Research Article

Multiple Positive Solutions for a System of Caputo Fractional p -Laplacian Boundary Value Problems

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In this work, we are pleased to investigate multiple positive solutions for a system of Caputo fractional p -Laplacian boundary value problems, and we also provide an example for illustrating our main results.

1. Introduction

In this work, we are pleased to discuss the positive solutions for the following system of Caputo fractional p -Laplacian boundary value problems:

$$\begin{cases} {}^c D_t^{\gamma_1} (\varphi_p({}^c D_t^{\mu_1} u(t))) + g_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\ {}^c D_t^{\gamma_2} (\varphi_p({}^c D_t^{\mu_2} v(t))) + g_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u(1) = 0, \gamma_1 u'(0) - \delta_1 u'(1) = 0, & {}^c D_t^{\mu_1} u(0) = 0, {}^c D_t^{\mu_1} u(1) = b_1 {}^c D_t^{\mu_1} u(\xi_1), \\ \alpha_2 v(0) - \beta_2 v(1) = 0, \gamma_2 v'(0) - \delta_2 v'(1) = 0, & {}^c D_t^{\mu_2} v(0) = 0, {}^c D_t^{\mu_2} v(1) = b_2 {}^c D_t^{\mu_2} v(\xi_2), \end{cases} \quad (1)$$

where ${}^c D_t^{\mu_i}, {}^c D_t^{\gamma_i}$ ($i = 1, 2$) are the fractional derivatives of Caputo sense with $\mu_i, \gamma_i \in (1, 2)$; φ_p is the p -Laplacian, i.e., $\varphi_p(x) = |x|^{p-2}x$ with $p > 1, x \in \mathbb{R}$; and the constants $\alpha_i, \beta_i, \gamma_i, \delta_i, b_i, \xi_i$ ($i = 1, 2$) and the functions g_i ($i = 1, 2$) satisfy the following conditions:

(H0) $\alpha_i > \beta_i > 0, \gamma_i > \delta_i > 0, b_i \geq 0, \xi_i \in (0, 1)$ with $b_i^{p-1} \xi_i \in [0, 1]$

(H1) g_i ($i = 1, 2$) are nonnegative continuous functions on $[0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+ := [0, +\infty)$

As a generalization of integer-order equations, fractional-order equations can effectively describe various materials and physical processes with memory and genetic properties. It has a large number of applications in our society, such as biology, chemical kinetics, electromagnetics, transmission and diffusion, and automatic control. Recently, there many researchers pay their attentions to studying the existence of solutions for various types of fractional-order equations by use of fixed point theorems, upper and lower solution methods, and monotone iterative techniques. For

instance, we refer the readers to [1–25] and references therein.

In [1], by using the method of upper and lower solutions and the Schauder fixed point theorem, Vong investigated the positive solutions for the following nonlocal fractional boundary value problem:

$$\begin{cases} {}^c D_t^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u'(0) = \dots = u^{(n-1)}(0) = 0, u(1) = \int_0^1 u(s) d\mu(s), \end{cases} \quad (2)$$

where $n \geq 2$, $\alpha \in (n-1, n)$, and $\mu(s)$ is a function of bounded variation. f may be singular at $t = 1$.

In [2], Wang and Yang studied the integral boundary value problem of Caputo sense:

$$\begin{cases} {}^c D_t^q \chi(t) = f(t, \chi(t)), & 0 < t < 1, \\ \alpha \chi(0) - \beta \chi(1) = \int_0^1 h(t) \chi(t) dt, \gamma \chi'(0) - \delta \chi'(1) = \int_0^1 g(t) \chi(t) dt. \end{cases} \quad (3)$$

By the Leggett–Williams fixed point theorem, they obtained multiple positive solutions when the nonlinearity f is bounded from below. Moreover, when f is asymptotically linear at infinity, they also obtained an existence theorem.

In [3], Wang et al. adopted the theory of mixed monotone operators to obtain a unique positive solution for the mixed fractional boundary value problems involving the p -Laplacian:

$$\begin{cases} {}^c D_t^\alpha \varphi_p(D_{0+}^\beta v(t)) + \lambda f(t, v(t), D_{0+}^{\beta_1} v(t), \dots, D_{0+}^{\beta_{n-2}} v(t)) = 0, & 0 < t < 1, \\ v(0) = D_{0+}^{\gamma_i} v(0) = D_{0+}^\beta v(0) = (\varphi_p(D_{0+}^\beta v(0)))' = 0, & i = 1, \dots, n-2, \\ D_{0+}^{r_0} v(1) = \lambda_1 \int_0^1 h_1(s) D_{0+}^{r_1} v(s) dA_1(s) + \lambda_2 \int_0^\eta h_2(s) D_{0+}^{r_2} v(s) dA_2(s) + \lambda_3 \sum_{j=1}^\infty \rho_j D_{0+}^{r_3} v(\eta_j), \end{cases} \quad (4)$$

where ${}^c D_t^\alpha$ is the Caputo fractional derivative and $D_{0+}^\beta, D_{0+}^{\beta_i}, D_{0+}^{\gamma_i}$, and $D_{0+}^{r_i}$ are the Riemann–Liouville fractional derivatives.

Recently, coupled systems of fractional differential equations have also been investigated by many authors.

Some results on the direction can be found in a series of papers [11–25] and the references cited therein.

In [11], Wang utilized the Guo–Krasnosel'skii fixed point theorem to investigate the multiple positive solutions for the mixed fractional p -Laplacian differential system:

$$\begin{cases} D_{0+}^{\beta_1} (\varphi_{p_1}({}^c D_t^{\alpha_1} u(t))) + f_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\ D_{0+}^{\beta_2} (\varphi_{p_2}({}^c D_t^{\alpha_2} v(t))) + f_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\ u'(0) = u''(0) = \dots = u^{(n-1)}(0) = 0, & u(1) = \mu_1 \int_0^1 a(s) v(s) dA_1(s), \\ v'(0) = v''(0) = \dots = v^{(m-1)}(0) = 0, & v(1) = \mu_2 \int_0^1 b(s) u(s) dA_2(s), \\ {}^c D_t^{\alpha_1} u(0) = 0, & {}^c D_t^{\alpha_1} u(1) = \varepsilon_1 {}^c D_t^{\alpha_1} u(\eta_1), \\ {}^c D_t^{\alpha_2} v(0) = 0, & {}^c D_t^{\alpha_2} v(1) = \varepsilon_2 {}^c D_t^{\alpha_2} v(\eta_2), \end{cases} \quad (5)$$

where $D_{0+}^{\beta_i}$ are the Riemann–Liouville fractional derivatives, ${}^c D_t^{\alpha_i}$ the Caputo fractional derivatives, and $\int_0^1 a(s) v(s) dA_1(s)$ and $\int_0^1 b(s) u(s) dA_2(s)$ are the Riemann–Stieltjes integrals.

In [12], Rao studied the system of fractional p -Laplacian differential equations:

$$\begin{cases} D_{0+}^{\beta_1} (\varphi_p(D_{0+}^{\alpha_1} u(t))) + f_1(t, u(t), v(t)) = 0, & a < t < b, \\ D_{0+}^{\beta_2} (\varphi_p(D_{0+}^{\alpha_2} v(t))) + f_2(t, u(t), v(t)) = 0, & a < t < b, \\ \xi u(a) - \eta u'(a) = 0, \gamma u(b) + \delta u'(b) = 0, & D_{0+}^{\alpha_1} u(a) = 0, \\ \xi v(a) - \eta v'(a) = 0, \gamma v(b) + \delta v'(b) = 0, & D_{0+}^{\alpha_2} v(a) = 0. \end{cases} \quad (6)$$

When the nonlinearities f_i satisfy some appropriate conditions, the author made use of the Avery–Henderson fixed point theorem and the six functionals' fixed point theorem to obtain some existence theorems of multiple positive solutions.

Inspired by the aforementioned results, in this work, we study the solvability for (1) and establish the existence results of multiple positive solutions via the six functional fixed point theorem under some bounded conditions for g_i ($i = 1, 2$). Finally, we also provide an example to illustrate our main results.

2. Preliminaries

In this section, we only recall the definition of Caputo fractional derivative, for more details, see the book [26].

Definition 1. The fractional derivative of f in the Caputo sense is defined as

$${}^c D_t^\mu f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t (t-s)^{n-\mu-1} f^{(n)}(s) ds, \quad n-1 < \mu < n, \quad (7)$$

where $n = [\mu] + 1$, $[\mu]$ denotes the integer part of the number μ .

Now, we calculate Green's functions associated with (1). Let

$$\begin{cases} \varphi_p({}^c D_t^{\mu_1} u(t)) = x(t), & 0 < t < 1, \\ \varphi_p({}^c D_t^{\mu_2} v(t)) = y(t), & 0 < t < 1, \end{cases} \quad (8)$$

and then by the boundary conditions in (1), we have

$$\begin{cases} x(0) = 0, & x(1) = b_1^{p-1} x(\xi_1), \\ y(0) = 0, & y(1) = b_2^{p-1} x(\xi_2). \end{cases} \quad (9)$$

Consequently, substituting (8) and (9) into (1), we obtain

$$\begin{cases} {}^c D_t^{\nu_1} x(t) + g_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\ {}^c D_t^{\nu_2} y(t) + g_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\ x(0) = 0, & x(1) = b_1^{p-1} x(\xi_1), \\ y(0) = 0, & y(1) = b_2^{p-1} x(\xi_2). \end{cases} \quad (10)$$

We next translate (10) into an equivalent system of integral equations. By the similar arguments as in [2], we have the following result.

Lemma 1. Problem (10) is equivalent to the following system of Hammerstein-type integral equations:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \int_0^1 H_1(t, s) g_1(s, u(s), v(s)) ds \\ \int_0^1 H_2(t, s) g_2(s, u(s), v(s)) ds \end{pmatrix}, \quad (11)$$

where

$$\begin{aligned} H_i(t, s) &= \overline{H}_i(t, s) + \frac{b_i^{p-1} t}{1 - b_i^{p-1} \xi_i} \overline{H}_i(\xi_i, s), \quad t, s \in [0, 1], \\ \overline{H}_i(t, s) &= \frac{1}{\Gamma(\nu_i)} \begin{cases} t(1-s)^{\nu_i-1} - (t-s)^{\nu_i-1}, & 0 \leq s \leq t \leq 1, \\ t(1-s)^{\nu_i-1}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \quad (12)$$

Proof. We only need to consider the case $i = 1$ (by the similar method, the case $i = 2$ can be easily proved). Using Lemma 2.5 of [2], we have

$$x(t) = c_0 + c_1 t - \int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds, \quad (13)$$

where $c_i \in \mathbb{R}$, $i = 0, 1$. $x(0) = 0$ implies that $c_0 = 0$, and by $x(1) = b_1^{p-1} x(\xi_1)$, we have

$$\begin{aligned} c_1 - \int_0^1 \frac{(1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds &= b_1^{p-1} \xi_1 c_1 - b_1^{p-1} \int_0^{\xi_1} \frac{(\xi_1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds, \\ c_1 &= \frac{1}{1 - b_1^{p-1} \xi_1} \int_0^1 \frac{(1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds - \frac{b_1^{p-1}}{1 - b_1^{p-1} \xi_1} \int_0^{\xi_1} \frac{(\xi_1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds. \end{aligned} \quad (14)$$

Therefore,

$$\begin{aligned}
x(t) &= \frac{1}{1 - b_1^{p-1} \xi_1} \int_0^1 \frac{t(1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds \\
&\quad - \frac{b_1^{p-1} t}{1 - b_1^{p-1} \xi_1} \int_0^{\xi_1} \frac{(\xi_1 - s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds - \int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds \\
&= \frac{1}{1 - b_1^{p-1} \xi_1} \int_0^1 \frac{t(1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds - \frac{b_1^{p-1} t}{1 - b_1^{p-1} \xi_1} \int_0^{\xi_1} \frac{(\xi_1 - s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds \\
&\quad - \int_0^1 \frac{t(1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds + \int_0^1 \frac{t(1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds \\
&\quad - \int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds = \int_0^1 \bar{H}_1(t, s) g_1(s, u(s), v(s)) ds \\
&\quad + \frac{b_1^{p-1} \xi_1}{1 - b_1^{p-1} \xi_1} \int_0^1 \frac{t(1-s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds - \frac{b_1^{p-1} t}{1 - b_1^{p-1} \xi_1} \int_0^{\xi_1} \frac{(\xi_1 - s)^{\nu_1-1}}{\Gamma(\nu_1)} g_1(s, u(s), v(s)) ds \\
&= \int_0^1 \bar{H}_1(t, s) g_1(s, u(s), v(s)) ds + \frac{b_1^{p-1} t}{1 - b_1^{p-1} \xi_1} \int_0^1 \bar{H}_1(\xi_1, s) g_1(s, u(s), v(s)) ds \\
&= \int_0^1 H_1(t, s) g_1(s, u(s), v(s)) ds.
\end{aligned} \tag{15}$$

This completes the proof.

Note from (8) and Lemma 1, we have

$$\begin{cases} \varphi_p({}^c D_t^{\mu_1} u(t)) = x(t) = \int_0^1 H_1(t, s) g_1(s, u(s), v(s)) ds, & 0 < t < 1, \\ \varphi_p({}^c D_t^{\mu_2} v(t)) = y(t) = \int_0^1 H_2(t, s) g_2(s, u(s), v(s)) ds, & 0 < t < 1, \end{cases} \tag{16}$$

and

$$\begin{cases} {}^c D_t^{\mu_1} u(t) = \varphi_p^{-1} \left(\int_0^1 H_1(t, s) g_1(s, u(s), v(s)) ds \right), & 0 < t < 1, \\ {}^c D_t^{\mu_2} v(t) = \varphi_p^{-1} \left(\int_0^1 H_2(t, s) g_2(s, u(s), v(s)) ds \right), & 0 < t < 1, \end{cases} \tag{17}$$

with the boundary conditions

$$\begin{cases} \alpha_1 u(0) - \beta_1 u(1) = 0, & \gamma_1 u'(0) - \delta_1 u'(1) = 0, \\ \alpha_2 v(0) - \beta_2 v(1) = 0, & \gamma_2 v'(0) - \delta_2 v'(1) = 0. \end{cases} \tag{18}$$

By Lemma 2.5 of [2], we have the result: $\square$

Lemma 2. Problems (17) and (18) is equivalent to the following Hammerstein-type integral equation:

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{bmatrix} \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \end{bmatrix}, \tag{19}$$

where

$$G_i(t, s) = \begin{cases} \frac{(t-s)^{\mu_i-1}}{\Gamma(\mu_i)} + \frac{\beta_i(1-s)^{\mu_i-1}}{(\alpha_i-\beta_i)\Gamma(\mu_i)} + \frac{\beta_i\delta_i(\mu_i-1)(1-s)^{\mu_i-2}}{(\alpha_i-\beta_i)(\gamma_i-\delta_i)\Gamma(\mu_i)} + \frac{\delta_i(\mu_i-1)t(1-s)^{\mu_i-2}}{(\gamma_i-\delta_i)\Gamma(\mu_i)}, & 0 \leq s \leq t \leq 1, \\ \frac{\beta_i(1-s)^{\mu_i-1}}{(\alpha_i-\beta_i)\Gamma(\mu_i)} + \frac{\beta_i\delta_i(\mu_i-1)(1-s)^{\mu_i-2}}{(\alpha_i-\beta_i)(\gamma_i-\delta_i)\Gamma(\mu_i)} + \frac{\delta_i(\mu_i-1)t(1-s)^{\mu_i-2}}{(\gamma_i-\delta_i)\Gamma(\mu_i)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (20)$$

Lemma 3 (see [2], Lemma 2.8). *The functions G_i, H_i ($i = 1, 2$) have the following properties:*

- (i) $G_i, H_i \in C([0, 1] \times [0, 1], \mathbb{R}^+)$
- (ii) $\mathcal{M}_i(s) \leq G_i(t, s) \leq (\alpha_i/\beta_i)\mathcal{M}_i(s)$, $\forall t, s \in [0, 1]$, where $\mathcal{M}_i(s) = (\beta_i(1-s)^{\mu_i-1}/(\alpha_i-\beta_i)\Gamma(\mu_i)) + (\beta_i\delta_i(\mu_i-1)(1-s)^{\mu_i-2}/(\alpha_i-\beta_i)(\gamma_i-\delta_i)\Gamma(\mu_i))$

Let $E := C[0, 1]$, $\|u\| := \max_{t \in [0, 1]} |u(t)|$ and $P := \{u \in E : u(t) \geq 0, \forall t \in [0, 1]\}$. Then, $(E, \|\cdot\|)$ is a real Banach space and P is a cone on E . Moreover, $E \times E$ is a Banach space with the norm $\|(u, v)\| = \|u\| + \|v\|$, and $P \times P$ a cone on $E \times E$. From Lemmas 1 and 2, we define operators T_i ($i = 1, 2$) and T as follows:

$$\begin{aligned} T_1(u, v)(t) &= \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds, \\ T_2(u, v)(t) &= \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds, \\ T(u, v)(t) &= (T_1, T_2)(u, v)(t), \quad \text{for } t \in [0, 1]. \end{aligned} \quad (21)$$

Then, we obtain that T_i ($i = 1, 2$): $P \times P \longrightarrow P$, $T: P \times P \longrightarrow P \times P$ are completely continuous operators, and if there exists $(\bar{u}, \bar{v}) \in P \times P \setminus \{0\}$ such that $T(\bar{u}, \bar{v}) = (\bar{u}, \bar{v})$, i.e., $T_1(\bar{u}, \bar{v}) = \bar{u}$, $T_2(\bar{u}, \bar{v}) = \bar{v}$, then $(\bar{u}, \bar{v})$ is a positive solution for (1).

Lemma 4 (see [27]). *Let P be a cone in a real Banach space E . Suppose that α, ψ , and ζ are nonnegative continuous concave functionals on P , β, σ , and θ are nonnegative continuous convex functionals on P , and there are nonnegative constants l, l', r, r', R , and R' such that $A: Q(\beta, R) \longrightarrow P$ is a completely continuous operator and*

- (B1) $Q(\beta, R)$ is a bounded set
- (B2) $Q(\sigma, l)$ and $Q(\alpha, \beta, r, R)$ are disjoint subset of $Q(\beta, R)$
- (B3) $\{u \in P: \theta(u) < r', r < \alpha(u), R' < \psi(u), \beta(u) < R\} \neq \emptyset$
- (B4) $\{u \in P: l' < \zeta(u), \sigma(u) < l\} \neq \emptyset$
- (B5) $\{u \in P: l < \sigma(u), \alpha(u) < r\} \neq \emptyset$

If the following claims hold:

- (C1) $\alpha(Au) > r$, $\forall u \in P$ with $\alpha(u) = r$, $\beta(u) \leq R$, and $r' < \theta(Au)$
- (C2) $\alpha(Au) > r$, $\forall u \in P$ with $\alpha(u) = r$, $\beta(u) \leq R$, and $\theta(u) \leq r'$
- (C3) $\beta(Au) < R$, $\forall u \in P$ with $r \leq \alpha(u)$, $\beta(u) = R$, and $\psi(Au) < R'$
- (C4) $\beta(Au) < R$, $\forall u \in P$ with $r \leq \alpha(u)$, $\beta(u) = R$, and $R' \leq \psi(u)$
- (C5) $\sigma(Au) < l$, $\forall u \in P$ with $\sigma(u) = l$ and $\zeta(Au) < l'$

$$(C6) \quad \sigma(Au) < l, \quad \forall u \in P \text{ with } \sigma(u) = l \text{ and } l' \leq \zeta(u)$$

Then, A has at least three fixed points u_1, u_2 , and u_3 in $Q(\beta, R)$ such that $\sigma(u_1) \leq l, r \leq \alpha(u_2)$ with $\beta(u_2) \leq R$, and $l < \sigma(u_3)$ with $\alpha(u_3) < r$.

3. Main Results

We first define some notations:

$$Q(\beta, R) = \{(u, v) \in P \times P: \beta(u, v) \leq R\},$$

$$Q(\alpha, \beta, r, R) = \{(u, v) \in P \times P: r \leq \alpha(u, v) \text{ and } \beta(u, v) \leq R\},$$

$$\alpha(u, v) = \min_{t \in [0, 1]} \{u(t) + v(t)\},$$

$$\psi(u, v) = \min_{t \in I} \{u(t) + v(t)\},$$

$$\zeta(u, v) = \min_{t \in I_1} \{u(t) + v(t)\},$$

$$\theta(u, v) = \max_{t \in [0, 1]} \{u(t) + v(t)\},$$

$$\beta(u, v) = \max_{t \in I} \{u(t) + v(t)\},$$

$$\sigma(u, v) = \max_{t \in I_1} \{u(t) + v(t)\},$$

(22)

where $I = [(1/4), (3/4)]$ and $I_1 = [(1/8), (7/8)]$. Then, α, ψ , and ζ are the concave functionals on P , and θ, β , and σ are the convex functionals on P .

Theorem 1. Suppose that there exist positive real numbers l, l', r, r', R , and R' such that $g_i (i = 1, 2)$ satisfy the following conditions:

(H2) $g_1(t, u(t), v(t)) < \varphi_p(l\beta_1/2\alpha_1\rho_1)$, $g_2(t, u(t), v(t)) < \varphi_p(l\beta_2/2\alpha_2\rho_2)$ for all $t \in [0, 1]$ and $(u, v) \in [0, (r/3)]$.

(H3) $g_1(t, u(t), v(t)) < \varphi_p(R\beta_1/2\alpha_1\rho_1)$, $g_2(t, u(t), v(t)) < \varphi_p(R\beta_2/2\alpha_2\rho_2)$ for all $t \in [0, 1]$ and $(u, v) \in [0, R]$.

(H4) $g_1(t, u(t), v(t)) < \varphi_p(r/2\rho_1)$, $g_2(t, u(t), v(t)) < \varphi_p(r/2\rho_2)$ for all $t \in [0, 1]$ and $(u, v) \in [r, (r/\eta)]$, where $\eta = \min\{(\beta_1/\alpha_1), (\beta_2/\alpha_2)\}$:

$$\begin{aligned} \rho_1 &= \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) d\tau \right) ds, \\ \rho_2 &= \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) d\tau \right) ds. \end{aligned} \quad (23)$$

Then, (1) has at least three positive solutions (u_1, v_1) , (u_2, v_2) , and (u_3, v_3) in $Q(\beta, R)$.

Proof. Note, from a standard calculus argument, we obtain the set $Q(\beta, R)$ is bounded. Since if $(u, v) \in Q(\beta, R)$, and then $\max_{t \in I} \{u(t) + v(t)\} < R$. Let $r' = (r/\eta)$, $R = (R'/\eta)$, $l = (l'/\eta)$, and $l = (r/3)$ with $R' \in ((2\eta^2 r' / (\eta + 1)), (2\eta r' / (\eta + 1)))$. Then, if $(u, v) \in Q(\sigma, l)$, we have $\max_{t \in I_1} \{u(t) + v(t)\} < l$. This means that

$$\begin{aligned} \alpha(u, v) &= \min_{t \in [0, 1]} \{u(t) + v(t)\} \leq \min_{t \in I_1} \{u(t) + v(t)\} \\ &\leq \max_{t \in I_1} \{u(t) + v(t)\} < l < r, \end{aligned} \quad (24)$$

and $Q(\sigma, l) \cap Q(\alpha, \beta, r, R) = \emptyset$. Also, it can easily be shown that

$$\begin{aligned} \left(\frac{R'}{2}, \frac{R}{2} \right) &\in \{(u, v) \in P: \theta(u, v) < r', r < \alpha(u, v), R' \\ &< \psi(u, v) \text{ and } \beta(u, v) < R\}, \\ \left(\frac{\eta l}{2}, \frac{l}{2} \right) &\in \{(u, v) \in P: l' < \zeta(u, v) \text{ and } \sigma(u, v) < l\}, \\ \left(\frac{r}{2}, \frac{l}{2} \right) &\in \{(u, v) \in P: l < \sigma(u, v) \text{ and } \alpha(u, v) < r\}. \end{aligned} \quad (25)$$

Hence, these sets are nonempty. As a result, (B1)–(B5) of Lemma 4 are satisfied.

Now, we verify the functional conditions. $\square$

Claim 1. $\alpha(T(u, v)) > r$, for all $(u, v) \in Q(\alpha, \beta, r, R)$ with $\alpha(u, v) = r$ and $r' < \theta(T(u, v))$. Note, from Lemma 3 (ii), we have

$$\begin{aligned} \alpha(T(u, v)) &= \min_{t \in [0, 1]} \{T_1(u, v)(t) + T_2(u, v)(t)\} \\ &= \min_{t \in [0, 1]} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\ &\geq \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &= \frac{\beta_1}{\alpha_1} \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \frac{\beta_2}{\alpha_2} \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\ &\geq \eta \max_{t \in [0, 1]} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\ &= \eta \theta(T(u, v)) \\ &> \eta r' \\ &= r. \end{aligned} \quad (26)$$

Claim 2. $\alpha(T(u, v)) > r$, for all $(u, v) \in \{(u, v) \in Q(\alpha, \beta, r, R): \theta(u, v) \leq r'\}$ with $\alpha(u, v) = r$. At this case, we

have $r \leq u(\tau) + v(\tau) \leq (r/\eta)$, for $\tau \in [0, 1]$, and from (H4), we obtain

$$\begin{aligned}
\alpha(T(u, v)) &= \min_{t \in [0,1]} \{T_1(u, v)(t) + T_2(u, v)(t)\} \\
&= \min_{t \in [0,1]} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
&\geq \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
&> \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) \varphi_p \left(\frac{r}{2\rho_1} \right) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) \varphi_p \left(\frac{r}{2\rho_2} \right) d\tau \right) ds \\
&= \frac{r}{2\rho_1} \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) d\tau \right) ds + \frac{r}{2\rho_2} \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) d\tau \right) ds \\
&= \frac{r}{2} + \frac{r}{2} \\
&= r.
\end{aligned} \tag{27}$$

Claim 3. $\beta(T(u, v)) < R$, for all $(u, v) \in Q(\alpha, \beta, r, R)$, $\beta(u, v) = R$, and $\psi(T(u, v)) < R'$. Then, we have

$$\begin{aligned}
\beta(T(u, v)) &= \max_{t \in I} \{T_1(u, v)(t) + T_2(u, v)(t)\} \\
&= \max_{t \in I} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
&\leq \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
&\leq \frac{1}{\eta} \left\{ \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
&\leq \frac{1}{\eta} \min_{t \in I} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
&= \frac{1}{\eta} \psi(u, v) \\
&< \frac{R'}{\eta} \\
&= R.
\end{aligned} \tag{28}$$

Claim 4. $\beta(T(u, v)) < R$, for all $(u, v) \in \{(u, v) \in Q(\alpha, \beta, r, R): R' \leq \psi(u, v)\}$ with $\beta(u, v) = R$. Therefore, by Lemma 3 (ii) and (H3), we have

$$\begin{aligned}
 \beta(T(u, v)) &= \max_{t \in I} \{T_1(u, v)(t) + T_2(u, v)(t)\} \\
 &= \max_{t \in I} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
 &\leq \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
 &< \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) \varphi_p \left(\frac{R\beta_1}{2\alpha_1\rho_1} \right) d\tau \right) ds + \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) \varphi_p \left(\frac{R\beta_2}{2\alpha_2\rho_2} \right) d\tau \right) ds \\
 &= \frac{R\beta_1}{2\alpha_1\rho_1} \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) d\tau \right) ds + \frac{R\beta_2}{2\alpha_2\rho_2} \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) d\tau \right) ds \\
 &= \frac{R}{2} + \frac{R}{2} \\
 &= R.
 \end{aligned} \tag{29}$$

Claim 5. $\sigma(T(u, v)) < l$, for all $(u, v) \in Q(\sigma, l)$ with $\sigma(u, v) = l$, and $\zeta(T(u, v)) < l'$. Then, we obtain

$$\begin{aligned}
 \sigma(T(u, v)) &= \max_{t \in I_1} \{T_1(u, v)(t) + T_2(u, v)(t)\} \\
 &= \max_{t \in I_1} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
 &\leq \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
 &\leq \frac{1}{\eta} \left\{ \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
 &\leq \frac{1}{\eta} \min_{t \in I_1} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
 &= \frac{1}{\eta} \zeta(T(u, v)) \\
 &< \frac{l'}{\eta} \\
 &= l.
 \end{aligned} \tag{30}$$

Claim 6. $\sigma(T(u, v)) < l$, for all $(u, v) \in \{Q(\sigma, l): l' \leq \zeta(u, v)\}$ with $\sigma(T(u, v)) < l$. Therefore, by Lemma 3 (ii) and (H2), we have

$$\begin{aligned}
 \sigma(T(u, v)) &= \max_{t \in I_1} \{T_1(u, v)(t) + T_2(u, v)(t)\} \\
 &= \max_{t \in I_1} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \right\} \\
 &\leq \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) g_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) g_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
 &< \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) \varphi_p \left(\frac{l\beta_1}{2\alpha_1\rho_1} \right) d\tau \right) ds + \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) \varphi_p \left(\frac{l\beta_2}{2\alpha_2\rho_2} \right) d\tau \right) ds \\
 &= \frac{l\beta_1}{2\alpha_1\rho_1} \int_0^1 \frac{\alpha_1}{\beta_1} \mathcal{M}_1(s) \varphi_p^{-1} \left(\int_0^1 H_1(s, \tau) d\tau \right) ds + \frac{l\beta_2}{2\alpha_2\rho_2} \int_0^1 \frac{\alpha_2}{\beta_2} \mathcal{M}_2(s) \varphi_p^{-1} \left(\int_0^1 H_2(s, \tau) d\tau \right) ds \\
 &= \frac{l}{2} + \frac{l}{2} \\
 &= l.
 \end{aligned} \tag{31}$$

Remark 1. Our Green's functions G_i ($i = 1, 2$) satisfy some special inequalities (see Lemma 3 (ii)). So, in the Claims 1 and 2 of Theorem 1, $\alpha(T(u, v)) = \min_{t \in [0, 1]} \{T_1(u, v)(t) + T_2(u, v)(t)\}$ cannot reach zero. Therefore, we use the Caputo derivative for the considered scheme. This also explains the

reason that we do not use other forms of fractional derivatives (including the Riemann–Liouville type and the Hadamard type).

Example 1. Consider the example:

$$\begin{cases} {}^c D_t^{3/2} (\varphi_p({}^c D_t^{3/2} u(t))) + g_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\ {}^c D_t^{3/2} (\varphi_p({}^c D_t^{3/2} v(t))) + g_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\ 2u(0) - u(1) = 0, 6u'(0) - 5u'(1) = 0, {}^c D_t^{3/2} u(0) = 0, {}^c D_t^{3/2} u(1) = {}^c D_t^{3/2} u\left(\frac{1}{4}\right), \\ 4v(0) - 3v(1) = 0, 8v'(0) - 7v'(1) = 0, {}^c D_t^{3/2} v(0) = 0, {}^c D_t^{3/2} v(1) = 2 {}^c D_t^{3/2} v\left(\frac{1}{8}\right), \end{cases} \tag{32}$$

where $p = 2$.

In order to obtain the constants in Theorem 1, we need to calculate

$$\begin{aligned}
 \int_0^1 \overline{H}_i(t, s) ds &= \frac{1}{\Gamma(\nu_i)} \int_0^1 t(1-s)^{\nu_i-1} ds - \frac{1}{\Gamma(\nu_i)} \int_0^t (t-s)^{\nu_i-1} ds \\
 &= \frac{t}{\nu_i \Gamma(\nu_i)} \left[-(1-s)^{\nu_i} \right]_0^1 + \frac{1}{\nu_i \Gamma(\nu_i)} (t-s)^{\nu_i} \Big|_0^t \\
 &= \frac{t-t^{\nu_i}}{\Gamma(\nu_i+1)},
 \end{aligned} \tag{33}$$

and we have

$$\begin{aligned}
 \int_0^1 H_i(t, s) ds &= \int_0^1 \overline{H}_i(t, s) ds + \frac{b_i^{p-1} t}{1-b_i^{p-1} \xi_i} \int_0^1 \overline{H}_i(\xi_i, s) ds \\
 &= \frac{t-t^{\nu_i}}{\Gamma(\nu_i+1)} + \frac{b_i^{p-1} t}{1-b_i^{p-1} \xi_i} \frac{\xi_i - \xi_i^{\nu_i}}{\Gamma(\nu_i+1)} \\
 &= \frac{(t-t^{\nu_i})(1-b_i^{p-1} \xi_i) + b_i^{p-1} t(\xi_i - \xi_i^{\nu_i})}{(1-b_i^{p-1} \xi_i) \Gamma(\nu_i+1)} \\
 &= \frac{t(1-b_i^{p-1} \xi_i^{\nu_i}) - t^{\nu_i}(1-b_i^{p-1} \xi_i)}{(1-b_i^{p-1} \xi_i) \Gamma(\nu_i+1)}.
 \end{aligned} \tag{34}$$

Moreover, we also obtain that $\eta = 0.5, b_i^{p-1}\xi_i = 0.25, b_1^{p-1}\xi_1^{\nu_1} = 0.125, b_2^{p-1}\xi_2^{\nu_2} = 0.088, \int_0^1 H_1(t, s)ds = 0.878t - 0.752t^{1.5}, \int_0^1 H_2(t, s)ds = 0.915t - 0.752t^{1.5}, \mathcal{M}_1(s) =$

$1.13(1-s)^{0.5} + 2.82(1-s)^{-0.5}, \mathcal{M}_2(s) = 3.39(1-s)^{0.5} + 11.85(1-s)^{-0.5}, \rho_1 = 0.9, \text{ and } \rho_2 = 4.29. \text{ If we put } l = 2, l' = 1, r = 6, r' = 12, R = 15.6, R' = 7.8, \text{ and}$

$$g_1(t, u, v) = \begin{cases} \frac{t}{60} + 0.25(u + v), & 0 \leq u + v \leq 2, t \in [0, 1], \\ \frac{t}{60} + 0.875(u + v) - 1.25, & 2 \leq u + v \leq 6, t \in [0, 1], \\ \frac{t}{60} + 4, & 6 \leq u + v \leq 12, t \in [0, 1], \\ \frac{t}{60} - 0.14(u + v) + 5.68, & 12 \leq u + v \leq 15.6, t \in [0, 1], \end{cases} \quad (35)$$

$$g_2(t, u, v) = \begin{cases} \frac{t}{70} + 0.08(u + v), & 0 \leq u + v \leq 2, t \in [0, 1], \\ \frac{t}{70} + 0.21(u + v) - 0.26, & 2 \leq u + v \leq 6, t \in [0, 1], \\ \frac{t}{70} + 1, & 6 \leq u + v \leq 12, t \in [0, 1], \\ \frac{t}{70} + 0.056(u + v) + 0.328, & 12 \leq u + v \leq 15.6, t \in [0, 1]. \end{cases}$$

Then, $g_i (i = 1, 2)$ satisfies the following conditions:

- (i) $g_1(t, u(t), v(t)) < 0.56, g_2(t, u(t), v(t)) < 0.18$ for all $t \in [0, 1]$ and $(u, v) \in [0, 2]$
- (ii) $g_1(t, u(t), v(t)) < 4.33, g_2(t, u(t), v(t)) < 1.37$ for all $t \in [0, 1]$ and $(u, v) \in [0, 15.6]$
- (iii) $g_1(t, u(t), v(t)) > 3.33, g_2(t, u(t), v(t)) > 0.7$ for all $t \in [0, 1]$ and $(u, v) \in [6, 12]$

Therefore, all assumptions in Theorem 1 are satisfied, and the system of (32) has at least triple positive solutions.

4. Conclusion

In this paper, we use the six functional fixed point theorem to study the system of (1) under some bounded conditions for $g_i (i = 1, 2)$. We first transform the original fractional differential system into the equivalent system of Hammerstein-type integral equations, and then with the help of properties of Green's functions, we provide some sufficient conditions to guarantee the existence of multiple positive solutions for our system. Finally, an example is presented to illustrate the effectiveness of the main result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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