Research Article

Hesitant Bipolar-Valued Fuzzy Soft Sets and Their Application in Decision Making

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Received 15 May 2020; Revised 21 June 2020; Accepted 22 July 2020; Published 17 August 2020

Guest Editor: Jing Na

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As an extension of fuzzy sets, hesitant bipolar-valued fuzzy set is a new mathematical tool for dealing with fuzzy problems, but it still has the problem with the inadequacy of the parametric tools. In order to further improve the accuracy of decision making, a new mixed mathematical model, named hesitant bipolar-valued fuzzy soft set, is constructed by combining hesitant bipolar-valued fuzzy sets with soft sets. Firstly, some related theories of hesitant bipolar-valued fuzzy sets are discussed. Secondly, the concept of hesitant bipolar-valued fuzzy soft set is given, and the algorithms of complement, union, intersection, “AND,” and “OR” are defined. Based on the above algorithms, the corresponding results of operation are analyzed and the relevant properties are discussed. Finally, a multiattribute decision-making method of hesitant bipolar-valued fuzzy soft sets is proposed by using the idea of score function and level soft sets. The effectiveness of the proposed method is illustrated by an example.

1. Introduction

The theory of fuzzy sets was put forward by Zadeh in 1965 [1], and the method of soft sets was presented even later by Molodtsov in 1999 [2]. Fuzzy sets as well as soft sets are complementary and important to deal with vague, imprecise, and uncertain knowledge. The integration of the two theories has become one of the popular research directions. In recent years, some scholars presented fuzzy soft sets and various kinds of extended models [3–11], such as fuzzy soft sets, intuitionistic fuzzy soft sets, interval-valued fuzzy soft sets, interval-valued intuitionistic fuzzy soft sets, hesitant fuzzy soft sets, interval-valued hesitant fuzzy soft sets, and generalized hesitant fuzzy soft sets. The corresponding basic procedures and properties are also discussed. Based on the above results, some scholars made a further study about the uncertainty measures [12–21] and the decision-making methods [22–25] of the extended model. For example, Majumdar and Samanta analyzed and compared the similarities and difference of different measures of the fuzzy soft sets [12]. Based on [12], Liu et al. introduced a novel approach to define the similarity measures and the distance measures between two fuzzy soft sets with different

parameter sets [13]. Jiang et al. generalized the uncertainty measures of fuzzy soft set, defined the distance measures between intuitionistic fuzzy soft sets, and provided the entropy between intuitionistic fuzzy soft sets and interval-valued fuzzy soft sets [14]. Based on [14], Muthukumar and Sai Sundara Krishnan proposed a novel similarity measure of intuitionistic fuzzy soft sets [15]. In recent years, some scholars have extended the uncertainty measures of fuzzy soft set to another direction. Beg and Rashid defined the distance measures between any two elements in hesitant fuzzy soft sets [16]. A novel distance measure and similarity measure of hesitant fuzzy soft sets were presented by K. Rezaei and H. Rezaei in [17]. Garg and Arora raised novel distance measures and similarity measures of dual hesitant fuzzy soft sets [18]. The correlation measures of hesitant fuzzy soft sets and interval-valued hesitant fuzzy soft sets were discussed by Das et al. in [19]. Garg and Arora made a further study about the weighted average and geometric operators of dual hesitant fuzzy soft sets [20]. Arora and Garg introduced a novel correlation coefficient measure of dual hesitant fuzzy soft sets [21]. Feng et al. introduced a novel way to fuzzy soft set by the idea of level soft sets [22]. Jiang et al. presented a novel adjustable way for decision-

making problems of intuitionistic fuzzy soft sets [23]. Xiao et al. studied the decision-making problems of interval-valued fuzzy soft sets in uncertain environment [24]. Zhao et al. introduced an innovative multicriteria ranking approach for decision-making problems of intuitionistic fuzzy soft sets [25]. With the continuous development of fuzzy set, soft set, and uncertainty theory, artificial intelligence has received extensive attention around the world. Related control schemes have also achieved fruitful results [26–38]. More detailed research on the uncertainty measures of fuzzy soft sets and other models can provide a reliable mathematical foundation for the development of cybernetics.

Although the research results of the soft set extension model have been very fruitful, the incompatible bipolarity is inevitable in the objective world. Just as an example of a psychology disease-bipolar disorder patient, who has symptoms of mania and depression, the sum of positive and negative symptoms value is more than one, or even both the two symptoms may reach extreme cases simultaneously. In recent years, the integration of bipolar-valued and hesitant fuzzy sets had drawn the attention of many scholars, and good results were obtained. Han put forward the perception of hesitant bipolar-valued fuzzy sets as well as introduced a hesitant bipolar fuzzy decision-making problem method based on TOPSIS [39]. Ullah et al. presented the concept of bipolar-valued hesitant fuzzy sets, and the aggregation operators were applied to deal with the decision-making problems [40]. Mandal and Ranadive proposed the perception of hesitant bipolar-valued fuzzy sets and bipolar-valued hesitant fuzzy sets. Both of them were applied to solve decision-making problems [41]. Wei discussed the decision-making problems of aggregation operators founded on hesitant bipolar-valued fuzzy information [42]. According to the Einstein operational laws, Mahmood et al. made a further study about the aggregation operators and the decision-making problems for bipolar-valued hesitant fuzzy information [43]. Xu and Wei considered the dual hesitant bipolar fuzzy sets and the decision-making problems [44]. A further study about a few aggregation operators for dual hesitant bipolar fuzzy information was discussed in [45]. Awang et al. presented the perception of hesitant bipolar-valued neutrosophic sets and introduced an original decision-making method based on the novel model [46]. Those models were superior to other fuzzy mathematical tools in describing abilities because of considering the bipolarity and nonuniqueness of the membership degree of an element simultaneously. But like other mathematical tools, it still has the inadequacy problem of the parametric tools. On this basis, the theory of parameters is introduced into the hesitant bipolar-valued fuzzy sets, and the concept of hesitant bipolar-valued fuzzy soft sets is defined. The model is a new mathematical tool for uncertainty problem. It can make up for the inadequacy of the parametric tools. Therefore, the model is more suitable to describe the uncertainty problem.

The rest of this paper is arranged as follows. In Section 2, we retrospect the basic concepts of soft sets and hesitant bipolar-valued fuzzy sets and supplement some correlated knowledge. In Section 3, we put forward the perception of hesitant bipolar-valued fuzzy soft sets and define 2a number

of basic procedures. A few corresponding properties are also considered. In Section 4, we propose a multiattribute decision-making approach based on the novel model, and the effectiveness of the proposed methodology is given by practical problems. Finally, the concluding is given.

2. Preliminaries

For convenience, throughout the paper, let U be an initial limited universe set, E be the set of all possible parameters with respect to U , and $D \subseteq E$.

Definition 1 (see [1]). Let $P(U)$ denote the power set of U . A pair (F, D) is called a soft set on U , where F is a mapping given by $F: D \rightarrow P(U)$.

Definition 2 (see [41]). A hesitant bipolar-valued fuzzy set $\mathbb{H}$ on U is defined as

$$\mathbb{H} = \{ \langle u, h_{\mathbb{H}}(u) = (h_{\mathbb{H}}^P(u), h_{\mathbb{H}}^N(u)) \rangle \mid u \in U \}, \quad (1)$$

where $h_{\mathbb{H}}^P(u)$, known as the hesitant fuzzy positive element, is a set of some values in $[0, 1]$ denoting the possible satisfaction degree of the $u \in U$ to the corresponding property to the $\mathbb{H}$; $h_{\mathbb{H}}^N(u)$, known as the hesitant fuzzy positive element, is a set of some values in $[-1, 0]$ denoting the possible satisfaction degree of the $u \in U$ to the opposite property to the $\mathbb{H}$; and $h_{\mathbb{H}}(u) = (h_{\mathbb{H}}^P(u), h_{\mathbb{H}}^N(u))$, known as the hesitant bipolar-valued fuzzy element, is a set of some values in $[0, 1] \times [-1, 0]$ to the $\mathbb{H}$, abbreviated as HBVFE.

As a matter of convenience, the HBVFE $h_{\mathbb{H}}(u) = (h_{\mathbb{H}}^P(u), h_{\mathbb{H}}^N(u))$ is abbreviated as $h_{\mathbb{H}} = (h_{\mathbb{H}}^P, h_{\mathbb{H}}^N)$ and all hesitant bipolar-valued fuzzy sets on U are abbreviated as HBVFS(U) in this study.

Definition 3 (see [41]). Let $h_1 = (h_1^P, h_1^N)$ and $h_2 = (h_2^P, h_2^N)$ be two HBVFEs; then,

(1) Complement:

$$h_1^c = \left(\bigcup_{\gamma_1^P \in h_1^P} \{1 - \gamma_1^P\}, \bigcup_{\gamma_1^N \in h_1^N} \{-1 - \gamma_1^N\} \right). \quad (2)$$

(2) Union:

$$h_1 \cup h_2 = \left(\bigcup_{\gamma_1^P \in h_1^P, \gamma_2^P \in h_2^P} \max\{\gamma_1^P, \gamma_2^P\}, \bigcup_{\gamma_1^N \in h_1^N, \gamma_2^N \in h_2^N} \min\{\gamma_1^N, \gamma_2^N\} \right). \quad (3)$$

(3) Intersection:

$$h_1 \cap h_2 = \left(\bigcup_{\gamma_1^P \in h_1^P, \gamma_2^P \in h_2^P} \min\{\gamma_1^P, \gamma_2^P\}, \bigcup_{\gamma_1^N \in h_1^N, \gamma_2^N \in h_2^N} \max\{\gamma_1^N, \gamma_2^N\} \right). \quad (4)$$

Theorem 1. Let $h_1 = (h_1^P, h_1^N)$, $h_2 = (h_2^P, h_2^N)$ and $h_3 = (h_3^P, h_3^N)$ be three HBVFEs; then, we have some operational laws, such as

(1) *Involutive:*

$$(\tilde{h}_1^c)^c = \tilde{h}_1. \quad (5)$$

(2) *Commutativity:*

$$\begin{aligned} \tilde{h}_1 \cup \tilde{h}_2 &= \tilde{h}_2 \cup \tilde{h}_1, \\ \tilde{h}_1 \cap \tilde{h}_2 &= \tilde{h}_2 \cap \tilde{h}_1. \end{aligned} \quad (6)$$

(3) *Associative:*

$$\begin{aligned} (\tilde{h}_1 \cup \tilde{h}_2) \cup \tilde{h}_3 &= \tilde{h}_1 \cup (\tilde{h}_2 \cup \tilde{h}_3), \\ (\tilde{h}_1 \cap \tilde{h}_2) \cap \tilde{h}_3 &= \tilde{h}_1 \cap (\tilde{h}_2 \cap \tilde{h}_3). \end{aligned} \quad (7)$$

(4) *Distributive:*

$$\begin{aligned} \tilde{h}_1 \cup (\tilde{h}_2 \cap \tilde{h}_3) &= (\tilde{h}_1 \cup \tilde{h}_2) \cap (\tilde{h}_1 \cup \tilde{h}_3), \\ \tilde{h}_1 \cap (\tilde{h}_2 \cup \tilde{h}_3) &= (\tilde{h}_1 \cap \tilde{h}_2) \cup (\tilde{h}_1 \cap \tilde{h}_3). \end{aligned} \quad (8)$$

(5) *De Morgan [41]:*

$$\begin{aligned} (\tilde{h}_1 \cup \tilde{h}_2)^c &= \tilde{h}_1^c \cap \tilde{h}_2^c, \\ (\tilde{h}_1 \cap \tilde{h}_2)^c &= \tilde{h}_1^c \cup \tilde{h}_2^c. \end{aligned} \quad (9)$$

$$\tilde{h}_1^c = \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P} \{1 - \gamma_1^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N} \{-1 - \gamma_1^N\} \right), \quad (10)$$

from Definition 3; then,

$$\begin{aligned} (\tilde{h}_1^c)^c &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P} \{1 - \gamma_1^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N} \{-1 - \gamma_1^N\} \right)^c \\ &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P} \{1 - (1 - \gamma_1^P)\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N} \{-1 - (-1 - \gamma_1^N)\} \right) \\ &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P} \{\gamma_1^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N} \{\gamma_1^N\} \right) \\ &= \tilde{h}_1. \end{aligned} \quad (11)$$

(2) It can be easily obtained from Definition 3.

(3) For three HBVFEs $\tilde{h}_1 = (\tilde{h}_1^P, \tilde{h}_1^N)$, $\tilde{h}_2 = (\tilde{h}_2^P, \tilde{h}_2^N)$, and $\tilde{h}_3 = (\tilde{h}_3^P, \tilde{h}_3^N)$, we have

$$\tilde{h}_1 \cup \tilde{h}_2 = \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_2^P \in \tilde{h}_2^P} \max\{\gamma_1^P, \gamma_2^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_2^N \in \tilde{h}_2^N} \min\{\gamma_1^N, \gamma_2^N\} \right), \quad (12)$$

Proof

(1) For a HBVFE $\tilde{h}_1 = (\tilde{h}_1^P, \tilde{h}_1^N)$, we have

and from Definition 3, we have

$$\begin{aligned} (\tilde{h}_1 \cup \tilde{h}_2) \cup \tilde{h}_3 &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_2^P \in \tilde{h}_2^P, \gamma_3^P \in \tilde{h}_3^P} \max\{\max\{\gamma_1^P, \gamma_2^P\}, \gamma_3^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_2^N \in \tilde{h}_2^N, \gamma_3^N \in \tilde{h}_3^N} \min\{\min\{\gamma_1^N, \gamma_2^N\}, \gamma_3^N\} \right) \\ &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_2^P \in \tilde{h}_2^P, \gamma_3^P \in \tilde{h}_3^P} \max\{\gamma_1^P, \gamma_2^P, \gamma_3^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_2^N \in \tilde{h}_2^N, \gamma_3^N \in \tilde{h}_3^N} \min\{\gamma_1^N, \gamma_2^N, \gamma_3^N\} \right) \\ &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_2^P \in \tilde{h}_2^P, \gamma_3^P \in \tilde{h}_3^P} \max\{\gamma_1^P, \max\{\gamma_2^P, \gamma_3^P\}\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_2^N \in \tilde{h}_2^N, \gamma_3^N \in \tilde{h}_3^N} \min\{\gamma_1^N, \min\{\gamma_2^N, \gamma_3^N\}\} \right) \\ &= \tilde{h}_1 \cup (\tilde{h}_2 \cup \tilde{h}_3). \end{aligned} \quad (13)$$

Similarly, we can also obtain

$$(\tilde{h}_1 \cap \tilde{h}_2) \cap \tilde{h}_3 = \tilde{h}_1 \cap (\tilde{h}_2 \cap \tilde{h}_3). \quad (14)$$

(4) For three HBVFEs $\tilde{h}_1$, $\tilde{h}_2$ and $\tilde{h}_3$, we have

$$\begin{aligned} \tilde{h}_1 \cup (\tilde{h}_2 \cap \tilde{h}_3) &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_2^P \in \tilde{h}_2^P, \gamma_3^P \in \tilde{h}_3^P} \max\{\gamma_1^P, \min\{\gamma_2^P, \gamma_3^P\}\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_2^N \in \tilde{h}_2^N, \gamma_3^N \in \tilde{h}_3^N} \min\{\gamma_1^N, \max\{\gamma_2^N, \gamma_3^N\}\} \right) \\ &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_2^P \in \tilde{h}_2^P, \gamma_3^P \in \tilde{h}_3^P} \min\{\max\{\gamma_1^P, \gamma_2^P\}, \max\{\gamma_1^P, \gamma_3^P\}\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_2^N \in \tilde{h}_2^N, \gamma_3^N \in \tilde{h}_3^N} \max\{\min\{\gamma_1^N, \gamma_2^N\}, \min\{\gamma_1^N, \gamma_3^N\}\} \right) \\ &= \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_2^P \in \tilde{h}_2^P} \max\{\gamma_1^P, \gamma_2^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_2^N \in \tilde{h}_2^N} \min\{\gamma_1^N, \gamma_2^N\} \right) \cap \left(\bigcup_{\gamma_1^P \in \tilde{h}_1^P, \gamma_3^P \in \tilde{h}_3^P} \max\{\gamma_1^P, \gamma_3^P\}, \bigcup_{\gamma_1^N \in \tilde{h}_1^N, \gamma_3^N \in \tilde{h}_3^N} \min\{\gamma_1^N, \gamma_3^N\} \right) \\ &= (\tilde{h}_1 \cup \tilde{h}_2) \cap (\tilde{h}_1 \cup \tilde{h}_3). \end{aligned} \quad (15)$$

Similarly, we can also obtain

$$h_1 \cap (h_2 \cup h_3) = (h_1 \cap h_2) \cup (h_1 \cap h_3). \quad (16)$$

Definition 4 (see [41]). Let $h = (h^P, h^N)$ be a HBVFE and the score function $\mathcal{S}(h)$ is defined as

$$\mathcal{S}(h) = \frac{1}{2} \left(\frac{1}{l_{h^P}} \sum_{\gamma^P \in h^P} \gamma^P - \frac{1}{l_{h^N}} \sum_{\gamma^N \in h^N} \gamma^N \right), \quad (17)$$

where l_{h^P} and l_{h^N} are the numbers of the elements in h^P and h^N , respectively.

Definition 5. Two special hesitant bipolar-valued fuzzy sets are defined as

(1) Empty set:

$$\emptyset = \{ \langle u, h_{\emptyset}(u) \rangle \mid u \in U \}, \quad \text{where } h_{\emptyset}(u) = (\{0\}, \{0\}). \quad (18)$$

(2) Full set:

$$\mathbb{U} = \{ \langle u, h_{\mathbb{U}}(u) \rangle \mid u \in U \}, \quad \text{where } h_{\mathbb{U}}(u) = (\{1\}, \{-1\}). \quad (19)$$

Definition 6. Let $\mathbb{H}_1, \mathbb{H}_2 \in \text{HBVFS}(U)$ and $\forall u \in U$; then,

- (1) If $\mathcal{S}(h_{\mathbb{H}_1}(u)) \leq \mathcal{S}(h_{\mathbb{H}_2}(u))$, then $\mathbb{H}_1$ is called as a hesitant bipolar-valued fuzzy quasisubset of $\mathbb{H}_2$, denoted by $\mathbb{H}_1 \subseteq \mathbb{H}_2$ or $\mathbb{H}_2 \supseteq \mathbb{H}_1$.
- (2) If $\mathcal{S}(h_{\mathbb{H}_1}(u)) = \mathcal{S}(h_{\mathbb{H}_2}(u))$, then $\mathbb{H}_1$ and $\mathbb{H}_2$ are called hesitant bipolar-valued fuzzy quasiequal, denoted by $\mathbb{H}_1 \approx \mathbb{H}_2$.
- (3) If $h_{\mathbb{H}_1}(u)$ and $h_{\mathbb{H}_2}(u)$ have the same elements, then $\mathbb{H}_1$ and $\mathbb{H}_2$ are equal, denoted by $\mathbb{H}_1 = \mathbb{H}_2$.

Definition 7. Let $\mathbb{H}_1, \mathbb{H}_2 \in \text{HBVFS}(U)$; then,

(1) Complement:

$$\mathbb{H}_1^c = \{ \langle u, h_{\mathbb{H}_1^c}(u) \rangle \mid u \in U \} \quad (20)$$

where $h_{\mathbb{H}_1^c} = h_{\mathbb{H}_1}^c$.

(2) Union:

$$\mathbb{H}_1 \cup \mathbb{H}_2 = \{ \langle u, h_{\mathbb{H}_1 \cup \mathbb{H}_2}(u) \rangle \mid u \in U \}, \quad (21)$$

where $h_{\mathbb{H}_1 \cup \mathbb{H}_2} = h_{\mathbb{H}_1} \cup h_{\mathbb{H}_2}$.

(3) Intersection:

$$\mathbb{H}_1 \cap \mathbb{H}_2 = \{ \langle u, h_{\mathbb{H}_1 \cap \mathbb{H}_2}(u) \rangle \mid u \in U \}, \quad (22)$$

where $h_{\mathbb{H}_1 \cap \mathbb{H}_2} = h_{\mathbb{H}_1} \cap h_{\mathbb{H}_2}$.

Theorem 2. Let $\mathbb{H}_1, \mathbb{H}_2, \mathbb{H}_3 \in \text{HBVFS}(U)$; then, we have some operational laws, such as

- (1) $(\mathbb{H}_1^c)^c = \mathbb{H}_1$.
- (2) $\mathbb{H}_1 \cup \mathbb{H}_2 = \mathbb{H}_2 \cup \mathbb{H}_1$, $\mathbb{H}_1 \cap \mathbb{H}_2 = \mathbb{H}_2 \cap \mathbb{H}_1$.
- (3) $\mathbb{H}_1 \cup (\mathbb{H}_2 \cap \mathbb{H}_3) = (\mathbb{H}_1 \cup \mathbb{H}_2) \cap \mathbb{H}_3$,
 $\mathbb{H}_1 \cap (\mathbb{H}_2 \cup \mathbb{H}_3) = (\mathbb{H}_1 \cap \mathbb{H}_2) \cup \mathbb{H}_3$.
- (4) $\mathbb{H}_1 \cup (\mathbb{H}_2 \cap \mathbb{H}_3) = (\mathbb{H}_1 \cup \mathbb{H}_2) \cap (\mathbb{H}_1 \cup \mathbb{H}_3)$,
 $\mathbb{H}_1 \cap (\mathbb{H}_2 \cup \mathbb{H}_3) = (\mathbb{H}_1 \cap \mathbb{H}_2) \cup (\mathbb{H}_1 \cap \mathbb{H}_3)$.
- (5) $(\mathbb{H}_1 \cup \mathbb{H}_2)^c = \mathbb{H}_1^c \cap \mathbb{H}_2^c$, $(\mathbb{H}_1 \cap \mathbb{H}_2)^c = \mathbb{H}_1^c \cup \mathbb{H}_2^c$.

Proof. It can be easily obtained from Theorem 1 and Definition 7. Thus, it is omitted here. $\square$

Theorem 3. Let $\mathbb{H}_1, \mathbb{H}_2 \in \text{HBVFS}(U)$; then, we have

- (1) $\mathbb{H}_1 \cap \mathbb{H}_2 \subseteq \mathbb{H}_1 \subseteq \mathbb{H}_1 \cup \mathbb{H}_2$.
- (2) $\mathbb{H}_1 \cap \mathbb{H}_2 \subseteq \mathbb{H}_2 \subseteq \mathbb{H}_1 \cup \mathbb{H}_2$.

Proof

- (1) From Definitions 3 and 7, we have

$$\mathcal{S}(h_{\mathbb{H}_1 \cap \mathbb{H}_2}) \leq \mathcal{S}(h_{\mathbb{H}_1}) \leq \mathcal{S}(h_{\mathbb{H}_1 \cup \mathbb{H}_2}). \quad (23)$$

Then, together with Definition 6, one has

$$\mathbb{H}_1 \cap \mathbb{H}_2 \subseteq \mathbb{H}_1 \subseteq \mathbb{H}_1 \cup \mathbb{H}_2. \quad (24)$$

- (2) It is similar to the proof of conclusion 1. $\square$

3. Hesitant Bipolar-Valued Fuzzy Soft Sets and Operation Properties

In this section, we propose the concept of hesitant bipolar-valued fuzzy soft sets as well as some basic operations.

Definition 8. A pair $(\tilde{\mathbb{J}}, D)$ is called as a hesitant bipolar-valued fuzzy soft set on U , where $\tilde{\mathbb{J}}$ is a mapping given by $\tilde{\mathbb{J}}: D \longrightarrow \text{HBVFS}(U)$ and

$$\tilde{\mathbb{J}}(\varepsilon) = \{ \langle u, h_{\tilde{\mathbb{J}}(\varepsilon)}(u) \rangle \mid u \in U \}, \quad \text{for } \forall \varepsilon \in D. \quad (25)$$

We will use the notation $\text{HBVFSS}(U)$ to denote all hesitant bipolar-valued fuzzy soft sets on U .

Note:

- (1) If D has only one value, that is $D = \{\varepsilon\}$, then $(\tilde{\mathbb{J}}, D)$ degenerates into a hesitant bipolar-valued fuzzy set.
- (2) If $h_{\tilde{\mathbb{J}}(\varepsilon)}^P$ and $h_{\tilde{\mathbb{J}}(\varepsilon)}^N$ have only one value, that is $h_{\tilde{\mathbb{J}}(\varepsilon)}^P = \mu^P$ and $h_{\tilde{\mathbb{J}}(\varepsilon)}^N = \mu^N$, then $(\tilde{\mathbb{J}}, D)$ degenerates into a bipolar-valued fuzzy soft set.
- (3) If the bipolarity is not considered, that is $h_{\tilde{\mathbb{J}}(\varepsilon)} = h_{\tilde{\mathbb{J}}(\varepsilon)}^P$, then $(\tilde{\mathbb{J}}, D)$ degenerates into a hesitant fuzzy soft set.

Example 1. Let $U = \{u_1, u_2, u_3\}$ be the set of houses, and $D_1 \subseteq E$, where $D_1 = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{\text{location; price; surrounding}$

environment}. $\tilde{\mathbb{J}}$ is a mapping given by $\tilde{\mathbb{J}}: D_1 \longrightarrow \text{HBVFS}(U)$, where

$$\begin{aligned}\tilde{\mathbb{J}}(\varepsilon_1) &= \{\langle u_1, (\{0.7, 0.8\}, \{-0.6, -0.8\}) \rangle, \langle u_2, (\{0.3, 0.4, 0.6\}, \{-0.7\}) \rangle, \langle u_3, (\{0.6, 0.7\}, \{-0.5, -0.6, -0.8\}) \rangle\}, \\ \tilde{\mathbb{J}}(\varepsilon_2) &= \{\langle u_1, (\{0.4, 0.7\}, \{-0.2, -0.4\}) \rangle, \langle u_2, (\{0.5, 0.6\}, \{-0.3, -0.5\}) \rangle, \langle u_3, (\{0.6, 0.7, 0.8\}, \{-0.3, -0.4\}) \rangle\}, \\ \tilde{\mathbb{J}}(\varepsilon_3) &= \{\langle u_1, (\{0.3, 0.4\}, \{-0.6, -0.7\}) \rangle, \langle u_2, (\{0.6, 0.7, 0.9\}, \{-0.4, -0.5\}) \rangle, \langle u_3, (\{0.8\}, \{-0.4, -0.6, -0.7\}) \rangle\}.\end{aligned}\quad (26)$$

Then, $(\tilde{\mathbb{J}}, D_1) \in \text{HBVFSS}(U)$.

Definition 9. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$ and $D_1, D_2 \subseteq E$, $(\tilde{\mathbb{J}}, D_1)$ is called as a hesitant bipolar-valued fuzzy soft subset of $(\tilde{\mathbb{K}}, D_2)$, if $D_1 \subseteq D_2$ and $\tilde{\mathbb{J}}(\varepsilon) \subseteq \tilde{\mathbb{K}}(\varepsilon)$ for $\forall \varepsilon \in D_1$.

In this case, we define $(\tilde{\mathbb{J}}, D_1) \sqsubseteq (\tilde{\mathbb{K}}, D_2)$ or $(\tilde{\mathbb{K}}, D_2) \supseteq (\tilde{\mathbb{J}}, D_1)$.

Example 2. Let $(\tilde{\mathbb{J}}, D_1) \in \text{HBVFSS}(U)$; it is defined as that in Example 1. Let $D_2 = \{\varepsilon_1, \varepsilon_2\} = \{\text{location; price}\}$ and $(\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; it is defined as

$$\begin{aligned}\tilde{\mathbb{K}}(\varepsilon_1) &= \{\langle u_1, (\{0.3, 0.4, 0.6\}, \{-0.5, -0.7\}) \rangle, \langle u_2, (\{0.1, 0.3\}, \{-0.2, -0.4\}) \rangle, \langle u_3, (\{0.6, 0.8\}, \{-0.3, -0.7\}) \rangle\}, \\ \tilde{\mathbb{K}}(\varepsilon_2) &= \{\langle u_1, (\{0.4, 0.6\}, \{-0.2, -0.3\}) \rangle, \langle u_2, (\{0.3\}, \{-0.2, -0.5, -0.6\}) \rangle, \langle u_3, (\{0.4, 0.5\}, \{-0.3, -0.4\}) \rangle\}.\end{aligned}\quad (27)$$

Obviously, $D_2 \subseteq D_1$; by calculating the score function, we can obtain

$$\begin{aligned}\mathcal{S}(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon_1)}(u_1)) &= 0.725, \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon_1)}(u_1)) &= 0.5167; \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon_1)}(u_2)) &= 0.5667, \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon_1)}(u_2)) &= 0.25; \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon_1)}(u_3)) &= 0.6417, \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon_1)}(u_3)) &= 0.6; \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon_2)}(u_1)) &= 0.425, \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon_2)}(u_1)) &= 0.375; \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon_2)}(u_2)) &= 0.475, \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon_2)}(u_2)) &= 0.3667; \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon_2)}(u_3)) &= 0.525, \\ \mathcal{S}(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon_2)}(u_3)) &= 0.4.\end{aligned}\quad (28)$$

Then, it has

$$s(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon_i)}(u_j)) \geq s(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon_i)}(u_j)). \quad (29)$$

Thus, we can have

$$\tilde{\mathbb{K}}(\varepsilon) \subseteq \tilde{\mathbb{J}}(\varepsilon), \quad \text{for } \forall \varepsilon \in D_2. \quad (30)$$

Hence, we can conclude $(\tilde{\mathbb{K}}, D_2) \subseteq (\tilde{\mathbb{J}}, D_1)$.

Definition 10. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; then,

- (1) If $(\tilde{\mathbb{J}}, D_1) \subseteq (\tilde{\mathbb{K}}, D_2)$ and $(\tilde{\mathbb{K}}, D_2) \subseteq (\tilde{\mathbb{J}}, D_1)$, then $(\tilde{\mathbb{J}}, D_1)$ and $(\tilde{\mathbb{K}}, D_2)$ are called hesitant bipolar-valued fuzzy soft equal, denoted by $(\tilde{\mathbb{J}}, D_1) \triangleq (\tilde{\mathbb{K}}, D_2)$.
- (2) If $D_1 = D_2$ and $\tilde{\mathbb{J}}(\varepsilon) = \tilde{\mathbb{K}}(\varepsilon)$ for $\forall \varepsilon \in D_1$, then $(\tilde{\mathbb{J}}, D_1)$ and $(\tilde{\mathbb{K}}, D_2)$ are equal, denoted by $(\tilde{\mathbb{J}}, D_1) = (\tilde{\mathbb{K}}, D_2)$.

Note: $(\tilde{\mathbb{J}}, D_1) = (\tilde{\mathbb{K}}, D_2)$ is different from $(\tilde{\mathbb{J}}, D_1) \triangleq (\tilde{\mathbb{K}}, D_2)$. Obviously, $(\tilde{\mathbb{J}}, D_1) = (\tilde{\mathbb{K}}, D_2)$ can be regarded as a special case of $(\tilde{\mathbb{J}}, D_1) \triangleq (\tilde{\mathbb{K}}, D_2)$. If $(\tilde{\mathbb{J}}, D_1) = (\tilde{\mathbb{K}}, D_2)$ holds, then $(\tilde{\mathbb{J}}, D_1) \triangleq (\tilde{\mathbb{K}}, D_2)$ holds; conversely, if $(\tilde{\mathbb{J}}, D_1) \triangleq (\tilde{\mathbb{K}}, D_2)$ holds, then $(\tilde{\mathbb{J}}, D_1) = (\tilde{\mathbb{K}}, D_2)$ may not hold. For example, let $U = \{u\}$ and $D_1 = D_2 = \{\varepsilon\}$, $\tilde{\mathbb{J}}(\varepsilon) = \{\langle u, (\{0.5, 0.6\}, \{-0.3, -0.4\}) \rangle\}$, $\tilde{\mathbb{K}}(\varepsilon) = \{\langle u, (\{0.6, 0.7\}, \{-0.2, -0.3\}) \rangle\}$. We can obtain $\mathcal{S}(\tilde{h}_{\tilde{\mathbb{J}}(\varepsilon)}(u)) = 0.45 = \mathcal{S}(\tilde{h}_{\tilde{\mathbb{K}}(\varepsilon)}(u))$, which implies $(\tilde{\mathbb{J}}, D_1) \triangleq (\tilde{\mathbb{K}}, D_2)$, but $(\tilde{\mathbb{J}}, D_1) \neq (\tilde{\mathbb{K}}, D_2)$.

Definition 11. Let $(\tilde{\mathbb{J}}, D) \in \text{HBVFSS}(U)$ and $u \in U$ be two special hesitant bipolar-valued fuzzy soft sets, which are defined as

- (1) $(\tilde{\mathbb{J}}, D)$ is called as an empty set, if $\tilde{\mathbb{J}}(\varepsilon) = \emptyset$ for $\forall \varepsilon \in D$, denoted by $(\emptyset, D)$.
- (2) $(\tilde{\mathbb{J}}, D)$ is called as a full set, if $\tilde{\mathbb{J}}(\varepsilon) = \mathbb{U}$ for $\forall \varepsilon \in D$, denoted by $(\mathbb{U}, D)$.

Definition 12. Let $(\tilde{\mathbb{J}}, D) \in \text{HBVFSS}(U)$, and $(\tilde{\mathbb{J}}^c, D)$ is called as the complementary set of the $(\tilde{\mathbb{J}}, D)$, denoted by $(\tilde{\mathbb{J}}, D)^c$, where $\tilde{\mathbb{J}}^c: D \longrightarrow \text{HBVFS}(U)$ and $\tilde{\mathbb{J}}^c(\varepsilon) = (\tilde{\mathbb{J}}(\varepsilon))^c$ for $\forall \varepsilon \in D$.

Example 3. Let $(\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; it is defined as that in Example 2; we obtain $(\tilde{\mathbb{K}}, D_2)^c$ as follows:

$$\begin{aligned}\tilde{\mathbb{K}}^c(\varepsilon_1) &= \{\langle u_1, (\{0.4, 0.6, 0.7\}, \{-0.3, -0.5\}) \rangle, \langle u_2, (\{0.7, 0.9\}, \{-0.6, -0.8\}) \rangle, \langle u_3, (\{0.2, 0.4\}, \{-0.3, -0.7\}) \rangle\}, \\ \tilde{\mathbb{K}}^c(\varepsilon_2) &= \{\langle u_1, (\{0.4, 0.6\}, \{-0.7, -0.8\}) \rangle, \langle u_2, (\{0.7\}, \{-0.4, -0.5, -0.8\}) \rangle, \langle u_3, (\{0.5, 0.6\}, \{-0.6, -0.7\}) \rangle\}.\end{aligned}\quad (31)$$

Theorem 4. Let $(\tilde{\mathcal{O}}, D), (\tilde{\mathcal{U}}, D) \in \text{HBVFSS}(U)$; then,

- (1) $(\tilde{\mathcal{O}}, D)^c = (\tilde{\mathcal{U}}, D)$.
- (2) $(\tilde{\mathcal{U}}, D)^c = (\tilde{\mathcal{O}}, D)$.

Then, $(\tilde{\mathbb{L}}, D)$ is called as the union of $(\tilde{\mathbb{J}}, D_1)$ and $(\tilde{\mathbb{K}}, D_2)$, denoted by

$$(\tilde{\mathbb{J}}, D_1) \sqcup (\tilde{\mathbb{K}}, D_2) = (\tilde{\mathbb{L}}, D) \quad (33)$$

Proof. It can be easily obtained from Definition 12. $\square$

Definition 13. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; if $D = D_1 \cup D_2$ and $\forall \varepsilon \in D$,

$$\tilde{\mathbb{L}}(\varepsilon) = \begin{cases} \tilde{\mathbb{J}}(\varepsilon), & \text{if } \varepsilon \in D_1 - D_2, \\ \tilde{\mathbb{K}}(\varepsilon), & \text{if } \varepsilon \in D_2 - D_1, \\ \tilde{\mathbb{J}}(\varepsilon) \sqcup \tilde{\mathbb{K}}(\varepsilon), & \text{if } \varepsilon \in D_1 \cap D_2. \end{cases} \quad (32)$$

Example 4. Let $U = \{u_1, u_2, u_3\}$ be the set of houses, and $D_1, D_2 \subseteq E$, where $D_1 = \{\varepsilon_1, \varepsilon_3\} = \{\text{location, surrounding environment}\}$ and $D_2 = \{\varepsilon_1, \varepsilon_2\} = \{\text{location, price}\}$. Let $(\tilde{\mathbb{J}}, D_1) \in \text{HBVFSS}(U)$; it is defined as

$$\begin{aligned}\tilde{\mathbb{J}}(\varepsilon_1) &= \{\langle u_1, (\{0.7, 0.8\}, \{-0.6, -0.8\}) \rangle, \langle u_2, (\{0.3, 0.4, 0.6\}, \{-0.7\}) \rangle, \langle u_3, (\{0.6, 0.7\}, \{-0.5, -0.6, -0.8\}) \rangle\}, \\ \tilde{\mathbb{J}}(\varepsilon_3) &= \{\langle u_1, (\{0.3, 0.4\}, \{-0.6, -0.7\}) \rangle, \langle u_2, (\{0.6, 0.7, 0.9\}, \{-0.4, -0.5\}) \rangle, \langle u_3, (\{0.8\}, \{-0.4, -0.6, -0.7\}) \rangle\}.\end{aligned}\quad (34)$$

Let $(\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; it is defined as that in Example 2; then, $(\tilde{\mathbb{J}}, D_1) \sqcup (\tilde{\mathbb{K}}, D_2) = (\tilde{\mathbb{L}}, D)$, where $D = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and

$$\begin{aligned}\tilde{\mathbb{L}}(\varepsilon_1) &= \{\langle u_1, (\{0.7, 0.8\}, \{-0.6, -0.7, -0.8\}) \rangle, \langle u_2, (\{0.3, 0.4, 0.6\}, \{-0.7\}) \rangle, \langle u_3, (\{0.6, 0.7, 0.8\}, \{-0.5, -0.6, -0.7, -0.8\}) \rangle\}, \\ \tilde{\mathbb{L}}(\varepsilon_2) &= \{\langle u_1, (\{0.4, 0.6\}, \{-0.2, -0.3\}) \rangle, \langle u_2, (\{0.3\}, \{-0.2, -0.5, -0.6\}) \rangle, \langle u_3, (\{0.4, 0.5\}, \{-0.3, -0.4\}) \rangle\}, \\ \tilde{\mathbb{L}}(\varepsilon_3) &= \{\langle u_1, (\{0.3, 0.4\}, \{-0.6, -0.7\}) \rangle, \langle u_2, (\{0.6, 0.7, 0.9\}, \{-0.4, -0.5\}) \rangle, \langle u_3, (\{0.8\}, \{-0.4, -0.6, -0.7\}) \rangle\}.\end{aligned}\quad (35)$$

Definition 14. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; if $D = D_1 \cap D_2$ and $\forall \varepsilon \in D$,

$$\tilde{\mathbb{L}}(\varepsilon) = \tilde{\mathbb{J}}(\varepsilon) \cap \tilde{\mathbb{K}}(\varepsilon). \quad (36)$$

Then, $(\tilde{\mathbb{L}}, D)$ is called as the intersection of $(\tilde{\mathbb{J}}, D_1)$ and $(\tilde{\mathbb{K}}, D_2)$, denoted by

$$(\tilde{\mathbb{J}}, D_1) \cap (\tilde{\mathbb{K}}, D_2) = (\tilde{\mathbb{L}}, D). \quad (37)$$

Example 5. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; it is defined as that in Example 4; then,

$$(\tilde{\mathbb{J}}, D_1) \cap (\tilde{\mathbb{K}}, D_2) = (\tilde{\mathbb{L}}, D), \quad (38)$$

where $D = \{\varepsilon_1\}$ and

$$\begin{aligned}\tilde{\mathbb{L}}(\varepsilon_1) &= \{\langle u_1, (\{0.3, 0.4, 0.6\}, \{-0.5, -0.6, -0.7\}) \rangle, \\ &\quad \langle u_2, (\{0.1, 0.3\}, \{-0.2, -0.4\}) \rangle, \\ &\quad \langle u_3, (\{0.6, 0.7\}, \{-0.3, -0.5, -0.6, -0.7\}) \rangle\}.\end{aligned}\quad (39)$$

Theorem 5. Let $(\tilde{\mathbb{J}}, D) \in \text{HBVFSS}(U)$; then,

- (1) $((\tilde{\mathbb{J}}, D)^c)^c = (\tilde{\mathbb{J}}, D)$.
- (2) $(\tilde{\mathbb{J}}, D) \sqcup (\tilde{\mathbb{J}}, D) = (\tilde{\mathbb{J}}, D)$.
- (3) $(\tilde{\mathbb{J}}, D) \cap (\tilde{\mathbb{J}}, D) = (\tilde{\mathbb{J}}, D)$.
- (4) $(\tilde{\mathbb{J}}, D) \sqcup (\tilde{\mathcal{O}}, D) = (\tilde{\mathbb{J}}, D)$.
- (5) $(\tilde{\mathbb{J}}, D) \cap (\tilde{\mathcal{O}}, D) = (\tilde{\mathcal{O}}, D)$.
- (6) $(\tilde{\mathbb{J}}, D) \sqcup (\tilde{\mathcal{U}}, D) = (\tilde{\mathbb{U}}, D)$.
- (7) $(\tilde{\mathbb{J}}, D) \cap (\tilde{\mathcal{U}}, D) = (\tilde{\mathbb{J}}, D)$.

Proof. It can be obtained from Definitions 12–14. $\square$

Theorem 6. Let $(\tilde{J}, D_1) \in \text{HBVFSS}(U)$; then,

- (1) $(\tilde{Q}, D) \sqcup (\tilde{J}, D_1) = (\tilde{J}, D_1)$ iff $D \subseteq D_1$.
- (2) $(\tilde{U}, D) \sqcup (\tilde{J}, D_1) = (\tilde{U}, D)$ iff $D_1 \subseteq D$.
- (3) $(\tilde{Q}, D) \sqcap (\tilde{J}, D_1) = (\tilde{Q}, D \cap D_1)$.
- (4) $(\tilde{U}, D) \sqcap (\tilde{J}, D_1) = (\tilde{J}, D \cap D_1)$.

Proof. It can be easily proved from Definitions 12–14. $\square$

Theorem 7. Let $(\tilde{J}, D), (\tilde{K}, D) \in \text{HBVFSS}(U)$; then,

- (1) $((\tilde{J}, D) \sqcup (\tilde{K}, D))^c = (\tilde{J}, D)^c \sqcap (\tilde{K}, D)^c$.
- (2) $((\tilde{J}, D) \sqcap (\tilde{K}, D))^c = (\tilde{J}, D)^c \sqcup (\tilde{K}, D)^c$.

Proof

- (1) Let $(\tilde{J}, D) \sqcup (\tilde{K}, D) = (\tilde{P}, D)$, where

$$\tilde{P}(\varepsilon) = \tilde{J}(\varepsilon) \sqcup \tilde{K}(\varepsilon), \quad \text{for } \forall \varepsilon \in D, \quad (40)$$

and

$$((\tilde{J}, D) \sqcup (\tilde{K}, D))^c = (\tilde{P}, D)^c = (\tilde{P}^c, D), \quad (41)$$

where

$$\tilde{P}^c(\varepsilon) = (\tilde{J}(\varepsilon) \sqcup \tilde{K}(\varepsilon))^c = \tilde{J}^c(\varepsilon) \sqcap \tilde{K}^c(\varepsilon), \quad \text{for } \forall \varepsilon \in D. \quad (42)$$

Let $(\tilde{J}, D)^c \sqcap (\tilde{K}, D)^c = (\tilde{J}^c, D) \sqcap (\tilde{K}^c, D) = (\tilde{Q}, D)$, where

$$\tilde{Q}(\varepsilon) = \tilde{J}^c(\varepsilon) \sqcap \tilde{K}^c(\varepsilon), \quad \text{for } \forall \varepsilon \in D. \quad (43)$$

Thus, one has $\tilde{P}^c(\varepsilon) = \tilde{Q}(\varepsilon)$ and $(\tilde{P}^c, D) = (\tilde{Q}, D)$.

Further, it yields

$$((\tilde{J}, D) \sqcup (\tilde{K}, D))^c = (\tilde{J}, D)^c \sqcap (\tilde{K}, D)^c. \quad (44)$$

- (2) It is similar to the proof of conclusion 1. $\square$

Theorem 8. Let $(\tilde{J}, D_1), (\tilde{K}, D_2) \in \text{HBVFSS}(U)$; then,

- (1) $((\tilde{J}, D_1) \sqcup (\tilde{K}, D_2))^c \subseteq (\tilde{J}, D_1)^c \sqcup (\tilde{K}, D_2)^c$.
- (2) $((\tilde{J}, D_1) \sqcap (\tilde{K}, D_2))^c \supseteq (\tilde{J}, D_1)^c \sqcap (\tilde{K}, D_2)^c$.
- (3) $((\tilde{J}, D_1) \sqcup (\tilde{K}, D_2))^c \supseteq (\tilde{J}, D_1)^c \sqcap (\tilde{K}, D_2)^c$.
- (4) $((\tilde{J}, D_1) \sqcap (\tilde{K}, D_2))^c \subseteq (\tilde{J}, D_1)^c \sqcup (\tilde{K}, D_2)^c$.

Proof

- (1) Let $(\tilde{J}, D_1) \sqcup (\tilde{K}, D_2) = (\tilde{P}, D_3)$, where $D_3 = D_1 \cup D_2$, $\forall \varepsilon \in D_3$, and

$$\tilde{P}(\varepsilon) = \begin{cases} \tilde{J}(\varepsilon), & \text{if } \varepsilon \in D_1 - D_2, \\ \tilde{K}(\varepsilon), & \text{if } \varepsilon \in D_2 - D_1, \\ \tilde{J}(\varepsilon) \sqcup \tilde{K}(\varepsilon), & \text{if } \varepsilon \in D_1 \cap D_2. \end{cases} \quad (45)$$

Then, it obtains

$$((\tilde{J}, D_1) \sqcup (\tilde{K}, D_2))^c = (\tilde{P}, D_3)^c = (\tilde{P}^c, D_3), \quad (46)$$

and $\forall \varepsilon \in D_3$; one has

$$\tilde{P}^c(\varepsilon) = \begin{cases} \tilde{J}^c(\varepsilon), & \text{if } \varepsilon \in D_1 - D_2, \\ \tilde{K}^c(\varepsilon), & \text{if } \varepsilon \in D_2 - D_1, \\ \tilde{J}^c(\varepsilon) \sqcap \tilde{K}^c(\varepsilon), & \text{if } \varepsilon \in D_1 \cap D_2. \end{cases} \quad (47)$$

Let $(\tilde{J}, D_1)^c \sqcap (\tilde{K}, D_2)^c = (\tilde{J}^c, D_1) \sqcap (\tilde{K}^c, D_2) = (\tilde{Q}, D_4)$, where $D_4 = D_1 \cup D_2$, $\forall \varepsilon \in D_4$, and

$$\tilde{Q}(\varepsilon) = \begin{cases} \tilde{J}^c(\varepsilon), & \text{if } \varepsilon \in D_1 - D_2, \\ \tilde{K}^c(\varepsilon), & \text{if } \varepsilon \in D_2 - D_1, \\ \tilde{J}^c(\varepsilon) \sqcap \tilde{K}^c(\varepsilon), & \text{if } \varepsilon \in D_1 \cap D_2. \end{cases} \quad (48)$$

From Theorem 3, we can obtain

$$\tilde{J}^c(\varepsilon) \sqcap \tilde{K}^c(\varepsilon) \subseteq \tilde{J}^c(\varepsilon) \sqcup \tilde{K}^c(\varepsilon), \quad \text{for } \forall \varepsilon \in D_1 \cap D_2. \quad (49)$$

Thus, we obtain

$$D_3 = D_4 = D_1 \cup D_2, \quad (50)$$

$$\tilde{P}^c(\varepsilon) \subseteq \tilde{Q}(\varepsilon), \quad \text{for } \forall \varepsilon \in D_1 \cup D_2.$$

Then, together with Definition 9, we have

$$((\tilde{J}, D_1) \sqcup (\tilde{K}, D_2))^c \subseteq (\tilde{J}, D_1)^c \sqcup (\tilde{K}, D_2)^c. \quad (51)$$

Similarly, we can also prove (2)–(4). $\square$

Definition 15. Let $(\tilde{J}, D_1), (\tilde{K}, D_2) \in \text{HBVFSS}(U)$; $(\tilde{J}, D_1)$ and $(\tilde{K}, D_2)$ are defined as $(\tilde{L}, D_1 \times D_2)$, where

$$\tilde{L}(\xi, \eta) = \tilde{J}(\xi) \sqcap \tilde{K}(\eta), \quad \text{for } \forall (\xi, \eta) \in D_1 \times D_2, \quad (52)$$

and $\tilde{L}$ is a mapping given by $\tilde{L}: D_1 \times D_2 \longrightarrow \text{HBVFS}(U)$.

In this case, define $(\tilde{J}, D_1) \overline{\wedge} (\tilde{K}, D_2)$.

Example 6. Let $(\tilde{J}, D_1), (\tilde{K}, D_2) \in \text{HBVFSS}(U)$. They are defined in Example 4; then, one gets

$$(\tilde{J}, D_1) \overline{\wedge} (\tilde{K}, D_2) = (\tilde{L}, D_1 \times D_2), \quad (53)$$

where

$$\begin{aligned}
\tilde{\mathbb{L}}(\varepsilon_1, \varepsilon_1) &= \{\langle u_1, (\{0.3, 0.4, 0.6\}, \{-0.5, -0.6, -0.7\}) \rangle, \langle u_2, (\{0.1, 0.3\}, \{-0.2, -0.4\}) \rangle, \langle u_3, (\{0.6, 0.7\}, \{-0.3, -0.5, -0.6, -0.7\}) \rangle\}, \\
\tilde{\mathbb{L}}(\varepsilon_1, \varepsilon_2) &= \{\langle u_1, (\{0.4, 0.6\}, \{-0.2, -0.3\}) \rangle, \langle u_2, (\{0.3\}, \{-0.2, -0.5, -0.6\}) \rangle, \langle u_3, (\{0.4, 0.5\}, \{-0.3, -0.4\}) \rangle\}, \\
\tilde{\mathbb{L}}(\varepsilon_3, \varepsilon_1) &= \{\langle u_1, (\{0.3, 0.4\}, \{-0.5, -0.6, -0.7\}) \rangle, \langle u_2, (\{0.1, 0.3\}, \{-0.2, -0.4\}) \rangle, \langle u_3, (\{0.6, 0.8\}, \{0.3, -0.4, -0.6, -0.7\}) \rangle\}, \\
\tilde{\mathbb{L}}(\varepsilon_3, \varepsilon_2) &= \{\langle u_1, (\{0.3, 0.4\}, \{-0.2, -0.3\}) \rangle, \langle u_2, (\{0.3\}, \{-0.2, -0.4, -0.5\}) \rangle, \langle u_3, (\{0.4, 0.5\}, \{-0.3, -0.4\}) \rangle\}.
\end{aligned} \tag{54}$$

Definition 16. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$. $(\tilde{\mathbb{J}}, D_1)$ or $(\tilde{\mathbb{K}}, D_2)$ is defined as $(\tilde{\mathbb{L}}, D_1 \times D_2)$, where

$$\tilde{\mathbb{L}}(\xi, \eta) = \tilde{\mathbb{J}}(\xi) \sqcup \tilde{\mathbb{K}}(\eta), \quad \text{for } \forall (\xi, \eta) \in D_1 \times D_2, \tag{55}$$

and $\tilde{\mathbb{L}}$ is a mapping given by $\tilde{\mathbb{L}}: D_1 \times D_2 \longrightarrow \text{HBVFS}(U)$.
In this case, define $(\tilde{\mathbb{J}}, D_1) \vee (\tilde{\mathbb{K}}, D_2)$.

Example 7. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$. They are defined as those in Example 4; then, it leads to

$$(\tilde{\mathbb{J}}, D_1) \vee (\tilde{\mathbb{K}}, D_2) = (\tilde{\mathbb{L}}, D_1 \times D_2), \tag{56}$$

where

$$\begin{aligned}
\tilde{\mathbb{L}}(\varepsilon_1, \varepsilon_1) &= \{\langle u_1, (\{0.7, 0.8\}, \{-0.6, -0.7, -0.8\}) \rangle, \langle u_2, (\{0.3, 0.4, 0.6\}, \{-0.7\}) \rangle, \langle u_3, (\{0.6, 0.7, 0.8\}, \{-0.5, -0.6, -0.7, -0.8\}) \rangle\}, \\
\tilde{\mathbb{L}}(\varepsilon_1, \varepsilon_2) &= \{\langle u_1, (\{0.7, 0.8\}, \{-0.6, -0.8\}) \rangle, \langle u_2, (\{0.3, 0.4, 0.6\}, \{-0.7\}) \rangle, \langle u_3, (\{0.6, 0.7\}, \{-0.5, -0.6, -0.8\}) \rangle\}, \\
\tilde{\mathbb{L}}(\varepsilon_3, \varepsilon_1) &= \{\langle u_1, (\{0.3, 0.4, 0.6\}, \{-0.6, -0.7\}) \rangle, \langle u_2, (\{0.6, 0.7, 0.9\}, \{-0.4, -0.5\}) \rangle, \langle u_3, (\{0.8\}, \{-0.4, -0.6, -0.7\}) \rangle\}, \\
\tilde{\mathbb{L}}(\varepsilon_3, \varepsilon_2) &= \{\langle u_1, (\{0.4, 0.6\}, \{-0.6, -0.7\}) \rangle, \langle u_2, (\{0.6, 0.7, 0.9\}, \{-0.4, -0.5, -0.6\}) \rangle, \langle u_3, (\{0.8\}, \{-0.4, -0.6, -0.7\}) \rangle\}.
\end{aligned} \tag{57}$$

Theorem 9. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2) \in \text{HBVFSS}(U)$; then,

- (1) $((\tilde{\mathbb{J}}, D_1) \overline{\wedge} (\tilde{\mathbb{K}}, D_2))^c = (\tilde{\mathbb{J}}, D_1)^c \vee (\tilde{\mathbb{K}}, D_2)^c$.
- (2) $((\tilde{\mathbb{J}}, D_1) \vee (\tilde{\mathbb{K}}, D_2))^c = (\tilde{\mathbb{J}}, D_1)^c \overline{\wedge} (\tilde{\mathbb{K}}, D_2)^c$.

- (1) $(\tilde{\mathbb{J}}, D_1) \overline{\wedge} ((\tilde{\mathbb{K}}, D_2) \overline{\wedge} (\tilde{\mathbb{L}}, D_3)) = ((\tilde{\mathbb{J}}, D_1) \overline{\wedge} (\tilde{\mathbb{K}}, D_2)) \overline{\wedge} (\tilde{\mathbb{L}}, D_3)$.
- (2) $(\tilde{\mathbb{J}}, D_1) \vee ((\tilde{\mathbb{K}}, D_2) \vee (\tilde{\mathbb{L}}, D_3)) = ((\tilde{\mathbb{J}}, D_1) \vee (\tilde{\mathbb{K}}, D_2)) \vee (\tilde{\mathbb{L}}, D_3)$.

Proof. (1) Let $(\tilde{\mathbb{J}}, D_1) \overline{\wedge} (\tilde{\mathbb{K}}, D_2) = (\tilde{\mathbb{P}}, D_1 \times D_2)$; then,

$$((\tilde{\mathbb{J}}, D_1) \overline{\wedge} (\tilde{\mathbb{K}}, D_2))^c = (\tilde{\mathbb{P}}^c, D_1 \times D_2). \tag{58}$$

Again let

$$(\tilde{\mathbb{J}}, D_1)^c \vee (\tilde{\mathbb{K}}, D_2)^c = (\tilde{\mathbb{J}}^c, D_1) \vee (\tilde{\mathbb{K}}^c, D_2) = (\tilde{\mathbb{Q}}, D_1 \times D_2). \tag{59}$$

Thus, it gets

$$\begin{aligned}
\tilde{\mathbb{P}}^c(\xi, \eta) &= (\tilde{\mathbb{J}}(\xi) \sqcap \tilde{\mathbb{K}}(\eta))^c = \tilde{\mathbb{J}}^c(\xi) \sqcup \tilde{\mathbb{K}}^c(\eta), \\
\tilde{\mathbb{Q}}(\xi, \eta) &= \tilde{\mathbb{J}}^c(\xi) \sqcup \tilde{\mathbb{K}}^c(\eta),
\end{aligned} \tag{60}$$

where $\forall (\xi, \eta) \in D_1 \times D_2$.

Obviously, we can obtain

$$\tilde{\mathbb{P}}^c(\xi, \eta) = \tilde{\mathbb{Q}}(\xi, \eta). \tag{61}$$

Hence, it yields

$$((\tilde{\mathbb{J}}, D_1) \overline{\wedge} (\tilde{\mathbb{K}}, D_2))^c = (\tilde{\mathbb{J}}, D_1)^c \vee (\tilde{\mathbb{K}}, D_2)^c. \tag{62}$$

□

Theorem 10. Let $(\tilde{\mathbb{J}}, D_1), (\tilde{\mathbb{K}}, D_2), (\tilde{\mathbb{L}}, D_3) \in \text{HBVFSS}(U)$; then,

Proof

- (1) Let $(\tilde{\mathbb{K}}, D_2) \overline{\wedge} (\tilde{\mathbb{L}}, D_3) = (\tilde{\mathbb{P}}, D_4)$; then, $D_4 = D_2 \times D_3$ and

$$\tilde{\mathbb{P}}(\eta, \zeta) = \tilde{\mathbb{K}}(\eta) \sqcap \tilde{\mathbb{L}}(\zeta), \quad \text{for } \forall (\eta, \zeta) \in D_4. \tag{63}$$

Let $(\tilde{\mathbb{J}}, D_1) \overline{\wedge} (\tilde{\mathbb{P}}, D_4) = (\tilde{\mathbb{Q}}, D_5)$; then, we get

$$D_5 = D_1 \times D_4 = D_1 \times (D_2 \times D_3),$$

$$\begin{aligned}
\tilde{\mathbb{Q}}(\xi, \eta, \zeta) &= \tilde{\mathbb{J}}(\xi) \sqcap \tilde{\mathbb{P}}(\eta, \zeta) = \tilde{\mathbb{J}}(\xi) \sqcap (\tilde{\mathbb{K}}(\eta) \sqcap \tilde{\mathbb{L}}(\zeta)), \\
&\text{for } \forall (\xi, \eta, \zeta) \in D_5.
\end{aligned} \tag{64}$$

Again let $(\tilde{\mathbb{J}}, D_1) \overline{\wedge} (\tilde{\mathbb{K}}, D_2) = (\tilde{\mathbb{M}}, D_6)$; then, it results in $D_6 = D_1 \times D_2$ and

$$\tilde{\mathbb{M}}(\xi, \eta) = \tilde{\mathbb{J}}(\xi) \sqcap \tilde{\mathbb{K}}(\eta), \quad \text{for } \forall (\xi, \eta) \in D_6. \tag{65}$$

Let $(\tilde{\mathbb{M}}, D_6) \overline{\wedge} (\tilde{\mathbb{L}}, D_3) = (\tilde{\mathbb{N}}, D_7)$; then, it deduces

$$D_7 = D_6 \times D_3 = (D_1 \times D_2) \times D_3,$$

$$\begin{aligned}
\tilde{\mathbb{N}}(\xi, \eta, \zeta) &= \tilde{\mathbb{M}}(\xi, \eta) \sqcap \tilde{\mathbb{L}}(\zeta) = (\tilde{\mathbb{J}}(\xi) \sqcap \tilde{\mathbb{K}}(\eta)) \sqcap \tilde{\mathbb{L}}(\zeta), \\
&\text{for } \forall (\xi, \eta, \zeta) \in D_7.
\end{aligned} \tag{66}$$

Thus, we can obtain

$$D_5 = D_7 = D_1 \times D_2 \times D_3, \quad (67)$$

$$\tilde{Q}(\xi, \eta, \zeta) = \tilde{N}(\xi, \eta, \zeta), \quad \text{for } \forall (\xi, \eta, \zeta) \in D_5 = D_7.$$

Hence, it has

$$(\tilde{Q}, D_5) = (\tilde{N}, D_7), \quad (68)$$

that is,

$$(\tilde{J}, D_1) \overline{\wedge} ((\tilde{K}, D_2) \overline{\wedge} (\tilde{L}, D_3)) = ((\tilde{J}, D_1) \overline{\wedge} (\tilde{K}, D_2)) \overline{\wedge} (\tilde{L}, D_3). \quad (69)$$

□

4. Hesitant Bipolar-Valued Fuzzy Soft Sets in Decision-Making Problems

In order to conveniently describe the content, throughout this section, let $U = \{u_1, u_2, \dots, u_n\}$, $D = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$, and $(\tilde{J}, D) \in \text{HBVFSS}(U)$.

Definition 17. The fuzzy soft set $(\tilde{J}_\delta, D)$ is called as the score matrix of the hesitant bipolar-valued fuzzy soft set $(\tilde{J}, D)$, where the membership degree of each element of the fuzzy set $\tilde{J}_\delta(\varepsilon_i)$ is the score function of each element of the hesitant bipolar-valued fuzzy set $\tilde{J}(\varepsilon_i)$ ($i = 1, 2, \dots, m$).

Definition 18 (see [22]). Let (F, D) be a fuzzy soft set on U and $\delta(D) = \{\delta(\varepsilon_1), \delta(\varepsilon_2), \dots, \delta(\varepsilon_m)\}$ be threshold vector on D ; the $\delta(D)$ -level soft set is defined as (F_δ, D) , where $F_\delta(\varepsilon) = \{u_j \in U: \mu_{F(\varepsilon)}(u_j) \geq \delta(\varepsilon)\}$, $\forall \varepsilon \in D$.

In Definition 18, by using different selection methods of attribute threshold, we may obtain different level soft sets. In [22], we can find some different selection methods of threshold such as maximum, minimum, and average. We determine the threshold with the method of average in this study, that is,

$$\delta(\varepsilon) = \frac{1}{n} \sum_{j=1}^n \mu_{F(\varepsilon)}(u_j) \quad \text{for } \forall \varepsilon \in D. \quad (70)$$

Next, a multiattribute decision-making approach for hesitant bipolar-valued fuzzy soft sets is given. This approach demonstrates that hesitant bipolar-valued fuzzy soft sets can be converted into simpler fuzzy soft sets by considering the score matrix. At the same time, the fuzzy soft set is converted into a classic soft set, and each attribute is set to be a threshold according to soft sets; then, the optimal plan can be acquired by calculating the choice value of plan correspondingly (Algorithm 1).

Note: for the calculation method of how to choose the value of $\mathcal{C}$ in the above algorithm, refer to [22].

Example 8. There are three members in Mr. Zhang's family; they want to buy a house, and they decide to choose one of the five houses after preliminary screening, $U = \{u_1, u_2, \dots, u_5\}$. The evaluation of the house is represented by five indicators in the hesitant bipolar-valued fuzzy soft set $(\tilde{J}, D)$, where $D = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_5\} = \{\text{location; price; surrounding environment; building structure;}$

Step 1 Input $(\tilde{J}, D)$.
 Step 2 Calculate the score matrix $(\tilde{J}_\delta, D)$ corresponding to $(\tilde{J}, D)$.
 Step 3 Calculating the average value assigned to each parameter, the threshold vector $\delta(D)$ can be obtained.
 Step 4 According to the threshold vector, calculate the average-level soft set $(\tilde{J}_{\delta\delta}, D)$ and the choice value $\mathcal{C}$ of every alternative.
 Step 5 Rank the objects in accordance with $\mathcal{C}$ and then select the optimal decision as u_k if $\mathcal{C}_k = \max \mathcal{C}_j$.
 Step 6 If the value of k is not the one in the previous step, then any one of u_k can be chosen for the optimal decision making.

ALGORITHM 1: Multiattribute decision making of HBVFSS.

property management}. It is shown in Table 1. Let us help them to make the optimal decision by using the above algorithm.

Calculate the score matrix $(\tilde{J}_\delta, D)$ corresponding to $(\tilde{J}, D)$, which is shown in Table 2.

Calculate the average value assigned to each parameter, that is,

$$\begin{aligned} \delta(\varepsilon_1) &= \frac{1}{5} \sum_{j=1}^5 \mu_{\tilde{J}_\delta(\varepsilon_1)}(u_j) = 0.4533, \\ \delta(\varepsilon_2) &= \frac{1}{5} \sum_{j=1}^5 \mu_{\tilde{J}_\delta(\varepsilon_2)}(u_j) = 0.4650, \\ \delta(\varepsilon_3) &= \frac{1}{5} \sum_{j=1}^5 \mu_{\tilde{J}_\delta(\varepsilon_3)}(u_j) = 0.4200, \\ \delta(\varepsilon_4) &= \frac{1}{5} \sum_{j=1}^5 \mu_{\tilde{J}_\delta(\varepsilon_4)}(u_j) = 0.4217, \\ \delta(\varepsilon_5) &= \frac{1}{5} \sum_{j=1}^5 \mu_{\tilde{J}_\delta(\varepsilon_5)}(u_j) = 0.4267. \end{aligned} \quad (71)$$

Then, we obtain the threshold vector as

$$\begin{aligned} \delta(D) &= \{\delta(\varepsilon_1), \delta(\varepsilon_2), \delta(\varepsilon_3), \delta(\varepsilon_4), \delta(\varepsilon_5)\} \\ &= \{0.4533, 0.4650, 0.4200, 0.4217, 0.4267\}. \end{aligned} \quad (72)$$

Calculate the average-level soft set $(\tilde{J}_{\delta\delta}, D)$ and the value of $\mathcal{C}$, which is shown in Table 3.

Ranking all the houses in accordance with the choice value $\mathcal{C}$, we can easily see $u_1 > u_5 > u_2 > u_3 = u_4$. Thus, select the optimal decision as u_1 .

The above examples not only consider the bipolarity and nonuniqueness of element membership but also solve the problem of insufficient parameter chemical tools. The conclusion is in line with objective reality and shows the superiority of the proposed algorithm.

TABLE 1: Hesitant bipolar-valued fuzzy soft set.

	ε_1	ε_2	ε_3	ε_4	ε_5
u_1	$(\{0.6, 0.5, 0.4\}, \{-0.3, -0.4, -0.5\})$	$(\{0.9, 0.8, 0.6\}, \{-0.3, -0.4\})$	$(\{0.8, 0.7, 0.4\}, \{-0.3, -0.4\})$	$(\{0.7, 0.6\}, \{-0.2, -0.4, -0.6\})$	$(\{0.6, 0.5, 0.4\}, \{-0.2, -0.3, -0.6\})$
u_2	$(\{0.8, 0.6\}, \{-0.2, -0.3, -0.6\})$	$(\{0.7, 0.6, 0.5\}, \{-0.1, -0.3\})$	$(\{0.4, 0.3\}, \{-0.3, -0.4, -0.6\})$	$(\{0.6, 0.4, 0.3\}, \{-0.1, -0.2\})$	$(\{0.9, 0.7\}, \{-0.1, -0.3, -0.4\})$
u_3	$(\{0.6, 0.4, 0.3\}, \{-0.1, -0.3, -0.6\})$	$(\{0.8, 0.7, 0.6\}, \{-0.1, -0.3\})$	$(\{0.6, 0.4, 0.3\}, \{-0.2, -0.4\})$	$(\{0.6, 0.2\}, \{-0.2, -0.4, -0.7\})$	$(\{0.8, 0.6, 0.5\}, \{-0.1, -0.4\})$
u_4	$(\{0.9, 0.7, 0.6\}, \{-0.1, -0.2\})$	$(\{0.6, 0.3\}, \{-0.1, -0.2, -0.3\})$	$(\{0.7, 0.6, 0.3\}, \{-0.1, -0.2\})$	$(\{0.8, 0.6, 0.5\}, \{-0.3, -0.4\})$	$(\{0.6, 0.4, 0.3\}, \{-0.1, -0.2, -0.4\})$
u_5	$(\{0.7, 0.6, 0.4\}, \{-0.3, -0.4\})$	$(\{0.9, 0.8\}, \{-0.1, -0.3, -0.6\})$	$(\{0.8, 0.7, 0.5\}, \{-0.3, -0.4\})$	$(\{0.7, 0.4, 0.3\}, \{-0.2, -0.3, -0.4\})$	$(\{0.3, 0.2\}, \{-0.3, -0.6, -0.7\})$

TABLE 2: Score matrix.

	ε_1	ε_2	ε_3	ε_4	ε_5
u_1	0.4500	0.5583	0.4917	0.5250	0.4333
u_2	0.5333	0.4000	0.3917	0.2917	0.5333
u_3	0.3833	0.4500	0.3667	0.4167	0.4417
u_4	0.4417	0.3250	0.3417	0.4917	0.3333
u_5	0.4583	0.5917	0.5083	0.3833	0.3917

TABLE 3: Average-level soft set and choice value.

	ε_1	ε_2	ε_3	ε_4	ε_5	$\mathcal{C}$
u_1	0	1	1	1	1	4
u_2	1	0	0	0	1	2
u_3	0	0	0	0	1	1
u_4	0	0	0	1	0	1
u_5	1	1	1	0	0	3

5. Conclusion

In this paper, the theory of hesitant bipolar-valued fuzzy soft sets is firstly proposed based on soft sets and hesitant bipolar-valued fuzzy sets. Then, some basic operations and corresponding properties are defined. A multiattribute decision-making approach for hesitant bipolar-valued fuzzy soft sets is proposed. By using the thought of score function and level soft set, the hesitant bipolar-valued fuzzy soft set is converted into a fuzzy soft set. Since the fuzzy soft set is transformed into a classic soft set, the optimal decision making is obtained. The method not only considers the bipolarity and nonuniqueness of element membership but also solves the insufficient parameterization problem. Finally, a practical problem is solved to demonstrate the capability of the proposed method. Our future work is to study the parameterization reduction or the uncertainty measurement of hesitant bipolar-valued fuzzy soft sets, such as distance, similarity, and entropy, in which a new multiattribute decision-making method will be constructed.

Data Availability

The simulation data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported in part by the National Natural Science Foundation of China under grant no. 61803190 and in part by the Liaoning Provincial Education Department Fund (JJL201915403).

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