Recursive State and Random Fault Estimation for Linear Discrete Systems under Dynamic Event-Based Mechanism and Missing Measurements

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Received 24 June 2020; Revised 27 August 2020; Accepted 20 September 2020; Published 10 November 2020

Academic Editor: Pietro De Lellis

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This paper is concerned with the event-based state and fault estimation problem for a class of linear discrete systems with randomly occurring faults (ROFs) and missing measurements. Different from the static event-based transmission mechanism (SETM) with a constant threshold, a dynamic event-based mechanism (DETM) is exploited here to regulate the threshold parameter, thus further reducing the amount of data transmission. Some mutually independent Bernoulli random variables are used to characterize the phenomena of ROFs and missing measurements. In order to simultaneously estimate the system state and the fault signals, the main attention of this paper is paid to the design of recursive filter; for example, for all DETM, ROFs, and missing measurements, an upper bound for the estimation error covariance is ensured and the relevant filter gain matrix is designed by minimizing the obtained upper bound. Moreover, the rigorous mathematical analysis is carried out for the exponential boundedness of the estimation error. It is clear that the developed algorithms are dependent on the threshold parameters and the upper bound together with the probabilities of missing measurements and ROFs. Finally, a numerical example is provided to indicate the effectiveness of the presented estimation schemes.

1. Introduction

The state estimation problem for the networked control systems has gained great concern in recent years owing to its huge potential applications in guidance, signal processing, navigation, econometrics, and so on [1–15]. As we all know, the performance of filters could often be affected by the networked-induced phenomena and a rich body of filtering results has been available [2, 4, 15–26]. For example, a time-varying state estimator has been designed in [16] in terms of the Riccati-like difference equations approach. The nonfragile filtering problem has been addressed [18] for discrete-time singular Markovian jump systems, where the uncertain probability of missing measurements was described by the norm-bounded uncertainties. It should be noted that some random variables obeying the Bernoulli distribution or certain probabilistic distributions are adopted to model the phenomenon of missing measurements.

Recently, the event-based transmission mechanism is well recognized to offer many advantages in decreasing the amount of data transfer and improving resource utilization compared with the time-triggered transmission mechanisms, which is because the event is triggered only if a given specified condition is met. As a result, the event-based state estimation problem has attracted extensive attention and many regarding results have been reported (see [16, 21, 27–32]). To mention a few, a recursive filtering problem has been studied in [30] for a class of discrete time-delayed stochastic nonlinear systems subject to event-based measurement transmissions and missing measurements. In [27], a novel event-based consensus Kalman filter algorithm has been proposed, in which the boundedness of the estimation error in mean-square sense has also been discussed.

It is noteworthy that most of the above references have relied on an implicit assumption that the threshold in event-based condition is a fixed scalar, which is called SETM. In
reality, however, the threshold parameter could be dynamically changed to ensure that the triggering instants are less than the static cases. Consequently, it is not surprising that much effort has devoted to the research on the dynamic event-triggered state estimation problems [33–39]. To be more specific, in [34], a distributed set-membership estimation problem has been addressed for linear discrete systems with unknown-but-bounded process and measurement noise, under the condition that the proposed dynamic scheme has been verified to cause larger average interevent times and thus less data transmission. With the aid of the Lyapunov-based analysis and the projection technique, a new dynamic event-based control scheme is developed and the stability analysis has been given. However, there are few studies focusing on the estimation issue of ROFs with the dynamic event-based scheme, which motivates our current investigation.

To maintain the safety and reliability of actual systems, in the past decade, fault detection and fault estimation problems have become hot research topics, and plenty of results could be found in [12, 40–47]. Generally, the model-based method is the common one; that is, a fault detection filter or observer is designed to generate a residual and compare it with a known threshold. When the residual evaluation function has a value larger than the threshold, an alarm is generated. For instance, in [40], the event-based fault detection filtering problem has been dealt with for networked switched control systems with random disturbance and repeated scalar nonlinearities. In the presence of packet dropouts, stochastic nonlinearities as well as randomly occurring uncertainties, the authors in [41] have presented a new fault estimation method to ensure that an optimized upper bound of the estimation error covariance exists and the explicit form of the estimator gain has been given. It is worth mentioning that the faults are taken as constants in most cases; however, the practical engineering systems are usually suffered from unpredictable parameter fluctuations or sudden structural changes, so the time-varying and random characteristics of faults should be considered in the engineering reality [48, 49]. Nevertheless, when it comes to the fault estimation issue in the networked control systems, ROFs has not yet gained attention, let alone the case where DETM and missing measurements are also taken into account. It is, therefore, another motivation for us to conduct this research to shorten such a gap.

Following the above discussion, the dynamic event-based state and fault estimation problem for linear discrete systems subject to missing measurements and ROFs is studied in this paper. Some random variables obeying the Bernoulli distribution are introduced to describe the phenomenon of missing measurements. Different from the constant fault, the fault considered in this paper satisfies the characteristics of time-varying and random occurring. The presented recursive state and fault estimation scheme is expressed by the solutions to two difference equations. The main innovations of this paper are (1) a dynamic event-based scheme including a new event generator function is utilized to reduce the transmission of data, and ROFs are taken into account to better reflect the engineering practice; (2) a new recursive joint state and fault estimation algorithm is proposed to attenuate the effects from the ROFs, missing measurements, and DETM onto the estimation performance; and (3) the exponentially boundedness of the estimation error is established under some mild conditions.

**Notation.** \( \| \cdot \| \) represents the \( l \) dimensional Euclidean space. \( \| F \| \) and \( F^T \) denote the norm and the transpose of a matrix \( F \), respectively. \( \text{tr} (F) \) stands for the trace of a matrix \( F \). \( I \) is the identity matrix. The expectation of the stochastic variable \( z \) is denoted by \( \mathbb{E} [z] \), \( \text{diag} \{ \cdot, \ldots, \cdot \} \) means a diagonal matrix. Other matrices in this paper are supposed to have suitable dimensions.

### 2. Problem Formulation

Consider the following linear discrete systems with ROFs and missing measurements:

\[
\begin{align*}
x_{i+1} &= H_i x_i + \xi_i G_i f_i + L_i w_i, \\
y_i &= \Lambda_i M_i x_i + v_i,
\end{align*}
\] (1)

where \( x_i \in \mathbb{R}^n \) is the state vector, \( y_i \in \mathbb{R}^m \) is the measurement output, and \( f_i \) represents fault signal. \( \xi_i = \text{diag}[\eta_{i1}, \eta_{i2}, \ldots, \eta_{im}] \) is applied to describe missing measurements, where \( \eta_{is} \) \((s = 1, 2, \ldots, m)\) are Bernoulli distributed random variables with \( \text{Prob} [\eta_{is} = 1] = \pi_{is} \) and \( \text{Prob} [\eta_{is} = 0] = 1 - \pi_{is} \). \( w_i \in \mathbb{R}^w \) is the unknown process noise with covariance \( Q_i \), and \( v_i \in \mathbb{R}^r \) represents zero-mean measurement noises with covariance \( R_i \). \( H_i, G_i, L_i, \) and \( M_i \) are known time-varying matrices with appropriate dimensions.

The initial state \( x_0 \) and \( w_0, v_0, \xi_0, \) and \( \eta_{is} \) are supposed to be mutually independent. Moreover, the statistical properties of \( x_0 \) are given as

\[
\begin{align*}
\mathbb{E}[x_0] &= \overline{x}_0, \\
\mathbb{E}[(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T] &= P_0.
\end{align*}
\] (2)

Note that the random variable \( \xi_0 \) is introduced here to describe the phenomenon of ROFs, and \( \xi_0 \) satisfies \( \text{Prob} [\xi_0 = 1] = \overline{\xi}_0, \text{Prob} [\xi_0 = 0] = 1 - \overline{\xi}_0 \). Meanwhile, the fault \( f_i \) satisfies the following dynamic behavior:

\[ f_{i+1} = H_{f,i} f_i, \] (3)

where \( H_{f,i} \) is a given matrix.

In this paper, the event-triggered condition of DETM satisfies the following relation:

\[ \rho (y_i, \lambda_i) = \| y_i - y_{i0} \| - \lambda_i > 0, \] (4)

where \( y_i \) are the current measurements, \( y_{i0} \) means latest transmitted measurements, \( \lambda_i = \mu + (\epsilon_i / \tau) \), and \( \mu \) and \( \tau \) are the known positive scalars. In addition, the dynamic variable \( \epsilon_i \) with initial value \( \epsilon_0 \) satisfies

\[ \epsilon_{i+1} = \alpha \epsilon_i + \mu - \| y_i - y_{i0} \|, \] (5)

where \( 0 < \alpha < 1 \) and \( \alpha \tau > 1 \). Consequently, the sequence of transmission instants \( t_j (0 \leq t_1 \leq t_2 \leq \cdots \leq t_f \leq \cdots) \) determined by condition (4) is given as
can dynamically change and their dynamic characteristics are described in (3). Such a description has been widely adopted in the existing literature (see [48–50]). In particular, the fault \( f_i \) becomes the constant one when \( H_{f,i} = I \). Also, according to the definition of \( X_i \), the estimated fault can be computed by \( \tilde{f}_i = [0, I] \tilde{X}_{i|i} \).

**Remark 2.** The static event-triggered condition is usually characterized by \((y_j - y_i)^T(y_j - y_i) - \mu > 0\), where the threshold \( \mu \) is a constant. In order to further improve the utilization of network resources, in this paper, the dynamic variable \( \xi_i \) is exploited in condition (4) to constantly adjust the threshold parameter. When \( \tau \to +\infty, \lambda_i \) reduces to \( \mu \). As stated in [34], if the dynamic variable \( \xi_i \) satisfies (5), the triggering instants under DETM are smaller than the static status. Hence, the description of DETM in (4) is quite general that includes the static event-triggered one as a special case.

**Remark 3.** By augmenting the original state and fault, new vector \( X_i \) is formed. Thus, the addressed system (1) can be rewritten as a stochastic parameterized one, where coefficient matrices \( H_i \) and \( \tilde{M}_i \) contains random variables \( \xi_i \) and \( \eta_{i,i} \). Similar to the classical optimal filter which minimizes the error covariance, the developed estimator of this paper aims to ensure that the estimation error covariance is upper bounded. Furthermore, the boundedness of the estimation error will be discussed in the sequel.

### 3. Main Results

#### 3.1. Recursive Estimator Design

In this section, the recursive estimator will be presented via the stochastic analysis approach. Before giving the main results, the following lemmas will be provided.

**Lemma 1.** Let \( D = \text{diag}[d_1, d_2, \ldots, d_n] \) be a diagonal stochastic matrix, and \( \Gamma \) is a real-valued matrix. Then,

\[
\mathbb{E}\left[ D\Gamma D^T \right] = \begin{bmatrix}
\mathbb{E}[d_1^2] & \mathbb{E}[d_1d_2] & \cdots & \mathbb{E}[d_1d_n] \\
\mathbb{E}[d_2d_1] & \mathbb{E}[d_2^2] & \cdots & \mathbb{E}[d_2d_n] \\
\vdots & \vdots & \ddots & \vdots \\
\mathbb{E}[d_nd_1] & \mathbb{E}[d_nd_2] & \cdots & \mathbb{E}[d_n^2]
\end{bmatrix} \Omega^T,
\]

where \( \ominus \) denotes the Hadamard product.

**Lemma 2.** Define \( \Phi_{1,j} \equiv \mathbb{E}[\Delta H_i X_i^T \Delta H_i^T] \) and \( \Phi_{2,j} \equiv \mathbb{E}[\Delta M_i X_i^T \Delta M_i^T] \), where \( \psi_i \equiv \mathbb{E}[X_i^T X_i] \) is the state covariance matrix. Then, the following results are given as

\[
\Phi_{1,j} = \xi_i^2 \Gamma_i \Gamma_i^T,
\]

\[
\Phi_{2,j} = \Omega_i \left( M_i \psi_i A_i^T M_i^T \right),
\]

where \( A = [I, 0] \) and \( \Omega_i = \text{diag}\{ \eta_{i,i}(1 - \eta_{i,i}), \ldots, \eta_{i,i}(1 - \eta_{i,i}) \} \).
Proof. Recalling the expression of $\Delta H_i$, $\Delta M_i$, and (1), (12)-(13) can be easily obtained. □

Lemma 3. The state second moment $\psi_i = \mathbb{E}[X_i X_i^T]$ of system (8) can be represented as

$$\psi_{i+1} = H_i \psi_i H_i^T + \Phi_{i+1} + L_i Q_i L_i^T,$$

with the initial value $\psi_0 = \begin{pmatrix} P_0 & X_0^T \\ X_0 & 0 \end{pmatrix}$. (14)

Proof. Based on the definition of $\psi_i = \mathbb{E}[X_i X_i^T]$, we have

$$\psi_{i+1} = \mathbb{E}\left[(H_i X_i + L_i u_i)(H_i^T X_i + L_i^T w_i)^T\right]$$

$$= \mathbb{E}\left[H_i X_i H_i^T + \varphi_{i+1}^T + L_i Q_i L_i^T\right].$$

Combining (12) and $\mathbb{E}[u_i w_i^T] = Q_i$, (14) holds. □

Lemma 4. Given $a_i > 0$ and $b_i > 0$, the covariance matrices $\Theta_i \triangleq \mathbb{E}[\epsilon_i^2]$ satisfies $\Theta_i \leq \overline{\Theta}_i$, where

$$\overline{\Theta}_{i+1} = \left[(1 + a_i)(1 + b_i)^2 - \frac{(1 + a_i^{-1})(1 + r)^2}{r^2}\right] \Theta_i$$

$$+ \left[(1 + a_i)(1 + b_i^{-1}) + (1 + a_i^{-1})(1 + r^{-1})\right] \mu^2,$$

with $\overline{\Theta}_0 = \epsilon_0^2$. (16)

Proof. The proof of (16) is similar to Lemma 4 in [29], so more details are omitted. □

Theorem 1. The dynamic equation of the prediction error covariance $O_{i+1|i}$ can be described as follows:

$$O_{i+1|i} = H_i O_i H_i^T + \Phi_{i+1} + L_i Q_i L_i^T.$$ (17)

Proof. Substituting $\bar{X}_{i+1|i}$ in (10) into $\bar{X}_{i+1|i}$, we can obtain

$$\bar{X}_{i+1|i} = H_i X_i + L_i w_i - H_i \bar{X}_{i|i}$$

$$= H_i X_i + \Delta H_i X_i + L_i w_i.$$

Since $\bar{X}_{i|i}$ and $X_i$ are uncorrelated with $w_i$, we have

$$O_{i+1|i} = \mathbb{E}\left[H_i \bar{X}_{i|i} H_i^T + \varphi_i + L_i Q_i L_i^T\right].$$ (19)

In light of Lemma 2 and $\mathbb{E}[\bar{X}_{i|i} \bar{X}_{i|i}^T] = O_{i|i}$, (17) can be obtained. □

Theorem 2. The dynamic equation of the filtering error covariance $O_{i+1|i}$ can be described as follows:

$$O_{i+1|i} = (I - J_{i+1} \bar{M}_{i+1}) O_{i+1|i} (I - J_{i+1} \bar{M}_{i+1})^T$$

$$+ J_{i+1} \mathbb{E}[\bar{y}_{i+1} \bar{y}_{i+1}^T] J_{i+1}^T$$

$$+ J_{i+1} \Phi_{i+1} J_{i+1}^T + J_{i+1} R_{i+1} K_{i+1}^T$$

$$+ \text{Sym} \left\{ \mathbb{E} \left[ (I - J_{i+1} \bar{M}_{i+1}) \bar{X}_{i+1|i} (\bar{y}_{i+1} - \bar{y}_{i+1}) J_{i+1}^T \right] \right\}$$

$$+ \mathbb{E} \left[ J_{i+1} (\bar{y}_{i+1} - \bar{y}_{i+1}) v_{i+1}^T J_{i+1}^T \right].$$ (20)

Proof. Substituting $\bar{X}_{i+1|i}$ in (10) into $\bar{X}_{i+1|i}$ yields

$$\bar{X}_{i+1|i} = \bar{X}_{i+1|i} - J_{i+1} (\bar{y}_{i+1} - \bar{M}_{i+1} \bar{X}_{i+1|i})$$

$$= \bar{X}_{i+1|i} - J_{i+1} (\bar{y}_{i+1} - \bar{y}_{i+1}) - J_{i+1} \Delta M_{i+1} X_{i+1} + v_{i+1}$$

$$- J_{i+1} \Delta M_{i+1} X_{i+1} + J_{i+1} v_{i+1},$$

where the term $\bar{y}_{i+1} - \bar{y}_{i+1}$ and

$$y_{i+1} - \bar{M}_{i+1} \bar{X}_{i+1|i} = \bar{M}_{i+1} X_{i+1} + v_{i+1} - \bar{M}_{i+1} \bar{X}_{i+1|i}$$

$$= \bar{M}_{i+1} \bar{X}_{i+1|i} + \Delta M_{i+1} X_{i+1} + v_{i+1}$$

have been added. Moreover, we have

$$\mathbb{E}\left[ (I - J_{i+1} \bar{M}_{i+1}) \bar{X}_{i+1|i} (J_{i+1} \Delta M_{i+1} X_{i+1})^T \right] = 0,$$

$$\mathbb{E}\left[ (I - J_{i+1} \bar{M}_{i+1}) \bar{X}_{i+1|i} v_{i+1}^T J_{i+1}^T \right] = 0,$$

$$\mathbb{E}\left[ J_{i+1} \Delta M_{i+1} X_{i+1} v_{i+1}^T J_{i+1}^T \right] = 0.$$ (23)

Therefore, by means of the definition of $O_{i+1|i}$, we can obtain

$$O_{i+1|i} = (I - J_{i+1} \bar{M}_{i+1}) O_{i+1|i} (I - J_{i+1} \bar{M}_{i+1})^T$$

$$+ J_{i+1} \mathbb{E}[\bar{y}_{i+1} \bar{y}_{i+1}^T] J_{i+1}^T$$

$$+ J_{i+1} \Phi_{i+1} J_{i+1}^T + J_{i+1} R_{i+1} K_{i+1}^T$$

$$+ \text{Sym} \left\{ \mathbb{E} \left[ (I - J_{i+1} \bar{M}_{i+1}) \bar{X}_{i+1|i} (\bar{y}_{i+1} - \bar{y}_{i+1}) J_{i+1}^T \right] \right\}$$

$$+ \mathbb{E} \left[ J_{i+1} (\bar{y}_{i+1} - \bar{y}_{i+1}) v_{i+1}^T J_{i+1}^T \right].$$ (24)

Therefore, (20) holds. □

Remark 4. It is worth mentioning that the estimation error covariances have been obtained in Theorems 1-2 for all ORFs, DETM, and missing measurements provided that
matrix equations (17) and (20) are solvable. However, the exact values of the estimation error covariances are difficult to be computed owing to the existence of $\bar{y}_{i+1} - y_{i+1}$ caused by DTM. In the following, we will find another way to solve this problem, i.e., an upper bound of the filtering error covariance will be obtained, and its trace will be minimized by designing the gain parameter.

**Theorem 3.** Let $e_1$ and $e_2$ be positive scalars and $Y_{00} = O_{00} > 0$. If the difference equations have solutions $\bar{y}_{i+1|j_i}$ and $Y_{i+1|j_i}$

\[
\begin{align*}
Y_{i+1|j_i} &= \bar{H}_{i} \bar{Y}_{i|j_i} \bar{H}_{i}^T + \Phi_{1j} + L_0 Q_{1} L_0^T,
Y_{i+1|j_i+1} &= (1 + e_1) (I - J_{i+1} M_{i+1}) Y_{i+1|j_i} (I - J_{i+1} \times M_{i+1})^T
+ (1 + e_1^{-1} + e_2) J_{i+1} \Theta_{i+1} J_{i+1} +
+ J_{i+1} \Phi_{2j} + (1 + e_2^{-1}) R_{i+1} J_{i+1},
\end{align*}
\]

then $Y_{i+1|j_i}$ is the upper bound of $O_{i+1|j_i}$ and $Y_{i+1|j_i+1}$ is the upper bound of $O_{i+1|j_i+1}$, i.e., $O_{i+1|j_i} \leq Y_{i+1|j_i}$ and $O_{i+1|j_i+1} \leq Y_{i+1|j_i+1}$ hold.

**Proof.** Obviously, we know $O_{00} \leq Y_{00}$. Suppose that $O_{i|j} \leq Y_{i|j}$, and we obtain

\[
O_{i+1|j} - Y_{i+1|j} = \bar{H}_{i} (O_{i|j} - Y_{i|j}) \bar{H}_{i}^T \leq 0,
\]

which implies $O_{i+1|j} \leq Y_{i+1|j}$.

Using the following inequality that

\[
f g^T + g f^T \leq \varepsilon f^T + e^{-1} g g^T (\varepsilon > 0).
\]

we obtain

\[
\begin{align*}
E\left[-(I - J_{i+1} M_{i+1}) \bar{X}_{i+1|j_1} (\bar{Y}_{i+1|j_1} - y_{i+1})^T J_{i+1}^T\right]
&+ E\left[-J_{i+1} (\bar{Y}_{i+1|j_1} - y_{i+1}) \bar{X}_{i+1|j_1} (I - J_{i+1} M_{i+1})^T\right]
\leq e_1 (I - J_{i+1} M_{i+1}) O_{i+1|j} (I - J_{i+1} M_{i+1})^T
+ e_1^{-1} J_{i+1} E\left[(\bar{Y}_{i+1|j_1} - y_{i+1}) (\bar{Y}_{i+1|j_1} - y_{i+1})^T\right] J_{i+1},
E\left[J_{i+1} (\bar{Y}_{i+1|j_1} - y_{i+1}) v_{i+1} v_{i+1}^T J_{i+1}^T\right]
+ E\left[J_{i+1} v_{i+1} (\bar{Y}_{i+1|j_1} - y_{i+1})^T J_{i+1}^T\right]
\leq e_2 J_{i+1} E\left[(\bar{Y}_{i+1|j_1} - y_{i+1}) (\bar{Y}_{i+1|j_1} - y_{i+1})^T\right] J_{i+1}^T
+ e_2^{-1} J_{i+1} E\left[v_{i+1} v_{i+1}^T\right] J_{i+1}^T.
\end{align*}
\]

Considering

\[
\begin{align*}
J_{i+1} E\left[(\bar{Y}_{i+1|j_1} - y_{i+1}) (\bar{Y}_{i+1|j_1} - y_{i+1})^T\right] J_{i+1}^T &\leq J_{i+1} \Theta_{i+1} J_{i+1}^T,
J_{i+1} E\left[v_{i+1} v_{i+1}^T\right] J_{i+1}^T &\leq J_{i+1} R_{i+1} J_{i+1}^T,
\end{align*}
\]

where $\Theta_{i+1} \approx (1 + r^{-1}) \mu^2 + (1 + r) (\Theta_{i+1}/r^2)$, we have

\[
O_{i+1|j_{i+1}} \leq (1 + e_1) (I - J_{i+1} M_{i+1}) O_{i+1|j_{i+1}} (I - J_{i+1} M_{i+1})^T
+ (1 + e_1^{-1} + e_2) J_{i+1} \Theta_{i+1} J_{i+1}^T
+ J_{i+1} \Phi_{2j} + (1 + e_2^{-1}) R_{i+1} J_{i+1}.
\]

Based on $O_{i+1|j_{i+1}} \leq Y_{i+1|j_{i+1}}$, it is not difficult to obtain $O_{i+1|j_{i+1}} \leq Y_{i+1|j_{i+1}}$.

**Theorem 4.** The trace of the upper bound $Y_{i+1|j_{i+1}}$ is minimal, if the estimator gain parameter $J_{i+1}$ satisfies

\[
J_{i+1} = (1 + e_1) Y_{i+1|j_{i+1}} \bar{M}_{i+1}^T \Pi_{i+1}^{-1},
\]

with

\[
\Pi_{i+1} = (1 + e_1) M_{i+1} Y_{i+1|j_{i+1}} \bar{M}_{i+1}^T + (1 + e_1^{-1} + e_2) \Theta_{i+1} I
+ \Phi_{2} J_{i+1} + (1 + e_2^{-1}) R_{i+1}.
\]

**Proof.** Taking the first variation to $\text{tr}(Y_{i+1|j_{i+1}})$ yields

\[
\frac{\partial \text{tr}(Y_{i+1|j_{i+1}})}{\partial J_{i+1}} = -2 (1 + e_1) (I - J_{i+1} \bar{M}_{i+1})
\times Y_{i+1|j_{i+1}} \bar{M}_{i+1}^T + 2 J_{i+1} \left[(1 + e_1^{-1} + e_2) \Lambda_{i+1} I
+ \Phi_{2} J_{i+1} + (1 + e_2^{-1}) R_{i+1}\right].
\]

Let $\frac{\partial \text{tr}(Y_{i+1|j_{i+1}})}{\partial J_{i+1}} = 0$; we can obtain

\[
J_{i+1} \Pi_{i+1} = (1 + e_1) Y_{i+1|j_{i+1}} \bar{M}_{i+1}^T.
\]

So, (31) can be immediately obtained from (34).

**Remark 5.** It should be noted that, in our main results, DTM, ROFs, and missing measurements are all dealt within a unified framework and are explicitly reflected in the estimator’s design. Specifically, $\bar{F}_{i}$ in the $\bar{H}_{i}$ is used to quantify the effects of ROFs, $\bar{n}_{j,i}$ in the $\bar{M}_{i}$ account for the phenomenon of missing measurements, and the $\Theta_{i+1}$ characterizes the parameter resulted from DTM. Obviously, the proposed state and fault estimation algorithms are different from some existing ones. For example, DTM and missing measurements have not been discussed in [50] despite ROFs also taken into account in the joint state and fault estimation problem. On the other hand, missing measurements have been discussed in [48] where the main attention has been devoted to the fault estimation for complex networks while DTM has not been considered, which provides further evidence that the presented estimation algorithms in this paper are not a simple generalization of existing results.

### 3.2. Performance Analysis

**Lemma 5** (see [51]). Given scalars $\varepsilon > 0$, $0 < b < 1$, and $c > 0$ such that the condition listed below holds:

\[
\begin{align*}
O_{i+1|j_{i+1}} &\leq (1 + e_1) (I - J_{i+1} \bar{M}_{i+1}) O_{i+1|j_{i+1}} (I - J_{i+1} \bar{M}_{i+1})^T
+ (1 + e_1^{-1} + e_2) J_{i+1} \Theta_{i+1} J_{i+1}^T
+ J_{i+1} \Phi_{2j} + (1 + e_2^{-1}) R_{i+1} J_{i+1}.
\end{align*}
\]
Based on (18) and (21), the filtering error square
\[
E\left\{\|\hat{X}_i\|^2\right\} \leq \varepsilon E\left\{\|\hat{X}_0\|^2\right\}b' + c. \tag{35}
\]

Then, the stochastic process \(\chi_i\) is exponentially bounded in mean square sense.

**Assumption 1.** There exist positive real numbers \(\tilde{h}, l, l_H, m, r, \bar{\psi}, \bar{\Omega}, \bar{r}, \bar{\tau}\), and \(m_0\), such that the following results hold:
\[
\|\tilde{H}\| \leq \tilde{h}, \\
\|l\| \leq l_H, \|l_H^T\| \leq \tilde{H}, \\
m \leq \|M_{i+1}\| \leq \bar{m}, \\
\|R_i\| \leq \bar{r}, \\
\text{tr}(\psi_i) \leq \bar{\psi}, \\
\|G_i\| \leq \bar{\Omega}, \\
\bar{\Omega}_i \subset \bar{\Theta}, \\
\|L_i\| \leq Q_i \leq \bar{r} I, \\
\|M_{i+1}\| \leq \bar{m}_0.
\]

Based on Lemma 2 and Assumption 1, we are in the position to analyze the boundedness of the estimation error.

**Theorem 5.** If the following inequalities hold
\[
\tilde{h} \bar{m} < 1, \\
(1 + \eta_i) \left[1 - \left(1 + \frac{\tilde{h}^2 \bar{m} \bar{\psi}}{l^2 \bar{r}^2}\right)^{-1}\right] < 1, \tag{37}
\]
then the estimation error is exponentially bounded in mean square.

**Proof.** Based on (18) and (21), the filtering error \(\tilde{X}_{i+1|i+1}\) can be rewritten as
\[
\tilde{X}_{i+1|i+1} = U_{i+1} \tilde{H} \tilde{X}_{i|i} + \beta_{i+1} + \zeta_{i+1}, \tag{38}
\]
where
\[
U_{i+1} = I - J_{i+1} \tilde{M}_{i+1}, \\
\beta_{i+1} = -J_{i+1}(\tilde{y}_{i+1} - \tilde{y}_{i+1}), \\
\zeta_{i+1} = U_{i+1} \Delta \tilde{h} X_i + U_{i+1} \tilde{L}_i \omega_i - J_{i+1} \Delta M_{i+1} X_{i+1} - J_{i+1} \tilde{v}_{i+1}. \tag{39}
\]

From (31) and Assumption 1, we can easily derive that
\[
\|J_{i+1}\| \leq (1 + \varepsilon_i) \frac{m}{(1 + \varepsilon_i) m^T} = \frac{m}{m^T}\bar{K} \varepsilon_i. \tag{40}
\]

Then, the norm of \(U_{i+1}\) can be deduced that
\[
\|U_{i+1}\| \leq 1 + \|J_{i+1}\| \|M_{i+1}\| \leq 1 + \frac{m^2}{m^T} \bar{\tau}. \tag{41}
\]

Meanwhile,
\[
E\{\beta^T_{i+1} \beta_{i+1}\} \leq \bar{\Theta} K \bar{\tau} = \bar{\beta},
\]
\[
E\{\zeta^T_{i+1} \zeta_{i+1}\} = E\left[\tilde{X}^T_{i+1} \Delta H^T_{i+1} U_{i+1} \Delta H_i X_i\right]
+ E\left[-\bar{L}_i^T U_{i+1} \Delta H_i \tilde{v}_{i+1}\right] + \bar{L}_i Q_i \bar{L}_i^T + \bar{\tau} I.
\]

Define
\[
\Omega_{i+1} = U_{i+1} \tilde{H} \Omega_i \tilde{H}^T U_{i+1} + \bar{L}_i Q_i \bar{L}_i^T + \bar{\tau} I. \tag{43}
\]

We can obtain the following results:
\[
\|\Omega_{i+1}\| \leq \tilde{h}^2 \|\Omega_i\| + \bar{r} \|\tilde{v}\| + \bar{\tau} + \delta \leq \left(\tilde{h}^2 \|\tilde{v}\|\right)^{i+1} \|\Omega_0\| + (\bar{r} + \delta) \sum_{j=0}^{i} \left(\tilde{h}^2 \|\tilde{v}\|\right)^j. \tag{44}
\]

If \(\tilde{h} \bar{m} < 1\), we further obtain
\[
\|\Omega_{i+1}\| \leq \|\Omega_0\| + \frac{\bar{r} + \delta}{1 - \tilde{h}^2 \|\tilde{v}\|}. \tag{45}
\]

Besides, it is easy to verify \(\Omega_{i+1} \geq \delta I\). Consequently, there exist \(\bar{v} > 0\) and \(\bar{\psi} > 0\) such that \(\bar{v} I \leq \Omega_i \leq \bar{v} I\) is true for all \(i \geq 0\).

Define \(V_i(\tilde{X}_{i|i}) = \tilde{X}_{i|i}^T \tilde{X}_{i|i}\); we obtain
\[
E\{V_i(\tilde{X}_{i+1|i+1}|\tilde{X}_{i|i})\} = E\{V_i(\tilde{X}_{i|i})\} = E\left[\left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)^T \Omega_i \left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)\right]
\]
\[
= E\left[\left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)^T \Omega_i \left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)\right]
\]
\[
+ E\left[\left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)^T \Omega_i \left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)\right]
\]
\[
+ E\left[\left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)^T \Omega_i \left(U_{i+1} \tilde{H} X_i + \beta_{i+1} + \zeta_{i+1}\right)\right].
\]
Utilizing (27) again, the above results can be computed as follows:

\[
E[V_{i+1}(\tilde{X}_{i+1|i}) | \tilde{X}_{i|i}] - (1 + \eta_a)V_i(\tilde{X}_{i|i}) \\
\leq (1 + \eta_a)\left\{ \tilde{X}_{i|i}^T \left[ \tilde{H}_i^T U_{i|i}^T \Omega_{i}^{-1} U_{i|i} \tilde{H}_i - \Omega_{i}^{-1} \right] \tilde{X}_{i|i} \right\} \\
+ (1 + \eta_a^{-1})E\left[ \mu_{i+1}^T \Omega_{i+1}^{-1} \mu_{i+1} + E\left[ \tilde{\xi}_{i,i}^T \Omega_{i,i}^{-1} \tilde{\xi}_{i,i} \right] \right].
\]

(47)

From the matrix inversion lemma, we can see that

\[
\tilde{H}_i^T U_{i|i}^T \Omega_{i}^{-1} U_{i|i} \tilde{H}_i - \Omega_{i}^{-1} = \tilde{H}_i^T U_{i|i}^T \left[ U_{i|i} \tilde{H}_i \Omega_{i}^{-1} U_{i|i}^T + L_i Q_i L_i^T + \delta I \right]^{-1} \tilde{H}_i^T U_{i|i}^T \Omega_{i}^{-1} U_{i|i} \tilde{H}_i
\]

\[
\times U_{i|i} \tilde{H}_i - \Omega_{i}^{-1} = \left[ 1 + \tilde{H}_i^T U_{i|i}^T \left( L_i Q_i L_i^T + \delta I \right)^{-1} U_{i|i} \tilde{H}_i \Omega_{i}^{-1} \right]^{-1} \Omega_{i}^{-1}
\]

\[
\leq \left( 1 + \frac{\tilde{H}_i^T U_{i|i}^T \left( L_i Q_i L_i^T + \delta I \right)^{-1} U_{i|i} \tilde{H}_i}{\Omega_{i}^{-1}} \right)^{-1} \Omega_{i}^{-1}.
\]

(48)

Therefore, it is clear that

\[
E[V_{i+1}(\tilde{X}_{k+1_{|k+1}}) | \tilde{X}_{i|i}] - (1 + \eta_a)V_i(\tilde{X}_{i|i}) \\
\leq - (1 + \eta_a) \left( 1 + \frac{\tilde{H}_i^T U_{i|i}^T \left( L_i Q_i L_i^T + \delta I \right)^{-1} U_{i|i} \tilde{H}_i}{\Omega_{i}^{-1}} \right)^{-1} V_i(\tilde{X}_{i|i}) \\
+ (1 + \eta_a^{-1}) \frac{\tilde{\nu}_a^2}{\Omega_{i}^{-1}} \frac{\tilde{\xi}_a^2}{\Omega_{i}^{-1}}.
\]

(49)

Rearranging the above formula, we further have

\[
E[V_{i+1}(\tilde{X}_{k+1_{|k+1}}) | \tilde{X}_{i|i}] \leq \nu V_i(\tilde{X}_{i|i}) + \phi.
\]

(50)

where \( \nu \equiv (1 + \eta_a) \left[ 1 - (1 + \tilde{H}_i^T U_{i|i}^T \Omega_{i}^{-1} U_{i|i} \tilde{H}_i)^{-1} \right] \) and \( \phi \equiv (1 + \eta_a^{-1}) \left( \tilde{H}_i^T U_{i|i}^T \Omega_{i}^{-1} U_{i|i} \tilde{H}_i \right)^{-1} \Omega_{i}^{-1} \left( \tilde{H}_i^T U_{i|i}^T \Omega_{i}^{-1} U_{i|i} \tilde{H}_i \right)^{-1} \Omega_{i}^{-1} \cdot \]

4. Illustrative Examples

In this section, a numerical example is provided to show the rationality of the designed algorithms.

Consider system (1) with the following coefficient matrices:

\[
H_i = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \\
L_i = \begin{bmatrix} 0.05 \\ 0.01 \end{bmatrix}, \\
M_i = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \\
G_i = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}.
\]

(51)

The initial values are assumed to be \( x_0 = [0.3, 0.2]^T, f(0) = 0.1, \tilde{X}_{1_{|0}} = [0.3, 0.2, 0.1]^T, R_0 = 0.01I_2, Q_0 = 0.01, \tilde{\xi}_i = 0.95, \tilde{\eta}_{i,j} = 0.95, \tilde{\eta}_{2,j} = 0.95, \psi_0 = 0.1I_3, H_{i,j} = 2 \sin(0.5t), \) and \( Y_{1_{|0}} = 4 \times 10^{-6}I_3. \) In addition, the other parameters are chosen as \( e_1 = e_2 = a_1 = a_2 = b_1 = b_2 = 0.01, \mu_1 = 0.01, \mu_2 = 0.01, \epsilon_0 = 0, \alpha_1 = 0.1, \alpha_2 = 0.1, \tau_1 = 10, \) and \( \tau_2 = 10. \)

To demonstrate the validity of the presented state and fault algorithms in this paper, Figures 1 and 2 give the curves of actual state and estimated ones and Figure 3 shows the curves of actual fault and its estimated fault. From the above figures, we can see that the designed estimators could track the original state or fault well. In particular, the
curves of actual estimation error variances and the corresponding upper bounds for the states and the fault are plotted in Figures 4–6. It is observed that the curves of upper bounds are above the curves of actual estimation error covariance, which indicate the rationality of our algorithm. All these simulation results illustrate that the estimation algorithms proposed in the present paper are efficient and feasible.

5. Conclusions

The recursive state and fault estimation problem has been studied in this work for an array of linear discrete systems subject to DETM, ROFs, and missing measurements. Some Bernoulli random variables have been used to characterize the phenomena of ROF and missing measurements. With the aid of the state augmentation, the system under investigation has been changed into a stochastic parameterized system, for which a recursive event-based estimator has been proposed to simultaneously estimate the state and the fault signal. Based on the stochastic analysis approach, an upper bound expressed by Riccati difference equations has been derived, which is then minimized by choosing the filter gain. Moreover, the existence of the exponential mean-square boundedness of the estimation error has also been analyzed. In the end, the effectiveness of the proposed joint estimation algorithm has been shown by a simulation example. Our future research topics would include the extension of developed method on the design of state and fault estimator for general complex systems with different event-triggered schemes.

Data Availability

No data were used to support this paper.
Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (61703050), Shandong Provincial Natural Science Foundation of China (ZR2016FQ16), and Binzhou University Project of China (BJXYL2016Y27 and BJXYL2017).

References


