\lambda}$.

Let $S_\lambda \in \ker \delta_{T|_{\text{ran } P_\lambda}} \cap \mathcal{F}_\lambda$, then $T_1 S_\lambda = S_\lambda T_1$. Choosing $S = S_\lambda \oplus 0$ implies $S \in \ker \delta_T \cap \mathcal{F} \subseteq \ker \tilde{\delta}_T \cap \mathcal{F}$. Thus, $TS^* = S^*T$ and therefore $T_1 S_\lambda^* = S_\lambda^* T_1$.

It follows from $\ker \delta_{T|_{\text{ran } P_\lambda}} \cap \mathcal{F}_\lambda \subseteq \ker \tilde{\delta}_{T|_{\text{ran } P_\lambda}} \cap \mathcal{F}_\lambda$ that $T|_{\text{ran } P_\lambda}$ is a normal operator and therefore

$$\sigma_{pr}(T|_{\text{ran } P_\lambda}) = \sigma_p(T|_{\text{ran } P_\lambda}) = \sigma(T|_{\text{ran } P_\lambda}), \quad (18)$$

which implies that $\sigma_{pr}(T) \subseteq \sigma_p(T) \cap \sigma_p(T^*)^*$ and $\sigma_{pr}(T)$ is not an emptyset.

(2) Let $\lambda \in \sigma_p(T) \cap \sigma_p(T^*)^*$, then there exist $x, y \in \mathcal{H} \setminus \{0\}$ such that $Tx = \lambda x$ and $T^*y = \bar{\lambda}y$. Hence, there is S such that

$$S = x \otimes y \in \ker \delta_T \cap \mathcal{F} \subseteq \ker \tilde{T} \cap \mathcal{F}, \quad (19)$$

Applying Fuglede–Putnam property [6], we get that $\text{ran } S$ reduces T and $T|_{\text{ran } S}$ is a normal operator. Since $x \in \ker (T|_{\text{ran } S} - \lambda)$, we derive that $T^*x = \bar{\lambda}x$ and therefore $\lambda \in \sigma_{pr}(T)$. The inclusion is proved.

Let us prove the reverse implication ($\impliedby$).

Let $\lambda \in \sigma_p(T) \cap \sigma_p(T^*)^*$, then there exist $x, y \in \mathcal{H} \setminus \{0\}$ such that $Tx = \lambda x$ and $T^*y = \bar{\lambda}y$. Setting $S = x \otimes y$, then $S \in \ker \delta_T \cap \mathcal{F}$. $\square$

Corollary 2. Let A be a normal operator in $B(\mathcal{H})$ with $\sigma_{pr}(A) = \emptyset$ and $S \notin \ker \delta_A$ be a positive operator in $C_1(\mathcal{H})$, then

$$\|\delta_A(X) + S\|_1 \geq \|S\|_1, \quad \text{for all } X \in C_1(\mathcal{H}). \quad (20)$$

Proof. Since S is positive, then $S = |S|$, hence $\text{tr}\|S\|_1 S = \text{tr}\|S\|_1 |S| = \|S\|_1^2$ so that there is $T = \|S\|_1 I \in \ker \delta_A$, and therefore by Corollary 1, isomorphisms (11) and (13), we get S is orthogonal to $\text{ran } \delta_A$ i.e.,

$$\|\delta_A(X) + S\|_1 \geq \|S\|_1, \quad \text{for all } X \in C_1(\mathcal{H}). \quad (21)$$

However, δ_A is a normal derivation satisfying $\ker \delta_A \subseteq \ker \tilde{\delta}_A$ and by the previous proposition, $\ker \delta \cap C_1(\mathcal{H}) = \{0\}$, and this means that $S \notin \ker \delta_A$.

We recall that the characterization of the points in $C_1(\mathcal{H})$ that are orthogonal to the range of the elementary operators has been carried out by certain authors for smooth case and for certain kind of elementary operators [9, 11]. In nonsmoothness case, S. Mecheri and M. Bounkhel [10] have obtained the following result. $\square$

Proposition 2 (see [10]). Let A, B be normal operators in $B(\mathcal{H})$ and let $S = U|S| \in C_1(\mathcal{H})$. Then, $S \in \ker \delta_{A,B}$ if and only if

$$\|d_{A,B}(X) + S\|_1 \geq \|S\|_1, \quad \forall X \in C_1(\mathcal{H}), \quad (22)$$

where $d_{A,B} = \Delta_{A,B}$ or $d_{A,B} = \delta_{A,B}$ for $\Delta_{A,B}(X) = AXB - X$ and $\delta_{A,B}(X) = AX - XB$.

Remark 2

(1) The previous result shows that $(\text{ran } d_{A,B})^\perp \cap C_1(cH) = \ker d_{A,B} \cap C_1(\mathcal{H})$. In what follows, we show that this is not true, i.e., $(\text{ran } \delta_A)^\perp \cap$

$C_1(\mathcal{H}) \not\subseteq \ker \delta_A \cap C_1(cH)$. Indeed, let $\mathcal{H} = l^2(\mathbb{Z})$, $A \in B(H)$: $Ae_{n+1} = e_n$; $n \in \mathbb{Z}$, and $S = e_1 \otimes e_1$, where $\{e_n; n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}$. Hence, A is a unitary operator (bilateral shift) with $\sigma_{\text{pr}}(A) = \emptyset$ and $S \in C_1(\mathcal{H})$. By simple calculus, we obtain

$$\begin{aligned} A(e_1 \otimes e_1) - (e_1 \otimes e_1)A &= e_0 \otimes e_1 - e_1 \otimes A^*e_1 \\ &= e_0 \otimes e_1 - e_1 \otimes e_2 \neq 0. \end{aligned} \quad (23)$$

Thus, $S \notin \ker \delta_A$.

On the contrary, $\phi_I(S) = \text{tr}(e_1 \otimes e_1) = 1$ and $\|\phi_I\| = \|I\| = 1$, where I is the identity operator on $\mathcal{H}$. Thus, $\phi_I \in D(S)$. Since $I \in \ker \delta_A$, we get by (13), $\phi_I \in \ker \delta_A^\dagger$ so that it follows from Corollary 1 that $S \perp \text{ran } \delta_A$.

- (2) Furthermore, Corollary 2 provides a class of operators as counterexample to the orthogonal decomposition $C_1(\mathcal{H}) = \text{ran } \delta_A \oplus \ker \delta_A$ so that the problem of characterization of operators in $C_1(\mathcal{H})$ that are orthogonal to the range of elementary operators is still open.

In this note, as an application of the previous section, we shall study this problem in different directions for arbitrary elementary operators.

Theorem 2. Let ξ be the elementary operator defined on $C_1(\mathcal{H})$ by

$\xi(X) = \sum_{i=1, \dots, n} A_i X B_i$, where $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ are n -tuples of operators in $B(\mathcal{H})^n$ and let $S \in C_1(\mathcal{H})$ with the condition $\ker \xi = \ker \tilde{\xi}$. Let $R = U|R| = S$ or S^* , then

- (1) If $\ker R$ is invariant under R^* , A_i and B_i^* for all $i = 1, \dots, n$, then there holds

$$\forall X \in C_1(\mathcal{H}): \|\xi(X) + R\|_1 \geq \|R\|_1 \implies \xi(U) = 0. \quad (24)$$

If $\ker R$ reduces R , A_i and B_i^* for all $i = 1, \dots, n$, then there holds

$$\xi(U) = 0 \iff \forall X \in C_1(\mathcal{H}): \|\xi(X) + R\|_1 \geq \|R\|_1. \quad (25)$$

- (2) If $\ker R$ is invariant under R^* , A_i and $[\text{ran } R], B_i] = 0$ for all $i = 1, \dots, n$, then there holds

$$\forall X \in C_1(\mathcal{H}): \|\xi(X) + R\|_1 \geq \|R\|_1 \implies \xi(R) = 0. \quad (26)$$

If $\ker R$ reduces R , A_i and $[\text{ran } R], B_i] = 0$ for all $i = 1, \dots, n$, then there holds

$$\xi(R) = 0 \iff \forall X \in C_1(\mathcal{H}): \|\xi(X) + R\|_1 \geq \|R\|_1. \quad (27)$$

Proof

- (1) We prove only the case when $R = S = U|S|$, the case $R = S^*$ is similar to the previous one.

If $\ker S$ is invariant under S^* , A_i and B_i^* for all $n = 1, \dots, n$, then with respect to the decomposition $\mathcal{H} = (\ker S)^\perp \oplus \ker S$, we get the following $S = S_1 \oplus 0$, $U = U_1 \oplus 0$ and

$$\begin{aligned} A_i &= \begin{bmatrix} A_i^1 & A_i^2 \\ 0 & A_i^3 \end{bmatrix}, \\ B_i &= \begin{bmatrix} B_i^1 & 0 \\ B_i^2 & B_i^3 \end{bmatrix}, \\ 1 &\leq i \leq n. \end{aligned} \quad (28)$$

It is clear that

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} \in C_1(\mathcal{H}) \implies X_1 \in C_1((\ker S)^\perp) \quad (29)$$

and $\xi(X) = \xi_1(X_1) \oplus 0$, where $\xi_1(X_1) = \sum_{i=1, \dots, n} A_i^1 X_1 B_i^1$. Hence,

$$\begin{aligned} \forall X \in C_1(\mathcal{H}): \|\xi(X) + S\|_1 \geq \|S\|_1 &\implies \|\xi_1(X_1) + S_1\|_1 \\ &\geq \|S_1\|_1, \forall X_1 \in C_1((\ker S)^\perp). \end{aligned} \quad (30)$$

Since S_1 is injective and $S_1 = U_1|S_1|$, then S_1 is a smooth point in $C_1((\ker S)^\perp)$. So, it follows by Corollary 1 and the uniqueness of T that $T = \|S_1\|_1 U_1^*$ and then by isomorphism (11), $\sum_{i=1, \dots, n} B_i^1 U_1^* A_i^1 = 0 = \sum_{i=1, \dots, n} A_i^1 U_1^* B_i^1$.

From the hypothesis $\ker \xi = \ker \tilde{\xi}$, we derive that $\ker \xi_1 = \ker \tilde{\xi}_1$ and then $\sum_{i=1, \dots, n} A_i^1 U_1 B_i^1 = 0$; therefore, $\xi(U) = \sum_{i=1, \dots, n} A_i U B_i = 0$ which means that $U \in \ker \xi$.

For the reverse implication, if $\xi(U) = 0$, then $\xi_1(U_1) = 0$, and since $\ker \xi_1 = \ker \tilde{\xi}_1$, we get $\xi_1(U_1) = 0 = \xi_1^\dagger(U_1^*)$. We derive from Corollary 1 that

$$\|\xi_1(X_1) + S_1\|_1 \geq \|S_1\|_1; \quad \forall X_1 \in C_1((\ker S)^\perp), \quad (31)$$

and since $\ker S$ reduces both A_i and B_i for all $i = 1, \dots, n$, we obtain, for $X = (X_k)_{k=1, \dots, 4} \in C_1(\mathcal{H})$ and $A_i = A_i^1 \oplus A_i^2$, $B_i = B_i^1 \oplus B_i^2$, that

$$\begin{aligned} \|\xi(X) + S\|_1 &= \left\| \begin{bmatrix} \xi_1(X_1) + S_1 & * \\ * & * \end{bmatrix} \right\|_1 \geq \|\xi_1(X_1) + S_1\|_1 \\ &\geq \|S_1\|_1 = \|S\|_1. \end{aligned} \quad (32)$$

- (2) If $\ker S$ is invariant under S^* , A_i and $[[S], B_i] = 0$ for all $i = 1, \dots, n$, then $\ker S$ reduces B , following the same steps as in the previous case, we get $\sum_{i=1, \dots, n} A_i U B_i = 0$. Hence, $[[S], B_i] = 0$ for all $i = 1, \dots, n$ implying $\xi(S) = 0$. Since $|S_1|$ is injective, then $\xi(U_1) = 0$. We argue similarly as in the previous case to finish the proof. $\square$

Remark 3. If S is a smooth point in $C_1(\mathcal{H})$, then $\ker S$ or $\ker S^*$ are trivial, then relations (11) and (13) hold for any elementary ξ satisfying $\ker \xi = \ker \tilde{\xi}$, and relation (13) hold under the conditions $[[S], B_i] = 0$ for all $i = 1, \dots, n$ for any operator $\xi(X) = \sum_{i=1, \dots, n} A_i X B_i$ with $\ker \xi = \ker \tilde{\xi}$.

Lemma 1. Let $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n)$ with A_i and B_i be operators in $B(\mathcal{H})$ such that $\sum_i A_i A_i^* \leq 1, \sum_i A_i^* A_i \leq 1, \sum_i B_i B_i^* \leq 1$, and $\sum_i B_i B_i^* \leq 1$. Let ξ be the elementary operator defined on $C_{p: 1 \leq p < \infty}(\mathcal{H})$ by

$$\xi(X) = \sum_{i=1, \dots, n} A_i X B_i - X. \quad (33)$$

Then,

- (i) $\ker \xi = \ker \tilde{\xi} \implies \ker \xi^\dagger = \ker \tilde{\xi}^\dagger$
(ii) If $\xi(S) = 0 = \tilde{\xi}(S)$ for some compact operator S , then $[[S], B_i] = 0$ for all $1 \leq i \leq n$

Proof

- (i) We have $\xi^\dagger(S) = 0 \iff \tilde{\xi}(S^*)$. Since $\ker \xi = \ker \tilde{\xi}$, it follows that

$$\tilde{\xi}(S) = 0 \iff \xi(S^*) = 0 \iff \tilde{\xi}^\dagger(S) = 0. \quad (34)$$

- (ii) See [11].

As an application of Theorem 2 and by using Lemma 1, we obtain the following result which generalizes that given by B. P. Duggal in the smooth case [9]. $\square$

Corollary 3. Let $\xi(X) = \sum_{i=1, \dots, n} A_i X B_i - X$ such that

$$\begin{aligned} \sum_i A_i A_i^* &\leq 1, \\ \sum_i A_i^* A_i &\leq 1, \\ \sum_i B_i^* B_i &\leq 1, \\ \sum_i B_i B_i^* &\leq 1, \end{aligned} \quad (35)$$

with $\ker \xi = \ker \tilde{\xi}$ and $\ker \xi_B = \ker \tilde{\xi}_B$. For all $S \in C_1(\mathcal{H})$, the following assertions hold:

- (i) If $\ker S$ reduces S , then

$$S \perp \text{ran } \xi \implies S \in \ker \xi. \quad (36)$$

- (ii) If $\ker S$ reduces S and $S \in \ker \xi_B$, then

$$S \perp \text{ran } \xi \implies S \in \ker \xi. \quad (37)$$

In what follows, for an elementary operator ξ defined on $C_1(\mathcal{H})$ by

$$\xi(X) = \sum_{i=1, \dots, n} A_i X B_i, \quad (38)$$

where $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ are n -tuples of operators in $B(\mathcal{H})^n$, we shall prove that there are operators, not necessary in the kernel, which are orthogonal to the range without the condition $\ker \xi \subseteq \ker \tilde{\xi}$.

Recall that if $\mathcal{X}$ is a Banach space, then the function $\psi_{x,y}$ defined by

$$\psi_{x,y}(t) = \frac{\|x + ty\| - \|x\|}{t} \quad (39)$$

is increasing on $(0, \infty)$, convex for all $x, y \in \mathcal{X}$, and $\lim_{t \rightarrow 0^+} \psi_{x,y}(t)$ exists for all $x, y \in \mathcal{X}$.

So, from the properties of $\psi_{x,y}$ and the fact that $\lim_{t \rightarrow 0^+} \psi_{x,y}(t) = \lim_{t \rightarrow 0^-} \psi_{x,-y}(t)$, it follows that

$$x \perp_{B-J} y \iff \lim_{t \rightarrow 0^-} \psi_{x,y}(t) \leq 0 \leq \lim_{t \rightarrow 0^+} \psi_{x,y}(t), \quad \forall t > 0. \quad (40)$$

Indeed, if $x \perp_{B-J} y$, then $\|x + \lambda y\| - \|x\| \geq 0$, for all $\lambda \in \mathbb{C}$. Therefore, by setting $\lambda = te^{i\theta}$, where $t = |\lambda|$ and $\theta \in \mathbb{R}$ and since $\psi_{x,y}$ is increasing on $(0, \infty)$, then $0 \leq \lim_{t \rightarrow 0^+} \psi_{x,y}(t)$. The converse holds from the convexity of $\psi_{x,y}$.

Lemma 2. Let $S, T \in C_p(1 \leq p < \infty)$ and $\mathcal{H}$ be a closed subspace of $\mathcal{H}$. If $\mathcal{H}$ reduces S , then

$$S|_{\mathcal{H}} \perp_{B-J} T|_{11} \text{ and } S|_{\mathcal{H}^\perp} \perp_{B-J} T|_{22} \implies S \perp_{B-J} T, \quad (41)$$

where $T|_{11}$ and $T|_{22}$ are the compression of T to the subspaces $\mathcal{H}$ and $\mathcal{H}^\perp$, respectively.

Proof. With respect to the decomposition $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}^\perp$, we can write $S = S|_{\mathcal{H}} \oplus S|_{\mathcal{H}^\perp}$ and $T = (T_{ij})_{i,j=1,2}$, then

$$S + \lambda T = \begin{bmatrix} S|_{\mathcal{H}} + \lambda T|_{11} & \lambda T|_{12} \\ \lambda T|_{21} & S|_{\mathcal{H}^\perp} + \lambda T|_{22} \end{bmatrix}, \quad \forall \lambda \in \mathbb{C}. \quad (42)$$

Since the p -norm of an operator matrix always dominates the p -norm of its diagonal parts [12], then

$$\begin{aligned} \|S + \lambda T\|_p &\geq \|S|_{\mathcal{H}} + \lambda T|_{11}\|_p + \|S|_{\mathcal{H}^\perp} + \lambda T|_{22}\|_p \\ \|S\|_p &= \|S|_{\mathcal{H}}\|_p + \|S|_{\mathcal{H}^\perp}\|_p. \end{aligned} \quad (43)$$

Setting $\lambda = te^{i\theta}$, where $t = |\lambda|$ and $\theta \in \mathbb{R}$, then

$$\begin{aligned} \|S + te^{i\theta} T\|_p - \|S\|_p &\leq \|S|_{\mathcal{H}} + te^{i\theta} T|_{11}\|_p - \|S|_{\mathcal{H}}\|_p \\ &\quad + \|S|_{\mathcal{H}^\perp} + te^{i\theta} T|_{22}\|_p - \|S|_{\mathcal{H}^\perp}\|_p. \end{aligned} \quad (44)$$

Thus,

$$\lim_{t \rightarrow 0^+} \psi_{S, e^{i\theta} T}(t) \leq \lim_{t \rightarrow 0^+} \psi_S|_{\mathcal{H}, e^{i\theta} T_{11}}(t) + \lim_{t \rightarrow 0^+} \psi_S|_{\mathcal{H}^\perp, e^{i\theta} T_{22}}(t). \quad (45)$$

So, the result follows from (40). $\square$

Proposition 3. Let ξ be the elementary operator defined on $C_1(\mathcal{H})$ by $\xi = L_A R_B$, where $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n)$ are n -tuples of operators in $B(\mathcal{H})$. If $S = U|S|$ such that $\ker S$ is invariant under S^* and A_i for all $i = 1, \dots, n$ (resp. $\ker S^*$ is invariant under S and A_i for all $i = 1, \dots, n$), then

$$\begin{aligned} L_{B_{11}} R_{A_{11}}(U_1^*) = 0 &\implies S \perp \text{ran } \xi \\ (\text{resp. } L_{B_{11}} R_{A_{11}}(U_1) = 0 &\implies S \perp \text{ran } \xi), \end{aligned} \quad (46)$$

such that $A_{11} = (A_{11}^1, A_{11}^2, \dots, A_{11}^n); A_k = (A_{i,j})_{i,j=1,2}, k = 1, \dots, n$ (with the same notations for B), where A_{11}^k (resp. B_{11}^k) is the compression of A_k (resp. B_k) to $(\ker S)^\perp$ and $S_1 = U_1|S_1| = S|_{(\ker S)^\perp}$.

Proof. Let $\mathcal{H} = (\ker S)^\perp \oplus \ker S$; since $\ker S$ is invariant under S^* and A_i for all $i = 1, \dots, n$, we obtain

$$S = S_1 \oplus 0, \quad A_k = \begin{bmatrix} A_{11}^k & 0 \\ A_{21}^k & A_{22}^k \end{bmatrix}, \quad (47)$$

$$B_k = (B_{i,j}^k)_{i,j=1,2},$$

for all $k = 1, \dots, n$. Let $X = (X_{i,j})_{i,j=1,2} \in C_1(\mathcal{H})$, by simple calculation, we obtain that the compression of $\xi(X)$ to the subspace $(\ker S)^\perp$ is

$$\xi_{11}(X_{11}) = \sum_{k=1}^n A_{11}^k X_{11} B_{11}^k = L_{A_{11}} R_{B_{11}}(X_{11}), \quad (48)$$

so that it follows from the previous lemma that if $S_1 \perp \text{ran } \xi_{11}$, then $S \perp \text{ran } \xi$. Since S_1 is a smooth point (injective operator) in $C_1((\ker S)^\perp)$ and $U_1|S_1|$ its polar decomposition, where U_1 is an isometry in $B((\ker S)^\perp)$. Applying Corollary 3 (ii), we get that if $L_{B_{11}} R_{A_{11}}(U_1^*) = 0$, then $S \perp \text{ran } \xi$. $\square$

Example 1. Let $\mathcal{H} = l_2(\mathbb{Z})$ and $Ae_{n+1} = e_n, n \in \mathbb{Z}$ and

$$Se_{n+1} = \begin{cases} \frac{1}{2^{n+1}} e_n, & \text{if } n \geq 0, \\ 0, & \text{if } n \leq -1. \end{cases} \quad (49)$$

We can assert that

$$S^* e_n = \begin{cases} \frac{1}{2^{n+1}} e_{n+1}, & \text{if } n \geq 0, \\ 0 & \text{if } n \leq -1. \end{cases} \quad (50)$$

A is unitary with $\sigma_{\text{pr}}(A) = \emptyset$ (see Remark 2), then $\ker \delta_A \cap C_1(\mathcal{H}) = \{0\}$. It is clear that $\ker S$ is spanned by the set $\{\dots, e_{-2}, e_{-1}, e_0\}$ and $\ker S^*$ is spanned by the set $\{\dots, e_{-2}, e_{-1}\}$, then $\ker S^* \subset \ker S$ so that $\ker S^*$ reduces S^* , also $A(\ker S^*) \subseteq \ker S^*$. Hence, $S = S|_{(\ker S^*)^\perp} \oplus 0$. If $S = U|S|$ and $U_1 = U|_{(\ker S^*)^\perp}$, then

$$\begin{aligned} Ue_n &= \begin{cases} e_{n-1}, & \text{if } n \geq 1, \\ 0 & \text{if } n \leq 0, \end{cases} \\ U_1 e_n &= \begin{cases} e_{n-1}, & \text{if } n \geq 1, \\ 0 & \text{if } n = 0. \end{cases} \end{aligned} \quad (51)$$

If A_1 is the compression of A to $(\ker S^*)^\perp$, then $A_1 e_n = \begin{cases} e_{n-1}, & \text{if } n \geq 1, \\ 0 & \text{if } n \leq 0. \end{cases}$

We have $A_1 U_1 e_n = U_1 A_1 e_n = e_{n-2}$ for all $n \geq 2$ and $A_1 U_1 e_n = U_1 A_1 e_n = 0$ for $n = 0, 1$. Then, $U_1 \ker \delta_A$, and therefore by Proposition 3, we get $S \perp \text{ran } \delta_A$. We have also $S^* e_0 = 1/2 e_1 \notin \ker S$ which means that $\ker S$ is not invariant under S^* .

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors thank the anonymous referees for their helpful remarks and extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through General Research Project grant number (G.R.P- 93-41).

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