Research Article

Complete Characterization of Resistance Distance for Linear Octagonal Networks

Jing Zhao,¹ Jia-Bao Liu ,¹ and Ali Zafari ²

¹School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China

²Department of Mathematics, Faculty of Science, Payame Noor University, P.O. Box 19395-4697, Tehran, Iran

Correspondence should be addressed to Jia-Bao Liu; liujiabaoad@163.com and Ali Zafari; zafari.math.pu@gmail.com

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Computing the resistance distance of a network is a fundamental and classical topic. In the aspects of considering the resistances between any two points of the lattice networks, there are many studies associated with the ladder networks and ladderlike networks. But the resistances between any two points for more complex structures than ladder networks or ladderlike networks are still unknown. In this paper, a rather complicated structure which is named linear octagonal network is considered. Treelike octagonal systems are cata-condensed systems of octagons, which represent a class of polycyclic conjugated hydrocarbons. A linear octagonal network is a cata-condensed octagonal system with no branchings. Moreover, the resistances between any two points of a linear octagonal network are first determined. One finds that the effective resistances between new inserted points and others points of a linear octagonal network can be given by the effective resistances between two initial points which are inherited from the linear polyomino network.

1. Introduction

Our convention in this paper follows [1] for notations that we omit here. The resistance distance Ω_{ij} between vertices i and j was introduced in 1993 by Klein and Randić [2]. Based on the electrical circuit theory, one can obtain the resistance distance between two points by replacing each edge of the network with a unit resistor. In the past two decades, the resistance distance has attracted experts' much attention. The most important reason for this heightened activity is that it goes beyond the practical applications of the traditional electrical ones, such as in chemistry and network science. The resistance distance is a good indicator to distinguish isomer or molecular networks with similar structures.

We say the electrical network is simple. It means that the resistance distance of each edge in an electrical network is the unit resistor [3]. Denoting Ω_e as the resistance distance of the edge e , then the conductance of the edge of e is $c_e = 1/\Omega_e$. The effective resistance between any points can be represented by the resistance distance between any points on

a network. For a general network, it is hard enough to determine the effective resistance between any points, unless the network is very small or one knows the complete information of the network, such as degree of each vertex and distances between any two vertices. Hence, such methods are proposed which only need some local information; examples include star-triangle transformation [4, 5], resistance distance local sum rules [6], and effective resistance sum rules [3]. For more information associated with the effective resistance or resistance distance, see [7–15].

For path P_k and its copy P'_k , let $V(P_k) = \{x_1, x_2, \dots, x_k\}$ and $V(P'_k) = \{x'_1, x'_2, \dots, x'_k\}$. If $k = Th + 1$ (T, h are all positive integers), we connect x_{Ti+1} to x'_{Ti+1} for $i = 0, 1, \dots, h$ and denote this graph by L_k . Obviously, if $T = 1$, L_k is the linear polymino network; if $T = 2$, L_k is the linear hexagonal network; and if $T = 3$, L_k is the linear octagonal network. Polyomino systems are widely studied in organic chemistry, especially in polycyclic aromatic compounds [16]. Treelike octagonal systems are cata-condensed systems of octagons, which represent a class of polycyclic conjugated hydrocarbons. A linear octagonal network is a cata-condensed

octagonal system with no branchings [17]. Also, a linear octagonal network can be constructed from a linear polyomino network by inserting some new points in the line with certain rules (see Figure 1). These networks are very crucial in theoretical chemistry because they are natural graph representations of some hydrocarbons.

The Kirchhoff index and multiplicative degree-Kirchhoff index are, respectively, defined as $Kf(G) = \sum_{i < j} \Omega_{ij}$ [2] and $Kf^*(G) = \sum_{i < j} d_i d_j \Omega_{ij}$ [18], where d_i is the degree of vertex i , which is equal to the number of edges connected to vertex i . These two indices are found to be well associated with the Laplacian eigenvalues λ_i and normalized Laplacian eigenvalues v_i . Putting them in other ways, $Kf(G) = n \cdot \sum_{i=2}^n 1/\lambda_i$ [2] and $Kf^*(G) = 2m \cdot \sum_{i=2}^n 1/v_i$ [19]. In most recent years, the study of Kirchhoff index and multiplicative degree-Kirchhoff index caught some scholars' eyes, especially for linear systems. The Laplacian and normalized Laplacian spectra of linear systems come from much symmetric tridiagonal matrices, like how Zhu [20] determined the Laplacian and normalized Laplacian spectra of linear octagonal networks and those two indices. The computation of the Laplacian and normalized Laplacian spectra of the corresponding symmetric tridiagonal matrices seems interesting but rather complicated. Hence, from the point of view of the definitions for the Kirchhoff index and multiplicative degree-Kirchhoff index, we aim to compute the effective resistances between any points of linear octagonal networks in this paper via circuit reduction theory and effective resistance sum rule. The effective resistances of other linear

systems and circuit reduction theory or star-triangle transformation can be referred in [4, 8, 21–23].

In this paper, some preliminaries are proposed in Section 2. The effective resistances between two initial points which are inherited from a linear polyomino network are determined in Section 3. The effective resistances between new inserted points and other points in a linear octagonal network are given in Section 4. The whole paper is concluded in Section 5.

2. Preliminaries

In this section, we put a lemma that is used in computing the effective resistances between new inserted points and other points in a linear octagonal network.

Lemma 1 (see [3]). *Suppose that $N = (G, C)$ is a electrical network. For any two points i and j , $i \neq j$. Then,*

$$c_i r_{ij} + \sum_{k \in N(i)} c_{ik} (r_{ik} - r_{jk}) = 2, \quad (1)$$

$$c_i = \sum_{k \in N(i)} c_{ik},$$

where $N(i)$ is the neighbor set of point i .

For the sake of convenience, we here define three functions:

$$\alpha = 4 - \sqrt{15},$$

$$H(x, y, n) = \frac{3(1 - \alpha^{x-y})[2 - \alpha^{x+y-1} + \alpha^{2y-1} + \alpha^{2n-2x+1}(1 - \alpha^{x-y} - 2\alpha^{x+y-1})]}{4\sqrt{15}(1 - \alpha^{2n})}, \quad (2)$$

$$G(x, y, n) = \frac{3(1 + \alpha^{x-y})[2 + \alpha^{x+y-1} + \alpha^{2y-1} + \alpha^{2n-2x+1}(1 + \alpha^{x-y} + 2\alpha^{x+y-1})]}{4\sqrt{15}(1 - \alpha^{2n})}.$$

According to the structures of the linear octagonal network, we can divide its effective resistances between any points into two cases. For any $i, j \in \{1, 2, \dots, n\}$, one has the following cases:

Case 1. Effective resistances between two initial points are

$$r_{L_n}(p_i, p_j), \quad (3)$$

$$r_{L_n}(q_i, p_j).$$

Case 2. Effective resistances between new inserted points and other points are

$$\begin{aligned} & r_{L_n}(s_i, s_j), \\ & r_{L_n}(s_i, t_j), \\ & r_{L_n}(s_i, p_j), \\ & r_{L_n}(s_i, q_j), \\ & r_{L_n}(u_i, u_j), \\ & r_{L_n}(u_i, v_j), \\ & r_{L_n}(s_i, u_j), \\ & r_{L_n}(u_i, p_j), \\ & r_{L_n}(u_i, q_j), \\ & r_{L_n}(s_i, v_j). \end{aligned} \quad (4)$$

FIGURE 1: Linear polyomino network Q_n and linear octagonal network L_n .

3. Effective Resistances between Two Initial Points

In this section, the effective resistances between two initial points which are inherited from a linear polyomino network are determined. Let the linear octagonal network be a simple electrical network. Putting it in another way, the resistance of each edge in a linear octagonal network is 1. One gets the linear polyomino network Q_n^* from the linear octagonal network L_n by using circuit reductions (see Figure 2).

Assume that $x_n = r_{Q_n^*}(p_n, p_1)$, $y_n = r_{Q_n^*}(p_n, q_1)$, and $z_n = r_{Q_n^*}(p_n, q_n)$. One obtains the network Q_{n-1}^* by making a circuit reduction for the linear polyomino network Q_n^* with respect to points p_{n-1} and q_{n-1} (see Figure 3).

According to the network Q_{n-1}^* , one obtains

$$\frac{1}{z_n} = 1 + \frac{1}{6 + z_{n-1}} = \frac{7 + z_{n-1}}{6 + z_{n-1}}, \quad (5)$$

$$z_1 = 1.$$

By computing equation (5), we obtain

$$z_n = \frac{(-3 + \sqrt{15})(-4 - \sqrt{15})^n + (3 + \sqrt{15})(-4 + \sqrt{15})^n}{(-4 - \sqrt{15})^n - (-4 + \sqrt{15})^n}$$

$$= -3 - \sqrt{15} + \frac{2\sqrt{15}(4 + \sqrt{15})}{(4 + \sqrt{15})^n - (4 - \sqrt{15})^n}. \quad (6)$$

In what follows, we first calculate $r_{Q_n^*}(p_n, p_1)$ and $r_{Q_n^*}(p_n, q_1)$. One obtains the network Q_{n+1}^* from the network Q_n^* through a circuit reduction or

star-triangle transformation about points p_n, q_n , and p_1 (see Figure 4).

On the one hand, one has $K + N = x_n$, $M + N = y_n$, and $K + M = z_n$ based on the network Q_{n+1}^* , namely,

$$K = \frac{x_n - y_n + z_n}{2},$$

$$M = \frac{-x_n + y_n + z_n}{2}, \quad (7)$$

$$N = \frac{x_n + y_n - z_n}{2}.$$

On the other hand, we have

$$x_{n+1} = \frac{(3 + K)(4 + M)}{7 + z_n} + N,$$

$$y_{n+1} = \frac{(3 + M)(4 + K)}{7 + z_n} + N. \quad (8)$$

Combining equations (7) and (8) yields

$$x_{n+1} = \frac{(6 + x_n - y_n + z_n)(8 - x_n + y_n + z_n)}{4(7 + z_n)} + \frac{x_n + y_n - z_n}{2}, \quad n \geq 1, \quad (9)$$

$$y_{n+1} = \frac{(6 - x_n + y_n + z_n)(8 + x_n - y_n + z_n)}{4(7 + z_n)} + \frac{x_n + y_n - z_n}{2}, \quad n \geq 1, \quad (10)$$

where $x_1 = 0$ and $y_1 = 1$.

Subtracting equation (10) from equation (9) leads to

FIGURE 2: Linear polyomino network Q_n^* . FIGURE 3: The network Q_{n-1}^* . FIGURE 4: The network Q_{n+1}^* .

$$x_{n+1} - y_{n+1} = - \prod_{k=1}^n \frac{1}{z_k + 7}, \quad (11) \quad y_{n+1} = y_n + \frac{\sqrt{15}}{1 - (4 - \sqrt{15})^{n+1}} - \frac{\sqrt{15}}{1 - (4 - \sqrt{15})^n} + \frac{3}{2}, \quad y_1 = 1. \quad (13)$$

According to equations (6) and (11), one arrives at

$$x_n - y_n = \frac{2\sqrt{15}}{(4 - \sqrt{15})^n - (4 + \sqrt{15})^n}. \quad (12)$$

Substituting equations (6) and (12) into equation (10), one obtains

Hence,

$$y_n = \frac{\sqrt{15}}{1 - (4 - \sqrt{15})^n} + \frac{3(n-2) - \sqrt{15}}{2}, \quad n \geq 1. \quad (14)$$

By equations (12) and (14), we obtain

$$x_n = \frac{\sqrt{15}}{1 + (4 - \sqrt{15})^n} + \frac{3(n-2) - \sqrt{15}}{2}, \quad n \geq 1. \quad (15)$$

Moreover,

$$\begin{aligned} x_n - y_n + z_n &= -3 - \sqrt{15} + \frac{\sqrt{15}}{1 + (4 - \sqrt{15})^n}, \\ -x_n + y_n + z_n &= -3 - \sqrt{15} + \frac{\sqrt{15}}{1 - (4 - \sqrt{15})^n}, \\ x_n + y_n - z_n &= 3(n-1). \end{aligned} \quad (16)$$

Next, we will compute $r_{Q_n^*}(p_n, p_i)$, $r_{Q_n^*}(p_n, q_i)$, and $r_{Q_n^*}(p_i, q_i)$, $n > i > 1$. We make a circuit reduction of the network Q_n^* with respect to points p_n , p_{i+1} , and q_{i+1} (see Figure 5).

It is easy to check that the upper and lower parts of the network Q_n^* are, respectively, the networks Q_{n-i}^* and Q_i^* in line with Figure 5. Also, one has $A + B = x_{n-i}$, $A + C = y_{n-i}$, and $B + C = z_{n-i}$.

According to equation (16), we arrive at

$$\begin{aligned} A &= \frac{x_{n-i} + y_{n-i} - z_{n-i}}{2} = \frac{3(n-i-1)}{2}, \\ B &= \frac{x_{n-i} - y_{n-i} + z_{n-i}}{2} = \frac{-3 - \sqrt{15}}{2} + \frac{\sqrt{15}}{2 + 2(4 - \sqrt{15})^{n-i}}, \\ C &= \frac{-x_{n-i} + y_{n-i} + z_{n-i}}{2} = \frac{-3 - \sqrt{15}}{2} + \frac{\sqrt{15}}{2 - 2(4 - \sqrt{15})^{n-i}}. \end{aligned} \quad (17)$$

By equation (17), we have

$$\begin{aligned} r_{Q_n^*}(p_n, p_i) &= A + \frac{(3+B)(C+3+z_i)}{6+z_{n-i}+z_i}, \\ r_{Q_n^*}(p_n, q_i) &= A + \frac{(3+C)(B+3+z_i)}{6+z_{n-i}+z_i}, \\ r_{Q_n^*}(p_i, q_i) &= \frac{z_i(z_{n-i}+6)}{6+z_{n-i}+z_i}. \end{aligned} \quad (18)$$

Substituting equations (6) and (17) into (18) yields

$$\begin{aligned} r_{Q_n^*}(p_n, p_i) &= \frac{3(n-i)}{2} + \frac{3(1-\alpha^{n-i})(2-2\alpha^{n+i}-\alpha^{n+i-1}-\alpha^{n-i+1}+\alpha^{2i-1}+\alpha)}{4\sqrt{15}(1-\alpha^{2n})}, \\ r_{Q_n^*}(p_n, q_i) &= \frac{3(n-i)}{2} + \frac{3(1+\alpha^{n-i})(2+2\alpha^{n+i}+\alpha^{n+i-1}+\alpha^{n-i+1}+\alpha^{2i-1}+\alpha)}{4\sqrt{15}(1-\alpha^{2n})}, \\ r_{Q_n^*}(p_i, q_i) &= \frac{1+\alpha^{2i-1}+\alpha^{2n-2i+1}+\alpha^{2n}}{\sqrt{15}(1-\alpha^{2n})}. \end{aligned} \quad (19)$$

It is not hard to verify that equation (19) holds for $n \geq i \geq 1$.

At this place, we are going to give the explicit formulas for $r_{L_n}(p_i, p_j)$ and $r_{L_n}(p_i, q_j)$. One makes a circuit reduction for the network Q_n^* with respect to points p_i , q_i , and p_j , as illustrated in Figure 6.

Based on Figure 6, one has

$$\begin{aligned} r_{Q_n^*}(p_i, p_j) &= D + E, \\ r_{Q_n^*}(q_i, p_j) &= E + F, \\ r_{Q_n^*}(p_i, q_i) &= D + F = z_i. \end{aligned} \quad (20)$$

Thus,

$$\begin{aligned} D &= \frac{r_{Q_n^*}(p_i, p_j) + z_i - r_{Q_n^*}(q_i, p_j)}{2}, \\ E &= \frac{r_{Q_n^*}(p_i, p_j) - z_i + r_{Q_n^*}(q_i, p_j)}{2}, \\ F &= \frac{-r_{Q_n^*}(p_i, p_j) + z_i + r_{Q_n^*}(q_i, p_j)}{2}. \end{aligned} \quad (21)$$

Computing equation (21), we have

$$\begin{aligned} r_{Q_n^*}(p_i, p_j) &= \frac{D(F+z_{n-i}+6)}{z_i+z_{n-i}+6} + E, \\ r_{Q_n^*}(q_i, p_j) &= \frac{F(D+z_{n-i}+6)}{z_i+z_{n-i}+6} + E. \end{aligned} \quad (22)$$

Combining equations (6), (19), (21), and (22), one obtains

$$\begin{aligned} r_{Q_n^*}(p_i, p_j) &= \frac{3(i-j)}{2} + H(i, j, n), \\ r_{Q_n^*}(q_i, p_j) &= \frac{3(i-j)}{2} + G(i, j, n). \end{aligned} \quad (23)$$

One can straightforwardly verify that the formulas of $r_{L_n}(p_i, p_j)$ and $r_{L_n}(q_i, p_j)$ also hold for $n \geq i \geq j \geq 1$. Moreover, we have

FIGURE 5: The network Q_n^* by a circuit reduction about points p_n, p_{i+1} , and q_{i+1} .

FIGURE 6: The network Q_n^* by a circuit reduction about points p_i, q_i , and p_j .

$$r_{L_n}(p_i, p_j) = r_{Q_n^*}(p_i, p_j) = \frac{3(i-j)}{2} + H(i, j, n), \quad (24)$$

$$r_{L_n}(q_i, p_j) = r_{Q_n^*}(q_i, p_j) = \frac{3(i-j)}{2} + G(i, j, n). \quad (25)$$

4. Effective Resistances between New Inserted Points and Others

In Section 3, the formulas of effective resistances between the initial points of the linear octagonal network are determined. At this point, we will give the complete information of effective

resistances between new inserted points and other points in the linear octagonal network by dividing them into 10 subcases. One obtains the network W_n^* by making a circuit reduction for the linear octagonal network L_n with respect to point sets $\{v_1, \dots, v_{n-1}\}$ and $\{u_1, \dots, u_{n-1}\}$ (see Figure 7).

In what follows, we will concentrate on computing $r_{W_n^*}(s_i, p_i)$ and $r_{W_n^*}(s_i, p_{i+1})$. According to Lemma 1 and Figure 7, we obtain

$$\begin{aligned} & \frac{3}{2}r_{W_n^*}(s_i, p_i) + r_{W_n^*}(s_i, p_i) - r_{W_n^*}(p_i, p_i) + \frac{1}{2}r_{W_n^*}(s_i, p_{i+1}) \\ & - \frac{1}{2}r_{W_n^*}(p_i, p_{i+1}) = 2, \end{aligned} \quad (26)$$

$$\begin{aligned} & \frac{3}{2}r_{W_n^*}(s_i, p_{i+1}) + r_{W_n^*}(s_i, p_i) - r_{W_n^*}(p_{i+1}, p_i) + \frac{1}{2}r_{W_n^*}(s_i, p_{i+1}) \\ & - \frac{1}{2}r_{W_n^*}(p_{i+1}, p_{i+1}) = 2. \end{aligned} \quad (27)$$

On the one hand, subtracting equation (27) from the equation which is obtained using equation (26) times 4 leads to

$$r_{W_n^*}(s_i, p_i) = \frac{1}{9} [6 + r_{W_n^*}(p_i, p_{i+1})]. \quad (28)$$

Moreover, $r_{W_n^*}(p_i, p_{i+1}) = r_{W_n^*}(p_{i+1}, p_i)$. Hence,

$$r_{W_n^*}(p_{i+1}, p_i) = \frac{3}{2} + H(i+1, i, n). \quad (29)$$

Substituting equation (29) into equation (28), one obtains

$$r_{W_n^*}(s_i, p_i) = \frac{5}{6} + \frac{1}{9}H(i+1, i, n). \quad (30)$$

On the other hand, subtracting equation (26) from the equation which is obtained using equation (27) times 5/2 yields

$$r_{W_n^*}(s_i, p_{i+1}) = \frac{2}{9} [3 + 2r_{W_n^*}(p_{i+1}, p_i)]. \quad (31)$$

Substituting equation (29) into equation (31), one has

$$r_{W_n^*}(s_i, p_{i+1}) = \frac{4}{3} + \frac{4}{9}H(i+1, i, n). \quad (32)$$

For the resistance distance between the point s_i and other points, we can obtain

$$\begin{aligned} & \frac{3}{2}r_{W_n^*}(s_i, p_j) + r_{W_n^*}(s_i, p_i) - r_{W_n^*}(p_j, p_i) + \frac{1}{2}r_{W_n^*}(s_i, p_{i+1}) - \frac{1}{2}r_{W_n^*}(p_j, p_{i+1}) = 2, \\ & \frac{3}{2}r_{W_n^*}(s_i, q_j) + r_{W_n^*}(s_i, p_i) - r_{W_n^*}(q_j, p_i) + \frac{1}{2}r_{W_n^*}(s_i, p_{i+1}) - \frac{1}{2}r_{W_n^*}(q_j, p_{i+1}) = 2, \\ & \frac{3}{2}r_{W_n^*}(s_i, s_j) + r_{W_n^*}(s_i, p_i) - r_{W_n^*}(s_j, p_i) + \frac{1}{2}r_{W_n^*}(s_i, p_{i+1}) - \frac{1}{2}r_{W_n^*}(s_j, p_{i+1}) = 2, \\ & \frac{3}{2}r_{W_n^*}(s_i, t_j) + r_{W_n^*}(s_i, p_i) - r_{W_n^*}(t_j, p_i) + \frac{1}{2}r_{W_n^*}(s_i, p_{i+1}) - \frac{1}{2}r_{W_n^*}(t_j, p_{i+1}) = 2. \end{aligned} \quad (33)$$

By calculations, one arrives at

$$\begin{aligned} r_{W_n^*}(s_i, p_j) &= \frac{5}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) + \frac{2}{3}H(i, j, n) + \frac{1}{3}H(i+1, j, n), \\ r_{W_n^*}(s_i, q_j) &= \frac{5}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) + \frac{2}{3}G(i, j, n) + \frac{1}{3}G(i+1, j, n), \\ r_{W_n^*}(s_i, s_j) &= \frac{2}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) - \frac{2}{9}H(j+1, j, n) + \frac{4}{9}H(i, j, n) + \frac{2}{9}H(i, j+1, n) + \frac{2}{9}H(i+1, j, n) + \frac{1}{9}H(i+1, j+1, n), \\ r_{W_n^*}(s_i, t_j) &= \frac{2}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) - \frac{2}{9}H(j+1, j, n) + \frac{4}{9}H(i, j, n) + \frac{2}{9}G(i, j+1, n) + \frac{2}{9}G(i+1, j, n) + \frac{1}{9}G(i+1, j+1, n). \end{aligned} \quad (34)$$

FIGURE 7: The network W_n^* .

One can easily check that $r_{W_n^*}(s_i, p_j) \neq r_{W_n^*}(s_j, p_i)$ and $r_{W_n^*}(s_i, q_j) \neq r_{W_n^*}(s_j, q_i)$. Thus, we have

$$\begin{aligned} \frac{3}{2}r_{W_n^*}(s_j, p_i) + r_{W_n^*}(s_j, p_j) - r_{W_n^*}(p_i, p_j) + \frac{1}{2}r_{W_n^*}(s_j, p_{j+1}) - \frac{1}{2}r_{W_n^*}(p_i, p_{j+1}) &= 2, \\ \frac{3}{2}r_{W_n^*}(s_j, q_i) + r_{W_n^*}(s_j, p_j) - r_{W_n^*}(q_i, p_j) + \frac{1}{2}r_{W_n^*}(s_j, p_{j+1}) - \frac{1}{2}r_{W_n^*}(q_i, p_{j+1}) &= 2. \end{aligned} \quad (35)$$

Putting them in another way,

$$\begin{aligned} r_{W_n^*}(s_j, p_i) &= -\frac{1}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(j+1, j, n) + \frac{2}{3}H(i, j, n) \\ &\quad + \frac{1}{3}H(i, j+1, n), \\ r_{W_n^*}(s_j, q_i) &= -\frac{1}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(j+1, j, n) \\ &\quad + \frac{2}{3}G(i, j, n) + \frac{1}{3}G(i, j+1, n). \end{aligned} \quad (36)$$

It turns out that the formulas for $r_{W_n^*}(s_i, p_j)$, $r_{W_n^*}(s_j, p_i)$, $r_{W_n^*}(s_i, q_j)$, $r_{W_n^*}(s_j, q_i)$, $r_{W_n^*}(s_i, s_j)$, and $r_{W_n^*}(s_i, t_j)$ hold for any $n \geq i \geq j \geq 1$. Besides, we have

$$\begin{aligned} r_{L_n}(s_j, s_i) &= r_{L_n}(s_i, s_j) = r_{W_n^*}(s_j, s_i) = r_{W_n^*}(s_i, s_j), \\ r_{L_n}(s_i, t_j) &= r_{L_n}(s_j, t_i) = r_{W_n^*}(s_i, t_j) = r_{W_n^*}(s_j, t_i), \\ r_{L_n}(s_i, p_j) &= r_{W_n^*}(s_i, p_j), \\ r_{L_n}(s_j, p_i) &= r_{W_n^*}(s_j, p_i), \\ r_{L_n}(s_i, q_j) &= r_{W_n^*}(s_i, q_j), \\ r_{L_n}(s_j, q_i) &= r_{W_n^*}(s_j, q_i). \end{aligned} \quad (37)$$

One obtains the network $W_n^\#$ by making a circuit reduction for the linear octagonal network L_n with respect to point sets $\{s_1, \dots, s_{n-1}\}$ and $\{t_1, \dots, t_{n-1}\}$ (see Figure 8).

In what follows, we will concentrate on computing $r_{W_n^\#}(u_i, p_i)$ and $r_{W_n^\#}(u_i, p_{i+1})$. According to Lemma 1 and Figure 8, we obtain

FIGURE 8: The network $W_n^\#$.

$$\frac{3}{2}r_{W_n^\#}(u_i, p_i) + \frac{1}{2}r_{W_n^\#}(u_i, p_i) - \frac{1}{2}r_{W_n^\#}(p_i, p_i) + r_{W_n^\#}(u_i, p_{i+1}) - r_{W_n^\#}(p_i, p_{i+1}) = 2, \quad (38)$$

$$\frac{3}{2}r_{W_n^\#}(u_i, p_{i+1}) + \frac{1}{2}r_{W_n^\#}(u_i, p_i) - \frac{1}{2}r_{W_n^\#}(p_{i+1}, p_i) + r_{W_n^\#}(u_i, p_{i+1}) - r_{W_n^\#}(p_{i+1}, p_{i+1}) = 2. \quad (39)$$

On the one hand, subtracting equation (39) from the equation which is obtained using equation (38) times 5/2 leads to

$$r_{W_n^\#}(u_i, p_i) = \frac{4}{3} + \frac{4}{9}H(i+1, i, n). \quad (40)$$

On the other hand, subtracting equation (38) from the equation which is obtained using equation (39) times 4 yields

$$r_{W_n^\#}(u_i, p_{i+1}) = \frac{5}{6} + \frac{1}{9}H(i+1, i, n). \quad (41)$$

Remark 1. Combining equations (32) and (40), we find that

$$r_{L_n}(p_i, u_i) = r_{L_n}(s_i, p_{i+1}), \quad (42)$$

for any $i < n$.

For the resistance distance between the point u_i and other points, we can obtain

$$\begin{aligned} \frac{3}{2}r_{W_n^\#}(u_i, p_j) + \frac{1}{2}r_{W_n^\#}(u_i, p_i) - \frac{1}{2}r_{W_n^\#}(p_j, p_i) + r_{W_n^\#}(u_i, p_{i+1}) - r_{W_n^\#}(p_j, p_{i+1}) &= 2, \\ \frac{3}{2}r_{W_n^\#}(u_i, q_j) + \frac{1}{2}r_{W_n^\#}(u_i, p_i) - \frac{1}{2}r_{W_n^\#}(q_j, p_i) + r_{W_n^\#}(u_i, p_{i+1}) - r_{W_n^\#}(q_j, p_{i+1}) &= 2, \\ \frac{3}{2}r_{W_n^\#}(u_i, u_j) + \frac{1}{2}r_{W_n^\#}(u_i, p_i) - \frac{1}{2}r_{W_n^\#}(u_j, p_i) + r_{W_n^\#}(u_i, p_{i+1}) - r_{W_n^\#}(u_j, p_{i+1}) &= 2, \\ \frac{3}{2}r_{W_n^\#}(u_i, v_j) + \frac{1}{2}r_{W_n^\#}(u_i, p_i) - \frac{1}{2}r_{W_n^\#}(v_j, p_i) + r_{W_n^\#}(u_i, p_{i+1}) - r_{W_n^\#}(v_j, p_{i+1}) &= 2. \end{aligned} \quad (43)$$

By calculations, one arrives at

$$\begin{aligned}
 r_{W_n^\#}(u_i, p_j) &= \frac{4}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) + \frac{1}{3}H(i, j, n) + \frac{2}{3}H(i+1, j, n), \\
 r_{W_n^\#}(u_i, q_j) &= \frac{4}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) + \frac{1}{3}G(i, j, n) + \frac{2}{3}G(i+1, j, n), \\
 r_{W_n^\#}(u_i, u_j) &= \frac{2}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) - \frac{2}{9}H(j+1, j, n) + \frac{1}{9}H(i, j, n) + \frac{2}{9}H(i, j+1, n) + \frac{2}{9}H(i+1, j, n) + \frac{4}{9}H(i+1, j+1, n), \\
 r_{W_n^\#}(u_i, v_j) &= \frac{2}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) - \frac{2}{9}H(j+1, j, n) + \frac{1}{9}H(i, j, n) + \frac{2}{9}G(i, j+1, n) + \frac{2}{9}G(i+1, j, n) + \frac{4}{9}G(i+1, j+1, n).
 \end{aligned} \tag{44}$$

One can easily check that $r_{W_n^\#}(u_i, p_j) \neq r_{W_n^*}(u_j, p_i)$ and $r_{W_n^\#}(u_i, q_j) \neq r_{W_n^*}(u_j, q_i)$. Thus, we have

$$\begin{aligned}
 \frac{3}{2}r_{W_n^\#}(u_j, p_i) + \frac{1}{2}r_{W_n^\#}(u_j, p_j) - \frac{1}{2}r_{W_n^\#}(p_i, p_j) + r_{W_n^\#}(u_j, p_{j+1}) - r_{W_n^\#}(p_i, p_{j+1}) &= 2, \\
 \frac{3}{2}r_{W_n^\#}(u_j, q_i) + \frac{1}{2}r_{W_n^\#}(u_j, p_j) - \frac{1}{2}r_{W_n^\#}(q_i, p_j) + r_{W_n^\#}(u_j, p_{j+1}) - r_{W_n^\#}(q_i, p_{j+1}) &= 2.
 \end{aligned} \tag{45}$$

Immediately,

$$\begin{aligned}
 r_{W_n^\#}(u_j, p_i) &= -\frac{2}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(j+1, j, n) + \frac{1}{3}H(i, j, n) + \frac{2}{3}H(i, j+1, n), \\
 r_{W_n^\#}(u_j, q_i) &= -\frac{2}{3} + \frac{3}{2}(i-j) - \frac{2}{9}H(j+1, j, n) + \frac{1}{3}G(i, j, n) + \frac{2}{3}G(i, j+1, n).
 \end{aligned} \tag{46}$$

In what follows, we will devote to calculate the values of $r_{L_n}(s_i, u_j)$ and $r_{L_n}(s_j, u_i)$. Then,

$$\begin{aligned}
 2r_{L_n}(s_i, u_j) + r_{L_n}(s_i, p_i) - r_{L_n}(u_j, p_i) + r_{L_n}(s_i, u_i) - r_{L_n}(u_j, u_i) &= 2, \\
 2r_{L_n}(s_i, u_i) + r_{L_n}(s_i, p_i) - r_{L_n}(u_i, p_i) + r_{L_n}(s_i, u_i) - r_{L_n}(u_i, u_i) &= 2.
 \end{aligned} \tag{47}$$

Namely,

$$\begin{aligned}
 r_{L_n}(s_i, u_i) &= \frac{5}{6} + \frac{1}{9}H(i+1, i, n), \\
 r_{L_n}(s_i, u_j) &= \frac{1}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(i+1, i, n) - \frac{2}{9}H(j+1, j, n) + \frac{2}{9}H(i, j, n) + \frac{4}{9}H(i, j+1, n) + \frac{1}{9}H(i+1, j, n) + \frac{2}{9}H(i+1, j+1, n).
 \end{aligned} \tag{48}$$

In the same way, one can obtain

$$2r_{L_n}(s_j, u_i) + r_{L_n}(s_j, p_j) - r_{L_n}(u_i, p_j) + r_{L_n}(s_j, u_j) - r_{L_n}(u_i, u_j) = 2. \quad (49)$$

The further form of equation (49) is as follows:

$$\begin{aligned} r_{L_n}(s_j, u_i) &= \frac{7}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(j+1, j, n) - \frac{2}{9}H(i+1, i, n) \\ &\quad + \frac{2}{9}H(i, j, n) + \frac{4}{9}H(i+1, j, n) + \frac{1}{9}H(i, j+1, n) \\ &\quad + \frac{2}{9}H(i+1, j+1, n). \end{aligned} \quad (50)$$

$$\begin{aligned} r_{L_n}(s_i, v_j) &= \frac{1}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(j+1, j, n) - \frac{2}{9}H(i+1, i, n) + \frac{1}{18}H(i, j, n) + \frac{1}{6}G(i, j, n) + \frac{4}{9}G(i, j+1, n) + \frac{1}{9}G(i+1, j, n) \\ &\quad + \frac{2}{9}G(i+1, j+1, n), \\ r_{L_n}(s_i, v_j) &= \frac{7}{6} + \frac{3}{2}(i-j) - \frac{2}{9}H(j+1, j, n) - \frac{2}{9}H(i+1, i, n) + \frac{1}{18}H(i, j, n) + \frac{1}{6}G(i, j, n) + \frac{4}{9}G(i+1, j, n) + \frac{1}{9}G(i, j+1, n) \\ &\quad + \frac{2}{9}G(i+1, j+1, n). \end{aligned} \quad (53)$$

It turns out that the formulas for $r_{W_n^\#}(u_i, p_j)$, $r_{W_n^\#}(u_j, p_i)$, $r_{W_n^\#}(u_i, q_j)$, $r_{W_n^\#}(u_j, q_i)$, $r_{W_n^\#}(u_i, u_j)$, and $r_{W_n^\#}(s_i, v_j)$ hold for any $n \geq i \geq j \geq 1$. Besides, we have

$$\begin{aligned} r_{L_n}(u_j, u_i) &= r_{L_n}(u_i, u_j) = r_{W_n^\#}(u_j, u_i) = r_{W_n^\#}(u_i, u_j), \\ r_{L_n}(u_i, v_j) &= r_{L_n}(u_j, v_i) = r_{W_n^\#}(u_i, v_j) = r_{W_n^\#}(u_j, v_i), \\ r_{L_n}(u_i, p_j) &= r_{W_n^\#}(u_i, p_j), \\ r_{L_n}(u_i, q_j) &= r_{W_n^\#}(u_i, q_j), \\ r_{L_n}(u_j, p_i) &= r_{W_n^\#}(u_j, p_i), \\ r_{L_n}(u_j, q_i) &= r_{W_n^\#}(u_j, q_i). \end{aligned} \quad (54)$$

Remark 2. Combining equations (30), (41), and (48), we find that

$$r_{L_n}(p_i, s_i) = r_{L_n}(s_i, u_i) = r_{L_n}(u_i, p_{i+1}), \quad (51)$$

for any $i < n$.

For $r_{L_n}(s_i, v_j)$ and $r_{L_n}(s_j, v_i)$, one has

$$\begin{aligned} 2r_{L_n}(s_i, v_j) + r_{L_n}(s_i, p_i) - r_{L_n}(v_j, p_i) + r_{L_n}(s_i, u_i) - r_{L_n}(v_j, u_i) &= 2, \\ 2r_{L_n}(s_j, v_i) + r_{L_n}(s_j, p_j) - r_{L_n}(v_i, p_j) + r_{L_n}(s_j, u_j) - r_{L_n}(v_i, u_j) &= 2. \end{aligned} \quad (52)$$

By solving equations (51) and (52), one arrives at

By using equations (24) and (25), one obtains that the effective resistances between new inserted points and others points can be given by the effective resistances between two initial points which are inherited from the linear polyomino network.

Theorem 1. For linear octagonal network $n \geq i \geq j \geq 1$, we have the following:

(i) Effective resistances between two initial points are

$$\begin{aligned} r_{L_n}(p_i, p_j) &= r_{Q_n^*}(p_i, p_j) = \frac{3(i-j)}{2} + H(i, j, n), \\ r_{L_n}(q_i, p_j) &= r_{Q_n^*}(q_i, p_j) = \frac{3(i-j)}{2} + G(i, j, n). \end{aligned} \quad (55)$$

(ii) Effective resistances between new inserted points and others are

$$\begin{aligned}
r_{L_n}(s_i, s_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{4}{9}r_{L_n}(p_i, p_j) + \frac{2}{9}r_{L_n}(p_i, p_{j+1}) + \frac{2}{9}r_{L_n}(p_{i+1}, p_j) + \frac{1}{9}r_{L_n}(p_{i+1}, p_{j+1}), \\
r_{L_n}(s_i, t_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{4}{9}r_{L_n}(p_i, p_j) + \frac{2}{9}r_{L_n}(p_i, q_{j+1}) + \frac{2}{9}r_{L_n}(p_{i+1}, q_j) + \frac{1}{9}r_{L_n}(p_{i+1}, q_{j+1}), \\
r_{L_n}(s_i, p_j) &= \frac{2}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) + \frac{2}{3}r_{L_n}(p_i, p_j) + \frac{1}{3}r_{L_n}(p_{i+1}, p_j), \\
r_{L_n}(s_j, p_i) &= \frac{2}{3} - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{2}{3}r_{L_n}(p_i, p_j) + \frac{1}{3}r_{L_n}(p_i, p_{j+1}), \\
r_{L_n}(s_i, q_j) &= \frac{2}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) + \frac{2}{3}r_{L_n}(p_i, q_j) + \frac{1}{3}r_{L_n}(p_{i+1}, q_j), \\
r_{L_n}(s_j, q_i) &= \frac{2}{3} - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{2}{3}r_{L_n}(p_i, q_j) + \frac{1}{3}r_{L_n}(p_i, q_{j+1}), \\
r_{L_n}(u_i, u_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{1}{9}r_{L_n}(p_i, p_j) + \frac{2}{9}r_{L_n}(p_i, p_{j+1}) + \frac{2}{9}r_{L_n}(p_{i+1}, p_j) + \frac{4}{9}r_{L_n}(p_{i+1}, p_{j+1}), \\
r_{L_n}(u_i, v_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{1}{9}r_{L_n}(p_i, q_j) + \frac{2}{9}r_{L_n}(p_i, q_{j+1}) + \frac{2}{9}r_{L_n}(p_{i+1}, q_j) + \frac{4}{9}r_{L_n}(p_{i+1}, q_{j+1}), \\
r_{L_n}(s_i, u_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{2}{9}r_{L_n}(p_i, p_j) + \frac{4}{9}r_{L_n}(p_i, p_{j+1}) + \frac{1}{9}r_{L_n}(p_{i+1}, p_j) + \frac{2}{9}r_{L_n}(p_{i+1}, p_{j+1}), \\
r_{L_n}(s_j, u_i) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{2}{9}r_{L_n}(p_i, p_j) + \frac{4}{9}r_{L_n}(p_{i+1}, p_j) + \frac{1}{9}r_{L_n}(p_i, p_{j+1}) + \frac{2}{9}r_{L_n}(p_{i+1}, p_{j+1}), \\
r_{L_n}(u_i, p_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) + \frac{1}{3}r_{L_n}(p_i, p_j) + \frac{2}{3}r_{L_n}(p_i, p_{j+1}), \\
r_{L_n}(u_j, p_i) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{1}{3}r_{L_n}(p_i, p_j) + \frac{2}{3}r_{L_n}(p_{i+1}, p_j), \\
r_{L_n}(u_i, q_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) + \frac{1}{3}r_{L_n}(p_i, q_j) + \frac{2}{3}r_{L_n}(p_{i+1}, q_j), \\
r_{L_n}(u_j, q_i) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{1}{3}r_{L_n}(p_i, q_j) + \frac{2}{3}r_{L_n}(p_i, p_{j+1}), \\
r_{L_n}(s_i, v_j) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{1}{18}r_{L_n}(p_i, p_j) + \frac{1}{6}r_{L_n}(p_i, q_j) + \frac{4}{9}r_{L_n}(p_i, q_{j+1}) + \frac{1}{9}r_{L_n}(p_{i+1}, q_j) + \frac{2}{9}r_{L_n}(p_{i+1}, q_{j+1}), \\
r_{L_n}(s_j, v_i) &= \frac{4}{3} - \frac{2}{9}r_{L_n}(p_i, p_{i+1}) - \frac{2}{9}r_{L_n}(p_j, p_{j+1}) + \frac{1}{18}r_{L_n}(p_i, p_j) + \frac{1}{6}r_{L_n}(p_i, q_j) + \frac{4}{9}r_{L_n}(p_{i+1}, q_j) + \frac{1}{9}r_{L_n}(p_i, q_{j+1}) + \frac{2}{9}r_{L_n}(p_{i+1}, q_{j+1}).
\end{aligned} \tag{56}$$

5. Conclusion

In this paper, we have determined the effective resistances between any points of a linear octagonal network. Our methods used in this paper are standard. On the one hand, one can use circuit reductions and effective resistance sum rules to study other linear networks, such as linear hexagonal networks, octagonal-quadrilateral networks, and generalized phenylenes networks. On the other hand, the resistance distance-based parameters, namely, Kirchhoff index, multiplicative degree-Kirchhoff index, and additive degree-Kirchhoff index, are well defined; we may compute those

resistance distance-based parameters in terms of the results obtained in this paper.

Data Availability

The data used to support the findings of this study are available from the authors upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] F. R. K. Chung, *Spectral Graph Theory*, American Mathematical Society, Providence, RI, USA, 1997.
- [2] D. J. Klein and M. Randić, "Resistance distance," *Journal of Mathematical Chemistry*, vol. 12, no. 1, pp. 81–95, 1993.
- [3] H. Chen, "Random walks and the effective resistance sum rules," *Discrete Applied Mathematics*, vol. 158, no. 15, pp. 1691–1700, 2010.
- [4] Z. Jiang and W. Yan, "Resistances between two nodes of a path network," *Applied Mathematics and Computation*, vol. 361, pp. 42–46, 2019.
- [5] Z. Jiang and W. Yan, "Some two-point resistances of the Sierpinski gasket network," *Journal of Statistical Physics*, vol. 172, no. 3, pp. 824–832, 2018.
- [6] H. Chen and F. Zhang, "Resistance distance local rules," *Journal of Mathematical Chemistry*, vol. 44, no. 2, pp. 405–417, 2008.
- [7] Y. Yang and D. J. Klein, "Two-point resistances and random walks on stellated regular graphs," *Journal of Physics A: Mathematical and Theoretical*, vol. 52, no. 7, Article ID 075201, 2019.
- [8] Z. Jiang and W. Yan, "Resistance between two nodes of a ring network," *Physica A: Statistical Mechanics and Its Applications*, vol. 484, pp. 21–26, 2017.
- [9] L. Ye and W. Yan, "Resistance between two vertices of almost complete bipartite graphs," *Discrete Applied Mathematics*, vol. 257, pp. 299–305, 2019.
- [10] S. V. Gervacio, "Resistance distance in complete n-partite graphs," *Discrete Applied Mathematics*, vol. 203, pp. 53–61, 2016.
- [11] W. Barrett, E. J. Evans, and A. E. Francis, "Resistance distance in straight linear 2-trees," *Discrete Applied Mathematics*, vol. 258, pp. 13–34, 2019.
- [12] J. Fan, J. Zhu, L. Tian, and Q. Wang, "Resistance distance in potting networks," *Physica A: Statistical Mechanics and Its Applications*, vol. 540, p. 123053, 2020.
- [13] R. B. Bapat and S. Gupta, "Resistance distance in wheels and fans," *Indian Journal of Pure and Applied Mathematics*, vol. 41, no. 1, pp. 1–13, 2010.
- [14] D. J. Klein, "Resistance-distance sum rules," *Croatica Chemica Acta*, vol. 75, no. 2, pp. 633–649, 2002.
- [15] Y. Yang and D. J. Klein, "Resistance distances in composite graphs," *Journal of Physics A: Mathematical and Theoretical*, vol. 47, no. 37, p. 375203, 2014.
- [16] Y. Pan, C. Liu, and J. Li, "Kirchhoff indices and numbers of spanning trees of molecular graphs derived from linear crossed polyomino chain," *Polycyclic Aromatic Compounds*, pp. 1–8, 2020.
- [17] S. Li and W. Wei, "Extremal octagonal chains with respect to the coefficients sum of the permanent polynomial," *Applied Mathematics and Computation*, vol. 328, pp. 45–57, 2018.
- [18] H. Chen and F. Zhang, "Resistance distance and the normalized Laplacian spectrum," *Discrete Applied Mathematics*, vol. 155, no. 5, pp. 654–661, 2007.
- [19] I. Gutman and B. Mohar, "The quasi-Wiener and the Kirchhoff indices coincide," *Journal of Chemical Information and Computer Sciences*, vol. 36, no. 5, pp. 982–985, 1996.
- [20] Q. Zhu, "Kirchhoff index, degree-Kirchhoff index and spanning trees of linear octagonal chains," *The Australasian Journal of Combinatorics*, 2019.
- [21] Z. Cinkir, "Effective resistances and Kirchhoff index of ladder graphs," *Journal of Mathematical Chemistry*, vol. 54, no. 4, pp. 955–966, 2016.
- [22] A. Carmona, A. M. Encinas, and M. Mitjana, "Effective resistances for ladder-like chains," *International Journal of Quantum Chemistry*, vol. 114, no. 24, pp. 1670–1677, 2014.
- [23] L. Shi and H. Chen, "Resistance distances in the linear polyomino chain," *Journal of Applied Mathematics and Computing*, vol. 57, no. 1-2, pp. 147–160, 2018.