Research Article

Dynamics and Complexity Analysis of Fractional-Order Chaotic Systems with Line Equilibrium Based on Adomian Decomposition

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In this paper, the Adomian decomposition method (ADM) is applied to solve the fractional-order system with line equilibrium. The dynamics of the system is analyzed by means of the Lyapunov exponent spectrum, bifurcations, chaotic attractor, and largest Lyapunov exponent diagram. At the same time, through the Lyapunov exponent spectrum and bifurcation graph of the system under the change of the initial value, the influence of fractional order q on the system state can be observed. That is, integer-order systems do not have the phenomenon of attractors coexistence, while fractional-order systems have it.

1. Introduction

Three hundred years ago, fractional-order calculation was proposed as a classical mathematical problem. Fractional calculation has no practical background and has not been applied in engineering. Therefore, researchers and scientists are not interested in fractional research. Recently, it has been found that many engineering and physical systems, electronic systems, etc. exhibit their own fractional-order characteristics, and the fractional order can more accurately reflect natural phenomena, such as material memory and damping characteristics [1–4]. In recent years, with the deepening and perfection of chaotic theory research, fractional-order chaotic systems have become a hot topic in the field. In particular, the complexity of fractional-order chaotic systems is not only related to the parameters of the system itself, but also related to the fractional order of the system. Many scholars have proposed several fractional-order chaotic systems based on integer-order chaotic systems: fractional-order Lorenz system [5], fractional-order

Lorenz hyperchaotic system [6], four-wing fractional-order chaotic system [7], and so on [8, 9].

In the numerical calculation of fractional chaotic systems, namely, discretization of fractional chaotic systems, many scholars have made some achievements based on frequency-domain method (FDM) [10], Adomian decomposition method (ADM) [5–7, 9, 11], and Adams–Bashforth–Moulton (ABM) algorithm [12, 13]. Among them, FDM uses high-dimensional system close to fractional-order system through Laplace change, but the error is relatively large [14]. ABM is the most commonly used method, but its operation speed is slow. ADM scheme, on the other hand, has more accuracy and requires less computational resources compared with ABM algorithm [5–7, 9]. Reference [15] shows that fractional-order chaotic systems show more complex chaotic behaviors than integer-order chaotic systems, which can be seen from complexity, Lyapunov index, bifurcation diagram, and other aspects. When the system order is smaller, the system is more complicated [9, 16].

On the other hand, hot research on chaotic systems mainly focuses on chaos control [17–20], Lyapunov exponent calculation and analysis [21, 22], and doubling and growth of attractors [23, 24], and chaotic systems with hidden attractors are also hot research topics, because such systems are extremely prone to multistable phenomenon, which is a common phenomenon in nature. Hidden attractors originate from several types of dynamical systems; the dynamical systems with one stable equilibrium [25], a line or plane equilibrium [26, 27], or no equilibrium [28] can all have hidden attractors. Hidden attractors are even found in some special chaotic systems, which have both unstable and stable equilibria. However, the fractional-order chaotic system with hidden attractor, which can be easily obtained by combining the hidden attractor with fractional-order systems, is rarely studied [29]. The system has high complexity, so it plays an important role in the field of communication encryption.

In this paper, we propose a new fractional-order chaotic system with line equilibrium. Discretization of the proposed system is performed using Adomian decomposition scheme. The rest of the paper is organized as follows. In Section 2, the equilibrium point and dynamic analysis of the integer-order chaotic equilibrium point system are presented. In Section 3, the fractional-order chaotic system is numerically calculated and simulated by Adomian decomposition method. In Section 4, the bifurcation diagram, phase diagrams, and Lyapunov exponents spectrum are employed to analyze the dynamics of the system. In Section 5, complexity of fractional-order chaotic system is analyzed. Finally, the obtained results are summarized in Section 6.

2. Integer-Order Chaotic System with Line Equilibrium

2.1. Analysis of Equilibrium Points of Systems. Reference [29] proposes a class of T chaotic systems defined by

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = cx - axz, \\ \dot{z} = xy - bz. \end{cases} \quad (1)$$

A 3D integer-order chaotic system is defined by

$$\begin{cases} \dot{x} = yz - ax, \\ \dot{y} = cx - axz, \\ \dot{z} = xy - bz, \end{cases} \quad (2)$$

where, a , b , and c are system (2) parameters and x , y , and z are three state variables of system (2). When parameters $a = 5$, $b = 2$, and $c = 34$, the system has two equilibrium points: $E_1((c\sqrt{ab})/a^2, \sqrt{ab}, (c/a))$, $E_2(-(c\sqrt{ab})/a^2, -\sqrt{ab}, (c/a))$, and $E_3(0, h, 0)$. The eigenvalues at E_1 and E_2 are

$\lambda_{11} = -6.0739$, $\lambda_{12} = -0.4631 + 21.3673i$, and $\lambda_{13} = 0$; $E_3(0, h, 0)$ is the line equilibrium point; and h is any real number. The Jacobian matrix linearized under equilibrium E_3 is as follows:

$$J = \begin{bmatrix} -a & 0 & h \\ c & 0 & 0 \\ h & 0 & -b \end{bmatrix}. \quad (3)$$

The characteristic equation of matrix is

$$\lambda^3 + (a + b)\lambda^2 + (ab - h^2)\lambda = 0. \quad (4)$$

The eigenvalue at the line equilibrium point E_3 is as follows:

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= -\frac{(a + b)}{2} - \frac{\sqrt{(a - b)^2 + 4h^2}}{2}, \\ \lambda_3 &= -\frac{(a + b)}{2} + \frac{\sqrt{(a - b)^2 + 4h^2}}{2}. \end{aligned} \quad (5)$$

Case 1. If $h^2 < ab$, then $\lambda_3 = -((a + b)/2) + (\sqrt{(a - b)^2 + 4h^2}/2) < 0$; according to Routh–Hurwitz criterion, the equilibrium E_3 is stable.

Case 2. If $h^2 > ab$, then $\lambda_3 = -((a + b)/2) + (\sqrt{(a - b)^2 + 4h^2}/2) > 0$; according to Routh–Hurwitz criterion, the equilibrium E_3 is unstable.

Case 3. If $h^2 = ab$, then $\lambda_3 = -((a + b)/2) + (\sqrt{(a - b)^2 + 4h^2}/2) = 0$; according to Routh–Hurwitz criterion, we obtain that the equilibrium E_3 is critical stable.

When $a = 5$, $b = 2$, and $c = 34$, using ODE45 function in MATLAB software to simulate (2), the chaotic attractor is shown in Figure 1. The orbits marked by the blue and red colors emerge from initial (1, 2, 3) and (1, 2, -1), and the red attractor is periodic. The blue attractors are shown in chaotic states.

2.2. Dynamics Analysis of the Chaotic System. Fix $a = 5$, $b = 2$, and $c = 34$; initial values of state variables $[x_0, y_0, z_0] = [1, 2, z_0]$; let the z_0 vary from -15 to 15 with step size of 0.01. Bifurcation diagram and Lyapunov exponents spectrum of system (2) are shown in Figure 2. As can be seen from the bifurcation diagram of the system in Figure 2(a), when $z_0 \in [-15, -0.6]$, the system is in a periodic state and

FIGURE 1: Phase diagrams in different projections of chaotic systems (2) from initial values $(1, 2, 3)$, red, and $(1, 2, -1)$. (a) Phase portraits in the x - y plane. (b) Phase portraits in the x - z plane. (c) Phase portraits in the y - z plane.

stable at $(0, 0, 0)$. When $z_0 \in (-0.6, 15]$, the system is in a chaos state. When $z_0 \in [-15, -0.6]$, the maximum Lyapunov exponent is equal to 0. When $z_0 \in (-0.6, 15]$, the largest Lyapunov exponent is positive, as shown in Figure 2(b).

Fix $a = 5$, $b = 2$, and $c = 34$; initial values of state variables $[x_0, y_0, z_0] = [1, 2, z_0]$; let the $z(0)$ vary from -15 to 15 with step size of 0.3 and $y(0)$ vary from -20 to 20 with step size of 0.4 . In the $z(0) - y(0)$ plane, the chaos diagram of the largest LEs is calculated and shown in Figure 3. In Figure 3 the chaotic region is green (labeled “C”) and the periodic

region is red (labeled “P”). We can see from Figure 3 that the system state is mainly affected by the initial condition z_0 .

3. Solution of Fractional Chaotic System Based on Adomian Decomposition

System (2) can generate complex chaotic attractors with line equilibrium point, and it has three typical parameter sets. Change the above system equations to the following fractional-order form:

FIGURE 2: Bifurcation diagram and Lyapunov exponents spectrum of system (2) with initial conditional $[x_0, y_0, z_0] = [1, 2, z_0]$. (a) Bifurcation diagram. (b) Lyapunov exponents (LEs).

FIGURE 3: The chaos diagram of the largest Lyapunov exponents with $z_0 \in [-15, 15]$ and $y_0 \in [-20, 20]$.

$$\begin{cases} \frac{dx^q}{d^q t} = yz - ax, \\ \frac{dy^q}{d^q t} = cx - axz, \\ \frac{dz^q}{d^q t} = xy - bz. \end{cases} \quad (6)$$

The initial condition is $\begin{cases} x^0 = x(t_0) = c_1^0 \\ y^0 = y(t_0) = c_2^0 \\ z^0 = z(t_0) = c_3^0 \end{cases}$. According to

domains decomposition method [30] and fractional calculus properties, the following properties are obtained:

$$\begin{cases} x^1 = (c_2^0 c_3^0 - ac_1^0) \frac{h^q}{\Gamma(q+1)}, \\ y^1 = [cc_1^0 - ac_1^0 c_3^0] \frac{h^q}{\Gamma(q+1)}, \\ z^1 = (c_1^0 c_2^0 - bc_3^0) \frac{h^q}{\Gamma(q+1)}, \end{cases} \quad (7)$$

where x^1, y^1, z^1 are the values of systems (3). $h_s = t - t_0$ is iteration step size. $\Gamma(\cdot)$ is Gamma function. The corresponding variables are assigned to the corresponding value; let

$$\begin{cases} c_1^1 = c_2^0 c_3^0 - a c_1^0, \\ c_2^1 = c c_1^0 - a c_1^0 c_3^0, \\ c_3^1 = c_1^0 c_2^0 - b c_3^0, \end{cases} \quad (8)$$

$$\begin{cases} c_1^2 = c_2^0 c_3^1 + c_2^1 c_3^0 - a c_1^1, \\ c_2^2 = c c_1^1 - a(c_1^1 c_3^0 + c_1^0 c_3^1), \\ c_3^2 = c_1^1 c_2^0 + c_1^0 c_2^1 - b c_3^1, \end{cases} \quad (9)$$

$$\begin{cases} c_1^3 = c_2^2 c_3^0 + c_2^0 c_3^2 + c_2^1 c_3^1 \frac{\Gamma(2q+1)}{\Gamma^2(q+1)} - a c_1^2, \\ c_2^3 = c c_1^2 - a c_1^2 c_3^0 - a c_1^0 c_3^2 - a c_1^1 c_3^1 \frac{\Gamma(2q+1)}{\Gamma^2(q+1)}, \\ c_3^3 = c_1^2 c_2^0 + c_1^0 c_2^2 + c_1^1 c_2^1 \frac{\Gamma(2q+1)}{\Gamma^2(q+1)} - b c_3^2, \end{cases} \quad (10)$$

$$\begin{cases} c_1^4 = c_2^3 c_3^0 + c_2^0 c_3^3 + (c_2^1 c_3^2 + c_2^2 c_3^1) \frac{\Gamma(3q+1)}{\Gamma(2q+1)\Gamma(q+1)} - a c_1^3, \\ c_2^4 = c c_1^3 - a c_1^3 c_3^0 - a c_1^0 c_3^3 - a(c_1^1 c_3^2 + c_1^2 c_3^1) \frac{\Gamma(3q+1)}{\Gamma(2q+1)\Gamma(q+1)}, \\ c_3^4 = c_1^3 c_2^0 + c_1^0 c_2^3 + (c_1^1 c_2^2 + c_1^2 c_2^1) \frac{\Gamma(3q+1)}{\Gamma(2q+1)\Gamma(q+1)} - b c_3^3, \end{cases} \quad (11)$$

$$\begin{cases} c_1^5 = c_2^4 c_3^0 + c_2^0 c_3^4 + (c_2^1 c_3^3 + c_2^2 c_3^1) \frac{\Gamma(4q+1)}{\Gamma(3q+1)\Gamma(q+1)} + c_2^2 c_3^2 \frac{\Gamma(4q+1)}{\Gamma^2(2q+1)} - a c_1^4, \\ c_2^5 = c c_1^4 - a c_1^4 c_3^0 - a c_1^0 c_3^4 - a(c_1^1 c_3^3 + c_1^2 c_3^1) \frac{\Gamma(4q+1)}{\Gamma(3q+1)\Gamma(q+1)} - a c_1^2 c_3^2 \frac{\Gamma(4q+1)}{\Gamma^2(2q+1)}, \\ c_3^5 = c_1^4 c_2^0 + c_1^0 c_2^4 + (c_1^1 c_2^3 + c_1^2 c_2^1) \frac{\Gamma(4q+1)}{\Gamma(3q+1)\Gamma(q+1)} + c_1^2 c_2^2 \frac{\Gamma(4q+1)}{\Gamma^2(2q+1)} - b c_3^4, \end{cases} \quad (12)$$

$$\begin{cases} c_1^6 = c_2^5 c_3^0 + c_2^0 c_3^5 + (c_2^1 c_3^4 + c_2^2 c_3^1) \frac{\Gamma(5q+1)}{\Gamma(3q+1)\Gamma(2q+1)} + (c_2^2 c_3^2 + c_2^3 c_3^1) \frac{\Gamma(5q+1)}{\Gamma(4q+1)\Gamma(q+1)} - a c_1^5, \\ c_2^6 = c c_1^5 - a c_1^5 c_3^0 - a c_1^0 c_3^5 - a(c_1^1 c_3^4 + c_1^2 c_3^1) \frac{\Gamma(5q+1)}{\Gamma(3q+1)\Gamma(2q+1)} - a(c_1^2 c_3^2 + c_1^3 c_3^1) \frac{\Gamma(5q+1)}{\Gamma(4q+1)\Gamma(q+1)}, \\ c_3^6 = c_1^5 c_2^0 + c_1^0 c_2^5 + (c_1^1 c_2^4 + c_1^2 c_2^1) \frac{\Gamma(5q+1)}{\Gamma(3q+1)\Gamma(2q+1)} + (c_1^2 c_2^2 + c_1^3 c_2^1) \frac{\Gamma(5q+1)}{\Gamma(4q+1)\Gamma(q+1)} - b c_3^4. \end{cases} \quad (13)$$

Thus, the solution of systems (5) is defined as

$$\begin{cases} x(t) = c_1^0 + c_1^1 \frac{h^q}{\Gamma(q+1)} + c_1^2 \frac{h^{2q}}{\Gamma(2q+1)} + c_1^3 \frac{h^{3q}}{\Gamma(3q+1)} + c_1^4 \frac{h^{4q}}{\Gamma(4q+1)} + c_1^5 \frac{h^{5q}}{\Gamma(5q+1)} + c_1^6 \frac{h^{6q}}{\Gamma(6q+1)}, \\ y(t) = c_2^0 + c_2^1 \frac{h^q}{\Gamma(q+1)} + c_2^2 \frac{h^{2q}}{\Gamma(2q+1)} + c_2^3 \frac{h^{3q}}{\Gamma(3q+1)} + c_2^4 \frac{h^{4q}}{\Gamma(4q+1)} + c_2^5 \frac{h^{5q}}{\Gamma(5q+1)} + c_2^6 \frac{h^{6q}}{\Gamma(6q+1)}, \\ z(t) = c_3^0 + c_3^1 \frac{h^q}{\Gamma(q+1)} + c_3^2 \frac{h^{2q}}{\Gamma(2q+1)} + c_3^3 \frac{h^{3q}}{\Gamma(3q+1)} + c_3^4 \frac{h^{4q}}{\Gamma(4q+1)} + c_3^5 \frac{h^{5q}}{\Gamma(5q+1)} + c_3^6 \frac{h^{6q}}{\Gamma(6q+1)}. \end{cases} \quad (14)$$

When $a = 5$, $b = 2$, and $c = 34$, system (5) has stable equilibrium with line $(0, h, 0)$. The analytical solution of the system can be obtained according to (14). Using MATLAB to simulate (14), the chaotic attractor is shown in Figure 4. The orbits marked by the blue and red colors emerge from initial $(1, 2, 3)$ and $(1, 2, -1)$, and the red attractor is periodic. The blue attractors are shown in chaotic states.

4. Dynamical Analysis

Based on the ADM solution of the fractional-order chaotic system, we focus on the analysis of the influence of parameters on the system. In this paper, the system bifurcation diagram and Lyapunov exponent are used to analyze the dynamic system. The bifurcation diagram is obtained by using the maximum value of state variables. In this section, the influence of three parameters on the system is studied.

4.1. Dynamics with $z(0)$ Varying. Fix $a = 5$, $b = 2$, $c = 34$, and $q = 0.8$; initial values of state variables $[x_0, y_0, z_0] = [1, 2, z_0]$; let the z_0 vary from -15 to 15 with step size of 0.01 . Bifurcation diagram and Lyapunov exponents spectrum of the system (5) are shown in Figure 5. As can be seen from the bifurcation diagram of the system in Figure 5(a), when $z_0 \in [-15, -0.7]$, the system is in a periodic state and stable at $(0, 0, 0)$. When $z_0 \in (-0.7, 15]$, the system is in a chaos state. When $z_0 \in [-15, -0.7]$, the largest Lyapunov exponents is equal to 0. When $z_0 \in (-0.7, 15]$, the largest Lyapunov exponents is positive, as shown in Figure 5(b).

4.2. Dynamics with q Varying. Fix $a = 5$, $b = 2$, and $c = 34$; let the derivative order q vary from 0.6 to 1 with step size of 0.002 and the initial values of state variables $[x_0, y_0, z_0] = [1, 2, 3]$. It can be seen from Figure 6(a) that, under the variation of derivative order q , the system chaos and periodic variation are repeated and crossed and are not a process from period to chaos directly. At the same time, Figure 6 shows that the system is chaotic over the interval $q \in [0.678, 0.824] \cap [0.876, 1]$. However, $q \in [0.876, 1]$ also has a periodic window. It can be seen clearly that the maximum Lyapunov exponent decreases with the increase of derivative order $q \in [0.678, 0.824]$. In this case, the minimum order for chaos is $q = 0.678$. The maximum Lyapunov exponent value at this point ($q = 0.678$) is the largest. With the change of parameter q , the bifurcation

diagram of the system is consistent with the Lyapunov exponent spectrum of the system.

For further research, influence of coexistence of attractors in fractional derivative q system. In this paper, the phase diagram is used to introduce. It is found that $a = 5$, $b = 2$, and $c = 34$. The integer-order system ($q = 1$) is different from the fractional-order system ($q = 0.98$). The attractors of the integer-order system are almost the same, while the attractors of the fractional-order system coexist. As shown in Figure 7, the red initial values are $[1, 2, 3]$. The blue initial values are $[1, 2, 20]$.

4.3. Dynamics with a and b Varying. The fractional chaotic system (5) has three system parameters besides the fractional derivative q . In this paper, the dynamic properties of the system parameters with a and b varying are analyzed. Firstly, bifurcation diagrams of the fractional-order system with $a \in [3, 10]$ is investigated as shown in Figure 8(a), where the step size of a is 0.05 , $q = 0.8$, $b = 2$, and the initial conditional is $[x_0, y_0, z_0] = [1, 2, 3]$. Obviously, it can be seen that $a \in [3.3, 4.1] \cup [4.3, 4.7] \cup [8.4, 8.5]$ is in a periodic state; phase diagrams of the systems with different a values are shown in Figure 9. Other regions are in chaotic state with periodic windows.

Secondly, bifurcation diagrams of the fractional-order system (5) with $b \in [0, 6]$ is investigated as shown in Figure 8(b), where the step size of b is 0.05 , $a = 5$, and the initial conditional is $[x_0, y_0, z_0] = [1, 2, 3]$. Obviously, it can be seen that $b \in [3.16, 3.38] \cup [3.92, 6]$ is in a periodic state, and other regions are in chaotic state with periodic windows.

QR decomposition method [6, 15] is effective in calculating the LEs, and its computational process is shown by

$$\begin{aligned} \text{qr}[J_m J_{m-1}, \dots, J_2 J_1] &= \text{qr}[J_m J_{m-1}, \dots, J_2 (J_1 Q_0)] \\ &= \text{qr}[J_m J_{m-1}, \dots, J_3 (J_2 Q_1)] \times [R_1] \\ &= \text{qr}[J_m J_{m-1}, \dots, J_4 (J_3 Q_2)] \times [R_2 R_1] \\ &= \dots \\ &= \text{qr}[J_m J_{m-1}, \dots, J_{i+1} (J_i Q_{i-1})] \\ &\quad \times [R_{i-1} R_{i-2}, \dots, R_1] \\ &= Q_m [R_m R_{m-1}, \dots, R_1] = Q_m R. \end{aligned} \quad (15)$$

Here, $\text{qr}(\cdot)$ represents QR decomposition function, m is the maximum iteration number, and J is the Jacobian matrix of

FIGURE 4: Phase diagrams in different projections of fractional-order chaotic systems from initial values (1, 2, 3), red, and (1, 2, -1). (a) Phase portraits in the (x)-(z) plane. (b) Phase portraits in the (x)-(y) plane. (c) Phase portraits in the (y)-(z) plane.

the given discrete formula as presented in (14). LCEs are calculated by

$$\lambda_k = \frac{1}{mh} \ln |\mathbf{R}_i(k, k)|. \quad (16)$$

Here, $k = 1, 2, n$ (system dimension); n is the maximum number of iterations; and h is the iteration time step. According to the above basic formula, make $n = 4$, $n = 7000$, and $h = 0.01$, and use MATLAB to draw the Lyapunov exponent under parameter changes.

Fix $b = 2$, $q = 0.8$; initial conditional is $[x_0, y_0, z_0] = [1, 2, 3]$; and let a vary. Lyapunov exponents spectrum of system (5) is shown in Figure 10(a), which is consistent with the bifurcation diagram; namely, $a \in [3.3, 4.1] \cup [4.3, 4.7] \cup [8.4, 8.5]$; the largest Lyapunov exponent of the system is equal to 0 (not positive), and the maximum Lyapunov exponent in other regions is greater than 0 (positive).

Fix $a = 5$, $q = 0.8$; initial conditional is $[x_0, y_0, z_0] = [1, 2, 3]$; and let b vary. Lyapunov exponents spectrum of system (5) is shown in Figure 10(b), which is consistent with the bifurcation

diagram; namely, $b \in [3.16, 3.38] \cup [3.92, 6]$; the largest Lyapunov exponent of the system is less than 0 (the phase diagram is a collection point), and the maximum Lyapunov exponent in other regions is greater than 0 (positive).

5. Complexity Analysis of the Fractional-Order System

5.1. Completeness Calculation. By calculating the energy distribution in the Fourier transform domain and combining it with the Shannon entropy, spectral entropy is obtained [31]. C0 complexity [32] is mainly to decompose the time series into regular and irregular parts and to test the proportion of irregular parts in the sequence. In practical applications, the chaos diagram of SE and C0 complexity can provide a basis to select parameter better.

Sample entropy (SampEn) [16] measures the complexity of time series by measuring the probability of generating new patterns in signals. The greater the probability of generation, the greater the value of the complexity of the sequence.

FIGURE 5: Bifurcation diagram and Lyapunov exponents spectrum of the fractional-order chaotic system (5) with initial conditional $[x_0, y_0, z_0] = [1, 2, z_0]$. (a) Bifurcation diagram. (b) Lyapunov exponents (LEs).

FIGURE 6: Bifurcation diagram and Lyapunov exponents spectrum of system (11) with $a = 5$, $b = 2$, and $c = 34$. (a) Bifurcation diagram. (b) Lyapunov exponents (LEs).

In general, for the specific calculation method of sample entropy of time series $\{x(n)\}$ composed of N data, the specific steps are as follows:

- (i) S1: according to the serial number to form a one-dimensional m vector sequence, $\{X_m(1), \dots, X_m(N - m + 1)\}$, where $X_m(1) = \{x(i), x(i + 1), \dots, x(i + m - 1)\}$, $1 \leq i \leq N - m + 1$. These vectors are m consecutive x values starting from point 1.
- (ii) S2: define the distance between $X_m(i)$ and $X_m(j)$ as $d[X_m(i), X_m(j)]$, as the difference between the maximum absolute values of the corresponding elements.
- (iii) S3: define the criterion of similarity r . Count the number of $d[X_m(i), X_m(j)] < r$ ($1 \leq j \leq N - m$, $1 \leq i \leq N - m$), and its ratio to total distance $N - m$ is denoted as $B_i^m(r) = B_i / (N - m - 1)$.

FIGURE 7: Phase diagrams in different projections of chaotic systems from initial values (1, 2, 20) and (1, 2, 3). (a) Phase diagrams in integer-order system (2). (b) Phase diagrams in fractional-order system (5).

FIGURE 8: Dynamical properties of the fractional-order system with a and b varying. (a) a varying. (b) b varying.

(iv) S4: the average value of $B_i^m(r)$ can be calculated as

$$B^m(r) = \frac{1}{N-m} \sum_{i=1}^{N-m} B_i^m(r). \quad (17)$$

(v) S5: similarly, change m to $m+1$ and repeat step one with step three; get $A^m(r)$.

(vi) S6: theoretically, the SampEn complexity can be calculated by

$$\text{SampEn}(m, r) = \lim_{N \rightarrow \infty} \left\{ -\ln \frac{A^m(r)}{B^m(r)} \right\}. \quad (18)$$

5.2. Complexity Analysis. Fix $a = 5$, $b = 2$, $c = 34$; let q change from 0.6 to 1; step size is 0.002; and the initial value of state variables $[x_0, y_0, z_0] = [1, 2, 3]$. The complexity of the system is shown in Figure 11(a). We can see that the complexity decreases with the increase of q , and the change of complexity is consistent with the change of Lyapunov exponent in Figure 6(b).

Fixed $b = 2$, $q = 0.8$; initial conditional is $[x_0, y_0, z_0] = [1, 2, 3]$; and let a vary. The complexity of the system is shown in Figure 11(b). The change of complexity is consistent with the change of Lyapunov exponent in Figure 10(a).

Fix $a = 2$, $q = 0.8$; initial conditional is $[x_0, y_0, z_0] = [1, 2, 3]$; and let b vary. The complexity of the system is shown in

FIGURE 9: Phase diagrams of the systems with different a values. (a) $a = 3.5$. (b) $a = 4.4$. (c) $a = 8.46$.

FIGURE 10: Lyapunov exponents spectrum of fractional order with a and b varying. (a) a varying. (b) b varying.

FIGURE 11: Complexity of the system with parameter varying. (a) q varying. (b) a varying. (c) b varying.

Figure 11(c). The change of complexity is consistent with the change of Lyapunov exponent in Figure 10(b).

Complexity is consistent with Lyapunov exponent, but Figure 11 also shows that sample entropy complexity is small; SE complexity is not obvious when some bifurcation points change; C_0 complexity is accurate; and C_0 complexity is faster than Lyapunov exponent, consumes less resources, and provides a good idea for analysis of chaotic systems.

Fix $a = 5$, $b = 2$, and $c = 34$; initial values of state variables $[x_0, y_0, z_0] = [1, 2, z_0]$; let the $z(0)$ vary from 0 to 25 with step

size of 0.2 and q vary from 0.6 to 1 with step size of 0.004. The C_0 complexity in the $z(0)$ - q plane is shown in Figure 12. It can be seen that, in the fractional-order region, the system complexity under the change of $z(0)$ is more complex than that of the integer order.

The complexity of the system is calculated by the time sequence of a certain state variable. Compared with Lyapunov exponent calculation, the calculation of complexity is faster and saves resources, but there are also some errors in complexity, especially when SE complexity is in a periodic state.

FIGURE 12: C_0 complexity in the $z(0)$ - q plane.

6. Conclusion

In this paper, the accurate approximate solution for the fractional-order system with line equilibrium is obtained based on the Adomian decomposition method. Dynamical behaviors of the systems are analyzed using the chaotic attractor, bifurcation diagram, Lyapunov exponent spectrum, and largest Lyapunov exponent diagram. Chaotic range and periodic windows are determined. Both the system parameter and the fractional order can be taken as bifurcation parameters, which shows that the fractional-order system has more complex and abundant dynamics than its integral-order counterpart. Besides, integer-order systems do not have the phenomenon of attractors coexistence, while fractional-order systems have it, so fractional-order systems have better effects when applied to communication security systems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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