

Research Article

Pontryagin's Maximum Principle for Optimal Control of Stochastic SEIR Models

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In this paper, we study the necessary conditions as well as sufficient conditions for optimality of stochastic SEIR model. The most distinguishing feature, compared with the well-studied SEIR model, is that the model system follows stochastic differential equations (SDEs) driven by Brownian motions. Hamiltonian function is introduced to derive the necessary conditions. Using the explicit formulation of adjoint variables, desired necessary conditions for optimal control results are obtained. We also establish a sufficient condition which is called verification theorem for the stochastic SEIR model.

1. Introduction

New corona viruses are very harmful to people. Especially, COVID-19 is currently being spread around the world. It has seriously affected people's lives. Many countries have made good efforts to deal with it and prevent it. At present, the COVID-19 epidemic in China has been basically controlled. It is well known that SEIR models are widely used to model the spreading of infectious diseases in a population. Up to now, many researchers from the world have achieved a variety of both theoretical results and applications, see [1, 2, 7, 9, 15, 16, 18, 20, 21] and the references therein.

However, it is worth pointing that out that SEIR models in the existing literatures are deterministic models. As we know the stochastic events are inevitable in practice, and the stochastic effects that may lead to significant changes, thus, the stochastic SEIR models maybe better to be applied to describe the COVID-19 epidemic. Motivated by the actual situation in reality and the lack of theory, this paper studies the optimal control of stochastic SEIR model. To the best of our knowledge, there were few literatures about the optimal control of epidemic model in the stochastic case. Our main objective is to derive necessary conditions for optimality of the stochastic SEIR model by using the stochastic maximum principle (SMP).

Stochastic optimal control problems have received considerable research attention in recent years due to wide applicability in a number of different fields such as physics, biology, economics, and management science. As it is well known, dynamic programming principle (DPP) and SMP are two main tools to study stochastic control problems. SMP, which provides a necessary condition of an optimal control in stochastic optimal control problems known as the stochastic version of Pontryagin's type [3–6, 8, 11–14, 19], has been the tool predominantly used to study the stochastic optimal control problems and some stochastic differential game problems.

For example, by using SMP, Xu and Shi [17] obtain the feedback form of optimal control for linear-quadratic-Gaussian (LQG) problems to study stochastic large population system with jump diffusion processes. The standard SMP involves solving the adjoint equation and minimizing the Hamiltonian function. We also followed this in our paper. We should point out that the SEIR model studied in [10] is deterministic case, while our SEIR model is stochastic case, that is, the main difference between our model and the model studied in [10].

The organization of this paper is as follows. Section 2 is devoted to the problem formulation and assumptions. Necessary conditions for optimality are introduced in

Section 3. Section 4 aims to prove that the necessary conditions presented in Section 3 are also the sufficient conditions for optimality. Finally, we end our work with some concluding remarks in Section 5.

2. Problem Statement and Assumptions

Throughout this paper, let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{F}; \mathbb{P})$ be a given complete filtered probability space, on which independent standard one-dimensional Brownian motions $\{W_i(t), 1 \leq i \leq 4\}_{0 \leq t \leq T}$ are defined. The superscript τ denotes the transpose of vectors or matrices. We suppose that the filtration $\mathcal{F}_t \geq 0$ is generated by the independent standard one-dimensional standard Brownian motions $\{W_i(t), 1 \leq i \leq 4\}_{0 \leq t \leq T}$.

Let $L^2(0, T; \mathbb{R})$ denote the set of Lebesgue measurable functions $\psi(\cdot): [0, T] \rightarrow \mathbb{R}$ such that $\int_0^T |\psi(t)|^2 dt < +\infty$. We write $\varphi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R})$ if $\varphi(\cdot): [0, T] \times \Omega \rightarrow \mathbb{R}$ is an \mathcal{F}_t -adapted square-integrable process (i.e., $\mathbb{E} \int_0^T |\varphi(t)|^2 dt < +\infty$). Let U be nonempty subsets of \mathbb{R} . We introduce the admissible control set as

$$\mathcal{U} := \left\{ \nu(\cdot) \mid \nu(t) \in U, \quad 0 \leq t \leq T; \nu(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \right\}. \quad (1)$$

Now, we introduce our stochastic SEIR model. Let $S(t), E(t), I(t)$, and $R(t)$ represent the number of individuals in the susceptible, exposed, infectious, and recovered compartments at time t , respectively. The total population is denoted by $N(t) = S(t) + E(t) + I(t) + R(t)$. Let $u(t)$ denote the fraction of susceptible individuals being vaccinated per unit of time.

We input the random disturbance proportionally to each variable value in the model and get the following dynamical system:

$$\begin{cases} dS(t) = (bN(t) - \mu S(t) - \beta S(t)I(t) - u(t)S(t))dt + \sigma_1 S(t)dW_1(t), \\ dE(t) = (\beta S(t)I(t) - (\mu + \epsilon)E(t))dt + \sigma_2 E(t)dW_2(t), \\ dI(t) = (\epsilon E(t) - (\mu + \gamma + a)I(t))dt + \sigma_3 I(t)dW_3(t), \\ dR(t) = (\gamma I(t) - \mu R(t) + u(t)S(t))dt + \sigma_4 R(t)dW_4(t), \\ S(0) = S_0, \\ E(0) = E_0, \\ I(0) = I_0, \\ R(0) = R_0. \end{cases} \quad (2)$$

The parameters in the disease transmission model is described as follows: b is the natural birth rate, μ represents the natural death rate, a denotes the death rate due to the disease, and β represents the incidence coefficient of horizontal transmission, and let ϵ be the rate at which the exposed individuals become infectious, and γ is removal rate. Note that the rate of transmission of the disease is $\beta S(t)I(t)$. In the above model, $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ denotes the random disturbance proportionally to each variable value in the model. The parameters in the model are supposed to be constants for simplicity. For more information about the

disease transmission model, we refer the reader to [2, 9, 13] and references within.

The expected cost functional is given by

$$J(u) = \mathbb{E} \left[\int_0^T (QI(t) + Gu^2(t))dt \right], \quad (3)$$

where Q and G are given constants.

The optimal control problem under consideration is as follows.

Problem (P): the objective of the control problem is to find admissible control $u^* \in \mathcal{U}$ such that

$$J(u^*) = \inf_{u \in \mathcal{U}} J(u). \quad (4)$$

A control that solves this problem is called optimal.

3. Necessary Conditions for Optimality

This section focuses on the necessary optimality conditions of Problem (P).

In order to apply the necessary conditions for optimal control in the form of maximum principle, we first introduce some notations. Assume that

$$\begin{aligned} x &:= \begin{pmatrix} S \\ E \\ I \\ R \end{pmatrix}, \\ A &:= \begin{pmatrix} b - \mu & b & b & b \\ 0 & -(\mu + \epsilon) & 0 & 0 \\ 0 & \epsilon & -(\mu + \gamma + a) & 0 \\ 0 & 0 & \gamma & -\mu \end{pmatrix}, \\ f_1(x) &:= \begin{pmatrix} -\beta SI \\ \beta SI \\ 0 \\ 0 \end{pmatrix}, \\ f_2(x) &:= \begin{pmatrix} -S \\ 0 \\ 0 \\ S \end{pmatrix}, \\ W &:= \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}, \end{aligned} \quad (5)$$

$$\sigma(x) := \text{diag}(\sigma_1 S, \sigma_2 E, \sigma_3 I, \sigma_4 R),$$

$$L = (0, 0, Q, 0).$$

Therefore, stochastic SEIR model (2) can be written as

$$\begin{cases} dx(t) = (Ax(t) + f_1(x(t)) + f_2(x(t))u(t))dt + \sigma(x(t))dW(t), \\ x(0) = (S_0, E_0, I_0, R_0)^T. \end{cases} \quad (6)$$

The corresponding cost functional is

$$J(u) = \mathbb{E} \left[\int_0^T (Lx(t) + Gu^2(t)) dt \right]. \quad (7)$$

Let (x^*, u^*) be the optimal pair of Problem P . The standard Hamiltonian function is given by

$$H(x, p, q, u) = \langle p, Ax + f_1(x) + f_2(x)u \rangle + \sigma(x)q + Lx + Gu^2, \quad (8)$$

where the adjoint variable $(p(\cdot), q(\cdot))$ satisfies

$$\begin{cases} dp(t) = -H_x[t]dt + q(t)dW(t), \\ p(T) = (0, 0, 0, 0), \end{cases} \quad (9)$$

where $H[t] := H(x^*(t), p(t), q(t), u^*(t))$.

Next, we want to obtain the adjoint variable $(p(\cdot), q(\cdot))$ explicitly. Let $p = (p_S, p_E, p_I, p_R)^T$ and $q = (q_S, q_E, q_I, q_R)^T$. According to (9), (p, q) are explicitly given by

$$\begin{cases} dp_S(t) = -(b - \mu - \beta I^*(t) - u^*(t))p_S(t)dt - \beta I^*(t)p_E(t)dt + \sigma_1 q_S(t)dW_1(t), \\ dp_E(t) = -(bp_S(t) - (\mu + \epsilon)p_E(t) + \epsilon p_I(t))dt + \sigma_2 q_E(t)dW_2(t), \\ dp_I(t) = -(bp_S(t) - \beta S^*(t)p_S(t) + \beta S^*(t)p_E(t) - (\mu + \gamma + a)p_I(t) + \gamma p_R(t) + Q)dt + \sigma_3 q_I(t)dW_3(t), \\ dp_R(t) = -(bp_S(t) - \mu p_R(t) + u^*(t)p_R(t))dt + \sigma_4 q_R(t)dW_4(t). \end{cases} \quad (10)$$

Next, we evaluate the necessary condition for the optimal control. By (8), we have

$$H_u(x, p, q, u) = -Sp_S + Sp_R + 2Gu = 0, \quad (11)$$

which means

$$u^*(t) = \frac{1}{2G} (p_S(t) - p_R(t))S^*(t). \quad (12)$$

Now, we summarize the above discussion with the main result of this article.

Theorem 1. *Let (x^*, u^*) be the optimal pair of Problem P with $x^* := (S^*, E^*, I^*, R^*)^T$. Then, $u^*(\cdot)$ fulfills (12), where $(p_S(\cdot), p_R(\cdot))$ admits (10).*

4. Sufficient Conditions for Optimality

In this section, we will establish the sufficient maximum principle (also called verification theorem) of Problem P . That is to say, $u^*(\cdot)$ given in (12) is also the sufficient condition of Problem P .

Theorem 2. *Assume that $u^*(\cdot)$ fulfills (12) with state trajectory $x^* := (S^*, E^*, I^*, R^*)^T$ which is given such that there exist solutions $(p_S(\cdot), p_R(\cdot))$ to the adjoint equation (10). Then, (x^*, u^*) is the optimal pair of Problem P .*

Proof. For any $u \in \mathcal{U}$, we consider

$$J(u) - J(u^*) = \mathbb{E} \left[\int_0^T (Q(I(t) - I^*(t)) + G(u^2(t) - (u^*(t))^2)) dt \right]. \quad (13)$$

Applying IÔ's formula to $\langle p(\cdot), x(\cdot) - x^*(\cdot) \rangle$, we obtain

$$\begin{aligned}
& \mathbb{E}[\langle p(T), x(T) - x^*(T) \rangle - \langle p(0), x(0) - x^*(0) \rangle] \\
&= \mathbb{E} \left[\int_0^T \langle p(t), d(x(t) - x^*(t)) \rangle + \int_0^T \langle x(t) - x^*(t), dp(t) \rangle + \int_0^T q(t)(\sigma(x(t)) - \sigma(x^*(t))) dt \right] \\
&= \mathbb{E} \left[\int_0^T \langle p(t), A(x(t) - x^*(t)) + f_1(x(t)) - f_1(x^*(t)) + f_2(x(t))u(t) - f_2(x^*(t))u^*(t) \rangle dt \right] \\
&\quad + \mathbb{E} \left[\int_0^T q(t)(\sigma(x(t)) - \sigma(x^*(t))) dt - \int_0^T \langle x(t) - x^*(t), H_x[t] \rangle dt \right] \\
&= \mathbb{E} \left[\int_0^T \left(H(x(t), p(t), q(t), u(t)) - Lx(t) - Gu^2(t) - (H[t] - Lx^*(t) - G(u^*(t))^2) \right) dt \right] \\
&\quad - \mathbb{E} \left[\int_0^T H(x(t), p(t), q(t), u(t)) - H[t] - \langle x(t) - x^*(t), H_x[t] \rangle dt \right] \\
&= \mathbb{E} \left[\int_0^T H(x(t), p(t), q(t), u(t)) - H[t] - \langle x(t) - x^*(t), H_x[t] \rangle dt \right] \\
&\quad - \mathbb{E} \left[\int_0^T \left(L(x(t) - x^*(t)) + G(u^2(t) - (u^*(t))^2) \right) dt \right].
\end{aligned} \tag{14}$$

Combining (6), (9), (13), with (14), one has

$$0 = \mathbb{E} \left[\int_0^T H(x(t), p(t), q(t), u(t)) - H[t] - \langle x(t) - x^*(t), H_x[t] \rangle dt \right] - J(u) + J(u^*) \geq -J(u) + J(u^*), \tag{15}$$

where, in the last step, we have used the condition of $u^*(\cdot)$ which fulfills (12).

Therefore, $J(u^*) \leq J(u)$. Hence, we draw the desired conclusion. \square

5. Conclusion

Maria do Rosário de et al. [10] considered an optimal control problem with mixed control-state constraint for a SEIR epidemic model of human infectious diseases. Motivated by their pioneering work and the lack of theory, this paper is concerned with the necessary conditions (also, the sufficient conditions) for optimality of the stochastic SEIR model. The model system follows SDEs driven by Brownian motions and with the corresponding cost. It is the first attempt to study this kind of control problem in our technical note, to the authors' knowledge. Using the explicit formulation of adjoint variables, we obtain the desired necessary conditions for optimal control results.

Some interesting topics deserve further investigations. On the one hand, one may determine the optimal control strategies for the stochastic delayed SIR model and compare it with that presented in this work. On the other hand, we shall investigate some more realistic but complex models, such as considering the effects of impulsive

perturbations on the system. We leave these investigations in our future work.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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