Research Article

Computing Minimal Doubly Resolving Sets and the Strong Metric Dimension of the Layer Sun Graph and the Line Graph of the Layer Sun Graph

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Let G be a finite, connected graph of order n with vertex set $V(G)$ and edge set $E(G)$. A set S of vertices of the graph G is a doubly resolving set for G if every two distinct vertices of G are doubly resolved by some two vertices of S . The minimal doubly resolving set of vertices of graph G is a doubly resolving set with minimum cardinality and is denoted by $\psi(G)$. In this paper, first, we construct a class of graphs of order $2n + \sum_{r=1}^{k-2} nm^r$, denoted by $LSG(n, m, k)$, and call these graphs as the layer Sun graphs with parameters n , m , and k . Moreover, we compute minimal doubly resolving sets and the strong metric dimension of the layer Sun graph $LSG(n, m, k)$ and the line graph of the layer Sun graph $LSG(n, m, k)$.

1. Introduction

In this paper, suppose G is a finite, simple connected graph of order n with vertex set $V(G)$ and edge set $E(G)$. If x and y are vertices in the graph G , then the distance x from y in G is denoted by $d_G(x, y)$ or simply $d(x, y)$, where $d(x, y)$ is the length of the shortest path from x to y . The line graph of a graph G is denoted by $L(G)$, with vertex set $V(L(G)) = E(G)$ and where two edges of G are adjacent in $L(G)$ if and only if they are incident in G , see [1]. Vertices x, y of the graph G are said to doubly resolve vertices u, v of G if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A set S of vertices of the graph G is a doubly resolving set of G if every two distinct vertices of G are doubly resolved by some two vertices of S . The minimal doubly resolving set of vertices of graph G is a doubly resolving set with minimum cardinality and is denoted by $\psi(G)$. The notion of a doubly resolving set of vertices of the graph G was introduced by Cáceres et al. [2]. A vertex w strongly resolves two vertices u and v if u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. A vertex set S of the graph G is a strong resolving

set of G if every two distinct vertices of G are strongly resolved by some vertex of S . A strong metric basis of G is denoted by $\text{sdim}(G)$ defined as the minimum cardinality of a strong resolving set of G . The notion of a strong metric dimension problem set of vertices of the graph G was introduced by A. Sebö and E. Tannier [3] and further investigated by O. R. Oellermann and Peters-Fransen [4]. The minimal doubly resolving sets for jellyfish and cocktail party graphs have been obtained in [5]. For more results related to these concepts, see [6–16]. In this paper, first, we construct a class of graphs of order $2n + \sum_{r=1}^{k-2} nm^r$, denoted by $LSG(n, m, k)$, and call these graphs as the layer Sun graphs with parameters n , m , and k , which is defined as follows:

Let n, m, k be integers such that $n, k \geq 3, m \geq 2$ and G be a graph with vertex set $V(G) = V_1 \cup V_2 \cup \dots \cup V_k$, where $V_1, V_2, \dots, V_k$ are called the layers of G such that $V_1 = V(C_n) = \{1, 2, \dots, n\}$, $V_2 = \{v_1, v_2, \dots, v_n\}$, and for $l \geq 3$, we have $V_l = \{B_{1_1}^{(l)}, B_{1_2}^{(l)}, \dots, B_{1_{m^{l-3}}}^{(l)}; B_{2_1}^{(l)}, B_{2_2}^{(l)}, \dots, B_{2_{m^{l-3}}}^{(l)}; \dots; B_{n_1}^{(l)}, B_{n_2}^{(l)}, \dots, B_{n_{m^{l-3}}}^{(l)}\}$, and let $B_{ij}^{(l)} = \{\bigcup_{t=1}^m (v_{ij}, t)^l\}$, such that every $(v_{ij}, t)^l$ is a vertex in the layer V_l , and $B_{ij}^{(l)} \cong$

$\overline{K_m}$ in the layer V_l , $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$, $1 \leq t \leq m$, where $\overline{K_m}$ is the complement of the complete graph on m vertices. Now, suppose that every vertex i in the cycle C_n is adjacent to exactly one vertex in the layer V_2 , say $v_i \in V_2$, and every vertex v_i in the layer V_2 is adjacent to exactly m vertices $(v_{i1}, 1)^3, (v_{i1}, 2)^3, \dots, (v_{i1}, m)^3 \in B_{i1}^{(3)} \in V_3$, in particular for $l \geq 3$, every vertex $(v_{ir}, t)^l \in B_{ir}^{(l)} \in V_l$ is adjacent to exactly m vertices $\bigcup_{t=1}^m (v_{ij}, t)^{l+1} \in B_{ij}^{(l+1)} \in V_{l+1}$, and then, the resulting graph is called the layer Sun graph $LSG(n, m, k)$ with parameters n, m , and k . Also, for $l \geq 3$, we recall that $B_{ij}^{(l)}$ as the components of the layer V_l , $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$. In particular, we say that two components $B_{ir}^{(l)}, B_{rs}^{(l)} \in V_l$, $1 \leq i, r \leq n$, $1 \leq j, s \leq m^{l-3}$ are fundamental if $i = r$ and $j \neq s$. It is natural to consider its vertex set of the layer Sun graph $LSG(n, m, k)$ as partitioned into k layers. The layers V_1 and V_2 consist of the vertices $\{1, 2, \dots, n\}$ and $\{v_1, v_2, \dots, v_n\}$, respectively. In particular, each layer V_l ($l \geq 3$) consists of the nm^{l-2} vertices. Note that, for each vertex i in the layer V_1 and every vertex $x \in B_{ij}^{(l)} \in V_l$, $l \geq 3$, $1 \leq j \leq m^{l-3}$, we have $d(i, x) = l - 1$. In this paper, we consider the problem of determining the cardinality $\psi(LSG(n, m, k))$ of minimal doubly resolving sets of the layer Sun graph $LSG(n, m, k)$. First, we find the metric dimension of the layer Sun graph $LSG(n, m, k)$; in fact, we prove that if $n, k \geq 3$ and $m \geq 2$, then the metric dimension of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$. Moreover, we consider the problem of determining the cardinality $\psi(LSG(n, m, k))$ of minimal doubly resolving sets of $LSG(n, m, k)$ and the strong metric dimension for the layer Sun graph $LSG(n, m, k)$ and the line graph of the layer Sun graph $LSG(n, m, k)$. The graph $LSG(3, 3, 4)$ is shown in Figure 1.

2. Definitions and Preliminaries

Definition 1. Let G be a graph. A vertex $x \in V(G)$ is said to resolve a pair $u, v \in V(G)$ if $d_G(u, x) \neq d_G(v, x)$. For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of vertices in the graph G and a vertex v of G , the metric representation of v with respect to W is the k -vector $r(v/W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. If every pair of distinct vertices of G has different metric representations, then the ordered set W is called a resolving set of G . If the set W is as small as possible, then it is called a metric basis of the graph G . We recall that the metric dimension of G , denoted by $\beta(G)$, is defined as the minimum cardinality of a resolving set for G .

Proposition 1. Let G be a graph. It is well known that a doubly resolving set is also a resolving set and $\beta(G) \leq \psi(G)$. In particular, every strong resolving set is a resolving set and $\beta(G) \leq sdim(G)$.

3. Main Results

3.1. Minimal Doubly Resolving Sets and the Strong Metric Dimension for the Layer Sun Graph $LSG(n, m, k)$

Theorem 1. Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Suppose that n, m, k are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the metric dimension of $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$.

Proof. Let $V(G) = V_1 \cup V_2 \cup \dots \cup V_k$, where $V_1, V_2, \dots, V_k$ are called the layers of vertices in the layer Sun graph $LSG(n, m, k)$, which is defined already. It is clear that if W is an ordered subset of the layers $V_1 \cup V_2 \cup \dots \cup V_{k-1}$, then W is not a resolving set in $LSG(n, m, k)$. We may assume that the layer V_k is equal to

$$V_k = \left\{ B_{i_1}^{(k)}, B_{i_2}^{(k)}, \dots, B_{i_{m^{k-3}}}^{(k)}; B_{2_1}^{(k)}, B_{2_2}^{(k)}, \dots, B_{2_{m^{k-3}}}^{(k)}; \dots; B_{n_1}^{(k)}, B_{n_2}^{(k)}, \dots, B_{n_{m^{k-3}}}^{(k)} \right\}, \quad (1)$$

where $B_{ij}^{(k)} = \left\{ \bigcup_{t=1}^m (v_{ij}, t)^k \right\}$, $1 \leq i \leq n$, $1 \leq j \leq m^{k-3}$. In the following cases, it can be shown that the metric dimension of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$.

Case 1: let W be an ordered subset of the layer V_k in the layer Sun graph $LSG(n, m, k)$ such that

$$W = \left\{ B_{i_2}^{(k)}, \dots, B_{i_{m^{k-3}}}^{(k)}; B_{2_1}^{(k)}, B_{2_2}^{(k)}, \dots, B_{2_{m^{k-3}}}^{(k)}; \dots; B_{n_1}^{(k)}, B_{n_2}^{(k)}, \dots, B_{n_{m^{k-3}}}^{(k)} \right\}. \quad (2)$$

Hence,

$$V(G) - W = \{V_1, V_2, \dots, V_{k-1}, B_{i_1}^{(k)}\}. \quad (3)$$

We know that the cardinality of W is $nm^{k-2} - m$ because $|B_{ij}^{(k)}| = m$, $1 \leq i \leq n$, $1 \leq j \leq m^{k-3}$. Therefore, the metric representation of all the vertices $(v_{i_1}, 1)^k, (v_{i_1}, 2)^k, \dots, (v_{i_1}, m)^k$ in the component $B_{i_1}^{(k)}$ is the same as $nm^{k-2} - m$ -vector with respect to W . Thus, W is not a resolving set in $LSG(n, m, k)$.

Case 2: let W be an ordered subset of the layer V_k in the layer Sun graph $LSG(n, m, k)$ such that

$$W = \left\{ B_{i_1}^{(k)} - \left\{ (v_{i_1}, 1)^k, (v_{i_1}, 2)^k \right\}, B_{i_2}^{(k)}, \dots, B_{i_{m^{k-3}}}^{(k)}; B_{2_1}^{(k)}, B_{2_2}^{(k)}, \dots, B_{2_{m^{k-3}}}^{(k)}; \dots; B_{n_1}^{(k)}, B_{n_2}^{(k)}, \dots, B_{n_{m^{k-3}}}^{(k)} \right\}. \quad (4)$$

Hence,

$$V(G) - W = \{V_1, V_2, \dots, V_{k-1}, (v_{i_1}, 1)^k, (v_{i_1}, 2)^k\}. \quad (5)$$

FIGURE 1: LSG(3,3,4).

We know that $|W| = nm^{k-2} - 2$. Therefore, the metric representation of two vertices $(v_{i_1}, 1)^k, (v_{i_1}, 2)^k$ in the component $B_{i_1}^{(k)}$ is the same as $nm^{k-2} - 2$ -vector with respect to W . Thus, W is not a resolving set in $LSG(n, m, k)$.

Case 3: let W be an ordered subset of the layer V_k in the layer Sun graph $LSG(n, m, k)$ such that

$$W = \left\{ B_{i_1}^{(k)} - (v_{i_1}, 1)^k, B_{i_2}^{(k)} - (v_{i_2}, 1)^k, \dots, B_{i_{m^{k-3}}}^{(k)} - (v_{i_{m^{k-3}}}, 1)^k, \right. \\ \left. - (v_{i_{m^{k-3}}}, 1)^k; \dots; B_{n_1}^{(k)} - (v_{n_1}, 1)^k, B_{n_2}^{(k)} - (v_{n_2}, 1)^k, \dots, B_{n_{m^{k-3}}}^{(k)} - (v_{n_{m^{k-3}}}, 1)^k \right\}. \quad (6)$$

Hence,

$$V(G) - W = \left\{ V_1, V_2, \dots, V_{k-1}, (v_{i_1}, 1)^k, \dots, (v_{i_{m^{k-3}}}, 1)^k, \dots, (v_{n_1}, 1)^k, \dots, (v_{n_{m^{k-3}}}, 1)^k \right\}. \quad (7)$$

We know that $|W| = nm^{k-2} - nm^{k-3}$. We can show that all the vertices in $V(G) - W$ have different representations with respect to W . Let u be the vertex of the layer $V_1 = V(C_n) = \{1, 2, \dots, n\}$. We can assume without loss of generality that $u = i$, $1 \leq i \leq n$. Hence, $d(u, (v_{i_j}, t)^k) = k - 1$, where $(v_{i_j}, t)^k \in B_{i_j}^{(k)}$, $1 \leq t \leq m$, $1 \leq j \leq m^{k-3}$; otherwise, if $u \neq i$, then

$d(u, (v_{i_j}, t)^k) > k - 1$. Now, let $u \in V_2 = \{v_1, v_2, \dots, v_n\}$. We can assume without loss of generality that $u = v_i$, $1 \leq i \leq n$. Hence, $d(u, (v_{i_j}, t)^k) = k - 2$, where $(v_{i_j}, t)^k \in B_{i_j}^{(k)}$, $1 \leq t \leq m$, $1 \leq j \leq m^{k-3}$; otherwise, if $u \neq v_i$, then $d(u, (v_{i_j}, t)^k) > k - 2$. In a similar way, we can show that all the vertices in the layers $V_3, \dots, V_{k-1}$ have different representations with respect to W . In particular, for every vertex $u \in \{(v_{i_1}, 1)^k, \dots, (v_{i_{m^{k-3}}}, 1)^k, \dots, (v_{n_1}, 1)^k, \dots, (v_{n_{m^{k-3}}}, 1)^k\}$, we have $d(u, (v_{i_j}, t)^k) = 2$, $2 \leq t \leq m$, if $u = (v_{i_j}, 1)^k$; otherwise, if $u \in \{(v_{i_1}, 1)^k, \dots, (v_{i_{m^{k-3}}}, 1)^k, \dots, (v_{n_1}, 1)^k, \dots, (v_{n_{m^{k-3}}}, 1)^k\}$ and $u \neq (v_{i_j}, 1)^k$, then $d(u, (v_{i_j}, t)^k) > 4$. Therefore, all the vertices in $V(G) - W$ have different representations with respect to W . This implies that W is a resolving set in $LSG(n, m, k)$. From the above-mentioned cases, we can be concluded that the minimum possible cardinality of a resolving set in $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$. $\square$

Theorem 2. Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Suppose that n, m, k are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the cardinality of minimum doubly resolving set of the $LSG(n, m, k)$ is nm^{k-2} .

Proof. In the following cases, it can be shown that the cardinality of minimum doubly resolving set of the layer Sun graph $LSG(n, m, k)$ is nm^{k-2} .

Case 1: we know that the ordered subset W of vertices in equation (8) in the layer V_k of $LSG(n, m, k)$ is a resolving set for $LSG(n, m, k)$ of cardinality $nm^{k-2} - nm^{k-3}$. We show that this subset is not a doubly resolving set for

LSG(n, m, k). Because if the vertex $u = (v_i, 1)^k \in B_{i_j}^{(k)} \in V_k$, $1 \leq i \leq n$, $1 \leq j \leq m^{k-3}$ is adjacent to a vertex $v \in V_{k-1}$, then for every $x, y \in W$, we have $d(u, x) - d(u, y) = d(v, x) - d(v, y)$.

$$W = \left\{ B_{1_1}^{(k)} - (v_{1_1}, 1)^k, B_{1_2}^{(k)} - (v_{1_2}, 1)^k, \dots, B_{1_{m^{k-3}}}^{(k)} - (v_{1_{m^{k-3}}}, 1)^k; \dots; B_{n_1}^{(k)} - (v_{n_1}, 1)^k, B_{n_2}^{(k)} - (v_{n_2}, 1)^k, \dots, B_{n_{m^{k-3}}}^{(k)} - (v_{n_{m^{k-3}}}, 1)^k \right\}. \quad (8)$$

Case 2: now, let the subset of vertices in LSG(n, m, k) be

$$W = \left\{ B_{1_1}^{(k)} - (v_{1_1}, 1)^k, B_{1_2}^{(k)}, \dots, B_{1_{m^{k-3}}}^{(k)}, \dots, B_{n_1}^{(k)}, B_{n_2}^{(k)}, \dots, B_{n_{m^{k-3}}}^{(k)} \right\}. \quad (9)$$

In a similar fashion as in Case 3 of Theorem 1, we can show that all the vertices in the layers $V_1, V_2, \dots, V_{k-1}$ of LSG(n, m, k) and the vertex $(v_{1_1}, 1)^k$ in the layer V_k of LSG(n, m, k) have different representations with respect to W . So, W is a resolving set in LSG(n, m, k) of cardinality $nm^{k-2} - 1$. Note that, in this case, by a similar way as in Case 1, we can show that this subset is not a doubly resolving set for LSG(n, m, k).

Case 3: finally, let the subset of vertices in LSG(n, m, k) be

$$W = \left\{ B_{1_1}^{(k)}, B_{1_2}^{(k)}, \dots, B_{1_{m^{k-3}}}^{(k)}; \dots; B_{n_1}^{(k)}, B_{n_2}^{(k)}, \dots, B_{n_{m^{k-3}}}^{(k)} \right\}. \quad (10)$$

In a similar fashion as in Theorem 1, we can show that all the vertices in the layers $V_1, V_2, \dots, V_{k-1}$ of LSG(n, m, k) have different representations with respect to W . So, this subset is also a resolving set in LSG(n, m, k) of cardinality nm^{k-2} . We show that this subset is a doubly resolving set for LSG(n, m, k). It is sufficient to prove that for two vertices u and v in LSG(n, m, k), there are vertices $x, y \in W$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Consider two vertices u and v in LSG(n, m, k). Then, we have the following.

Case 3.1: suppose that both vertices u and v lie in the layer V_1 . Hence, there are $r, s \in \{1, 2, \dots, n\}$ such that $u = r$ and $v = s$. Moreover, we know that the layer Sun graph LSG(n, m, k) has the property that, for each vertex r in the layer V_1 , there is some vertex such as $x = (v_r, t)^k$ in the component $B_{r_j}^{(k)}$, $1 \leq j \leq m^{k-3}$ in the layer V_k , at distance $k-1$ from u ; in fact, $d(u, x) = k-1$. In the same way, there is some vertex such as $y = (v_s, t)^k$ in the component $B_{s_j}^{(k)}$, in the layer V_k , at distance $k-1$ from v . In particular, it is easy to prove that $d(u, x) - d(u, y) < 0$ because $d(u, y) \geq k$. Also, $d(v, x) - d(v, y) > 0$ because $d(v, x) \geq k$.

Case 3.2: now, suppose that both vertices u and v lie in the layer V_2 . In a similar way as in Case 3.1, we can show that there are vertices $x, y \in W$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Case 3.3: suppose that both vertices u and v lie in the layer V_l , $l \geq 3$ such that these vertices lie in the one component of the layer V_l , say $B_{i_j}^{(l)}$, $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$. In this case, $d(u, v) = 2$. Moreover, we know that the layer Sun graph LSG(n, m, k) has the property that, for each vertex $u \in B_{i_j}^{(l)}$ in the layer V_l , there is a component of the layer V_k , say $B_{i_r}^{(k)}$, $1 \leq i \leq n$, $1 \leq r \leq m^{k-3}$ such that for any vertex $x \in B_{i_r}^{(k)}$, we have $d(u, x) = k-l$. In the same way, for the vertex $v \in B_{i_j}^{(l)}$ in the layer V_l , there is a component of the layer V_k , say $B_{i_s}^{(k)}$, $1 \leq i \leq n$, $1 \leq s \leq m^{k-3}$, $r \neq s$ such that, for any vertex $y \in B_{i_s}^{(k)}$, we have $d(v, y) = k-l$. Thus, $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, y) = k-l+2$ and $d(v, x) = k-l+2$.

Case 3.4: suppose that both vertices u and v lie in the layer V_l , $l \geq 3$ such that these vertices lie in the two distinct components of the layer V_l . We can assume without loss of generality that $u \in B_{p_{j_1}}^{(l)}$ and $v \in B_{q_{j_2}}^{(l)}$, $1 \leq p, q \leq n$, and $1 \leq j_1, j_2 \leq m^{l-3}$. Moreover, we know that the layer Sun graph LSG(n, m, k) has the property that, for each vertex $u \in B_{p_{j_1}}^{(l)}$ in the layer V_l , there is a component of the layer V_k , say $B_{p_r}^{(k)}$, $1 \leq r \leq m^{k-3}$ such that for any vertex $x \in B_{p_r}^{(k)}$, we have $d(u, x) = k-l$. In the same way, for the vertex $v \in B_{q_{j_2}}^{(l)}$ in the layer V_l , there is a component of the layer V_k , say $B_{q_s}^{(k)}$, $1 \leq s \leq m^{k-3}$, such that for any vertex $y \in B_{q_s}^{(k)}$, we have $d(v, y) = k-l$. In the following, let two components $B_{p_{j_1}}^{(l)}$ and $B_{q_{j_2}}^{(l)}$ be fundamental; indeed, $p = q$. Hence, $d(u, v) = 2l-4$, $d(u, y) = d(v, x) = k+l-4$. Thus, $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Now, let two components $B_{p_{j_1}}^{(l)}$ and $B_{q_{j_2}}^{(l)}$ not be fundamental; indeed, $p \neq q$. Hence, $d(u, v) = 2l-2 + d_{C_n}(p, q)$, $d(u, y) = d(v, x) = k+l-2 + d_{C_n}(p, q)$. Thus, $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Case 3.5: suppose that vertices u and v lie in distinct layers V_a, V_b , respectively. Note that if $a = 1$ and $b = 2$, $a = 1$ and $b > 2$, or $a = 2$ and $b > 2$, there is nothing to do. Now, let $3 \leq a < b$. Hence, there is a component of the layer V_a , say $B_{i_j}^{(a)}$, $1 \leq i \leq n$, $1 \leq j \leq m^{a-3}$ such that $u \in B_{i_j}^{(a)}$. Also, there is a component of the layer V_b , say $B_{p_q}^{(b)}$, $1 \leq p \leq n$, $1 \leq q \leq m^{b-3}$ such that $v \in B_{p_q}^{(b)}$. In particular, there is a component of the layer V_k , say $B_{i_r}^{(k)}$, $1 \leq i \leq n$, $1 \leq r \leq m^{k-3}$ such that for any vertex $x \in B_{i_r}^{(k)}$, we have $d(u, x) = k-a$. Now, let $i = p$; if we consider $y \in B_{z_s}^{(k)}$, $z \neq i$, $1 \leq z \leq n$, and $1 \leq s \leq m^{k-3}$, then we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Because $d(u, y) = k+a-2 + d_{C_n}(i, z)$, $d(v, y) = k+b-2 + d_{C_n}(i, z)$ and $d(u, x) \neq d(v, x)$. Note that if $i \neq p$, then there is a component of the layer V_k , say $B_{p_s}^{(k)}$, $1 \leq p \leq n$, $1 \leq s \leq m^{k-3}$ such that for any vertex $y \in B_{p_s}^{(k)}$, we have $d(v, y) = k-b$, and then, we have

$d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, x) - d(u, y) < 0$ and $d(v, x) - d(v, y) > 0$. Thus, from the abovementioned cases, we can conclude that the cardinality of the minimum doubly resolving set of the layer Sun graph $LSG(n, m, k)$ is nm^{k-2} . $\square$

Theorem 3. Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Suppose that n, m, k are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the strong metric dimension of $LSG(n, m, k)$ is $nm^{k-2} - 1$.

Proof. In the following cases, it can be seen that the cardinality of the minimum strong resolving set of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - 1$.

$$W = \left\{ B_{1_1}^{(k)} - (v_{1_1}, 1)^k, B_{1_2}^{(k)} - (v_{1_2}, 1)^k, \dots, B_{1_{m^{k-3}}}^{(k)} - (v_{1_{m^{k-3}}}, 1)^k; \dots; B_{n_1}^{(k)} - (v_{n_1}, 1)^k, B_{n_2}^{(k)} - (v_{n_2}, 1)^k, \dots, B_{n_{m^{k-3}}}^{(k)} - (v_{n_{m^{k-3}}}, 1)^k \right\}, \quad (11)$$

Case 2: on the other hand, we can show that the subset W of vertices in equation (12) in the graph G is a resolving set for graph G . We show that this subset is a strong resolving set in graph G . It is sufficient to prove

$$W = \left\{ B_{1_1}^{(k)} - (v_{1_1}, 1)^k, B_{1_2}^{(k)}, \dots, B_{1_{m^{k-3}}}^{(k)}, \dots, B_{n_1}^{(k)}, B_{n_2}^{(k)}, \dots, B_{n_{m^{k-3}}}^{(k)} \right\}. \quad (12)$$

Case 2.1: suppose that both vertices u and v lie in the layer V_1 . Hence, there are $r, s \in \{1, 2, \dots, n\}$ such that $u = r$ and $v = s$. Moreover, we know that the layer Sun graph $LSG(n, m, k)$ has the property that, for each vertex r in the layer V_1 , there is a component $B_{r_j}^{(k)}$, $1 \leq j \leq m^{k-3}$ in the layer V_k such that, for every vertex such as $w \in B_{r_j}^{(k)}$, we have $d(u, w) = k - 1$ and $d(v, w) > k - 1$, and hence, u belongs to a shortest $w - v$ path.

Case 2.2: now suppose that both vertices u and v lie in the layer V_2 . In a similar way as in Case 2.1, we can show that the vertices u and v are strongly resolved by a vertex $w \in W$.

Case 2.3: suppose that both vertices u and v lie in the layer V_l , $l \geq 3$ such that these vertices lie in the one component of the layer V_l , say $B_{i_j}^{(l)}$, $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$. In this case, $d(u, v) = 2$. Moreover, we know that the layer Sun graph $LSG(n, m, k)$ has the property that, for each vertex $u \in B_{i_j}^{(l)}$ in the layer V_l , there is a component of the layer V_k , say $B_{i_r}^{(k)}$, $1 \leq i \leq n$, $1 \leq r \leq m^{k-3}$ such that for any vertex $w \in B_{i_r}^{(k)}$, we have $d(u, w) = k - l$, and hence, u belongs to a shortest $w - v$ path.

Case 1: we know that the ordered subset W of vertices in equation (11) in the layer V_k of the layer Sun graph $LSG(n, m, k)$ is a resolving set for $LSG(n, m, k)$ of cardinality $nm^{k-2} - nm^{k-3}$. Now, let $N = V_k - W = \left\{ (v_{1_1}, 1)^k, \dots, (v_{1_{m^{k-3}}}, 1)^k, \dots, (v_{n_1}, 1)^k, \dots, (v_{n_{m^{k-3}}}, 1)^k \right\}$. By considering distinct vertices $u, v \in N$, we can show that there is not a vertex $w \in W$ such that u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path because the valency of every vertex in the layer V_k is one. So, this subset is not a strong resolving set for G . Thus, we can conclude that if W is a strong resolving set for graph G , then $|W| \geq nm^{k-2} - 1$ because $|N|$ must be less than 2.

that every two distinct vertices $u, v \in V(G) - W$ are strongly resolved by a vertex $w \in W$. Then, we have the following:

Case 2.4: suppose that both vertices u and v lie in the layer V_l , $l \geq 3$ such that these vertices lie in the two distinct components of the layer V_l . We can assume without loss of generality that $u \in B_{p_{j_1}}^{(l)}$ and $v \in B_{q_{j_2}}^{(l)}$, $1 \leq p, q \leq n$, and $1 \leq j_1, j_2 \leq m^{l-3}$. Moreover, we know that the layer Sun graph $LSG(n, m, k)$ has the property that, for each vertex $u \in B_{p_{j_1}}^{(l)}$ in the layer V_l , there is a component of the layer V_k , say $B_{p_r}^{(k)}$, $1 \leq r \leq m^{k-3}$ such that for any vertex $w \in B_{p_r}^{(k)}$, we have $d(u, w) = k - l$, and hence, u belongs to a shortest $w - v$ path.

Case 2.5: suppose that vertices u and v lie in distinct layers V_a, V_b , respectively. Note that if $a = 1$ and $b = 2$, $a = 1$ and $b > 2$, or $a = 2$ and $b > 2$, there is nothing to do. Now, let $3 \leq b < a$. Hence, there is a component of the layer V_a , say $B_{i_j}^{(a)}$, $1 \leq i \leq n$, $1 \leq j \leq m^{a-3}$ such that $u \in B_{i_j}^{(a)}$. Also, there is a component of the layer V_b , say $B_{p_q}^{(b)}$, $1 \leq p \leq n$, $1 \leq q \leq m^{b-3}$ such that $v \in B_{p_q}^{(b)}$. In particular, there is a component of the layer V_k , say $B_{i_r}^{(k)}$, $1 \leq i \leq n$, $1 \leq r \leq m^{k-3}$ such that, for any vertex $w \in B_{i_r}^{(k)}$, we have $d(u, w) = k - a$, and hence, u belongs to a shortest $w - v$ path.

Case 2.6: let u, v be two distinct vertices in $V(G) - W$ such that $u = (v_1, 1)^k \in V_k$ and $v \in V_l$, $l \geq 3$. So, there is a component $B_{r_{j_1}}^{(l)}$ in the layer V_l , $1 \leq r \leq n$ and $1 \leq j_1 \leq m^{l-3}$ such that $v \in B_{r_{j_1}}^{(l)}$. Thus, there is some vertex in the component $B_{r_{j_2}}^{(k)}$, $1 \leq j_2 \leq m^{k-3}$ say w such that $d(w, v) = k - l$, $d(w, u) > k - l$, and v belongs to a shortest $w - u$ path.

Case 2.7: let u, v be two distinct vertices in $V(G) - W$ such that $u = (v_1, 1)^k \in V_k$ and $v \in V_1$. So, there is some $i \in V_1 = \{1, 2, \dots, n\}$ such that $v = i$. If $i = 1$, indeed $d(u, v) = k - 1$, and then, there is a component $B_{r_j}^{(k)}$ in the layer V_k , $r \neq 1$ and $1 \leq j \leq m^{k-3}$ such that, for every vertex such as w in the component $B_{r_j}^{(k)}$, we have $d(w, v) \geq k$, $d(w, u) \geq 2k - 1$, and v belongs to a shortest $w - u$ path. Now, let $i \neq 1$; indeed, $d(u, v) \geq k$, and hence, there is a component $B_{i_j}^{(k)}$ in the layer V_k , $1 \leq j \leq m^{k-3}$ such that, for every vertices such as w in the components $B_{i_j}^{(k)}$, we have $d(w, v) = k - 1$, $d(w, u) \geq 2k - 1$, and v belongs to a shortest $w - u$ path.

Thus, from the abovementioned cases, we can be concluded that the cardinality of minimum strong resolving set of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - 1$. $\square$

3.2. Minimal Doubly Resolving Sets and the Strong Metric Dimension for the Line Graph of Layer Sun Graph $LSG(n, m, k)$. Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Now, let H be a graph with vertex set $V(H) = U_1 \cup U_2 \cup \dots \cup U_k$, where $U_1, U_2, \dots, U_k$ are called the layers of H which is defined as follows:

Let $U_1 = V(C_n) = \{1, 2, \dots, n\}$ and $U_2 = \{u_1, u_2, \dots, u_n\}$, and for $l \geq 3$, we have

$$U_l = \left\{ D_{1_1}^{(l)}, D_{1_2}^{(l)}, \dots, D_{1_{m^{l-3}}}^{(l)}; D_{2_1}^{(l)}, D_{2_2}^{(l)}, \dots, D_{2_{m^{l-3}}}^{(l)}; \dots; D_{n_1}^{(l)}, D_{n_2}^{(l)}, \dots, D_{n_{m^{l-3}}}^{(l)} \right\}, \quad (13)$$

and let $D_{ij}^{(l)} = \left\{ \bigcup_{t=1}^m (u_{ij}, t)^l \right\}$ such that every $(u_{ij}, t)^l$ is a vertex in the layer U_l and $D_{ij}^{(l)} \cong K_m$ in the layer U_l , $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$, $1 \leq t \leq m$, where K_m is the complete graph on m vertices. Now, suppose that every vertex $i \in \{2, 3, \dots, n\}$ in the cycle C_n or the layer U_1 is adjacent to exactly two vertices in the layer U_2 say $u_i, u_{i-1} \in U_2$. In particular, for the vertex 1 in the layer U_1 , we have 1 adjacent to exactly two vertices in the layer U_2 , say $u_1, u_n \in U_2$. Also, every vertex u_i in the layer U_2 is adjacent to exactly m vertices $(u_{i1}, 1)^3, (u_{i1}, 2)^3, \dots, (u_{i1}, m)^3 \in D_{i1}^{(3)} \in U_3$, in particular for $l \geq 3$, every vertex $(u_{ir}, t)^l \in D_{i_r}^{(l)} \in U_l$ is adjacent to exactly m vertices $\bigcup_{t=1}^m (u_{ij}, t)^{l+1} \in D_{i_j}^{(l+1)} \in U_{l+1}$, and then, the resulting graph is isomorphic with the line graph of the layer Sun graph $LSG(n, m, k)$ with parameters n, m , and k ; in fact, $L(G) \cong H$. Note that simply we use refinement of the natural relabelling of the line graph of the graph $LSG(n, m, k)$. Also, for $l \geq 3$, we recall $D_{ij}^{(l)}$ as the components of U_l , $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$. In particular, we say

that two components $D_{ij}^{(l)}, D_{rs}^{(l)}$, $1 \leq i, r \leq n$, $1 \leq j, s \leq m^{l-3}$ are fundamental if $i = r$ and $j \neq s$. It is natural to consider its vertex set of the line graph of the layer Sun graph $LSG(n, m, k)$ is also as partitioned into k layers. The layers U_1 and U_2 consist of the vertices $\{1, 2, \dots, n\}$ and $\{u_1, u_2, \dots, u_n\}$, respectively. In particular, each layer U_l ($l \geq 3$) consists of the nm^{l-2} vertices. Note that, for each vertex i in the layer U_1 and every vertex $x \in D_{ij}^{(l)} \in U_l$, $l \geq 3$, $1 \leq j \leq m^{l-3}$, we have $d(i, x) = l - 1$. In this section, we consider the problem of determining the cardinality $\psi(L(G))$ of minimal doubly resolving sets of the line graph of the layer Sun graph $LSG(n, m, k)$. We find the minimal doubly resolving set for the line graph of the layer Sun graph $LSG(n, m, k)$, and in fact, we prove that if $n, k \geq 3$ and $m \geq 2$, then the minimal doubly resolving set of the line graph of the layer Sun graph $LSG(n, m, k)$ is $nm^{k-2} - nm^{k-3}$. Figure 2 shows the line graph of the graph $LSG(3, 3, 4)$. Note that simply we use refinement of the natural relabelling of the line graph of the graph $LSG(3, 3, 4)$.

Theorem 4. Let $G = LSG(3, 3, 4)$ be the layer Sun graph which is defined already. Suppose that n, m, k are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the cardinality of minimum doubly resolving set in the line graph of the graph G is $nm^{k-2} - nm^{k-3}$.

Proof. Let W be an ordered subset of the layer U_k in the line graph of the graph G such that

$$W = \left\{ D_{1_1}^{(k)} - (u_1, 1)^k, D_{1_2}^{(k)} - (u_2, 1)^k, \dots, D_{1_{m^{k-3}}}^{(k)} - (u_{1_{m^{k-3}}}, 1)^k; \dots; D_{n_1}^{(k)} - (u_{n_1}, 1)^k, D_{n_2}^{(k)} - (u_{n_2}, 1)^k, \dots, D_{n_{m^{k-3}}}^{(k)} - (u_{n_{m^{k-3}}}, 1)^k \right\}. \quad (14)$$

Hence,

$$V(L(G)) - W = \left\{ U_1, U_2, \dots, U_{k-1}, (u_1, 1)^k, \dots, (u_{1_{m^{k-3}}}, 1)^k, \dots, (u_{n_1}, 1)^k, \dots, (u_{n_{m^{k-3}}}, 1)^k \right\}. \quad (15)$$

We know that $|W| = nm^{k-2} - nm^{k-3}$. In a similar way as in Theorem 1, we can show that this subset is a minimal resolving set for the line graph of the graph G . We prove that this subset is a doubly resolving set for the line graph of the graph G , and hence, $\beta(L(G)) = \psi(L(G))$. It is sufficient to prove that, for any two vertices u and v in $L(G)$, there are vertices $x, y \in W$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Consider two vertices u and v in $L(G)$. Then, we have the following:

Case 1: suppose that both vertices u and v lie in the layer U_1 . Hence, there are $r, s \in \{1, 2, \dots, n\}$ such that $u = r$ and $v = s$. Moreover, we know that the line graph of the graph G has the property that, for each vertex r in

FIGURE 2: The line graph LSG(3,3,4).

the layer U_1 there is some vertex such as $x = (u_r, t)^k$ in the component $D_{r_j}^{(k)}$, $1 \leq j \leq m^{k-3}$ in the layer U_k , at distance $k-1$ from u , and in fact, $d(u, x) = k-1$. In the same way, there is some vertex such as $y = (u_s, t)^k$ in the component $D_{s_j}^{(k)}$, in the layer U_k , at distance $k-1$ from v . In particular, it is easy to prove that $d(u, x) - d(u, y) < 0$ because $d(u, y) \geq k$. Also, $d(v, x) - d(v, y) > 0$ because $d(v, x) \geq k$.

Case 2: now, suppose that both vertices u and v lie in the layer U_2 . In a similar way as in Case 1, we can show that there are vertices $x, y \in W$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Case 3: suppose that both vertices u and v lie in the layer U_l , $l \geq 3$ such that these vertices lie in the one component of the layer U_l , say $D_{i_j}^{(l)}$, $1 \leq i \leq n$, $1 \leq j \leq m^{l-3}$. In this case, $d(u, v) = 1$. Moreover, we know that the line graph of the graph G has the property that, for each vertex $u \in D_{i_j}^{(l)}$ in the layer U_l , there is a component of the layer U_k , say $D_{i_r}^{(k)}$, $1 \leq i \leq n$, $1 \leq r \leq m^{k-3}$ such that, for every vertex $x \in D_{i_r}^{(k)}$, we have $d(u, x) = k-l$. In the same way, for the vertex $v \in D_{i_j}^{(l)}$ in the layer U_l , there is a component of the layer U_k , say $D_{i_s}^{(k)}$, $1 \leq i \leq n$, $1 \leq s \leq m^{k-3}$, $r \neq s$ such that, for every vertex $y \in D_{i_s}^{(k)}$, we have $d(v, y) = k-l$. Thus, $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, y) = k-l+1$ and $d(v, x) = k-l+1$.

Case 4: suppose that both vertices u and v lie in the layer U_l , $l \geq 3$ such that these vertices lie in the two distinct components of the layer U_l . We can assume without loss of generality that $u \in D_{p_{j_1}}^{(l)}$ and $v \in D_{q_{j_2}}^{(l)}$, $1 \leq p, q \leq n$, and $1 \leq j_1, j_2 \leq m^{l-3}$. Moreover, we know that the line graph of the graph G has the property that, for each vertex $u \in D_{p_{j_1}}^{(l)}$ in the layer U_l , there is a component of the layer U_k , say $D_{p_r}^{(k)}$, $1 \leq r \leq m^{k-3}$ such that, for every vertex $x \in D_{p_r}^{(k)}$, we have $d(u, x) = k-l$. In the same

way, for the vertex $v \in D_{q_{j_2}}^{(l)}$ in the layer U_l , there is a component of the layer U_k , say $D_{q_s}^{(k)}$, $1 \leq s \leq m^{k-3}$, such that, for every vertex $y \in D_{q_s}^{(k)}$, we have $d(v, y) = k-l$. In the following, let two components $D_{p_{j_1}}^{(l)}$ and $D_{q_{j_2}}^{(l)}$ be fundamental; indeed, $p = q$. Hence, $d(u, v) = 2l-5$, $d(u, y) = d(v, x) = k+l-5$. Thus, $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Now, let two components $D_{p_{j_1}}^{(l)}$ and $D_{q_{j_2}}^{(l)}$ not be fundamental; indeed, $p \neq q$. Hence, $d(u, v) = 2l-4 + d_{U_2}(u_p, u_q)$ and $d(u, y) = d(v, x) = k+l-4 + d_{U_2}(u_p, u_q)$. Thus, $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Case 5: suppose that vertices u and v lie in distinct layers U_a, U_b , respectively. Note that if $a = 1$ and $b = 2$, $a = 1$ and $b > 2$, or $a = 2$ and $b > 2$, there is nothing to do. Now, let $3 \leq a < b$. Hence, there is a component of the layer U_a , say $D_{i_j}^{(a)}$, $1 \leq i \leq n$, $1 \leq j \leq m^{a-3}$ such that $u \in D_{i_j}^{(a)}$. Also, there is a component of the layer U_b , say $D_{p_q}^{(b)}$, $1 \leq p \leq n$, $1 \leq q \leq m^{b-3}$ such that $v \in D_{p_q}^{(b)}$. In particular, there is a component of the layer U_k , say $D_{i_r}^{(k)}$, $1 \leq i \leq n$, $1 \leq r \leq m^{k-3}$ such that, for any vertex $x \in D_{i_r}^{(k)}$, we have $d(u, x) = k-a$. Now, let $i = p$; if we consider $y \in D_{z_s}^{(k)}$, $z \neq i$, $1 \leq z \leq n$, and $1 \leq s \leq m^{k-3}$, then we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, y) = k+a-4 + d_{U_2}(u_i, u_z)$, $d(v, y) = k+b-4 + d_{U_2}(u_i, u_z)$, and $d(u, x) \neq d(v, x)$. Note that if $i \neq p$, then there is a component of the layer U_k , say $D_{p_s}^{(k)}$, $1 \leq p \leq n$, $1 \leq s \leq m^{k-3}$ such that, for any vertex $y \in D_{p_s}^{(k)}$, we have $d(v, y) = k-b$, and then, we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$.

Thus, from the abovementioned cases, we conclude that the cardinality of minimum doubly resolving set in the line graph of the graph G is $nm^{k-2} - nm^{k-3}$. $\square$

Theorem 5. Let $G = LSG(n, m, k)$ be the layer Sun graph which is defined already. Suppose that n, m, k are integers such that $n, k \geq 3$ and $m \geq 2$. Then, the strong metric dimension in the line graph of the graph G is $nm^{k-2} - 1$.

Proof. In a similar way as in the proof of Theorem 3, we can show that the subset

$$W = \left\{ D_{l_1}^{(k)} - (v_{l_1}, 1)^k, D_{l_2}^{(k)}, \dots, D_{l_{mk-3}}^{(k)}, \dots, D_{n_1}^{(k)}, D_{n_2}^{(k)}, \dots, D_{n_{mk-3}}^{(k)} \right\} \quad (16)$$

of vertices in the line graph of the graph G is a minimal resolving set of $L(G)$. $\square$

4. Conclusions

In this paper, we have constructed a layer Sun graph $LSG(n, m, k)$, discussed this graph, and computed the minimum cardinality of the doubly resolving set and strong resolving set of layer Sun graph $LSG(n, m, k)$ and the line graph of the layer Sun graph $LSG(n, m, k)$. We deduce that, by this way, we can construct a layer jellyfish graph $JFG(n, m, k)$, of order $n + \sum_{r=1}^{k-1} nm^r$, where the jellyfish graph is $JFG(n, m)$, which is defined in [5], and by a similar way, we can obtain and compute the minimum cardinality of doubly resolving set and strong resolving set of layer jellyfish graph $JFG(n, m, k)$ and the line graph of the layer jellyfish graph $JFG(n, m, k)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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