Research Article

Global Existence and Decay of Solutions for Coupled Nondegenerate Kirchhoff System with a Time Varying Delay Term

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This paper deals with the global existence of solutions in a bounded domain for nonlinear viscoelastic Kirchhoff system with a time varying delay by using the energy and Faedo–Galerkin method with respect to the delay term weight condition in the feedback and the delay speed. Furthermore, by using some convex functions properties, we prove a uniform stability estimate.

1. Introduction

1.1. Model. Consider the following viscoelastic Kirchhoff system:

$$\begin{cases} |u_t|^l u_{tt} + \alpha v - M(\|\nabla u\|^2) \Delta u - \Delta u_{tt} + \int_0^t h_1(t-s) \Delta u(s) ds - \mu_1 \Delta u_t(x, t - \tau(t)) = 0, & \text{in } \Omega \times]0, +\infty[, \\ |v_t|^l v_{tt} + \alpha u - M(\|\nabla v\|^2) \Delta v - \Delta v_{tt} + \int_0^t h_2(t-s) \Delta v(s) ds - \mu_2 \Delta v_t(x, t - \tau(t)) = 0, & \text{in } \Omega \times]0, +\infty[, \\ u(x, t) = v(x, t) = 0, & \text{on } \partial\Omega \times]0, +\infty[, \\ (u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), (u_t(x, 0), v_t(x, 0)) = (u_1(x), v_1(x)), & \text{in } \Omega, \\ (u_t(x, t - \tau(0)), v_t(x, t - \tau(0))) = (f_0(x, t - \tau(0)), g_0(x, t - \tau(0))), & \text{in } \Omega \times]0, \tau(0)[, \end{cases} \quad (1)$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}^*$, with a smooth boundary $\partial\Omega$, $l > 0$, α, μ_1 and μ_2 are positive real numbers, h_1 and h_2 are positive functions which decay exponentially, $\tau(t) > 0$ is a time varying delay, and the initial data $(u_0, v_0, u_1, v_1, f_0, g_0)$ are in a suitable function space.

$M(r) = a + br^\gamma$ is a C^1 -function for $r \geq 0$, with $a, b > 0$, and $\gamma \geq 1$.

Time delay is often present in applications and practical problems. In last few years, the control of PDEs with time delay effects has become an active area of research (see, for

example, [1–4] and the references therein). In [5], the authors showed that a small delay in a boundary control could turn a well-behaved hyperbolic system into a wild one and therefore delay becomes a source of instability. However, sometimes it can also improve the performance of the system.

By using the Faedo–Galerkin method, Wu in [6] proved the result of local existence and established the decay result by suitable Lyapunov functionals according to appropriate conditions on μ_1 , μ_2 and on the kernel h .

Daewook [7] studied the following viscoelastic Kirchhoff equation with nonlinear source term and varying time delay:

$$\begin{aligned} u_{tt} - M(x, t, \|\nabla u\|^2) \Delta u + \int_0^t h(t-s) \operatorname{div}(a(x) \nabla u(s)) ds \\ + |u|^m u + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau(t)) = 0, \\ \text{in } \Omega \times]0, +\infty[, \end{aligned} \quad (2)$$

$$\begin{cases} |u_t|^l u_{tt} - M(\|\nabla u\|^2) \Delta u - \Delta u_{tt} + \int_0^t h(t-s) \Delta u(s) ds + \mu_1 g_1(u_t(x, t)) + \mu_2 g_2(u_t(x, t - \tau(t))) = 0, & \text{in } \Omega \times]0, +\infty[, \\ u(x, t) = 0, & \text{on } \partial\Omega \times [0, +\infty[, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & \text{in } \Omega, \\ u_t(x, t - \tau(0)) = f_0(x, t - \tau(0)), & \text{in } \Omega \times]0, \tau(0)[, \end{cases} \quad (3)$$

with respect to some proposed assumptions. Under assumption setting on g_1 , g_2 , σ , and τ , the authors have obtained the global existence of solution and the decay rate of energy.

Recently, Mezouar and Boulaaras [1] have studied viscoelastic nondegenerate Kirchhoff equation with varying delay term in the internal feedback.

In the present paper, we extend our recent published paper in [1] for a coupled system (3). The famous technique using the presence of delay in PDE's problem is to set a new variable defined by velocity depending on delay, which will give us new problem equivalent to our studied problem but the last one is a coupled system without delay. After this, we can prove the existence of global solutions in suitable Sobolev spaces by combining the energy method with the Faedo–Galerkin procedure, and under a choice of a suitable Lyapunov functional, we establish an exponential decay result.

The outline of the paper is as follows. In Section 2, some hypotheses related to problem are given and we state our main result. Then, in Section 3, the global existence of weak solutions is proven. Finally, in Section 4, we give the uniform energy decay.

which is a description of axially moving viscoelastic materials. According to the smallness condition taking into account of Kirchhoff coefficient and the relaxation function and by summing $0 \leq m \leq (2/(n-2))$ if $n > 2$ or $0 \leq m$ if $n \leq 2$, he got the uniform decay rate of the Kirchhoff type energy.

Very recently, in [1], we have proved the global existence and energy decay of solutions of the following viscoelastic nondegenerate Kirchhoff equation:

1.2. Preliminaries and Assumptions. We denote by $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$.

Now, we introduce, as in [8], the new variables

$$\begin{aligned} z_1(x, \rho, t) &= u_t(x, t - \rho\tau(t)), \quad x \in \Omega, \rho \in (0, 1), t > 0, \\ z_2(x, \rho, t) &= v_t(x, t - \rho\tau(t)), \quad x \in \Omega, \rho \in (0, 1), t > 0. \end{aligned} \quad (4)$$

Then, we have

$$\begin{aligned} \tau(t) z_1'(x, \rho, t) + (1 - \rho\tau'(t)) \frac{\partial}{\partial \rho} z_1(x, \rho, t) &= 0, \\ \text{in } \Omega \times (0, 1) \times (0, +\infty). \end{aligned} \quad (5)$$

Similarly, we have

$$\begin{aligned} \tau(t) z_2'(x, \rho, t) + (1 - \rho\tau'(t)) \frac{\partial}{\partial \rho} z_2(x, \rho, t) &= 0, \\ \text{in } \Omega \times (0, 1) \times (0, +\infty). \end{aligned} \quad (6)$$

Therefore, problem (3) is equivalent to

$$\left\{ \begin{array}{ll}
|u_t|^l u_{tt} + \alpha v - M(\|\nabla u\|^2) \Delta u - \Delta u_{tt} + \int_0^t h_1(t-s) \Delta u(s) ds - \mu_1 \Delta z_1(x, 1, t) = 0, & \text{in } \Omega \times]0, +\infty[, \\
|v_t|^l u_{tt} + \alpha u - M(\|\nabla v\|^2) \Delta v - \Delta v_{tt} + \int_0^t h_2(t-s) \Delta v(s) ds - \mu_2 \Delta z_2(x, 1, t) = 0, & \text{in } \Omega \times]0, +\infty[, \\
\tau(t) z_1'(x, \rho, t) + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_1(x, \rho, t) = 0, & \text{in } \Omega \times (0, 1) \times (0, +\infty), \\
\tau(t) z_2'(x, \rho, t) + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_2(x, \rho, t) = 0, & \text{in } \Omega \times (0, 1) \times (0, +\infty), \\
u(x, t) = v(x, t) = 0, & \text{on } [\partial\Omega \times [0, \infty, \\
(z_1(x, 0, t), z_2(x, 0, t)) = (u_t(x, t), v_t(x, t)), & \text{on } \Omega \times]0, \infty[, \\
(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), (u_t(x, 0), (v_t(x, 0)) = (u_1(x), v_1(x)), & \text{in } \Omega, \\
(z_1(x, \rho, 0), z_2(x, \rho, 0)) = (f_0(x, -\rho\tau(0)), g_0(x, -\rho\tau(0))), & \text{in } \Omega \times]0, 1[.
\end{array} \right. \quad (7)$$

To state and prove our result, we need some assumptions.

Assumption 1. Assume that $0 < l \leq \gamma$ satisfies

$$\left\{ \begin{array}{ll}
\gamma \leq \frac{2}{n-2}, & \text{if } n > 2, \\
\gamma < \infty, & \text{if } n \leq 2.
\end{array} \right. \quad (8)$$

Assumption 2. For the relaxation functions, $h_i: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are bounded C^1 functions such that

$$a - \int_0^\infty h_i(s) ds \geq k > 0, \quad (9)$$

and suppose that there exist positive constants ζ_i satisfying

$$h_i'(t) \leq -\zeta_i h_i(t), \quad (10)$$

for $i = 1, 2$.

Assumption 3. τ is a function in $W^{2,\infty}([0, T])$, $T > 0$, such that

$$\begin{aligned}
0 < \tau_0 \leq \tau'(t) \leq \tau_1, \quad \forall t > 0, \\
\tau'(t) \leq d < 1, \quad \forall t > 0.
\end{aligned} \quad (11)$$

We define the energy associated to the solution of system (7) by

$$\begin{aligned}
E(t) = & \frac{1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) + 2\alpha \int_\Omega u(x, t) v(x, t) dx \\
& + \frac{b}{2(\gamma+1)} \left(\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)} \right) \\
& + \frac{1}{2} \left(a - \int_0^t h_1(s) ds \right) \|\nabla u\|^2 + \frac{1}{2} \left(a - \int_0^t h_2(s) ds \right) \\
& \cdot \|\nabla v\|^2 + \frac{1}{2} \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) \\
& + \frac{1}{2} (h_1 \circ \nabla u)(t) + \frac{1}{2} (h_2 \circ \nabla v)(t) \\
& + \xi \tau(t) \int_0^1 \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) d\rho,
\end{aligned} \quad (12)$$

where ξ is a positive constant such that

$$\frac{\max\{\mu_1, \mu_2\}}{2(1-d)} < \xi, \quad (13)$$

$$(h_i \circ w)(t) = \int_0^t h_i(t-s) \|w(., t) - w(., s)\|^2 ds, \quad \text{for } i = 1, 2. \quad (14)$$

Theorem 1 (global existence). *Let $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, s , and $(f_0, g_0) \in (H_0^1(\Omega, H^1(0, 1)))^2$ satisfy the compatibility condition*

$$(f_0(., 0), g_0(., 0)) = (u_1, v_1). \quad (15)$$

Assume that Assumptions 1–3 hold. Then, problem (3) admits a weak solution such that $u, v \in L^\infty((0, \infty; H^2(\Omega) \cap H_0^1(\Omega)))$, $u_t, v_t \in L^\infty(0, \infty; H_0^1(\Omega))$, and $u_{tt}, v_{tt} \in L^2(0, \infty; H_0^1(\Omega))$.

Theorem 2 (decay rates of energy). Assume that Assumptions 1–3 hold. Then, for every $t_0 > 0$ there exist positive constants K and c' such that the energy defined by (12) possesses the following decay:

$$E(t) \leq K e^{-c't}, \quad \forall t \geq t_0. \quad (16)$$

2. Preliminaries

Lemma 1 (Sobolev – Poincaré inequality). Let q be a number with

$$2 \leq q < +\infty \ (n = 1, 2) \text{ or } 2 \leq q \leq \frac{2n}{(n-2)} \ (n \geq 3). \quad (17)$$

Then, there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_q \leq C_s \|\nabla u\|_2 \text{ for } u \in H_0^1(\Omega). \quad (18)$$

The following lemma states an important property of the convolution operator.

Lemma 2 (see [9]). For $h, \varphi \in C^1([0, +\infty[, \mathbb{R})$, we have

$$\begin{aligned} \int_{\Omega} h * \varphi_t dx &= -\frac{1}{2} h(t) \|\varphi(t)\|^2 + \frac{1}{2} (h' \circ \varphi)(t) \\ &\quad - \frac{1}{2} \frac{d}{dt} \left[(h \circ \varphi)(t) - \left(\int_0^t h(s) ds \right) \|\varphi\|^2 \right]. \end{aligned} \quad (19)$$

Lemma 3. Let (u, v, z_1, z_2) be a solution of problem (7). Then, the energy functional defined by (12) satisfies

$$\begin{aligned} E(t) &\leq -\beta \left(\|\nabla z_1(x, 1, t)\|^2 + \|\nabla z_2(x, 1, t)\|^2 \right) \\ &\quad + \lambda \left(\|\nabla u_t(x, t)\|^2 + \|\nabla v_t(x, t)\|^2 \right) \\ &\quad + \frac{1}{2} [(h_1' \circ \nabla u)(t) + (h_2' \circ \nabla v)(t)], \end{aligned} \quad (20)$$

where $\lambda = \xi + (\mu/2)$, $\beta = \xi(1-d) - \mu/2$, and $\mu = \max\{\mu_1, \mu_2\}$ are positive.

Proof. Multiplying the first equation in (7) by u_t , integrating over Ω , and using integration by parts, we get

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{l+2} \|u_t\|_{l+2}^{l+2} + \frac{b}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} + \frac{1}{2} a \|\nabla u\|^2 + \frac{1}{2} \|\nabla u_t\|^2 \right] \\ - \int_{\Omega} \int_0^t h_1(t-s) \nabla u(s) \nabla u_t(t) ds dx \\ + \alpha \int_{\Omega} u_t(x, t) v(x, t) dx + \mu_1 \int_{\Omega} \nabla u_t(x, t) \nabla z_1(x, 1, t) dx = 0. \end{aligned} \quad (21)$$

Consequently, by applying Lemma 2, equation (21) becomes

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{l+2} \|u_t\|_{l+2}^{l+2} + \frac{b}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} + \frac{1}{2} \left(a - \int_0^t h_1(s) ds \right) \right. \\ \left. \cdot \|\nabla u\|^2 + \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} (h_1 \circ \nabla u)(t) \right] \\ + \frac{1}{2} h_1(t) \|\nabla u(t)\|^2 - \frac{1}{2} (h_1' \circ \nabla u)(t) + \alpha \int_{\Omega} u_t(x, t) v(x, t) dx \\ + \mu_1 \int_{\Omega} \nabla u_t(x, t) \nabla z_1(x, 1, t) dx = 0. \end{aligned} \quad (22)$$

Similarly by multiplying the second equation in (7) by v_t , integrating over Ω , and using integration by parts, we get

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{l+2} \|v_t\|_{l+2}^{l+2} + \frac{b}{2(\gamma+1)} \|\nabla v\|^{2(\gamma+1)} + \frac{1}{2} \left(a - \int_0^t h_2(s) ds \right) \right. \\ \left. \cdot \|\nabla v\|^2 + \frac{1}{2} \|\nabla v_t\|^2 + \frac{1}{2} (h_2 \circ \nabla v)(t) \right] \\ + \frac{1}{2} h_2(t) \|\nabla v(t)\|^2 - \frac{1}{2} (h_2' \circ \nabla v)(t) + \alpha \int_{\Omega} u(x, t) v_t(x, t) dx \\ + \mu_2 \int_{\Omega} \nabla v_t(x, t) \nabla z_2(x, 1, t) dx = 0. \end{aligned} \quad (23)$$

Multiply the third equation in (7) by $\xi \Delta z_1$ and integrate the result over $\Omega \times (0, 1)$ to obtain

$$\begin{aligned} \xi \tau(t) \int_{\Omega} \int_0^1 z_1'(x, \rho, t) \Delta z_1(x, \rho, t) \Omega \times (0, 1) d\rho dx \\ = -\xi \int_{\Omega} \int_0^1 (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_1(x, \rho, t) \Delta z_1(x, \rho, t) d\rho dx. \end{aligned} \quad (24)$$

Consequently,

$$\begin{aligned}
& \frac{d}{dt} \left(\xi \tau(t) \int_{\Omega} \int_0^1 |\nabla z_1(x, \rho, t)|^2 d\rho dx \right) \\
&= \xi \tau'(t) \int_{\Omega} \int_0^1 |\nabla z_1(x, \rho, t)|^2 d\rho dx \\
&\quad - \xi \int_{\Omega} \int_0^1 (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} |\nabla z_1(x, \rho, t)|^2 d\rho dx \\
&= -\xi \int_{\Omega} \int_0^1 \frac{\partial}{\partial \rho} ((1 - \rho \tau(t)) |\nabla z_1(x, \rho, t)|^2) d\rho dx \\
&= -\xi (1 - \tau'(t)) \int_{\Omega} |\nabla z_1(x, 1, t)|^2 dx + \xi \int_{\Omega} |\nabla u_t(x, t)|^2 dx.
\end{aligned} \tag{25}$$

Similarly, we get

$$\begin{aligned}
& \frac{d}{dt} \left(\xi \tau(t) \int_{\Omega} \int_0^1 |\nabla z_2(x, \rho, t)|^2 d\rho dx \right) \\
&= -\xi (1 - \tau'(t)) \int_{\Omega} |\nabla z_2(x, 1, t)|^2 dx + \xi \int_{\Omega} |\nabla v_t(x, t)|^2 dx.
\end{aligned} \tag{26}$$

Combining (22)–(26), we obtain

$$\begin{aligned}
E'(t) &= -\xi (1 - \tau'(t)) \int_{\Omega} (|\nabla z_1(x, 1, t)|^2 + |\nabla z_2(x, 1, t)|^2) dx \\
&\quad + \xi \int_{\Omega} (|\nabla u_t(x, t)|^2 + |\nabla v_t(x, t)|^2) dx \\
&\quad - \frac{1}{2} [h_1(t) \|\nabla u(t)\|^2 + h_2(t) \|\nabla v(t)\|^2] \\
&\quad + \frac{1}{2} [(h_1' \circ \nabla u)(t) + (h_2' \circ \nabla v)(t)] \\
&\quad - \mu_1 \int_{\Omega} \nabla u_t(x, t) \nabla z_1(x, 1, t) dx \\
&\quad - \mu_2 \int_{\Omega} \nabla v_t(x, t) \nabla z_2(x, 1, t) dx.
\end{aligned} \tag{27}$$

From Assumption 3, we get

$$\begin{aligned}
E'(t) &\leq -\left(\xi(1-d) - \frac{\mu_1}{2}\right) \int_{\Omega} |\nabla z_1(x, 1, t)|^2 dx \\
&\quad - \left(\xi(1-d) - \frac{\mu_2}{2}\right) \int_{\Omega} |\nabla z_2(x, 1, t)|^2 dx \\
&\quad - \frac{1}{2} h_1(t) \|\nabla u_t(t)\|^2 + \left(\xi + \frac{\mu_1}{2}\right) \|\nabla u_t(t)\|^2 \\
&\quad - \frac{1}{2} h_2(t) \|\nabla v_t(t)\|^2 + \left(\xi + \frac{\mu_2}{2}\right) \|\nabla v_t(t)\|^2 \\
&\quad + \frac{1}{2} [(h_1' \circ \nabla u)(t) + (h_2' \circ \nabla v)(t)].
\end{aligned} \tag{28}$$

Using (13), this completes the proof. $\square$

3. Global Existence (Proof of Theorem 1)

Throughout this section we assume $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1, v_1 \in H_0^1(\Omega)$, and $f_0, g_0 \in H_0^1(\Omega, H^1(0, 1))$. We will use the Faedo–Galerkin method to prove the existence of global solutions. Let $T > 0$ be fixed and let $w^k, k \in \mathbb{N}$ be a basis of $H^2(\Omega) \cap H_0^1(\Omega)$ and V_k be the space generated by w^k . Now, we define, for $1 \leq j \leq k$, the sequence $\phi^j(x, \rho)$ as follows:

$$\phi^j(x, 0) = w^j. \tag{29}$$

Then, we may extend $\phi^j(x, 0)$ by $\phi^j(x, \rho)$ over $L^2(\Omega \times (0, 1))$ such that $(\phi^j)_j$ forms a basis of $L^2(\Omega, H^1(0, 1))$ and denote Z_k the space generated by $\{\phi^k\}$. We construct approximate solutions (u^k, v^k, z_1^k, z_2^k) , $k = 1, 2, 3, \dots$, in the form

$$\begin{aligned}
u^k(t) &= \sum_{j=1}^k a^{jk}(t) w^j, \\
v^k(t) &= \sum_{j=1}^k b^{jk}(t) w^j, \\
z_1^k(t) &= \sum_{j=1}^k c^{jk}(t) \phi^j, \\
z_2^k(t) &= \sum_{j=1}^k d^{jk}(t) \phi^j,
\end{aligned} \tag{30}$$

where a^{jk}, b^{jk}, c^{jk} , and d^{jk} ($j = 1, 2, \dots, k$) are determined by the following ordinary differential equations:

$$\begin{cases}
\left(|u_t^k|^l |u_{tt}^k, w^j \right) + \alpha(v^k, w^j) + M\left(\|\nabla u^k(t)\|^2 \right) (\nabla u^k, \nabla w^j) \\
\quad + (\nabla u_{tt}^k, \nabla w^j) - \int_0^t h_1(t-s) (\nabla u^k(s), \nabla w^j) ds \\
\quad + \mu_1 (\nabla z_1^k(\cdot, 1), \nabla w^j) = 0, 1 \leq j \leq k, \\
\left(|v_t^k|^l |v_{tt}^k, w^j \right) + \alpha(u^k, w^j) + M\left(\|\nabla v^k(t)\|^2 \right) (\nabla v^k, \nabla w^j) \\
\quad + (\nabla v_{tt}^k, \nabla w^j) - \int_0^t h_2(t-s) (\nabla v^k(s), \nabla w^j) ds \\
\quad + \mu_2 (\nabla z_2^k(\cdot, 1), \nabla w^j) = 0, 1 \leq j \leq k, \\
z_1^k(x, 0, t) = u_t^k(x, t), z_2^k(x, 0, t) = v_t^k(x, t),
\end{cases} \tag{31}$$

$$\begin{aligned}
u^k(0) &= u_0^k = \sum_{j=1}^k (u_0, w^j) w^j \longrightarrow u_0, \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \\
&\longrightarrow +\infty,
\end{aligned} \tag{32}$$

$$\begin{aligned} v^k(0) = v_0^k &= \sum_{j=1}^k (v_0, w^j) w^j \longrightarrow v_0, \\ &\text{in } H^2(\Omega) \cap H_0^1(\Omega) \text{ as } k \longrightarrow +\infty, \end{aligned} \quad (33)$$

$$\begin{aligned} u_t^k(0) = u_1^k &= \sum_{j=1}^k (u_1, w^j) w^j \longrightarrow u_1, \\ &\text{in } H_0^1(\Omega) \text{ as } k \longrightarrow +\infty, \end{aligned} \quad (34)$$

$$\begin{aligned} v_t^k(0) = v_1^k &= \sum_{j=1}^k (v_1, w^j) w^j \longrightarrow v_1, \\ &\text{in } H_0^1(\Omega) \text{ as } k \longrightarrow +\infty, \end{aligned} \quad (35)$$

$$\begin{cases} \left(\tau(t) \frac{\partial}{\partial t} z_1^k + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_1^k, \phi^j \right) = 0, & 1 \leq j \leq k, \\ \left(\tau(t) \frac{\partial}{\partial t} z_2^k + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_2^k, q \phi^j \right) = 0, & 1 \leq j \leq k, \end{cases} \quad (36)$$

$$\begin{aligned} z_1^k(\rho, 0) &= \sum_{j=1}^k (f_0, \phi^j) \phi^j \longrightarrow f_0, \\ &\text{in } H_0^1(\Omega, H^1(0, 1)) \text{ as } k \longrightarrow +\infty, \end{aligned} \quad (37)$$

$$\begin{aligned} z_2^k(\rho, 0) &= \sum_{j=1}^k (g_0, \phi^j) \phi^j \longrightarrow g_0, \\ &\text{in } H_0^1(\Omega, H^1(0, 1)) \text{ as } k \longrightarrow +\infty. \end{aligned} \quad (38)$$

Noting that $l/2(l+1) + 1/2(l+1) + 1/2 = 1$, from the generalized Hölder inequality, we obtain

$$\begin{aligned} \left(|u_t^k|^l |u_{tt}^k|, w_j \right) &= \int_{\Omega} |u_t^k|^l |u_{tt}^k| w_j dx \leq \left(\int_{\Omega} |u_t^k|^{2(l+1)} dx \right)^{l/(2(l+1))} \\ &\quad \cdot \|u_{tt}^k\|_{2(l+1)} \|w_j\|_2. \end{aligned} \quad (39)$$

Since Assumption 2 holds, according to Sobolev embedding, the nonlinear terms $(|u_t^k|^l |u_{tt}^k|, w_j)$ and $(|v_t^k|^l |v_{tt}^k|, w_j)$ in (31) make sense.

The standard theory of ODE guarantees that systems (31)–(38) have a unique solution in $[0, t_k]$, with $0 < t_k < T$. In the next step, we obtain a priori estimates for the solution of systems (31)–(38), so that it can be extended outside $[0, t_k]$ to obtain one solution defined for all $t > 0$, using a standard compactness argument for the limiting procedure.

3.1. First Estimate. Since the sequences $u_0^k, v_0^k, u_1^k, v_1^k, z_1^k(\rho, 0)$, and $z_2^k(\rho, 0)$ converge and from Lemma 3 with employing Gronwall's lemma, we can find a positive constant C_1 independent of k such that

$$E^k(t) + \beta \int_0^t \left(\|\nabla z_1^k(x, 1, s)\|^2 + \|\nabla z_2^k(x, 1, s)\|^2 \right) ds \leq C_1, \quad (40)$$

where

$$\begin{aligned} E^k(t) &= \frac{1}{l+2} \left(\|u_t^k\|_{l+2}^{l+2} + \|v_t^k\|_{l+2}^{l+2} \right) + 2\alpha \int_{\Omega} u^k v^k dx \\ &\quad + \frac{b}{2(\gamma+1)} \left(\|\nabla u^k\|^{2(\gamma+1)} + \|\nabla v^k\|^{2(\gamma+1)} \right) \\ &\quad + \frac{1}{2} \left(a - \int_0^t h_1(s) ds \right) \|\nabla u^k\|^2 \\ &\quad + \frac{1}{2} \left(a - \int_0^t h_2(s) ds \right) \|\nabla v^k\|^2 + \frac{1}{2} \left(\|\nabla u_t^k\|^2 + \|\nabla v_t^k\|^2 \right) \\ &\quad + \frac{1}{2} \left[(h_1 \circ \nabla u^k)(t) + (h_2 \circ \nabla v^k)(t) \right] \\ &\quad + \xi \tau(t) \int_0^1 \left(\|\nabla z_1^k(x, \rho, t)\|^2 + \|\nabla z_2^k(x, \rho, t)\|^2 \right) d\rho. \end{aligned} \quad (41)$$

Noting Assumption 1 and estimate (40) yields that

$$\begin{aligned} u^k, v^k &\text{ are bounded in } L_{\text{loc}}^{\infty}(0, \infty, H_0^1(\Omega)), \\ u_t^k, v_t^k &\text{ are bounded in } L_{\text{loc}}^{\infty}(0, \infty, H_0^1(\Omega)), \\ z_1^k(x, \rho, t), z_2^k(x, \rho, t) &\text{ are bounded in } L_{\text{loc}}^{\infty}(0, \infty, L^1(0, 1, H_0^1(\Omega))). \end{aligned} \quad (42)$$

3.2. Second Estimate. Multiplying the first equation (respectively, the second equation) in (31) by a_{tt}^{jk} (respectively, by b_{tt}^{jk}) and summing over j from 1 to k , it follows that

$$\begin{cases} \int_{\Omega} |u_t^k|^l |u_{tt}^k|^2 dx - \alpha \int_{\Omega} v_t^k u_t^k dx + \int_{\Omega} M(\|\nabla u^k\|^2) \nabla u^k \nabla u_{tt}^k dx \\ \quad + \|\nabla u_{tt}^k\|^2 = \int_0^t h_1(t-s) \int_{\Omega} \nabla u^k(s) \nabla u_{tt}^k(t) dx ds \\ \quad - \mu_1 \int_{\Omega} \nabla u_{tt}^k \nabla (z_1^k(x, 1, t)) dx, \\ \int_{\Omega} |v_t^k|^l |v_{tt}^k|^2 dx - \alpha \int_{\Omega} v_t^k u_t^k dx + \int_{\Omega} M(\|\nabla v^k\|^2) \nabla v^k \nabla v_{tt}^k dx \\ \quad + \|\nabla v_{tt}^k\|^2 = \int_0^t h_2(t-s) \int_{\Omega} \nabla v^k(s) \nabla v_{tt}^k(t) dx ds \\ \quad - \mu_2 \int_{\Omega} \nabla v_{tt}^k \nabla (z_2^k(x, 1, t)) dx. \end{cases} \quad (43)$$

Differentiating (36) with respect to t , we get

$$\begin{cases} \left(\left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \frac{\partial}{\partial t} z_1^k + \frac{\tau(t)}{(1-\rho\tau'(t))} \frac{\partial^2}{\partial t^2} z_1^k + \frac{\partial^2}{\partial t \partial \rho} z_1^k, \phi^j \right) = 0, \\ \left(\left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \frac{\partial}{\partial t} z_2^k + \frac{\tau(t)}{(1-\rho\tau'(t))} \frac{\partial^2}{\partial t^2} z_2^k + \frac{\partial^2}{\partial t \partial \rho} z_2^k, \phi^j \right) = 0. \end{cases} \quad (44)$$

Multiplying the first equation by c_t^{jk} (respectively, the second equation by d_t^{jk}) and summing over j from 1 to k , it follows that

$$\begin{cases} \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 + \frac{1}{2} \frac{\tau(t)}{(1-\rho\tau'(t))} \frac{d}{dt} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 + \frac{1}{2} \frac{d}{d\rho} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 = 0, \\ \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 + \frac{1}{2} \frac{\tau(t)}{(1-\rho\tau'(t))} \frac{d}{dt} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 + \frac{1}{2} \frac{d}{d\rho} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 = 0, \end{cases} \quad (45)$$

and then we have

$$\begin{cases} \frac{1}{2} \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 \right) \\ + \frac{1}{2} \frac{d}{d\rho} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 = 0, \\ \frac{1}{2} \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 \right) \\ + \frac{1}{2} \frac{d}{d\rho} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 = 0. \end{cases} \quad (46)$$

Integrating over $(0, 1)$ with respect to ρ , we obtain

$$\begin{cases} \frac{1}{2} \int_0^1 \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho \right) + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_1^k(x, 1, t) \right\|^2 - \frac{1}{2} \left\| u_{tt}^k(x, t) \right\|^2 = 0, \\ \frac{1}{2} \int_0^1 \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho \right) + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_2^k(x, 1, t) \right\|^2 - \frac{1}{2} \left\| v_{tt}^k(x, t) \right\|^2 = 0. \end{cases} \quad (47)$$

Summing (43) and (47) and as $M(r) \geq a$, we get

$$\begin{cases} \int_{\Omega} |u_t^k|^l |u_{tt}^k|^2 dx + \left\| \nabla u_{tt}^k \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho \right) + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_1^k(x, 1, t) \right\|^2 \leq -a \int_{\Omega} \nabla u^k \nabla u_{tt}^k dx + \frac{1}{2} \left\| u_{tt}^k(x, t) \right\|^2 \\ + \alpha \int_{\Omega} v_t^k u_t^k dx - \frac{1}{2} \int_0^1 \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho + \int_0^t h_1(t-s) \int_{\Omega} \nabla u^k(s) \nabla u_{tt}^k(t) dx ds - \mu_1 \int_{\Omega} \nabla u_{tt}^k \nabla z_1^k(x, 1, t) dx, \\ \int_{\Omega} |v_t^k|^l |v_{tt}^k|^2 dx + \left\| \nabla v_{tt}^k \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{(1-\rho\tau'(t))} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho \right) + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_2^k(x, 1, t) \right\|^2 \leq -a \int_{\Omega} \nabla v^k \nabla v_{tt}^k dx + \frac{1}{2} \left\| v_{tt}^k(x, t) \right\|^2 \\ + \alpha \int_{\Omega} v_t^k u_t^k dx - \frac{1}{2} \int_0^1 \left(\frac{\tau(t)}{(1-\rho\tau'(t))} \right)' \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho + \int_0^t h_2(t-s) \int_{\Omega} \nabla v^k(s) \nabla v_{tt}^k(t) dx ds - \mu_2 \int_{\Omega} \nabla v_{tt}^k \nabla z_2^k(x, 1, t) dx. \end{cases} \quad (48)$$

By Young's inequality, the right hand side of (48) can be estimated as follows:

$$\begin{cases} \left| \int_{\Omega} a \nabla u^k \nabla u_{tt}^k dx \right| \leq \eta \|\nabla u_{tt}^k\|^2 + \frac{a^2}{4\eta} \|\nabla u^k\|^2, \\ \left| \int_{\Omega} a \nabla v^k \nabla v_{tt}^k dx \right| \leq \eta \|\nabla v_{tt}^k\|^2 + \frac{a^2}{4\eta} \|\nabla v^k\|^2. \end{cases} \quad (49)$$

Using Cauchy-Schwarz inequality and Sobolev-Poincaré inequality, we obtain

$$\alpha \int_{\Omega} v_t^k u_t^k dx \leq \frac{\alpha}{2} (\|v_t^k\|^2 + \|u_t^k\|^2) \leq \frac{C_s \alpha}{2} (\|\nabla v_t^k\|^2 + \|\nabla u_t^k\|^2), \quad (50)$$

$$\begin{aligned} & \left| \int_0^t h_1(t-s) \int_{\Omega} \nabla u^k(s) \nabla u_{tt}^k(t) dx ds \right| \leq \eta \|\nabla u_{tt}^k\|^2 \\ & \quad + \frac{1}{4\eta} \int_{\Omega} \left(\int_0^t h_1(t-s) \nabla u^k(s) ds \right)^2 dx \\ & \leq \eta \|\nabla u_{tt}^k\|^2 + \frac{1}{4\eta} \int_0^t h(s) ds \int_{\Omega} \int_0^t h_1(t-s) |\nabla u^k(s)|^2 ds dx \\ & \leq \eta \|\nabla u_{tt}^k\|^2 + \frac{1}{4\eta} (a-k) \int_{\Omega} \int_0^t h_1(t-s) |\nabla u^k(s)|^2 ds dx \\ & \leq \eta \|\nabla u_{tt}^k\|^2 + \frac{1}{4\eta} (a-k) \int_0^t h_1(t-s) \|\nabla u^k(s)\|^2 ds \\ & \leq \eta \|\nabla u_{tt}^k\|^2 + \frac{1}{4\eta} (a-k) h_1(0) \int_0^t \|\nabla u^k(s)\|^2 ds. \end{aligned} \quad (51)$$

Similarly,

$$\begin{aligned} & \left| \int_0^t h_2(t-s) \int_{\Omega} \nabla v^k(s) \nabla v_{tt}^k(t) dx ds \right| \leq \eta \|\nabla v_{tt}^k\|^2 \\ & \quad + \frac{1}{4\eta} (a-k) h_2(0) \int_0^t (\nabla v^k(s))^2 ds, \end{aligned} \quad (52)$$

$$\begin{cases} \left| \mu_1 \int_{\Omega} \nabla u_{tt}^k \nabla z_1^k(x, 1, t) dx \right| \leq \eta \mu_1^2 \|\nabla u_{tt}^k\|^2 + \frac{1}{4\eta} \|\nabla z_1^k(x, 1, t)\|^2, \\ \left| \mu_2 \int_{\Omega} \nabla v_{tt}^k \nabla z_2^k(x, 1, t) dx \right| \leq \eta \mu_2^2 \|\nabla v_{tt}^k\|^2 + \frac{1}{4\eta} \|\nabla z_2^k(x, 1, t)\|^2. \end{cases} \quad (53)$$

By using (49)–(53) in (48), we deduce

$$\begin{aligned} & \int_{\Omega} |u_t^k|^4 |u_{tt}^k|^2 dx + \|\nabla u_{tt}^k\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho \right) \\ & \quad + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_1^k(x, 1, t) \right\|^2 \leq \left(\eta(\mu_1^2 + 2) + \frac{C_s^2}{2} \right) \|\nabla u_{tt}^k\|^2 \\ & \quad + \frac{C_s \alpha}{2} (\|\nabla v_t^k\|^2 + \|\nabla u_t^k\|^2) + \frac{a^2}{4\eta} \|\nabla u^k\|^2 + \frac{1}{4\eta} \|\nabla z_1^k(x, 1, t)\|^2 \\ & \quad + \frac{1}{4\eta} (a-k) h_1(0) \int_0^t \|\nabla u^k(s)\|^2 ds \\ & \quad - \frac{1}{2} \int_0^1 \left(\frac{\tau(t)}{1-\rho\tau'(t)} \right)' \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho, \\ & \int_{\Omega} |v_t^k|^4 |v_{tt}^k|^2 dx + \|\nabla v_{tt}^k\|^2 + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{1-\rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho \right) \\ & \quad + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_2^k(x, 1, t) \right\|^2 \leq \left(\eta(\mu_2^2 + 2) + \frac{C_s^2}{2} \right) \|\nabla v_{tt}^k\|^2 \\ & \quad + \frac{C_s \alpha}{2} (\|\nabla v_t^k\|^2 + \|\nabla u_t^k\|^2) + \frac{a^2}{4\eta} \|\nabla v^k\|^2 + \frac{1}{4\eta} \|\nabla z_2^k(x, 1, t)\|^2 \\ & \quad + \frac{1}{4\eta} (a-k) h_2(0) \int_0^t \|\nabla v^k(s)\|^2 ds \\ & \quad - \frac{1}{2} \int_0^1 \left(\frac{\tau(t)}{1-\rho\tau'(t)} \right)' \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho. \end{aligned} \quad (54)$$

By using Assumption 3 and taking the first estimate (40) into account, we infer

$$\left\{ \begin{aligned} & \int_{\Omega} |u_t^k|^l |u_{tt}^k|^2 dx + \left(1 - \left(\eta(\mu_1^2 + 2) + \frac{C_s^2}{2} \right) \right) \|\nabla u_{tt}^k\|^2 \\ & + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho \right) \\ & + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_1^k(x, 1, t) \right\|^2 \leq C_2 + \frac{1}{4\eta} (a - k_1) h_1(0) C_1 T \\ & + a_1 \int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho, \\ & \int_{\Omega} |v_t^k|^l |v_{tt}^k|^2 dx + \left(1 - \left(\eta(\mu_2^2 + 2) + \frac{C_s^2}{2} \right) \right) \|\nabla v_{tt}^k\|^2 \\ & + \frac{1}{2} \frac{d}{dt} \left(\int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho \right) \\ & + \frac{1}{2} \left\| \frac{\partial}{\partial t} z_2^k(x, 1, t) \right\|^2 \leq C_2 + \frac{1}{4\eta} (a - k_2) h_2(0) C_1 T \\ & + a_2 \int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho, \end{aligned} \right. \quad (55)$$

where C_2 is a positive constant which depends on $\eta, a, C_1, C_s, \alpha$.

Integrating (55) over $(0, t)$, we obtain

$$\left\{ \begin{aligned} & \int_0^t \int_{\Omega} |u_t^k|^l |u_{tt}^k|^2 dx dt + \left(1 - \left(\eta(\mu_1^2 + 2) + \frac{C_s^2}{2} \right) \right) \\ & \int_0^t \|\nabla u_{tt}^k(s)\|^2 ds + \int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_1^k \right\|^2 d\rho \\ & + \frac{1}{2} \int_0^1 \left\| \frac{\partial}{\partial t} z_1^k(x, 1, t) \right\|^2 dt \leq C_2 T + \frac{1}{4\eta} (a - k) h_1(0) C_1 T^2 \\ & + a_1 \int_0^t \int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \left\| \frac{\partial}{\partial s} z_1^k \right\|^2 d\rho ds, \\ & \int_0^t \int_{\Omega} |v_t^k|^l |v_{tt}^k|^2 dx dt + \left(1 - \left(\eta(\mu_2^2 + 2) + \frac{C_s^2}{2} \right) \right) \\ & \int_0^t \|\nabla v_{tt}^k(s)\|^2 ds + \int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 d\rho \\ & + \frac{1}{2} \int_0^1 \left\| \frac{\partial}{\partial t} z_2^k(x, 1, t) \right\|^2 dt \leq C_2 T + \frac{1}{4\eta} (a - k) h_2(0) C_1 T^2 \\ & + a_2 \int_0^t \int_0^1 \frac{\tau(s)}{1 - \rho\tau'(s)} \left\| \frac{\partial}{\partial s} z_2^k \right\|^2 d\rho ds. \end{aligned} \right. \quad (56)$$

For a suitable $\eta > 0$ such that $1 - (\eta(\mu_i^2 + 2)t + nC_s^2/2) > 0$ for $i = 1, 2$ and using Gronwall's lemma, we obtain the second estimate

$$\int_0^t \left(\|\nabla u^{k_{tt}}(s)\|^2 + \|\nabla v^{k_{tt}}(s)\|^2 \right) ds + \int_0^1 \frac{\tau(t)}{1 - \rho\tau'(t)} \cdot \left(\left\| \frac{\partial}{\partial t} z_1^k \right\|^2 + \left\| \frac{\partial}{\partial t} z_2^k \right\|^2 \right) d\rho \leq C_3. \quad (57)$$

We observe from estimates (40) and (57) that there exist subsequences (u^m) of (u^k) and (v^m) of (v^k) such that

$$(u^m, v^m) \rightharpoonup (u, v) \text{ weakly star in } L^\infty(0, T, H_0^1(\Omega)), \quad (58)$$

$$(u_t^m, v_t^m) \rightharpoonup (u_t, v_t) \text{ weakly star in } L^\infty(0, T, H_0^1(\Omega)), \quad (59)$$

$$(u_{tt}^m, v_{tt}^m) \rightharpoonup (u_{tt}, v_{tt}) \text{ weakly in } L^2(0, T, H_0^1(\Omega)), \quad (60)$$

$$(z_1^m, z_2^m) \rightharpoonup (z_1, z_2) \text{ weakly star in } L^\infty(0, T, H_0^1(\Omega, L^2(0, 1))), \quad (61)$$

$$\left(\frac{\partial}{\partial t} z_1^m, \frac{\partial}{\partial t} z_2^m \right) \rightharpoonup \left(\frac{\partial}{\partial t} z_1, \frac{\partial}{\partial t} z_2 \right) \text{ weakly star in } L^\infty(0, T, L^2(\Omega \times (0, 1))), \quad (62)$$

In the following, we will treat the nonlinear term. From the first estimate (40) and Lemma 1, we deduce

$$\begin{aligned} \left\| |u_t^k|^l u_t^k \right\|_{L^2(0, T, L^2(\Omega))} &= \int_0^T \|u_t^k\|_{2^{(l+1)}}^{2^{(l+1)}} dt \\ &\leq C_s^{2^{(l+1)}} \int_0^T \|\nabla u_t^k\|_2^{2^{(l+1)}} dt \leq C_s^{2^{(l+1)}} C_1^{(l+1)} T. \end{aligned} \quad (63)$$

On the other hand, from Aubin–Lions theorem (see [10]), we deduce that there exists a subsequence of (u^m) , still denoted by (u^m) such that

$$u_t^m \longrightarrow u_t \text{ strongly in } L^2(0, T, L^2(\Omega)), \quad (64)$$

which implies

$$u_t^m \longrightarrow u_t \text{ almost everywhere in } \mathcal{A}. \quad (65)$$

Hence,

$$|u_t^m|^l u_t^m \longrightarrow |u_t|^l u_t \text{ almost everywhere in } \mathcal{A}, \quad (66)$$

where $\mathcal{A} = \Omega \times (0, T)$. Thus, using (64) and (66) and Lions lemma, we derive

$$|u_t^m|^l u_t^m \rightharpoonup |u_t|^l u_t \text{ weakly in } L^2(0, T, L^2(\Omega)). \quad (67)$$

Similarly,

$$|v_t^m|^l v_t^m \rightharpoonup |v_t|^l v_t \text{ weakly in } L^2(0, T, L^2(\Omega)), \quad (68)$$

$$(z_1^m, z_2^m) \longrightarrow (z_1, z_2) \text{ strongly in } L^2(0, T, L^2(\Omega)), \quad (69)$$

which implies $(z_1^m, z_2^m) \rightarrow (z_1, z_2)$ almost everywhere in $\mathcal{A}$.

By multiplying (31) and (36) by $\theta(t) \in \mathcal{D}(0, T)$ and by integrating over $(0, T)$, it follows that

$$\left\{ \begin{aligned} & -\frac{1}{l+1} \int_0^T \left(|u_t^k(t)|^l u_t^k(t), w^j \right) \theta'(t) dt + \int_0^T M(\|\nabla u^k(t)\|^2) (\nabla u^k(t), \nabla w^j) \theta(t) dt + \int_0^T (\nabla u_{tt}^k, \nabla w^j) \theta(t) dt \\ & - \int_0^T \int_0^t h_1(t-s) (\nabla u^k(s), \nabla w^j) \theta(t) ds dt + \mu_1 \int_0^T (\nabla z_1^k(\cdot, 1), \nabla w^j \theta(t)) dt = 0, \\ & -\frac{1}{l+1} \int_0^T \left(|v_t^k(t)|^l v_t^k(t), w^j \right) \theta'(t) dt + \int_0^T M(\|\nabla v^k(t)\|^2) (\nabla v^k(t), \nabla w^j) \theta(t) dt + \int_0^T (\nabla v_{tt}^k, \nabla w^j) \theta(t) dt \\ & - \int_0^T \int_0^t h_2(t-s) (\nabla v^k(s), \nabla w^j) \theta(t) ds dt + \mu_2 \int_0^T (\nabla z_2^k(\cdot, 1), \nabla w^j \theta(t)) dt = 0, \\ & \int_0^T \int_0^1 \int_\Omega \left(\tau(t) \frac{\partial}{\partial t} z_1^k + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_1^k \right) \phi^j \theta(t) dx d\rho dt = 0, \\ & \int_0^T \int_0^1 \int_\Omega \left(\tau(t) \frac{\partial}{\partial t} z_2^k + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_2^k \right) \phi^j \theta(t) dx d\rho dt = 0, \end{aligned} \right. \quad (70)$$

for all $j = 1 \dots k$.

The convergence condition in (58)–(62), (68) and (67) are sufficient, thus we can pass to the limit in (70). Then, we have

$$\left\{ \begin{aligned} & \int_0^T \int_\Omega \left(|u_t|^l u_{tt} - M(\|\nabla u(t)\|^2) \Delta u - \Delta u_{tt} + \int_0^t h_1(t-s) \Delta u(s) ds - \mu_1 \Delta z_1(\cdot, 1) \right) w^j \theta(t) dx dt = 0, \\ & \int_0^T \int_\Omega \left(|v_t|^l v_{tt} - M(\|\nabla v(t)\|^2) \Delta v - \Delta v_{tt} + \int_0^t h_2(t-s) \Delta v(s) ds - \mu_2 \Delta z_2(\cdot, 1) \right) w^j \theta(t) dx dt = 0, \\ & \int_0^T \int_0^1 \int_\Omega \left(\tau(t) \frac{\partial}{\partial t} z_1 + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_1 \right) \phi^j \theta(t) dx d\rho dt = 0, \\ & \int_0^T \int_0^1 \int_\Omega \left(\tau(t) \frac{\partial}{\partial t} z_2 + (1 - \rho \tau'(t)) \frac{\partial}{\partial \rho} z_2 \right) \phi^j \theta(t) dx d\rho dt = 0, \end{aligned} \right. \quad (71)$$

for all $j = 1 \dots k$. This completes the proof of Theorem 1.

4. Uniform Decay of the Energy (Proof of Theorem 2)

In this section we study the solution's asymptotic behavior of system (3).

To prove our main result, we construct a Lyapunov functional equivalent to E . For this, we define some functionals which allow us to obtain the desired estimate.

Lemma 4. *Let (u, v, z_1, z_2) be a solution of problem (7). Then, the functional*

$$I(t) = \tau(t) \int_0^1 e^{-2\tau(t)\rho} \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_1(x, \rho, t)\|^2 \right) d\rho, \quad (72)$$

satisfies the estimate

- (i) $|I(t)| \leq (1/\xi)E(t)$.
- (ii) $I'(t) \leq -2\tau(t)e^{-2\tau_1} \int_0^1 (\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2) d\rho - (1-d)e^{-2\tau_1} (\|\nabla z_1(x, 1, t)\|^2 + \|\nabla z_2(x, 1, t)\|^2) + \|\nabla u_t(x, t)\|^2 + \|\nabla v_t(x, t)\|^2$.

Proof. (ii) Differentiating (72) with respect to t and using (5)-(6) and Assumption 3, we get

$$\begin{aligned}
\frac{d}{dt} I(t) &= \tau'(t) \int_0^1 e^{-2\tau(t)\rho} \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) d\rho \\
&\quad + \tau(t) \int_{\Omega} \int_0^1 \left[e^{-2\tau(t)\rho} \frac{\partial \left(|\nabla z_1(x, \rho, t)|^2 + |\nabla z_2(x, \rho, t)|^2 \right)}{\partial t} \right. \\
&\quad \left. - 2\tau'(t)\rho e^{-2\tau(t)\rho} \left(|\nabla z_1(x, \rho, t)|^2 + |\nabla z_2(x, \rho, t)|^2 \right) \right] d\rho dx \\
&= \int_{\Omega} \int_0^1 e^{-2\tau(t)\rho} \left[\tau'(t) \left(|\nabla z_1(x, \rho, t)|^2 + |\nabla z_2(x, \rho, t)|^2 \right) + \tau(t) \frac{\partial \left(|\nabla z_1(x, \rho, t)|^2 + |\nabla z_2(x, \rho, t)|^2 \right)}{\partial t} \right] d\rho dx \\
&\quad - 2 \int_{\Omega} \int_0^1 \tau(t) \tau'(t) \rho e^{-2\tau(t)\rho} \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) d\rho dx \\
&= - \int_0^1 e^{-2\tau(t)\rho} \frac{\partial}{\partial \rho} \left((1 - \rho\tau'(t)) \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) \right) d\rho \\
&\quad - 2 \int_0^1 \tau(t) \tau'(t) \rho e^{-2\tau(t)\rho} \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) d\rho \tag{73} \\
&= - \int_0^1 \left[\frac{\partial}{\partial \rho} \left(e^{-2\tau(t)\rho} (1 - \tau'(t)\rho) \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) \right) \right. \\
&\quad \left. - 2\tau(t) e^{-2\tau(t)\rho} (1 - \tau'(t)\rho) \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) \right] d\rho \\
&\quad - 2\tau(t) \tau'(t) \int_0^1 \rho e^{-2\tau(t)\rho} \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) d\rho \\
&= \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) - e^{-2\tau(t)\rho} (1 - \tau'(t)) \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) \\
&\quad - 2\tau(t) \int_{\Omega} \int_0^1 [(1 - \tau'(t)\rho) + \tau'(t)\rho] e^{-2\tau(t)\rho} \left(|\nabla z_1(x, \rho, t)|^2 + |\nabla z_2(x, \rho, t)|^2 \right) d\rho dx \\
&\leq -2I(t) + \|\nabla u_t(x, t)\|^2 + \|\nabla v_t(x, t)\|^2 - e^{-2\tau(t)\rho} (1 - d) \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right).
\end{aligned}$$

Since $e^{-2\tau(t)\rho}$ is a decreasing function for $\rho \in [0, 1]$ and $\tau(t) \in [\tau_0, \tau_1]$, we deduce

$$I(t) \geq \tau(t) \int_0^1 e^{-2\tau_1} \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) d\rho. \tag{74}$$

Thus, our proof is completed. $\square$

Lemma 5. Let (u, v, z_1, z_2) be a solution of problem (7). Then, the functional

$$\begin{aligned}
\phi(t) &= \frac{1}{l+1} \int_{\Omega} \left(|u_t|^l u_t u + |v_t|^l v_t v \right) dx \\
&\quad + \int_{\Omega} \nabla u_t \nabla u dx + \int_{\Omega} \nabla v_t \nabla v dx, \tag{75}
\end{aligned}$$

satisfies the estimate

- (i) $|\phi(t)| \leq (1/(l+2)) (\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}) + ((l+1)^{-1}/(l+2)) c_s^{l+2} + (c/2) (\|\nabla u\|_{l+2}^{l+2} + \|\nabla v\|_{l+2}^{l+2}) + (1/2) (\|\nabla u\|^2 + \|\nabla v\|^2).$
- (ii) $\phi'(t) \leq (1/(l+1)) (\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}) + (\eta(a-k+1) + \alpha C_s - k) (\|\nabla u\|^2 + \|\nabla v\|^2) + (1/4\eta) [(h_1 \circ \nabla u)(t) + (h_1 \circ \nabla v)(t)] + (\mu_1^2/4\eta) \|\nabla z_1(x, 1, t)\|^2 + (\mu_2^2/4\eta) \|\nabla z_2(x, 1, t)\|^2 + \|\nabla u_t\|^2 + \|\nabla v_t\|^2.$

Proof

- (i) From Young's inequality, the Sobolev embedding, and $L^{l+2} \longrightarrow L^2$, we deduce

$$\begin{aligned}
|\phi(t)| &\leq \frac{1}{l+2} \|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} \|u\|_{l+2}^{l+2} + \frac{1}{l+2} \|v_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} \|v\|_{l+2}^{l+2} \\
&\quad + \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\nabla v_t\|^2 + \frac{1}{2} \|\nabla v\|^2 \\
&\leq \frac{1}{l+2} \|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} c_s^{l+2} \|\nabla u\|^{l+2} + \frac{1}{l+2} \|v_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} c_s^{l+2} \|\nabla v\|^{l+2} \\
&\quad + \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\nabla v_t\|^2 + \frac{1}{2} \|\nabla v\|^2 \\
&\leq \frac{1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) + \left(\frac{(l+1)^{-1}}{l+2} c_s^{l+2} + \frac{c}{2} \right) \left(\|\nabla u\|^{l+2} + \|\nabla v\|^{l+2} \right) + \frac{1}{2} \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right).
\end{aligned} \tag{76}$$

(i) Differentiating $\phi(t)$ with respect to t and using the first and second equations of (7), we get

$$\begin{aligned}
\phi'(t) &= \frac{1}{l+1} \int_{\Omega} \left(|u_t|^l u_t \right)' u dx + \frac{1}{l+1} \int_{\Omega} |u_t|^{l+2} dx + \frac{1}{l+1} \int_{\Omega} \left(|v_t|^l v_t \right)' v dx + \frac{1}{l+1} \int_{\Omega} |v_t|^{l+2} dx \\
&\quad + \int_{\Omega} \nabla u_{tt} \nabla u dx + \int_{\Omega} \nabla u_t \nabla u_t dx + \int_{\Omega} \nabla v_{tt} \nabla v dx + \int_{\Omega} \nabla v_t \nabla v_t dx \\
&= \int_{\Omega} \left[|u_t|^l u_t \right]' u dx + \frac{1}{l+1} \|u_t\|_{l+2}^{l+2} + \int_{\Omega} \left[|v_t|^l v_t \right]' v dx + \frac{1}{l+1} \|v_t\|_{l+2}^{l+2} \\
&\quad - \int_{\Omega} \nabla u_{tt} u dx + \|\nabla u_t\|^2 - \int_{\Omega} \nabla v_{tt} v dx + \|\nabla v_t\|^2 \\
&= \frac{1}{l+1} \left(\|u_t\|_{l+2}^{l+2} + \|u_t\|_{l+2}^{l+2} \right) + \int_{\Omega} \left[|u_t|^l u_t - \Delta u_{tt} \right] u dx + \int_{\Omega} \left[|v_t|^l v_t - \Delta v_{tt} \right] v dx + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \\
&= \frac{1}{l+1} \left(\|u_t\|_{l+2}^{l+2} + \|u_t\|_{l+2}^{l+2} \right) + \int_{\Omega} \left[-\alpha v + M(\|\nabla u\|^2) \Delta u - \int_0^t h_1(t-s) \Delta u(s) ds + \mu_1 \Delta z_1(x, 1, t) \right] u dx \\
&\quad + \int_{\Omega} \left[-\alpha u + M(\|\nabla v\|^2) \Delta v - \int_0^t h_1(t-s) \Delta v(s) ds + \mu_2 \Delta z_2(x, 1, t) \right] v dx + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 \\
&= \frac{1}{l+1} \left(\|u_t\|_{l+2}^{l+2} + \|u_t\|_{l+2}^{l+2} \right) - M(\|\nabla u_t\|^2) \|\nabla v_t\|^2 + \int_{\Omega} \nabla u(t) \int_0^t h_1(t-s) \Delta u(s) ds dx - \mu_1 \int_{\Omega} \nabla z_1(x, 1, t) \nabla u dx \\
&\quad - M(\|\nabla v\|^2) \|\nabla v\|^2 + \int_{\Omega} \nabla v(t) \int_0^t h_2(t-s) \Delta v(s) ds dx - \mu_2 \int_{\Omega} \nabla z_2(x, 1, t) \nabla v dx + \|\nabla u_t\|^2 + \|\nabla v_t\|^2 - 2\alpha \int_{\Omega} u v dx.
\end{aligned} \tag{77}$$

As $M(r) \geq a$ and making use Young's inequality, we obtain

$$\begin{aligned}
\phi'(t) &\leq \frac{1}{l+1} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) - a \|\nabla u\|^2 + \int_{\Omega} \nabla u(t) \int_0^t h_1(t-s) \nabla u(s) ds dx + \frac{\mu_1^2}{4\eta} \|\nabla z_1(x, 1, t)\|^2 + \eta \|\nabla u\|^2 \\
&\quad - a \|\nabla v\|^2 + \int_{\Omega} \nabla v(t) \int_0^t h_2(t-s) \nabla v(s) ds dx + \frac{\mu_2^2}{4\eta} \|\nabla z_2(x, 1, t)\|^2 + \eta \|\nabla v\|^2 + \|\nabla u_t\|^2 + \|\nabla v_t\|^2.
\end{aligned} \tag{78}$$

By use of Young's inequality, we can estimate the third term in the right side as follows:

$$\begin{aligned}
\int_{\Omega} \nabla u(t) \int_0^t h_1(t-s) \nabla u(s) ds dx &\leq \int_0^t h(t-s) \int_{\Omega} |\nabla u(t) \nabla u(s) - \nabla u(t)| dx ds + \|\nabla u(t)\|^2 \int_0^t h_1(t-s) ds \\
&\leq \eta \|\nabla u(t)\|^2 \int_0^t h_1(s) ds + \frac{1}{4\eta} \int_0^t h_1(t-s) \|\nabla u(s) - \nabla u(t)\|^2 ds \\
&\quad + \|\nabla u(t)\|^2 \int_0^t h_1(s) ds \\
&\leq (1+\eta)(a-k) \|\nabla u(t)\|^2 + \frac{1}{4\eta} (h_1 \circ \nabla u)(t).
\end{aligned} \tag{79}$$

Similarly,

$$\begin{aligned}
\int_{\Omega} \nabla v(t) \int_0^t h_2(t-s) \nabla v(s) ds dx &\leq (1+\eta)(a-k) \|\nabla v(t)\|^2 + \frac{1}{4\eta} (h_2 \circ \nabla v)(t), \\
-2\alpha \int_{\Omega} uv dx &\leq \alpha C_s (\|\nabla u\|^2 + \|\nabla v\|^2).
\end{aligned} \tag{80}$$

Thus, our proof is completed. $\square$

Lemma 6. Let (u, z) be a solution of problem (7). Then, the functional

$$\psi(t) = \int_{\Omega} \left(\Delta u_t - \frac{1}{l+1} |u_t|^l u_t \right) \int_0^t h_1(t-s) (u(t) - u(s)) ds dx + \int_{\Omega} \left(\Delta v_t - \frac{1}{l+1} |v_t|^l v_t \right) \int_0^t h_2(t-s) (v(t) - v(s)) ds dx, \tag{81}$$

satisfies the estimates

- (i) $|\psi(t)| \leq (1/2) (\|\nabla u_t\|^2 + \|\nabla v_t\|^2) + (1/2)(a-k)(1 + ((l+1)^{-1}/(l+2))(a-k)^l c_s^{l+2}) [(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)] + ((l+1)^{-1}/(l+2))(a-k)^{l+2} c_s^{l+2} 2^{2l+1} (\|\nabla u\|^{2(l+1)} + \|\nabla v\|^{2(l+1)}) + (1/(l+2)) (\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2})$.
- (ii) $\psi(t) \leq \delta [(a-k) + ((l+1)^{-1}/(l+2))(h_1(0))^{l+2} c_s^{l+2} 2^{2(l+1)}] M (\|\nabla u\|^2) \|\nabla u\|^2 + ((\alpha C_s/2) + 2\delta(a-k)^2) \|\nabla u\|^2 + ((M(\|\nabla u\|^2)/4\delta) + ((\alpha C_s/2) + 2\delta + (1/2\delta))(a-k))(h_1 \circ \nabla v)(t) - (h_1(0)/4\delta)(1 + ((l+1)^{-1}/(l+2))(h_1(0))^l c_s^{l+2}) (h_1' \circ \nabla v)(t) + (\delta - \int_0^t h_1(s) ds) \|\nabla u_t\|^2 + \mu_1^2 \delta \|\nabla z_1(x, 1, t)\|^2 + (1/(l+1))(1 - \int_0^t h_1(s) ds) \|u_t\|_{l+2}^{l+2} + \delta [(a-k) + ((l+1)^{-1}/(l+2))(h_2(0))^{l+2} c_s^{l+2} 2^{2(l+1)}] M (\|\nabla v\|^2) \|\nabla v\|^2 + ((\alpha C_s/2) + 2\delta(a-k)^2) \|\nabla v\|^2 + ((M(\|\nabla v\|^2)/4\delta) + ((\alpha C_s/2) + 2\delta +$

$$\begin{aligned}
& (1/2\delta))(a-k))(h_2 \circ \nabla v)(t) - (h_2(0)/4\delta)(1 + ((l+1)^{-1}/(l+2))(h_2(0))^l c_s^{l+2}) (h_2' \circ \nabla v)(t) + (\delta - \int_0^t h_2(s) ds) \|\nabla v_t\|^2 + \mu_2^2 \delta \|\nabla z_2(x, 1, t)\|^2 + (1/(l+1))(1 - \int_0^t h_2(s) ds) \|v_t\|_{l+2}^{l+2},
\end{aligned}$$

where $\delta > 0$ and c_s is the Sobolev embedding constant.

Proof

$$\begin{aligned}
\text{(i)} \quad \psi(t) &= - \int_{\Omega} \nabla u_t \int_0^t h_2(t-s) (\nabla u(t) - \nabla u(s)) ds dx - \int_{\Omega} (1/(l+1)) |u_t|^l u_t \int_0^t h_2(t-s) (u(t) - u(s)) ds dx - \int_{\Omega} \nabla v_t \int_0^t h_2(t-s) (\nabla v(t) - \nabla v(s)) ds dx - \int_{\Omega} (1/(l+1)) |v_t|^l v_t \int_0^t h_2(t-s) (v(t) - v(s)) ds dx.
\end{aligned}$$

We use Young's inequality with the conjugate exponents $p = (l+2)/(l+1)$ and $q = l+2$, and the second term in the right hand side can be estimated as

$$\begin{aligned}
& \left| -\int_{\Omega} \frac{1}{l+1} |u_t|^l u_t \int_0^t h_1(t-s)(u(t)-u(s)) ds dx \right| \leq \frac{1}{l+1} \left| \int_{\Omega} \left(|u_t|^l u_t \right) \left(\int_0^t h_1((t-s)u(t)-u(s)) ds \right) dx \right| \\
& \leq \frac{1}{l+1} \left[\frac{1}{p} \int_{\Omega} |u_t|^l |u_t|^p dx + \frac{1}{q} \int_{\Omega} \left| \int_0^t h_1((t-s)u(t)-u(s)) ds \right|^q dx \right] \\
& \leq \frac{1}{l+1} \left[\frac{1}{p} \int_{\Omega} |u_t|^{l+1} dx + \frac{1}{q} \int_{\Omega} \int_0^t (h_1((t-s)u(t)-u(s)))^q ds dx \right] \\
& \leq \frac{1}{l+2} \|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} \int_{\Omega} \left[\int_0^t (h_1(t-s))^{\frac{l+1}{l+2}} \left((h_1(t-s))^{\frac{1}{l+2}} |u(t)-u(s)| \right) ds \right]^{l+2} dx.
\end{aligned} \tag{82}$$

We have by Hölder's inequality

$$\begin{aligned}
& \int_{\Omega} \left[\int_0^t (h_1(t-s))^{((l+1)/(l+2))} \left((h_1(t-s))^{(1/(l+2))} |u(t)-u(s)| \right) ds \right]^{l+2} dx \\
& \leq \int_{\Omega} \left[\left(\int_0^t ((h_1(t-s))^{((l+1)/(l+2))})^p ds \right)^{1/p} \left(\int_0^t ((h_1(t-s))^{(1/(l+2))} |u(t)-u(s)|)^q ds \right)^{1/q} \right]^{l+2} dx \\
& \leq \int_{\Omega} \left(\int_0^t h_1(t-s) ds \right)^{((l+1)/(l+2))} \left[\left(\int_0^t h_1(t-s) |u(t)-u(s)|^{l+2} ds \right)^{1/(l+2)} \right]^{l+2} dx \\
& \leq \left(\int_0^t h_1(t-s) ds \right)^{l+1} \int_0^t h_1(t-s) \|u(t)-u(s)\|_{l+2}^{l+2} ds \\
& \leq (a-k)^{l+1} c_s^{l+2} \int_0^t \sqrt{h_1(t-s)} \sqrt{h_1(t-s)} \|\nabla u(t) - \nabla u(s)\|^{l+1} \|\nabla u(t) - \nabla u(s)\| ds \\
& \leq (a-k)^{l+1} c_s^{l+2} \left(\frac{1}{2} \int_0^t h_1(t-s) \|\nabla u(t) - \nabla u(s)\|^{2l+2} ds + \frac{1}{2} \int_0^t h_1(t-s) \|\nabla u(t) - \nabla u(s)\|^2 ds \right) \\
& \leq (a-k)^{l+1} c_s^{l+2} \left(\frac{1}{2} \int_0^t h_1(t-s) \|2\nabla u(t)\|^{2l+2} ds + \frac{1}{2} (h_1 \circ \nabla u)(t) \right) \\
& \leq (a-k)^{l+1} c_s^{l+2} \left(2^{2l+1} (a-k) \|\nabla u(t)\|^{2(l+1)} + \frac{1}{2} (h_1 \circ \nabla u)(t) \right).
\end{aligned} \tag{83}$$

Combining (83) with (82), we obtain

$$\begin{aligned}
& \left| -\int_{\Omega} \frac{1}{l+1} |u_t|^l u_t \int_0^t h_1(t-s)(u(t)-u(s)) ds dx \right| \leq \frac{1}{l+2} \|u_t\|_{l+2}^{l+2} \\
& + \frac{(l+1)^{-1}}{l+2} \left[(a-k)^{l+1} c_s^{l+2} \left(2^{2l+1} (a-k) \|\nabla u(t)\|^{2(l+1)} + \frac{1}{2} (h_1 \circ \nabla u)(t) \right) \right].
\end{aligned} \tag{84}$$

Similarly, we get

$$\begin{aligned} \left| -\int_{\Omega} \nabla u_t \int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s)) ds dx \right| &\leq \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \int_{\Omega} \left(\int_0^t h_1(t-s) |\nabla u(t) - \nabla u(s)| ds \right)^2 dx \\ &\leq \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} (a-k)(h_1 \circ \nabla u)(t). \end{aligned} \quad (85)$$

Similarly,

$$\begin{cases} \left| -\int_{\Omega} \frac{1}{l+1} |v_t|^l v_t \int_0^t h_2(t-s)(v(t) - v(s)) ds dx \right| \leq \frac{1}{l+2} \|v_t\|^{l+2} \\ \quad + \frac{(l+1)^{-1}}{l+2} \left[(a-k)^{l+1} c_s^{l+2} \left(2^{2l+1} (a-k) \|\nabla v(t)\|^{2(l+1)} + \frac{1}{2} (h_2 \circ \nabla v)(t) \right) \right] \\ \left| -\int_{\Omega} \nabla v_t \int_0^t h_2(t-s)(\nabla v(t) - \nabla v(s)) ds dx \right| \leq \frac{1}{2} \|\nabla v(t)\|^2 + \frac{1}{2} (a-k)(h_2 \circ \nabla v)(t). \end{cases} \quad (86)$$

Combining (84)–(86), we deduce (i).

(ii) Using the Leibniz formula and the first and second equations of (7), we have

$$\begin{aligned} \psi'(t) &= \int_{\Omega} \left(\Delta u_{tt} - |u_t|^l u_{tt} \right) \int_0^t h_1(t-s)(u(t) - u(s)) ds dx \\ &\quad + \int_{\Omega} \left(\Delta u_t - \frac{1}{l+1} |u_t|^l u_t \right) \left(\int_0^t (h_1'(t-s)(u(t) - u(s))) + h_1(t-s) u_t(t) ds \right) dx \\ &\quad + \int_{\Omega} \left(\Delta v_{tt} - |v_t|^l v_{tt} \right) \int_0^t h_2(t-s)(v(t) - v(s)) ds dx \\ &\quad + \int_{\Omega} \left(\Delta v_t - \frac{1}{l+1} |v_t|^l v_t \right) \left(\int_0^t (h_2'(t-s)(v(t) - v(s))) + h_2(t-s) v_t(t) ds \right) dx \\ &= -\alpha \int_{\Omega} v(t) \int_0^t h_1(t-s)(u(t) - u(s)) ds dx + \int_{\Omega} M(\|\nabla u\|)^2 \nabla u(t) \int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\ &\quad - \int_{\Omega} \int_0^t h_1(t-s) \nabla u(s) ds \int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\ &\quad + \mu_1 \int_{\Omega} \nabla z_1(x, 1, t) \int_0^t h_1(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\ &\quad - \int_{\Omega} \nabla u_t \int_0^t h_1'(t-s)(\nabla u(t) - \nabla u(s)) ds dx - \frac{1}{l+1} \int_{\Omega} |u_t|^l u_t \int_0^t h_1'(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\ &\quad - \|\nabla u_t\|^2 \int_0^t h_1(s) ds - \frac{1}{l+1} \|u_t\|_{l+2}^{l+2} \int_0^t h_1(s) ds \end{aligned}$$

$$\begin{aligned}
& -\alpha \int_{\Omega} u(t) \int_0^t h_2(t-s)(v(t)-v(s))ds dx + \int_{\Omega} M(\|\nabla v\|^2) \nabla v(t) \int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
& - \int_{\Omega} \int_0^t h_2(t-s) \nabla v(s) ds \int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
& + \mu_2 \int_{\Omega} \nabla z_2(x, 1, t) \int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
& - \int_{\Omega} \nabla v_t \int_0^t h_2'(t-s)(\nabla v(t)-\nabla v(s))ds dx - \frac{1}{l+1} \int_{\Omega} |v_t|^l v_t \int_0^t h_2'(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
& - \|\nabla v_t\|^2 \int_0^t h_2(s)ds - \frac{1}{l+1} \|v_t\|^{l+2} \int_0^t h_2(s)ds \\
& = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 - \|\nabla u_t\|^2 \int_0^t h_1(s)ds - \frac{1}{l+1} \|u_t\|^{l+2} \int_0^t h_1(s)ds \\
& - \|\nabla v_t\|^2 \int_0^t h_2(s)ds - \frac{1}{l+1} \|v_t\|^{l+2} \int_0^t h_2(s)ds,
\end{aligned} \tag{87}$$

where

$$\left\{ \begin{aligned}
I_1 &= \int_{\Omega} M(\|\nabla u\|^2) \nabla u(t) \int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds dx + \int_{\Omega} M(\|\nabla v\|^2) \nabla v(t) \int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
I_2 &= - \int_{\Omega} \int_0^t h_1(t-s) \nabla u(s) ds \int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds dx \\
& - \int_{\Omega} \int_0^t h_2(t-s) \nabla v(s) ds \int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
I_3 &= \mu_1 \int_{\Omega} \nabla z_1(x, 1, t) \int_0^t h_1(t-s)(\nabla u(t)-\nabla u(s))ds dx + \mu_2 \int_{\Omega} \nabla z_2(x, 1, t) \int_0^t h_2(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
I_4 &= - \int_{\Omega} \nabla u_t \int_0^t h_1'(t-s)(\nabla u(t)-\nabla u(s))ds dx - \int_{\Omega} \nabla v_t \int_0^t h_2'(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
I_5 &= -\frac{1}{l+1} \int_{\Omega} |u_t|^l u_t \int_0^t h_1'(t-s)(\nabla u(t)-\nabla u(s))ds dx - \frac{1}{l+1} \int_{\Omega} |v_t|^l v_t \int_0^t h_2'(t-s)(\nabla v(t)-\nabla v(s))ds dx \\
I_6 &= -\alpha \int_{\Omega} v(t) \int_0^t h_1'(t-s)(\nabla u(t)-\nabla u(s))ds dx - \alpha \int_{\Omega} u(t) \int_0^t h_2'(t-s)(\nabla v(t)-\nabla v(s))ds dx.
\end{aligned} \right. \tag{88}$$

In what follows we will estimate $I_1, \dots, I_6$.

For I_1 , we use Hölder and Young's inequalities with $p = q = 2$, and we get

$$\begin{aligned}
|I_1| &\leq M(\|\nabla u\|^2) \int_{\Omega} |\nabla u(t)| \left(\int_0^t h_1(s) ds \right)^{1/2} \left(\int_0^t h_1(s) |\nabla u(t) - \nabla u(s)|^2 ds \right)^{1/2} dx \\
&\quad + M(\|\nabla v\|^2) \int_{\Omega} |\nabla v(t)| \left(\int_0^t h_2(s) ds \right)^{1/2} \left(\int_0^t h_2(s) |\nabla v(t) - \nabla v(s)|^2 ds \right)^{1/2} dx \\
&\leq M(\|\nabla u\|^2) \left[\delta \int_{\Omega} |\nabla u(t)|^2 \left(\int_0^t h_1(s) ds \right) dx + \frac{1}{4\delta} \int_{\Omega} \int_0^t h_1(t-s) |\nabla u(t) - \nabla u(s)|^2 ds dx \right] \\
&\quad + M(\|\nabla v\|^2) \left[\delta \int_{\Omega} |\nabla v(t)|^2 \left(\int_0^t h_2(s) ds \right) dx + \frac{1}{4\delta} \int_{\Omega} \int_0^t h_2(t-s) |\nabla v(t) - \nabla v(s)|^2 ds dx \right] \\
&\leq M(\|\nabla u\|^2) \left(\delta(a-k) \|\nabla u(t)\|^2 + \frac{1}{4\delta} (h_1 \circ \nabla u)(t) \right) + M(\|\nabla v\|^2) \left(\delta(a-k) \|\nabla v(t)\|^2 + \frac{1}{4\delta} (h_2 \circ \nabla v)(t) \right).
\end{aligned} \tag{89}$$

Similarly,

$$\begin{aligned}
|I_2| &\leq \delta \int_{\Omega} \left(\int_0^t h_1(t-s) |\nabla u(s)| ds \right)^2 dx + \frac{1}{4\delta} \int_{\Omega} \left(\int_0^t h_1(t-s) |\nabla u(t) - \nabla u(s)| ds \right)^2 dx \\
&\quad + \delta \int_{\Omega} \left(\int_0^t h_2(t-s) |\nabla v(s)| ds \right)^2 dx + \frac{1}{4\delta} \int_{\Omega} \left(\int_0^t h_2(t-s) |\nabla v(t) - \nabla v(s)| ds \right)^2 dx \\
&\leq \delta \int_{\Omega} \left(\int_0^t h_1(t-s) (|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|) ds \right)^2 dx + \frac{1}{4\delta} \left(\int_0^t h_1(s) ds (h_1 \circ \nabla u)(t) \right) \\
&\quad + \delta \int_{\Omega} \left(\int_0^t h_2(t-s) (|\nabla v(s) - \nabla v(t)| + |\nabla v(t)|) ds \right)^2 dx + \frac{1}{4\delta} \left(\int_0^t h_2(s) ds (h_2 \circ \nabla v)(t) \right) \\
&\leq 2\delta \|\nabla u(t)\|^2 \left(\int_0^t h_1(t) ds^2 dx + \left(2\delta + \frac{1}{4\delta} \right) \int_0^t h_1(s) ds (h_1 \circ \nabla u)(t) \right) \\
&\quad + 2\delta \|\nabla v(t)\|^2 \left(\int_0^t h_2(t) ds^2 dx + \left(2\delta + \frac{1}{4\delta} \right) \int_0^t h_2(s) ds (h_2 \circ \nabla v)(t) \right) \\
&\leq 2\delta \|\nabla u(t)\|^2 (a-k)^2 + \left(2\delta + \frac{1}{4\delta} \right) (a-k) (h_1 \circ \nabla u)(t) + 2\delta \|\nabla v(t)\|^2 (a-k)^2 + \left(2\delta + \frac{1}{4\delta} \right) (a-k) (h_2 \circ \nabla v)(t), \\
|I_3| &\leq \delta \left(\mu_1^2 \|\nabla z_1(x, 1, t)\|^2 + \mu_2^2 \|\nabla z_2(x, 1, t)\|^2 \right) + \frac{(a-k)}{4\delta} (h_1 \circ \nabla u)(t) + \frac{(a-k)}{4\delta} (h_2 \circ \nabla v)(t),
\end{aligned} \tag{90}$$

$$\begin{aligned}
|I_4| &\leq \delta \int_{\Omega} |\nabla u_t|^2 dx + \frac{1}{4\delta} \int_{\Omega} \left(\int_0^t |h_1'(t-s)| |\nabla u(t) - \nabla u(s)| ds \right)^2 dx \\
&\quad + \delta \int_{\Omega} |\nabla v_t|^2 dx + \frac{1}{4\delta} \int_{\Omega} \left(\int_0^t |h_2'(t-s)| |\nabla v(t) - \nabla v(s)| ds \right)^2 dx \\
&\leq \delta \|\nabla u_t\|^2 + \frac{1}{4\delta} \int_0^t (-h_1'(t-s)) ds \int_{\Omega} \int_0^t (-h_1'(t-s)) |\nabla u(t) - \nabla u(s)|^2 ds dx \\
&\quad + \delta \|\nabla v_t\|^2 + \frac{1}{4\delta} \int_0^t (-h_2'(t-s)) ds \int_{\Omega} \int_0^t (-h_2'(t-s)) |\nabla v(t) - \nabla v(s)|^2 ds dx \\
&\leq \delta \|\nabla u_t\|^2 - \frac{h_1(0)}{4\delta} (h_1' \circ \nabla u)(t) + \delta \|\nabla v_t\|^2 - \frac{h_2(0)}{4\delta} (h_2' \circ \nabla v)(t),
\end{aligned} \tag{91}$$

and since $l \leq \gamma$, we follow almost the same steps to obtain $|\psi(t)|$

$$\begin{aligned}
|I_5| &\leq \frac{1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) + \frac{(l+1)^{-1}}{l+2} \left[(h_1(0))^{l+1} \int_0^t (-h_1'(t-s)) \|u(t) - u(s)\|_{l+2}^{l+2} ds \right. \\
&\quad \left. + (h_2(0))^{l+1} \int_0^t (-h_2'(t-s)) \|v(t) - v(s)\|_{l+2}^{l+2} ds \right] \\
&\leq \frac{1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) + \frac{(l+1)^{-1}}{l+2} c_s^{l+2} \left[(h_1(0))^{l+1} \int_0^t (-h_1'(t-s)) \|\nabla u(t) - \nabla u(s)\|^{l+2} ds \right. \\
&\quad \left. + (h_2(0))^{l+1} \int_0^t (-h_2'(t-s)) \|\nabla v(t) - \nabla v(s)\|^{l+2} ds \right] \\
&\leq \frac{1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) + \frac{(l+1)^{-1}}{l+2} c_s^{l+2} (h_1(0))^{l+1} \left[\delta 2^{2(l+1)} h_1(0) \|\nabla u(t)\|^{2(l+1)} - \frac{1}{4\delta} (h_1' \circ \nabla u)(t) \right] \\
&\quad + \frac{(l+1)^{-1}}{l+2} c_s^{l+2} (h_2(0))^{l+1} \left[\delta 2^{2(l+1)} h_2(0) \|\nabla v(t)\|^{2(l+1)} - \frac{1}{4\delta} (h_2' \circ \nabla v)(t) \right] \\
&\leq \frac{1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) + \frac{(l+1)^{-1}}{l+2} c_s^{l+2} (h_1(0))^{l+1} \left[\delta 2^{2(l+1)} h_1(0) M(\|\nabla u(t)\|^2) \|\nabla u(t)\|^2 - \frac{1}{4\delta} (h_1' \circ \nabla u)(t) \right] \\
&\quad + \frac{(l+1)^{-1}}{l+2} c_s^{l+2} (h_2(0))^{l+1} \left[\delta 2^{2(l+1)} h_2(0) M(\|\nabla v(t)\|^2) \|\nabla v(t)\|^2 - \frac{1}{4\delta} (h_2' \circ \nabla v)(t) \right], \\
|I_6| &\leq \frac{\alpha C_s}{2} (\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + (a-k)[(h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)]). \tag{94}
\end{aligned}$$

Combining (87) and (89)–(94), we finish the proof. $\square$

Proof of Theorem 1.2. Now, for $M, \varepsilon_1 > 0$, we introduce the following functional:

$$F(t) = ME(t) + \varepsilon_1 \phi(t) + \psi(t) + I(t) \sim E(t). \tag{95}$$

Indeed, to prove $F(t) \sim E(t)$, we show that there exist two positive constants κ_1 and κ_2 such that

$$\kappa_1 E(t) \leq F(t) \leq \kappa_2 E(t). \tag{96}$$

From (i) of Lemmas 4–6 and recalling the fact that $l \leq \gamma$, we get

$$\begin{aligned}
|\varepsilon_1 \phi(t) + \psi(t) + I(t)| &\leq \frac{\varepsilon_1 + 1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) + \frac{\varepsilon_1 + 1}{2} \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) \\
&\quad + \left(\frac{\varepsilon_1 c}{2} + \frac{(l+1)^{-1}}{(l+2)} c_s^{l+2} (\varepsilon_1 + 2^{2l+1} (a-k)^{l+2}) \right) (\|\nabla u\|^{2(\gamma+1)} + \|\nabla v\|^{2(\gamma+1)}) \\
&\quad + \frac{a-k}{2} \left(1 + \frac{(l+1)^{-1}}{(l+2)} (a-k)^l c_s^{l+2} \right) ((h_1 \circ \nabla u)(t) + (h_2 \circ \nabla v)(t)) + \frac{1}{\xi} E(t) \\
&\leq \kappa E(t), \tag{97}
\end{aligned}$$

where $\kappa > 0$ depending on $\varepsilon_1, a, b, l, c, c_s, k, \xi$. If we choose $M = \kappa + \varepsilon$, we can obtain our result.

By recalling Lemmas 3–6 and by Assumption 2, we deduce that for $t \geq t_0 > 0$,

$$\begin{aligned}
F'(t) &= ME'(t) + \varepsilon_1 \phi'(t) + \psi'(t) + I'(t) \\
&\leq \left(\mu^2 \delta + \varepsilon_1 \frac{\mu^2}{4\eta} - (1-d)e^{-2\tau_1} - M\beta \right) \left(\|\nabla z_1(x, 1, t)\|^2 + \|\nabla z_2(x, 1, t)\|^2 \right) \\
&\quad - 2\tau(t)e^{-2\tau_1} \int_0^1 \left(\|\nabla z_1(x, p, t)\|^2 + \|\nabla z_2(x, p, t)\|^2 \right) d\rho \\
&\quad - \left(\varepsilon_1 [k - \eta(a - k + 1)] - \frac{\alpha C_S}{2} - 2\delta(a - k)^2 - \delta \left[(a - k) + \frac{(l+1)^{-1}}{(l+2)} (h_2 c_s)^{l+2} 2^{2(l+1)} \right] M_0 \right) (\|\nabla u\|^2 + \|\nabla v\|^2) \\
&\quad + \left[\left(\frac{M}{2} - \frac{h_1}{4\delta} \left(1 + \frac{(l+1)^{-1}}{(l+2)} h_1^l c_s^{l+2} \right) - \frac{1}{\zeta} \left(\frac{\varepsilon_1}{4\eta} + \frac{M_0}{4\delta} + \left(\frac{\alpha C_S}{2} + 2\delta + \frac{1}{2\delta} \right) (a - k) \right) \right] (h_1' o \nabla u)(t) + (h_2' o \nabla v)(t) \right. \\
&\quad \left. - \frac{1}{l+1} (h_0 - 1 - \varepsilon_1) \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) - (h_0 - \delta - M\lambda - 1 - \varepsilon_1) \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) \right],
\end{aligned} \tag{98}$$

where $M_0 = \max\{M(\|\nabla u\|^2), M(\|\nabla v\|^2)\}$, $h_0 = \min\{\int_0^{t_0} h_1(s)ds, \int_0^{t_0} h_2(s)ds\}$, $h_1 = \min\{h_1(0), h_2(0)\}$, $h_2 = \min\{h_1(0), h_2(0)\}$, and $\zeta = \max\{\zeta_1, \zeta_2\}$.

Let $\epsilon > 0$ be sufficiently small so M is fixed; we take $h_0 - M\lambda - 1 > \varepsilon_1$ and δ small enough such that

$$\begin{aligned}
a_3 &= h_0 - 1 - \varepsilon_1 > 0, \\
a_4 &= h_0 - \delta - M\lambda - 1 - \varepsilon_1.
\end{aligned} \tag{99}$$

Further, we choose η, α small enough such that

$$\begin{aligned}
a_1 &= \mu_1^2 \delta + \varepsilon_1 \frac{\mu^2}{4\eta} - (1-d)e^{-2\tau_1} - M\beta > 0, \\
a_2 &= \varepsilon_1 [k - \eta(a - k + 1)] - \frac{\alpha C_S}{2} - 2\delta(a - k)^2 - \delta \left((a - k) + \frac{(l+1)^{-1}}{(l+2)} (h_2 c_s)^{l+2} 2^{2(l+1)} \right) M_0 > 0, \\
a_5 &= \frac{M}{2} - \frac{h_1}{4\delta} \left(1 + \frac{(l+1)^{-1}}{(l+2)} h_1^l c_s^{l+2} \right) - \frac{1}{\zeta} \left(\frac{\varepsilon_1}{4\eta} + \frac{M_0}{4\delta} + \left(\frac{\alpha C_S}{2} + 2\delta + \frac{1}{2\delta} \right) (a - k) \right) < 0.
\end{aligned} \tag{100}$$

Thus,

$$\begin{aligned}
F'(t) &\leq -a_3 \frac{1}{l+2} \left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2} \right) - a_2 (\|\nabla u\|^2 + \|\nabla v\|^2) - 2\tau(t)e^{-2\tau_1} \int_0^1 \left(\|\nabla z_1(x, \rho, t)\|^2 + \|\nabla z_2(x, \rho, t)\|^2 \right) d\rho \\
&\quad + a_1 \left(\|\nabla z_1(x, 1, t)\|^2 + \|\nabla z_2(x, 1, t)\|^2 \right) - a_4 \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2 \right) + a_5 [(h_1' o \nabla u)(t) + (h_2' o \nabla v)(t)] \\
&\leq -mE(t) - cE'(t),
\end{aligned} \tag{101}$$

where $m = \min\{(2e^{-2\tau_1}/\xi), 2(a_2/a), a_3\}$ and $(a_1/\beta) \leq c \leq \min\{(a_4/\lambda), -2a_5\}$.

Let $L(t) = F(t) + cE(t) \sim E(t)$. From (101), we get

$$L'(t) \leq -c' L(t), \quad \forall t \geq t_0, \tag{102}$$

for some $c' > 0$. A simple integration over (t_0, t) yields

$$L(t) \leq L(t_0) e^{-c'(t-t_0)}, \quad \forall t \geq t_0. \tag{103}$$

Thanks to equivalence between L and E , we obtain (16). $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to this study. All authors read and approved the final manuscript.

References

- [1] N. Mezouar and S. Boulaaras, "Global existence and decay of solutions for a class of viscoelastic Kirchhoff equation," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 43, no. 1, pp. 725–755, 2020.
- [2] N. Mezouar and S. Boulaaras, "Global existence of solutions to a viscoelastic non-degenerate Kirchhoff equation," *Applicable Analysis*, 2018, in Press.
- [3] S. Nicaise and C. Pignotti, "Interior feedback stabilization of wave equations with time dependent delay," *Electronic Journal of Differential Equations*, vol. 41, pp. 1–20, 2011.
- [4] Q. C. Zhong, *Robust Control of Time-Delay Systems*, Springer, London, UK, 2006.
- [5] E. Fridman, S. Nicaise, and J. Valein, "Stabilization of second order evolution equations with unbounded feedback with time-dependent delay," *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 5028–5052, 2010.
- [6] S.-T. Wu, "Asymptotic behavior for a viscoelastic wave equation with a delay term," *Taiwanese Journal of Mathematics*, vol. 17, no. 3, pp. 765–784, 2013.
- [7] K. Daewook, "Asymptotic behavior for the viscoelastic Kirchhoff type equation with an internal time varying delay term," *East Asian Mathematical Journal*, vol. 32, no. 3, pp. 399–412, 2016.
- [8] S. Nicaise and C. Pignotti, "Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks," *SIAM Journal on Control and Optimization*, vol. 45, no. 5, pp. 1561–1585, 2006.
- [9] J. Y. Park and J. R. Kang, "Global existence and uniform decay for a nonlinear viscoelastic equation with damping," *Acta Applicandae Mathematicae*, vol. 110, no. 3, pp. 1393–1406, 2010.
- [10] J. L. Lions, *Quelques Methodes de Resolution des Problemes Aux Limites Non Lineaires*, Dunod, Paris, France, in French, 1969.