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Research Article

Finite-Time Synchronization for a Class of Multiweighted Complex Networks with Markovian Switching and Time-Varying Delay

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In this paper, we investigate the finite-time synchronization control for a class of nonlinear coupled multiweighted complex networks (NCMWCNs) with Markovian switching and time-varying delay analytically and quantitatively. The value of this study lies in four aspects: First, it designs the finite-time synchronization controller to make the NCMWCNs with Markovian switching and time-varying delay achieve global synchronization in finite time. Second, it derives two kinds of finite-time estimation approaches by analyzing the impact of the nonlinearity of nonlinear coupled function on synchronization dynamics and synchronization convergence time. Third, it presents the relationship between Markovian switching parameters and synchronization problems of subsystems and the overall system. Fourth, it provides some numerical examples to demonstrate the effectiveness of the theoretical results.

1. Introduction

In the past several decades, since the pioneer work of Watts and Strogatz [1], complex networks have received extensive concern in various fields of science and engineering [2–5]. As a matter of fact, complex networks are composed of lots of interconnected nodes, in which each node is a fundamental unit and may have specific dynamics [6–11], and exist everywhere in the real world, such as sensor networks, smartphone networks, industrial control system networks, neural networks, and communication networks [12–16]. Therefore, the research of complex networks can help us to further comprehend the functions and effects of the real-world networks.

It is worthy to note that the applications of complex networks heavily depend on the dynamics of them, for example, passivity dynamics [17, 18], synchronization dynamics [19–22], and stabilization dynamics [23, 24], and hence, exploring dynamics of complex networks is necessary and significant. Especially, synchronization, as one of the most collective dynamics of complex networks, has gained widespread concern, and many valuable theoretical results have been obtained [23–31]. And numerous studies have been conducted to establish some feasible control methods such as event-triggered sliding mode control [32], adaptive sliding mode fault-tolerant control [33], and finite-time control [34, 35], to make the addressed systems get the desired dynamics behaviors.

Furthermore, in some practical engineering fields, the desired dynamic behavior is often required to be achieved in finite time interval [36, 37]. Therefore, recently, more researchers have increasingly drawn attention to finite-time synchronization dynamics control problem for complex networks [38–44]. In [38], an asynchronous switching feedback controller based on the derived sufficient conditions was designed to realize finite-time synchronization for a class of uncertain coupled switched networks. In [39], the authors gave some sufficient conditions which ensure finite-time synchronization for a class of switched coupled neural networks with discontinuous or continuous activations. In [40], under the framework of Filippov solution, the authors studied finite-time synchronization for two classes of coupled Markovian discontinuous neural networks with mixed delays. It should be noted that in [38–42, 44], the proposed complex networks were entangled by linear coupling. In fact, many physical networks are entangled by nonlinear coupling. For example, in Kuramoto oscillator network, there exist nonlinear interactions among different oscillators [45]. In electrical gird dynamical networks, different electrical elements are coupled by nonlinear interactions [46]. Obviously, linear function is a special case of nonlinear function, and nonlinear coupled complex networks are more general and their synchronization dynamics may be more complicated and unpredictable [2, 6, 25, 47–51]. In [47], under finite-time pinning control, the authors investigated the finite-time cluster synchronization problem of nonlinearly coupled and discontinuous Lur'e networks. In [48], the authors derived some sufficient conditions to ensure finite-time synchronization of the considered linear coupled and nonlinear coupled complex networks. Unfortunately, there are merely a few results on finite-time synchronization control of nonlinear coupled complex networks. And the impact of the nonlinearity of the coupled function on synchronization dynamics cannot be reflected by the derived settling finite time t^* [47, 48]. This is because that the finite-time control method cannot process how nonlinearity of nonlinear coupled function impacts synchronization dynamics of the addressed complex networks. So, there was no function relationship between t^* and the nonlinearity of the coupled function. Actually, every element attribute in a dynamical system may impact synchronization dynamics and synchronization convergence time, which means that nonlinearity of the coupled function, as one of the coupling function attributes, may affect synchronization dynamics and synchronization convergence time of the considered nonlinear coupled complex networks. Therefore, it is important to further design more reasonable and feasible finite-time control method to deal with the existing issue.

It should be mentioned that most of existing research results on synchronization control problems of complex networks focused on synchronization dynamics of complex networks with single weight (for example, [42, 44–47, 49–52] and references therein). But in reality, for example, people can contact each other by mail, telephone, MSN, e-mail, and so on; suppose every piece of contact information is of different weight, so human connection network is a complex network with multiweights, where the nodes are connected by more than one weight [53]. Indeed, this is a well-established way of introducing environmental noise into dynamic networks. Recently, synchronization dynamics of complex networks with multiweights have increasingly attracted interests [17, 18, 31, 34, 41–43, 48, 52–58]. Of them, Huang et al. [17] established several finite-time passivity criteria for several classes of linear coupled MWCNs (LCMWNCs). Qiu et al. [41] proposed finite-time synchronization control methods to make the addressed LCMWNCs. He et al. [52] considered four classes of LCMWNCs and investigated the global synchronization of them. Particularly, in [48, 55], the authors studied the finite-time synchronization control of nonlinear coupled MWCNs (NCMWCNs) and analyzed how the nonlinearity of nonlinear coupled function impacts synchronization dynamics and synchronization convergence time.

Motivated by the above discussion and analysis, in this paper, we will focus on the finite-time synchronization control for a class of NCMWCNs with Markovian switching and time-varying delay and propose that the finite-time estimation approaches based on the designed controllers can reflect how nonlinearity of nonlinear coupled function impacts synchronization dynamics and synchronization convergence time of the addressed network.

The rest of this paper is organized as follows. In Section 2, we present the model derivations and preliminaries. In Section 3, we provide the main results of the present paper. In Section 4, we give some numerical results to show the complicated dynamics of the system. In Section 5, we provide a brief discussion and summarize our main results.

2. Problem Description and Preliminaries

For any vector $x(t) \in \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n\times m}$, we denote the following:

$$\|x(t)\|_1 = \sum_{i=1}^{n} |x_i(t)|, \quad \|A\|_2 = \sqrt{\lambda_{\max}(A^T A)}, \tag{1}$$

where $\lambda_{\max}(A^T A)$ denotes the maximal eigenvalue of $A^T A$ and T represents the matrix transposition. diag$(\cdot)$ represents a block diagonal matrix. The number N represents a positive integer. The Kronecker product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{M\times N}$ is a matrix in $\mathbb{R}^{mM\times nN}$ denoted as $A \otimes B$. $I_n \in \mathbb{R}^{n\times n}$ means an n-dimensional identity matrix. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null and is right continuous). $\mathbb{E}(x)$ means the expectation of the random variable x.

Consider the following NCMWCNs with Markovian switching and time-varying delay:

$$\begin{aligned}\dot{\tilde{y}}_i(t) &= \tilde{A}\tilde{f}(\tilde{y}_i(t)) + \sum_{k=1}^{m}\sum_{j=1}^{N} \tilde{a}_k \tilde{g}_{ij}^k(\tilde{r}(t))\tilde{\Gamma}_k \tilde{h}(\tilde{y}_j(t)) \\ &\quad + \sum_{k=1}^{m}\sum_{j=1}^{N} \hat{a}_k \tilde{g}_{ij}^k(\tilde{r}(t))\hat{\Gamma}_k \tilde{h}(\tilde{y}_j(t-\tau(t))),\end{aligned} \tag{2}$$

where $\tilde{y}_i(t) = [\tilde{y}_{i1}(t), \tilde{y}_{i2}(t), \ldots, \tilde{y}_{in}(t)]^T \in \mathbb{R}^n$, $\tilde{f}(y_i(t)) = [\tilde{f}(\tilde{y}_{i1}(t)), \tilde{f}(\tilde{y}_{i2}(t)), \ldots, \tilde{f}(\tilde{y}_{in}(t))]^T \in \mathbb{R}^n$ denotes the activation function of the i th node $\tilde{y}_i(t)$, $\tilde{h}(\cdot): \mathbb{R}^n \longrightarrow \mathbb{R}^n$ represents nonlinear coupled function, $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{a}_k > 0$ and $\hat{a}_k > 0\,(k = 1, 2, \ldots, m)$ are coupled strength, $\tilde{G}_k(\tilde{r}(t)) = (\tilde{g}_{ij}^k(\tilde{r}(t)))_{N\times N}$ stands for the k th outer-coupled weight matrix, $\tilde{\Gamma}_k > 0$ and $\hat{\Gamma}_k > 0$ are the inner-coupled matrix, $\tau(t)$ is the coupled time-varying delay, and $\tilde{r}(t)$ is a right-continuous Markov chain with known transition rate on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ taking values in a finite state space $\widehat{S} = \{1, 2, \ldots, \widehat{s}\}$ with a generator $\Upsilon = (\delta_{pq})_{\widehat{s}\times\widehat{s}(p,q\in\widehat{S})}$ given by

$$P\{\widetilde{r}(t+\Delta t) = q \mid \widetilde{r}(t) = p\} = \begin{cases} \delta_{pq}\Delta t + o(\Delta t), & p \neq q, \\ 1 + \delta_{pq}\Delta t + o(\Delta t), & p = q, \end{cases} \quad (3)$$

where $\Delta t > 0$, $\lim_{\Delta t \longrightarrow 0}(o(\Delta t)/\Delta t) = 0$, $\delta_{pq} > 0$ $(\forall p \neq q)$ is the transition rate from mode p at time t to mode q at time $t + \Delta t$, and $\delta_{pp} = -\sum_{q=1,p\neq q}^{\widehat{s}} \delta_{pq} < 0$. For notation simplicity, we denote $\widetilde{G}_k(\widetilde{r}(t))$, $\widetilde{g}_{ij}^k(\widetilde{r}(t))$, and $\widetilde{r}(t)$ as $\widetilde{G}_k^{\widetilde{r}}$, $\widetilde{g}_{ij}^{k,\widetilde{r}}$, and $\widetilde{r}$, respectively.

Remark 1. From the discussion in Section 1 about multi-weighted complex networks (MWCNs) and their synchronization dynamics, it can be seen that many real-world networks can be more accurately modeled by some classes of MWCNs; some potential applications of MWCNs are closely related to their dynamics such as passivity dynamics and synchronization dynamics. And dynamics problems of MWCNs have witnessed an increasing interest [17, 18, 31, 34, 41, 48, 53–58]. It needs to be pointed out in [17, 18, 31, 34, 41, 48, 53–58]; the addressed MWCNs are LCMWNCs. And there are few research results about finite-time synchronization control of MWCNs [41, 48, 55].

Remark 2. In [48, 55], the authors investigated finite-time synchronization control of NCMWCNs and analyzed how the nonlinearity of nonlinear coupled function impacts synchronization dynamics and synchronization convergence time. However, in [48], the derived settling finite-time t^* based on the designed finite-time control scheme cannot reflect how the nonlinearity of nonlinear coupled function affects synchronization dynamics in finite time and synchronization convergence time. In [55], though the derived settling finite time t^* can reflect the impact of the nonlinearity of nonlinear coupled function on synchronization dynamics in finite time and synchronization convergence time of the proposed NCMWCNs with switching topology, the designed finite-time control method cannot process time-varying delay. That is, in [55], if time-varying delay is considered into the addressed NCMWCNs with switching topology, the derived finite-time control method is invalid. In the present paper, we will establish the finite-time control method which can effectively process time-varying delay of the addressed network. Furthermore, we will show that the impact of synchronization dynamics in finite time and synchronization convergence time can be reflected by the obtained settling finite time t^*.

Remark 3. In network (2), there is no diffusive coupling condition restriction: $\widetilde{g}_{ii}^{k,\widetilde{r}} = -\sum_{j=1,j\neq i}^{N} \widetilde{g}_{ij}^{k,\widetilde{r}}$. Therefore, compared with the existing multiweighted diffusive coupling complex networks [18, 31, 34, 41, 53–58], the coupled method among different nodes of network (2) is more flexible. Actually, the coupled method of the network (2) has been adopted in [17, 48, 55].

From network (2), we can get

$$\begin{aligned} \dot{s}(t) &= \widetilde{A}\widetilde{f}(s(t)) + \sum_{k=1}^{m}\sum_{j=1}^{N} \widetilde{a}_k \widetilde{g}_{ij}^{k,\widetilde{r}} \widetilde{\Gamma}_k \widetilde{h}(s(t)) \\ &\quad + \sum_{k=1}^{m}\sum_{j=1}^{N} \widehat{a}_k \widetilde{g}_{ij}^{k,\widetilde{r}} \widetilde{\Gamma}_k \widetilde{h}(s(t-\tau(t))), \end{aligned} \quad (4)$$

where $s(t)$ is the synchronization state of network (2).

Subtracting (4) from (2), we can get the error system of network (2):

$$\begin{aligned} \dot{\widetilde{e}}_i(t) &= \widetilde{A}\widetilde{F}(\widetilde{e}_i(t)) + \sum_{k=1}^{m}\sum_{j=1}^{N} \widetilde{a}_k \widetilde{g}_{ij}^{k,\widetilde{r}} \widetilde{\Gamma}_k \widetilde{H}(\widetilde{e}_j(t)) \\ &\quad + \sum_{k=1}^{m}\sum_{j=1}^{N} \widehat{a}_k \widetilde{g}_{ij}^{k,\widetilde{r}} \widetilde{\Gamma}_k \widetilde{H}(\widetilde{e}_j(t-\tau(t))), \end{aligned} \quad (5)$$

where $\widetilde{e}_i(t) = \widetilde{y}_i(t) - s(t)$, $\widetilde{F}(\widetilde{e}_i(t)) = \widetilde{f}(\widetilde{y}_i(t)) - \widetilde{f}(s(t))$, $\widetilde{H}(\widetilde{e}_j(t)) = \widetilde{h}(\widetilde{y}_j(t)) - \widetilde{h}(s(t))$, and $\widetilde{H}(\widetilde{e}_j(t-\tau(t))) = \widetilde{h}\widetilde{y}_j(t-\tau(t)) - \widetilde{h}(s(t-\tau(t)))$.

In order to make network (2) realize finite-time synchronization, we take the finite-time controller $u_i(t,\widetilde{r})$ as follows:

$$u_i(t,\widetilde{r}) = \begin{cases} -\sum_{k=1}^{m} \iota_i^{k,\widetilde{r}} \widetilde{e}_i(t) - \sum_{k=1}^{m} \widetilde{\beta}(\chi(\widetilde{h}))\widehat{\eta}_i^{k,\widetilde{r}} \widehat{Q}^{(\widetilde{\alpha}-1/2)} \mathrm{diag}\Big(\mathrm{sgn}\big(\widetilde{e}_i(t)\big)\Big)\big|\widetilde{e}_i(t)\big|^{\widetilde{\alpha}}, & \\ -m\widehat{Q}^{-1} \sum_{k=1}^{m} \widetilde{\beta}(\chi(\widetilde{h}))\widehat{\eta}_i^{k,\widetilde{r}} \left(\frac{\widehat{a}_k \widetilde{\theta}^{\widetilde{r}}}{1-\rho} \int_{t-\tau(t)}^{t} \widetilde{e}_i^{T}(s)\widehat{F}_i^{k}\widetilde{e}_i(s)\mathrm{d}s \right)^{((\widetilde{\alpha}+1)/2)} \dfrac{\widetilde{e}_i(t)}{\left\|\widetilde{e}_i(t)\right\|_1^2}, & \text{if } \left\|\widetilde{e}_i(t)\right\|_1 \neq 0 \\ 0, & \text{if } \left\|\widetilde{e}_i(t)\right\|_1 = 0 \end{cases} \quad (6)$$

where $\iota_i^{k,\widetilde{r}}$, $\widehat{\eta}_i^{k,\widetilde{r}}$, $\widehat{\theta}^{\widetilde{r}}$, $\widetilde{\beta}(\chi(\widetilde{h})) > 0$, $0 < \widetilde{\alpha} < 1$, $0 \leq \rho < 1$, $\widehat{Q} = \mathrm{diag}(\widehat{q}_1, \widehat{q}_2, \ldots, \widehat{q}_n) > 0$, $\widehat{F}_i^{k} \in R^{n\times n} > 0$, $\mathrm{sgn}(\cdot)$ is the sign function, $\mathrm{diag}(\mathrm{sgn}(\widetilde{e}_i(t))) = \mathrm{diag}(\mathrm{sgn}(\widetilde{e}_{i1}(t)), \mathrm{sgn}(\widetilde{e}_{i2}(t)), \ldots, \mathrm{sgn}(\widetilde{e}_{in}(t)))$, $\widetilde{e}_i(t) = (\widetilde{e}_{i1}(t), \widetilde{e}_{i2}(t), \ldots, \widetilde{e}_{in}(t))^T$, and $|\widetilde{e}_i(t)|^{\widetilde{\alpha}} = (|\widetilde{e}_{i1}t)|^{\widetilde{\alpha}}, |\widetilde{e}_{i2}(t)|^{\widetilde{\alpha}}, \ldots, |\widetilde{e}_{in}(t)|^{\widetilde{\alpha}})^T$.

Next, we give some definitions, assumptions, and lemmas, which are used in the analysis of main results.

Definition 1 (see [19, 20]). Network (2) with controller (6) is said to be synchronized in finite time if there exists a constant $t^* > 0$ which depends on the initial state vector value $\tilde{y}(0) = (\tilde{y}_1^T(0), \tilde{y}_2^T(0), \ldots, \tilde{y}_N^T(0))^T$ and $\tilde{y}_i(0) = (\tilde{y}_{i1}(0), \tilde{y}_2(0), \ldots, \tilde{y}_n(0))^T$, such that

$$\lim_{t\longrightarrow t^*}\sum_{i=1}^{N}\left\|\tilde{y}_i(t) - s(t)\right\|_1 = 0, \qquad \sum_{i=1}^{N}\left\|\tilde{y}_i(t) - s(t)\right\|_1 = 0, \tag{7}$$

for $t \geq t^*$, where $\tilde{y}_i(t) = (\tilde{y}_{i1}(t), \tilde{y}_2(t), \ldots, \tilde{y}_n(t))^T$ and $s(t) = (s_1(t), s_2(t), \ldots, s_n(t))^T$.

Definition 2 (see [47, 48]). The nonlinearity of $\tilde{h}(\cdot)$ is defined as follows:

$$\chi(\tilde{h}) = \frac{\|\tilde{h}(x) - \tilde{h}(y)\|_1}{\|x - y\|_1}, \tag{8}$$

where $x, y \in \mathbb{R}^n$.

In order to study the finite-time synchronization control for NCMWCNs with Markovian switching and time-varying delay, the following assumptions are needed.

Assumption 1. $\tilde{f}(\cdot)$ and $\tilde{h}(\cdot)$ of network (2) satisfy the Lipschitz conditions, i.e., there exist constants $L > 0$ and $\tilde{L} > 0$, such that $\|f(x) - f(y)\|_1 \leq L\|x - y\|_1$ and $\|\tilde{h}(x) - \tilde{h}(y)\|_1 \leq \tilde{L}\|x - y\|_1$.

Remark 4. From Definition 2, there is $\chi(\tilde{h}) = (\|\tilde{h}(x) - \tilde{h}(y)\|_1/\|x - y\|_1) \geq 0$. When x and y are fixed, $\chi(\tilde{h})$ is a positive proportion function with $\|\tilde{h}(x) - \tilde{h}(y)\|_1$. Hence, let $\tilde{H}(\Delta e) = \tilde{h}(x) - \tilde{h}(y)$, where $\Delta e = x - y$; then, $\chi(\tilde{h})$ decreases as $\|\tilde{h}(x) - \tilde{h}(y)\|_1$ decreases, and hence, $\|\tilde{H}(\Delta e)\|_1$ decreases as $\chi(\tilde{h})$ decreases. And we can conclude that $\chi(\tilde{h})$ can reflect the amplitude variation of the nonlinear function $\tilde{h}(\cdot)$.

Remark 5. In the finite-time controller $u_i(t, \tilde{r})$ (6), there is a *sign* function sgn. It is well known that traditional finite-time control techniques are based on sliding mode controllers, which utilize *sign* function and give rise to the phenomenon of chattering. How can we avoid this phenomenon in (6)? From [6–9], chatting will occur when the control in the addressed system adopts switching function. In [6], although the *sign* function in the switching control term was used, the switching control term can be softened to be a smooth signal by using low-pass filter technique. In [7], Tang addressed that some "smooth" functions must be used instead of the *sign* function in order to eliminate chatting of the sliding mode control system. Despite there is the *sign* function in the controller (6), the phenomenon of chatting cannot occur because the switching control term $\tilde{\beta}(\chi(\tilde{h}))\tilde{\eta}_i^{k,\tilde{r}}\hat{Q}^{(\alpha-1)/2}\text{diag}(\text{sgn}(\tilde{e}_i(t)))|\tilde{e}_i(t)|^{\alpha}$ is a smooth function when $\tilde{e}_i(t) > 0$ and $\tilde{e}_i(t) < 0$. The analysis is as follows: Assume that $\tilde{e}_i(t)$ satisfies $\|\lim_{\triangle\longrightarrow 0}(\tilde{e}_i(t + \triangle) - \tilde{e}_i(t))\| = C_i$ and $C_i > 0$; then, $\|(\text{d}\tilde{e}_i(t)/\text{d}t)\| = \|\lim_{\triangle\longrightarrow 0}\tilde{e}_i(t + \triangle) - \tilde{e}_i(t)/\triangle\| = \|\lim_{\triangle\longrightarrow 0}C_i/\triangle\| = +\infty$. Because functions $\tilde{F}(\tilde{e}_i(t))$ and $\tilde{H}(\tilde{e}_j(t - \tau(t)))$ satisfy Assumption 1, these two functions are bounded. Thus, combining (5), we can know that $\| - \sum_{k=1}^{m}\iota_i^{k,\tilde{r}}\tilde{e}_i(t) - \sum_{k} = 1^m\tilde{\beta}(\chi(\tilde{h}))\tilde{\eta}_i^{k,\tilde{r}}\hat{Q}^{(\tilde{\alpha}-1)/2}\text{diag}(\text{sgn}(\tilde{e}_i(t)))|\tilde{e}_i(t)|^{\tilde{\alpha}}\| \longrightarrow +\infty$, which shows that $|\tilde{e}_i(t)| \longrightarrow +\infty$ and $|\tilde{e}_i(t)|^{\tilde{\alpha}} \longrightarrow +\infty$. Obviously, this is wrong. In order to make $\|(\text{d}\tilde{e}_i(t)/\text{d}t)\|$ bounded, we have $\|(\text{d}\tilde{e}_i(t)/\text{d}t)\| = \|\lim_{\triangle\longrightarrow 0}(\tilde{e}_i(t + \triangle) - \tilde{e}_i(t)/\triangle)\| = \|\lim_{\triangle\longrightarrow 0}(C_i/\triangle)\| \leq \chi < +\infty$. Thus, $\|\lim_{\triangle\longrightarrow 0}(\tilde{e}_i(t + \triangle) - \tilde{e}_i(t))\| = 0$. This shows that $\tilde{e}_i(t)$ is a smooth function. Therefore, $\tilde{e}_i(t)^{\tilde{\alpha}}$ (in the case of $\tilde{e}_i(t) > 0$ and $\text{sgn}(\tilde{e}_i(t)) = 1$) and $-\tilde{e}_i(t)^{\tilde{\alpha}}$ (in the case of $\tilde{e}_i(t) < 0$ and $\text{sgn}(\tilde{e}_i(t)) = -1$) are smooth functions. Thus, when $\tilde{e}_i(t) > 0$ and $\tilde{e}_i(t) < 0$, we can conclude that $\tilde{\beta}(\chi(\tilde{h}))\tilde{\eta}_i^{k,\tilde{r}}\hat{Q}^{(\tilde{\alpha}-1)/2}\text{diag}(\text{sgn}(\tilde{e}_i(t)))|\tilde{e}_i(t)|^{\tilde{\alpha}}$ is a smooth function. Simply speaking, in the finite-time controller $u_i(t, \tilde{r})$ (6), there is no phenomenon of chattering.

Assumption 2. $\tau(t)$ satisfies $0 \leq \tau(t) \leq \tau_M$ and $0 \leq \dot{\tau}(t) \leq \rho < 1$.

Lemma 1 (see [59]). *Let $x(t)$ be an n-dimensional It $\hat{o}$ process on $t \geq 0$ with the following stochastic differential:*

$$\text{d}x(t) = f(x(t), t, \tilde{r}(t))\text{d}t + g(x(t), t, \tilde{r}(t))\text{d}B(t), \tag{9}$$

where $f\colon \mathbb{R}^n \times \mathbb{R}_+ \times \hat{S} \longrightarrow \mathbb{R}^n$ and $g\colon \mathbb{R}^n \times \mathbb{R}_+ \times \hat{S} \longrightarrow \mathbb{R}^{n\times m}$ are continuous differentiable functions. Let $B(t) = (B_t^1, B_t^2, \ldots, B_t^m)^T$ be an m-dimensional Brownian motion defined on the probability space. According to It $\hat{o}$ formula, if $V(x(t), t, p) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \hat{S}; \mathbb{R}_+)$, we define an operator $\mathscr{L}V$ from $\mathbb{R}^n \times \mathbb{R}_+ \times \hat{S}$ to $\mathbb{R}$ by

$$\begin{aligned}\mathscr{L}V(x(t), t, p) &= V_t(x(t), t, p) + V_x(x(t), t, r(t))f(x(t), t, p) \\ &\quad + \frac{1}{2}\text{trace}\left[g^T(x(t), t, p)V_{xx}g(x(t), t, p)\right] \\ &\quad + \sum_{q=1}^{\hat{s}}\delta_{pq}V(x(t), t, p),\end{aligned} \tag{10}$$

where $p, q \in \hat{S}$, $\tilde{r}(t)$ is a right-continuous Markov chain which is given in formula (3), $V_t(x(t), t, p) = (\partial V(x(t), t, p)/\partial t)$, $V_{xx}(x(t), t, p) = (\partial^2 V(x(t), t, p)/\partial x_i \partial x_j)_{n\times n}$, and $V_x(x(t), t, p) = ((\partial V(x(t), t, p)/\partial x_1), \ldots, (\partial V(x(t), t, p)/\partial x_n))$.

Lemma 2 (see [60]). *Let $\nu_1, \nu_2, \ldots, \nu_n \geq 0$ and $0 < p \leq 1$; then,*

$$\left(\sum_{i=1}^{n}\nu_i\right)^p \leq \sum_{i=1}^{n}\nu_i^p. \tag{11}$$

Lemma 3 (see [61]). *For $\forall x, y \in R^n$ and $\psi \in R^{n\times n} > 0$, then*

$$2x^T y \leq x^T\psi^{-1}x + y^T\psi y. \tag{12}$$

Lemma 4 (see [62]). *If a Lyapunov function $V(t)\colon [0, \infty) \longrightarrow [0, \infty)$ is differentiable (i.e., right derivative) and*

$$\frac{dV(t)}{dt} \leq -\kappa V^{\varsigma}(t), \tag{13}$$

where $\kappa > 0$ and $0 < \varsigma < 1$, $V(t)$ will reach zero at finite time $t^* \leq V^{1-\varsigma}(0)/\kappa(1-\varsigma)$ and $V(t) = 0$ for all $t \geq t^*$.

3. Main Results

In this section, we will focus on the sufficient conditions for ensuring that network (2) with controller (6) is finite-time synchronized. Furthermore, based on the designed controller (6), we will refine more feasible controllers.

Theorem 1. *Network (2) with controller (6) can achieve synchronization within finite-time t^* if Assumptions 1 and 2 and the following conditions hold:*

(i) If $q \neq p$, then $(\tilde{v}_q \hat{\theta}^q/1-\rho) - \hat{\varrho}_p \leq 0$; if $q = p$, then $(\tilde{v}_q \hat{\theta}^q/1-\rho) - \hat{\varrho}_p \geq 0$, where $p, q \in \hat{S}$, $\hat{\varrho}_q > 0$, $\tilde{v}_q \geq 1$, $\hat{\theta}^q > 0$, and $0 \leq \rho < 1$.

(ii) The following inequalities are satisfied:

$$\begin{aligned}\tilde{\Theta}_p^{(1)} = {} & \left(L^2\|\psi\|_2 + \tilde{L}^2\left\|\tilde{\psi}_p\right\|_2 \sum_{k=1}^m \tilde{a}_k\right) I_N \otimes I_N + \frac{\hat{\theta}^p}{1-\rho}\sum_{k=1}^m \hat{a}_k \hat{F}^k \\ & + I_N \otimes (\hat{Q}\tilde{A})\psi^{-1}(\hat{Q}\tilde{A})^T \\ & + \sum_{q=1}^{\hat{s}} \frac{\delta_{pq}\tilde{v}_q}{\tilde{v}_p} I_N \otimes \hat{Q} + \sum_{k=1}^m \tilde{a}_k\left(\tilde{G}_k^p \otimes \hat{Q}\tilde{\Gamma}_k\right)\tilde{\psi}_p^{-1}\left(\tilde{G}_k^p \otimes \hat{Q}\tilde{\Gamma}_k\right)^T \\ & + \sum_{k=1}^m \hat{a}_k\left(\hat{G}_k^p \otimes \hat{Q}\hat{\Gamma}_k\right)\hat{\psi}_p^{-1} \\ & \cdot\left(\hat{G}_k^p \otimes \hat{Q}\hat{\Gamma}_k\right)^T - 2\sum_{k=1}^m \Xi_p^k \otimes \hat{Q} \leq 0,\end{aligned} \tag{14}$$

$$\tilde{\Theta}_p^{(2)} = \sum_{k=1}^m \hat{a}_k\left(\tilde{L}^2\left\|\hat{\psi}_p\right\|_2 (I_N \otimes I_N) - \hat{\theta}^p \hat{F}^k\right) \leq 0, \tag{15}$$

where $p = 1, 2, \ldots, \hat{s}$, $\psi \in \mathbb{R}^{n\times n} > 0$, $\tilde{\psi}_p \in \mathbb{R}^{Nn\times Nn} > 0$, $\hat{\psi}_p \in \mathbb{R}^{Nn\times Nn} > 0$, $\Gamma_p^k = diag(\iota_1^{k,p}, \iota_2^{k,p}, \ldots, \iota_N^{k,p})$, $\hat{Q} = diag(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n) > 0$, and $\hat{F}^k = diag(\hat{F}_1^k, \hat{F}_2^k, \ldots, \hat{F}_N^k) \in \mathbb{R}^{Nn\times Nn} > 0$.

(iii) The settling finite time t^* satisfies $t^* \leq ([V(\tilde{e}(0), 0, \tilde{r}(0))]^{(1-\tilde{\alpha})/2}/\tilde{\beta}(\chi(\tilde{h}))m\hat{\eta}(1-\tilde{\alpha}))$, where $0 < \tilde{\alpha} < 1$, $\tilde{\beta}(\chi(\tilde{h})) > 0$, $\hat{\eta}^{k,p} = \min_{i\in\{1,2,\ldots,N\}}\{\hat{\eta}_i^{k,p}\} > 0$, $\hat{\eta}^p = \min_{k\in\{1,2,\ldots,m\}}\{\hat{\eta}^{k,p}\}$, $\hat{\eta} = \min\{\hat{\eta}^1, \ldots, \hat{\eta}^{\tilde{s}}\}$, $V(\tilde{e}(0), 0, \tilde{r}(0)) = \tilde{v}_{\tilde{r}(0)} \sum_{i=1}^N \tilde{e}_i^T(0)\hat{Q}e_i(0)$, and $\tilde{v}_{\tilde{r}(0)} \geq 1$.

Proof. Construct a Lyapunov functional for network (2) with controller (6) as follows:

$$\begin{aligned}V(\tilde{e}(t), t, p) = {} & \tilde{v}_p\left[\sum_{i=1}^N \tilde{e}_i^T(t)\hat{Q}\tilde{e}_i(t) + \frac{\hat{\theta}^p}{1-\rho}\sum_{k=1}^m \hat{a}_k\right. \\ & \left.\int_{t-\tau(t)}^t \tilde{e}^T(s)\hat{F}^k\tilde{e}(s)ds\right],\end{aligned} \tag{16}$$

where $p \in \hat{S}$, $\hat{a}_k > 0$, $\tilde{v}_p \geq 1$, $\hat{\theta}^p > 0$, $\hat{Q} = diag(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n) > 0$, and $\hat{F}^k = diag(F_1^k, F_2^k, \ldots, F_N^k) \in \mathbb{R}^{Nn\times Nn} > 0$.

Substituting (6) into error system (5) of network (2), we can obtain the closed-loop system of error system (5) as follows:

$$\begin{aligned}\mathscr{E}(\tilde{e}_i(t)) := {} & \tilde{A}\tilde{F}(\tilde{e}_i(t)) + \sum_{k=1}^m\sum_{j=1}^N \tilde{a}_k \tilde{g}_{ij}^{k,p}\tilde{\Gamma}_k\tilde{H}\left(\tilde{e}_j(t)\right) \\ & + \sum_{k=1}^m\sum_{j=1}^N \hat{a}_k \hat{g}_{ij}^{k,p}\hat{\Gamma}_k\tilde{H}\left(\tilde{e}_j(t-\tau(t))\right) - \sum_{k=1}^m \iota_i^{k,p}\tilde{e}_i(t) \\ & - \sum_{k=1}^m \tilde{\beta}(\tilde{L})\hat{\eta}_i^{k,p}\hat{Q}^{(\tilde{\alpha}-1/2)}\mathrm{diag}(\mathrm{sgn}(\tilde{e}_i(t)))\left|\tilde{e}_i(t)\right|^{\tilde{\alpha}} \\ & - m\hat{Q}^{-1}\sum_{k=1}^m \tilde{\beta}(\tilde{L})\hat{\eta}_i^{k,p} \\ & \cdot\left(\frac{\hat{a}_k\hat{\theta}^p}{1-\rho}\int_{t-\tau(t)}^t \tilde{e}_i^T(s)\hat{F}_i^k\tilde{e}_i(s)ds\right)^{(\tilde{\alpha}+1/2)} \frac{\tilde{e}_i(t)}{\left\|\tilde{e}_i(t)\right\|_1^2}.\end{aligned} \tag{17}$$

Then, from Lemma 1, we can compute the derivative $\mathscr{L}V(\tilde{e}(t), t, p)$ along the trajectory of closed-loop system (17) as follows:

$$\begin{aligned}\mathscr{L}V(\tilde{e}(t), t, p) = {} & V_t(\tilde{e}(t), t, p) + V_{\tilde{e}(t)}(\tilde{e}(t), t, p)\left[\tilde{A}\tilde{F}(\tilde{e}_i(t)) + \sum_{k=1}^m\sum_{j=1}^N \tilde{a}_k\tilde{g}_{ij}^{k,p}\tilde{\Gamma}_k\tilde{H}\left(\tilde{e}_j(t)\right)\right. \\ & + \sum_{k=1}^m\sum_{j=1}^N \hat{a}_k\hat{g}_{ij}^{k,p}\hat{\Gamma}_k\tilde{H}\left(\tilde{e}_j(t-\tau(t))\right) - \sum_{k=1}^m \iota_i^{k,p}\tilde{e}_i(t) - \sum_{k=1}^m \tilde{\beta}(\tilde{L})\hat{\eta}_i^{k,p}\hat{Q}^{(\tilde{\alpha}-1/2)}\mathrm{diag}(\mathrm{sgn}(\tilde{e}_i(t)))\left|\tilde{e}_i(t)\right|^{\tilde{\alpha}} \\ & \left. - m\hat{Q}^{-1}\sum_{k=1}^m \tilde{\beta}(\tilde{L})\hat{\eta}_i^{k,p}\left(\frac{\hat{a}_k\hat{\theta}^p}{1-\rho}\int_{t-\tau(t)}^t \tilde{e}_i^T(s)\hat{F}_i^k\tilde{e}_i(s)ds\right)^{(\tilde{\alpha}+1/2)}\frac{\tilde{e}_i(t)}{\left\|\tilde{e}_i(t)\right\|_1^2}\right] + \sum_{q=1}^{\hat{s}}\delta_{pq}(V\tilde{e}(t), t, q).\end{aligned} \tag{18}$$

From Assumption 2, we have

$$\begin{aligned} V_t(\tilde{e}(t), t, p) &= \frac{\tilde{v}_p \hat{\theta}^p}{1-\rho} \sum_{k=1}^{m} \hat{a}_k \Big[\tilde{e}^T(t) \hat{F}^k \tilde{e}(t) - (1-\dot{\tau}(t)) \tilde{e}^T(t-\tau(t)) \hat{F}^k \tilde{e}(t-\tau(t)) \Big] \\ &\le \frac{\tilde{v}_p \hat{\theta}^p}{1-\rho} \sum_{k=1}^{m} \hat{a}_k \tilde{e}^T(t) \hat{F}^k \tilde{e}(t) - \tilde{v}_p \hat{\theta}^p \sum_{k=1}^{m} \hat{a}_k \tilde{e}^T(t-\tau(t)) \hat{F}^k \tilde{e}(t-\tau(t)). \end{aligned} \tag{19}$$

By Lemma 3 and Assumption 1, we obtain

$$\begin{aligned} V_{\tilde{e}(t)}(\tilde{e}(t), t, p) \tilde{A} \tilde{F}(\tilde{e}_i(t)) &= 2\tilde{v}_p \sum_{i=1}^{N} \tilde{e}_i^T(t) \hat{Q} \tilde{A} \tilde{F}(\tilde{e}_i(t)) \\ &\le \tilde{v}_p \sum_{i=1}^{N} \Big[\tilde{e}_i^T(t) (\hat{Q}\tilde{A}) \psi^{-1} (\hat{Q}\tilde{A})^T \tilde{e}_i(t) + \tilde{F}^T(\tilde{e}_i(t)) \psi \tilde{F}(\tilde{e}_i(t)) \Big] \\ &\le \tilde{v}_p \sum_{i=1}^{N} \Big[\tilde{e}_i^T(t) (\hat{Q}\tilde{A}) \psi^{-1} (\hat{Q}\tilde{A})^T \tilde{e}_i(t) + L^2 \|\psi\|_2 \tilde{e}_i^T(t) \tilde{e}_i(t) \Big], \end{aligned} \tag{20}$$

$$\begin{aligned} V_{\tilde{e}(t)}(\tilde{e}(t), t, p) \sum_{k=1}^{m} \sum_{j=1}^{N} \tilde{a}_k \tilde{g}_{ij}^{k,p} \tilde{\Gamma}_k \tilde{H}\big(\tilde{e}_j(t)\big) &= 2\tilde{v}_p \sum_{i=1}^{N} \tilde{e}_i^T(t) \hat{Q} \sum_{k=1}^{m} \sum_{j=1}^{N} \tilde{a}_k \tilde{g}_{ij}^{k,p} \tilde{\Gamma}_k \tilde{H}\big(\tilde{e}_j(t)\big) \\ &= 2\tilde{v}_p \sum_{k=1}^{m} \tilde{a}_k \Big[\big(\tilde{G}_k^p \otimes \hat{Q}\tilde{\Gamma}_k \big)^T \tilde{e}(t) \Big]^T \tilde{H}(\tilde{e}(t)) \\ &\le \tilde{v}_p \sum_{k=1}^{m} \tilde{a}_k \Big[\tilde{e}^T(t) \big(\tilde{G}_k^p \otimes \hat{Q}\tilde{\Gamma}_k \big) \tilde{\psi}_p^{-1} \big(\tilde{G}_k^p \otimes \hat{Q}\tilde{\Gamma}_k \big)^T \tilde{e}(t) + \tilde{H}^T(\tilde{e}(t)) \tilde{\psi}_p \tilde{H}(\tilde{e}(t)) \Big] \\ &\le \tilde{v}_p \sum_{k=1}^{m} \tilde{a}_k \Big[\tilde{e}^T(t) \big(\tilde{G}_k^p \otimes \hat{Q}\tilde{\Gamma}_k \big) \tilde{\psi}_p^{-1} \big(\tilde{G}_k^p \otimes \hat{Q}\tilde{\Gamma}_k \big)^T \tilde{e}(t) + \tilde{L}^2 \big\|\tilde{\psi}_p\big\|_2 \tilde{e}^T(t) \tilde{e}(t) \Big], \end{aligned} \tag{21}$$

$$\begin{aligned} V_{\tilde{e}(t)}(\tilde{e}(t), t, p) \sum_{k=1}^{m} \sum_{j=1}^{N} \hat{a}_k \tilde{g}_{ij}^{k,p} \hat{\Gamma}_k \tilde{H}\big(\tilde{e}_j(t-\tau(t))\big) &= 2\tilde{v}_p \sum_{i=1}^{N} \tilde{e}_i^T(t) \hat{Q} \sum_{k=1}^{m} \sum_{j=1}^{N} \hat{a}_k \tilde{g}_{ij}^{k,p} \hat{\Gamma}_k \tilde{H}\big(\tilde{e}_j(t-\tau(t))\big) \\ &= 2\tilde{v}_p \sum_{k=1}^{m} \hat{a}_k \Big[\big(\tilde{G}_k^p \otimes \hat{Q}\hat{\Gamma}_k \big)^T \tilde{e}(t) \Big]^T \tilde{H}(\tilde{e}(t-\tau(t))) \\ &\le \tilde{v}_p \sum_{k=1}^{m} \hat{a}_k \Big[\tilde{e}^T(t) \big(\tilde{G}_k^p \otimes \hat{Q}\hat{\Gamma}_k \big) \hat{\psi}_p^{-1} \big(\tilde{G}_k^p \otimes \hat{Q}\hat{\Gamma}_k \big)^T \tilde{e}(t) + \tilde{H}^T(\tilde{e}(t-\tau(t))) \hat{\psi}_p \tilde{H}(\tilde{e}(t-\tau(t))) \Big] \\ &\le \tilde{v}_p \sum_{k=1}^{m} \hat{a}_k \Big[\tilde{e}^T(t) \big(\tilde{G}_k^p \otimes \hat{Q}\hat{\Gamma}_k \big) \hat{\psi}_p^{-1} \big(\tilde{G}_k^p \otimes \hat{Q}\hat{\Gamma}_k \big)^T \tilde{e}(t) + \tilde{L}^2 \big\|\hat{\psi}_p\big\|_2 \tilde{e}^T(t-\tau(t)) \tilde{e}(t-\tau(t)) \Big], \end{aligned} \tag{22}$$

where

$$\begin{aligned}
\tilde{e}(t) &= \left[\tilde{e}_1^T(t), \tilde{e}_2^T(t), \ldots, \tilde{e}_N^T(t)\right]^T, \\
\tilde{e}_i(t) &= \left[\tilde{e}_1(t), \tilde{e}_2(t), \ldots, \tilde{e}_n(t)\right]^T, \\
\tilde{H}(\tilde{e}(t)) &= \left[\tilde{H}^T(\tilde{e}_1(t)), \tilde{H}^T(\tilde{e}_2(t)), \ldots, \tilde{H}^T(\tilde{e}_N(t))\right]^T, \\
\tilde{H}(\tilde{e}_i(t)) &= \left[\tilde{H}(\tilde{e}_1(t)), \tilde{H}(\tilde{e}_2(t)), \ldots, \tilde{H}(\tilde{e}_n(t))\right]^T, \\
\tilde{H}(\tilde{e}(t-\tau(t))) &= \left[\tilde{H}^T(\tilde{e}_1(t-\tau(t))), \tilde{H}^T(\tilde{e}_2(t-\tau(t))), \ldots, \tilde{H}^T(\tilde{e}_N(t-\tau(t)))\right]^T, \\
\tilde{H}(\tilde{e}_i(t-\tau(t))) &= \left[\tilde{H}(\tilde{e}_1(t-\tau(t))), \tilde{H}(\tilde{e}_2(t-\tau(t))), \ldots, \tilde{H}(\tilde{e}_n(t-\tau(t)))\right]^T, \\
-V_{\tilde{e}(t)}(\tilde{e}(t), t, p)\sum_{k=1}^{m}\iota_i^{k,p}\tilde{e}_i(t) &= -2\tilde{\upsilon}_p\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}\sum_{k=1}^{m}\iota_i^{k,p}\tilde{e}_i(t) = -\tilde{\upsilon}_p\tilde{e}^T(t)\left(2\sum_{k=1}^{m}\Xi_p^k\otimes\hat{Q}\right)\tilde{e}(t),
\end{aligned} \tag{23}$$

where $\Xi_p^k = \mathrm{diag}(\iota_1^{k,p}, \iota_2^{k,p}, \ldots, \iota_N^{k,p})$.

Because $\sum_{p\in\hat{S},q=1,2,\ldots,\hat{s}}^{\hat{s}}\delta_{pq} = 0$, then $\sum_{p\in\hat{S},q=1,2,\ldots,\hat{s}}^{\hat{s}}\delta_{pq} = 0$, where $\hat{\varrho}_p > 0$. Thus, by condition (I) of Theorem 1, we get

$$\begin{aligned}
\sum_{q=1}^{\hat{s}}\delta_{pq}V(\tilde{e}(t), t, q) &= \sum_{q=1}^{\hat{s}}\delta_{pq}\tilde{\upsilon}_q\left[\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}\tilde{e}_i(t) + \frac{\hat{\theta}^q}{1-\rho}\sum_{k=1}^{m}\hat{a}_k\int_{t-\tau(t)}^{t}\tilde{e}^T(s)\hat{F}^k\tilde{e}(s)\mathrm{d}s\right] \\
&= \sum_{q=1}^{\hat{s}}\delta_{pq}\tilde{\upsilon}_q\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}\tilde{e}_i(t) + \sum_{q=1}^{\hat{s}}\delta_{pq}\left(\frac{\tilde{\upsilon}_q\hat{\theta}^q}{1-\rho} - \hat{\varrho}_p\right)\sum_{k=1}^{m}\hat{a}_k\int_{t-\tau(t)}^{t}\tilde{e}^T(s)\hat{F}^k\tilde{e}(s)\mathrm{d}s \\
&\le \sum_{q=1}^{\hat{s}}\delta_{pq}\tilde{\upsilon}_q\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}\tilde{e}_i(t).
\end{aligned} \tag{24}$$

Let $\hat{\eta}^{k,p} = \min_{i\in\{1,2,\ldots,N\}}\{\hat{\eta}_i^{k,p}\}$ and $\hat{\eta}^p = \min_{k\in\{1,2,\ldots,m\}}\{\hat{\eta}^{k,p}\}$; then, according to Lemma 2, we can obtain

$$\begin{aligned}
&-V_{\tilde{e}(t)}(\tilde{e}(t), t, p)\sum_{k=1}^{m}\tilde{\beta}(\chi(\tilde{h}))\hat{\eta}_i^{k,p}\hat{Q}^{(\tilde{\alpha}-1/2)}\mathrm{diag}(\mathrm{sgn}(\tilde{e}_i(t)))|\tilde{e}_i(t)|^{\tilde{\alpha}} \\
&= -2\tilde{\upsilon}_p\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}\sum_{k=1}^{m}\tilde{\beta}(\chi(\tilde{h}))\hat{\eta}_i^{k,p}\hat{Q}^{(\tilde{\alpha}-1/2)}\mathrm{diag}(\mathrm{sgn}(\tilde{e}_i(t)))|\tilde{e}_i(t)|^{\tilde{\alpha}} \\
&\le -2\tilde{\upsilon}_p\sum_{k=1}^{m}\tilde{\beta}(\chi(\tilde{h}))\hat{\eta}^{k,p}\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}^{(1+\tilde{\alpha}/2)}\mathrm{diag}(\mathrm{sgn}(\tilde{e}_i(t)))|\tilde{e}_i(t)|^{\tilde{\alpha}} \\
&\quad -2\tilde{\beta}(\chi(\tilde{h}))\sum_{k=1}^{m}\hat{\eta}^{k,p}\left(\tilde{\upsilon}_p\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}\tilde{e}_i(t)\right)^{(1+\tilde{\alpha}/2)} \\
&\le -2\tilde{\beta}(\chi(\tilde{h}))m\hat{\eta}^p\left(\tilde{\upsilon}_p\sum_{i=1}^{N}\tilde{e}_i^T(t)\hat{Q}\tilde{e}_i(t)\right)^{(1+\tilde{\alpha}/2)},
\end{aligned} \tag{25}$$

$$
\begin{aligned}
&- V_{\widetilde{e}(t)}(\widetilde{e}(t),t,p)m\widehat{Q}^{-1}\sum_{k=1}^{m}\widetilde{\beta}(\chi(\widetilde{h}))\widehat{\eta}_i^{k,p}\left(\frac{\widehat{a}_k\widehat{\theta}^p}{1-\rho}\int_{t-\tau(t)}^{t}\widetilde{e}_i^T(s)\widehat{F}_i^k\widetilde{e}_i(s)\mathrm{d}s\right)^{(\widetilde{\alpha}+1/2)}\frac{\widetilde{e}_i(t)}{\left\|\widetilde{e}_i(t)\right\|_1^2}\\
&\le -2\widetilde{\beta}(\chi(\widetilde{h}))m\sum_{k=1}^{m}\widehat{\eta}^{k,p}\left[\sum_{i=1}^{N}\left(\frac{\widetilde{v}_p\widehat{a}_k\widehat{\theta}^p}{1-\rho}\int_{t-\tau(t)}^{t}\widetilde{e}_i^T(s)\widehat{F}_i^k\widetilde{e}_i(s)\mathrm{d}s\right)^{(\widetilde{\alpha}+1/2)}\right]\\
&\le -2\widetilde{\beta}(\chi(\widetilde{h}))m\widehat{\eta}^p\sum_{k=1}^{m}\left(\frac{\widetilde{v}_p\widehat{a}_k\widehat{\theta}^p}{1-\rho}\int_{t-\tau(t)}^{t}\widetilde{e}^T(s)\widehat{F}^k\widetilde{e}(s)\mathrm{d}s\right)^{(\widetilde{\alpha}+1/2)}\\
&\le -2\widetilde{\beta}(\chi(\widetilde{h}))m\widehat{\eta}^p\left(\frac{\widetilde{v}_p\widehat{\theta}^p}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\int_{t-\tau(t)}^{t}\widetilde{e}^T(s)\widehat{F}^k\widetilde{e}(s)\mathrm{d}s\right)^{(\widetilde{\alpha}+1/2)}.
\end{aligned}
\tag{26}
$$

Combining Lemma 2 and inequalities (25) and (26), we can obtain

$$
\begin{aligned}
&-2\widetilde{\beta}(\chi(\widetilde{h}))m\widehat{\eta}^p\left[\left(\widetilde{v}_p\sum_{i=1}^{N}\widetilde{e}_i^T(t)\widehat{Q}\widetilde{e}_i(t)\right)^{(1+\widetilde{\alpha}/2)}+\left(\frac{\widetilde{v}_p\widehat{\theta}^p}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\int_{t-\tau(t)}^{t}\widetilde{e}^T(s)\widehat{F}^k\widetilde{e}(s)\mathrm{d}s\right)^{(1+\widetilde{\alpha}/2)}\right]\\
&\le -2\widetilde{\beta}(\chi(\widetilde{h}))m\widehat{\eta}^p\left(\widetilde{v}_p\sum_{i=1}^{N}\widetilde{e}_i^T(t)\widehat{Q}\widetilde{e}_i(t)+\frac{\widetilde{v}_p\widehat{\theta}^p}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\int_{t-\tau(t)}^{t}\widetilde{e}^T(s)\widehat{F}^k\widetilde{e}(s)\mathrm{d}s\right)^{(1+\widetilde{\alpha}/2)}\\
&\le -2\widetilde{\beta}(\chi(\widetilde{h}))m\widehat{\eta}(V(\widetilde{e}(t),t,p))^{(1+\widetilde{\alpha}/2)},
\end{aligned}
\tag{27}
$$

where $\widehat{\eta}=\min\left\{\widehat{\eta}^1,\ldots,\widehat{\eta}^{\widetilde{s}}\right\}$.

Substituting (19)–(24) and (27) into (18), we can get

$$
\mathscr{L}V(\widetilde{e}(t),t,p)\le\mathscr{L}V^{(1)}(\widetilde{e}(t),t,p)+\mathscr{L}V^{(2)}(\widetilde{e}(t),t,p), \tag{28}
$$

where

$$
\mathscr{L}V^{(1)}(\widetilde{e}(t),t,p)=\widetilde{v}_p\left[\widetilde{e}^T(t)\widetilde{\Theta}_p^{(1)}\widetilde{e}(t)+\widetilde{e}^T(t-\tau(t))\widetilde{\Theta}_p^{(2)}\widetilde{e}(t-\tau(t))\right], \tag{29}
$$

$$
\begin{aligned}
\widetilde{\Theta}_p^{(1)}={}&\left(L^2\|\psi\|_2+\widetilde{L}^2\left\|\widetilde{\psi}_p\right\|_2\sum_{k=1}^{m}\widetilde{a}_k\right)I_N\otimes I_N+\frac{\widehat{\theta}^p}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\widehat{F}^k+I_N\otimes(\widehat{Q}\widetilde{A})\psi^{-1}(\widehat{Q}\widetilde{A})^T\\
&+\sum_{q=1}^{\widehat{s}}\frac{\delta_{pq}\widetilde{v}_q}{\widetilde{v}_p}I_N\otimes\widehat{Q}\sum_{k=1}^{m}\widetilde{a}_k\left(\widetilde{G}_k^p\otimes\widehat{Q}\widetilde{\Gamma}_k\right)\widetilde{\psi}_p^{-1}\left(\widetilde{G}_k^p\otimes\widehat{Q}\widetilde{\Gamma}_k\right)^T\\
&+\sum_{k=1}^{m}\widehat{a}_k\left(\widetilde{G}_k^p\otimes\widehat{Q}\widetilde{\Gamma}_k\right)\widehat{\psi}_p^{-1}\left(\widetilde{G}_k^p\otimes\widehat{Q}\widetilde{\Gamma}_k\right)^T-2\sum_{k=1}^{m}\Xi_p^k\otimes\widehat{Q},
\end{aligned}
\tag{30}
$$

$$
\widetilde{\Theta}_p^{(2)}=\sum_{k=1}^{m}\widehat{a}_k\left(\widetilde{L}^2\left\|\widehat{\psi}_p\right\|_2(I_N\otimes I_N)-\widehat{\theta}^p\widehat{F}^k\right), \tag{31}
$$

$$
\mathscr{L}V^{(2)}(\widetilde{e}(t),t,p)=-2\widetilde{\beta}(\chi(\widetilde{h}))m\widehat{\eta}(V(\widetilde{e}(t)),t,p))^{(\widetilde{\alpha}+1/2)}. \tag{32}
$$

Thus, taking the expectation on both sides of (28) and using condition (II) of Theorem 1, we have

$$\begin{aligned}\mathbb{E}[\mathscr{L}V(\tilde{e}(t),t,\tilde{r})] &\le \mathbb{E}\left[\mathscr{L}V^{(1)}(\tilde{e}(t),t,\tilde{r})\right] \\ &\quad + \mathbb{E}\left[\mathscr{L}V^{(2)}(\tilde{e}(t),t,\tilde{r})\right] \\ &\le \mathbb{E}\left[-2\tilde{\beta}(\chi(\tilde{h}))m\tilde{\eta}(V(\tilde{e}(t),t,\tilde{r}))^{(\tilde{\alpha}+1/2)}\right],\end{aligned} \tag{33}$$

where $\tilde{r}$ is a right-continuous Markov chain with known transition rate δ_{pq} which is located in formula (3) above.

According to Lemma 2 and inequality (33), we get

$$\begin{aligned}\mathbb{E}[\mathscr{L}V(\tilde{e}(t),t,\tilde{r})] &\le \mathbb{E}\left[-2\tilde{\beta}(\chi(\tilde{h}))m\tilde{\eta}(V(\tilde{e}(t),t,\tilde{r}))^{(\tilde{\alpha}+1/2)}\right] \\ &\le -2\tilde{\beta}(\chi(\tilde{h}))m\tilde{\eta}\{\mathbb{E}[V(\tilde{e}(t),t,\tilde{r})]\}^{(\tilde{\alpha}+1/2)}.\end{aligned} \tag{34}$$

Then, there is a real number $T > 0$ such that

$$V(\tilde{e}(t),t,\tilde{r}) \le \left[V(\tilde{e}(0),0,\tilde{r}(0))^{(1-\tilde{\alpha}/2)} - 2\tilde{\beta}(\chi(\tilde{h}))m\tilde{\eta}\frac{1-\tilde{\alpha}}{2}t\right]^{(2/1-\tilde{\alpha})}, \quad \forall t \in [0,T]. \tag{35}$$

Moreover, by Lemma 4 and inequality (35), T can be chosen as

$$T := t^* \le \frac{[V(\tilde{e}(0),0,\tilde{r}(0))]^{(1-\tilde{\alpha}/2)}}{\tilde{\beta}(\chi(\tilde{h}))m\tilde{\eta}(1-\tilde{\alpha})} := T_0. \tag{36}$$

Indeed, it is obvious that $V(\tilde{e}(t),t,\tilde{r}) = 0$ at $t = T_0$. If $t^* > T_0$, then $V(\tilde{e}(t),t*,\tilde{r}) = 0$ from the definition of t^* in (35), which contradicts (34) because $V(\tilde{e}(t),T_0,\tilde{r}) = 0$.

Therefore, if $t \ge t^*$, then $\mathbb{E}(V(\tilde{e}(t),t,\tilde{r})) = 0$. Combining equality (16) and Definition 1, we can know that if $t \ge t^*$, then $\|\tilde{y}_i(t) - s(t)\|_1 = 0$, where $i = 1,2,\ldots,N$. Hence, within finite time t^*, network (2) with controller (6) can achieve synchronization. □

Remark 6. From the proof of Theorem 1 and inequality (26), we can obtain

$$\sum_{q=1}^{\hat{s}} \delta_{pq}\left(\frac{\tilde{v}_q\hat{\theta}^q}{1-\rho} - \hat{\varrho}_p\right)\sum_{k=1}^{m}\hat{a}_k\int_{t-\tau(t)}^{t}\tilde{e}^T(s)\hat{F}^k\tilde{e}(s)\mathrm{d}s \le 0, \tag{37}$$

by condition (I). Here, $p,q \in \{1,2,\ldots,\hat{s}\}$, $\tilde{v}_q \ge 1$, $\hat{\theta}^q > 0$, $\hat{a}_k > 0$, and $\hat{F}^k > 0$.

Thus, the item

$$\sum_{q=1}^{\hat{s}}\frac{\delta_{pq}\tilde{v}_q\hat{\theta}^q}{1-\rho}\sum_{k=1}^{m}\hat{a}_k\int_{t-\tau(t)}^{t}\tilde{e}^T(s)\hat{F}^k\tilde{e}(s)\mathrm{d}s. \tag{38}$$

in $\sum_{q=1}^{\hat{s}}\delta_{pq}V(\tilde{e}(t),t,q)$ can be removed. Otherwise, if condition (I) in Theorem 1 is invalid, there must exist $\sum_{q=1}^{\hat{s}}(\delta_{pq}\tilde{v}_q\hat{\theta}^q/1-\rho)\sum_{k=1}^{m}\hat{a}_k\int_{t-\tau(t)}^{t}\tilde{e}^T(s)\hat{F}^k\tilde{e}(s)\mathrm{d}s$ in $\mathscr{L}V(\tilde{e}(t),t,p)$, which makes it difficult to obtain the settling finite time t^* given in condition (III) of Theorem 1.

According to condition (II) of Theorem 1, parameter $\iota_i^{k,\tilde{r}} > 0$ and the matrices $\hat{F}_i^k > 0$ and $\hat{Q} > 0$ of controller (6) are designed, where $\tilde{r} = 1,2,\ldots,\hat{s}$, $k = 1,2,\ldots,m$ and $i = 1,2,\ldots,N$. The settling finite time t^* can be obtained from condition (III) of Theorem 1, where $0 < \tilde{\alpha} < 1$, $\tilde{\beta}(\chi(\tilde{h})) > 0$, and $\tilde{\eta}^{k,p} = \min_{i\in\{1,2,\ldots,N\}}\{\hat{\eta}_i^{k,p}\} > 0$.

In addition, it is worthy to point out that when condition (I) of Theorem 1 is used to process the item $\sum_{q=1}^{\hat{s}}(\delta_{pq}\tilde{v}_q\hat{\theta}^q/1-\rho)\sum_{k=1}^{m}\hat{a}_k\int_{t-\tau(t)}^{t}\tilde{e}^T(s)\hat{F}^k\tilde{e}(s)\mathrm{d}s$, $0 \le \rho < 1$, $\hat{\theta}^q > 0$, and $\tilde{v}_q \ge 1$ must hold.

Remark 7. How to eliminate the synchronization error $\tilde{e}(t)$? Actually, if the synchronization error $\tilde{e}(t)$ is eliminated, then $\tilde{e}(t) = 0$. According to the proof of Theorem 1, we can use the following four steps to derive the synchronization error $\tilde{e}(t)$:

(i) Step 1: the Lyapunov functional $V(\tilde{e}(t),t,p)$ for network (2) with controller (6) is constructed, where $V(\tilde{e}(t),t,p) \ge 0$, $p \in \{1,2,\ldots,\hat{s}\}$

(ii) Step 2: by using generalised Itô formula, $\mathscr{L}V(\tilde{e}(t),t,p)$ is derived

(iii) Step 3: some inequality techniques are utilized to make

$$\begin{aligned}\mathscr{L}V(\tilde{e}(t),t,p) &\le -2\tilde{\beta}(\chi(\tilde{h}))m\tilde{\eta}(V(\tilde{e}(t),t,p)^{(\tilde{\alpha}+1/2)} \\ \mathbb{E}[\mathscr{L}V(\tilde{e}(t),t,p)] &\le -2\tilde{\beta}(\chi(\tilde{h}))m\tilde{\eta}\{\mathbb{E}(V(\tilde{e}(t),t,p)\}^{(\tilde{\alpha}+1/2)},\end{aligned} \tag{39}$$

hold, where $\tilde{r} = 1,2,\ldots,\hat{s}$, $\tilde{\beta}(\chi(\tilde{h})) > 0$, $m > 0$, and $\tilde{\eta} > 0$.

(iv) Step 4: according to finite-time stability theory given in Lemma 4, the settling finite time t^* can be obtained.

Remark 8. In the design procedure of controller (6), there exist some constraints which include $\iota_i^{k,\tilde{r}} > 0$, $\hat{\eta}_i^{k,\tilde{r}} > 0$, $\hat{\theta}^{\tilde{r}} > 0$, $\tilde{\beta}(\chi(\tilde{h})) > 0$, $\hat{Q} = \mathrm{diag}(\hat{q}_1,\hat{q}_2,\ldots,\hat{q}_n) > 0$, $\hat{F}_i^k \in R^{n\times n} > 0$, $0 < \tilde{\alpha} < 1$, $0 \le \rho < 1$, $\hat{a}_k > 0$, and m is a positive integer. If $0 \le t < \tau(t)$ and $\|\tilde{e}_i(t)\|_1 = 0$ hold, there exist $\int_{t-\tau(t)}^{t}\tilde{e}_i^T(s)\hat{F}_i^k\tilde{e}_i(s)\mathrm{d}s = 0$ and $u_i(t,r) = 0$.

It should be noted that the constraints above in controller (6) are necessary. If not, Theorem 1 may not hold. For example, in order to use Lemma 4 to derive the settling finite time t^*, there must be $0 < \tilde{\alpha} < 1$. If the parameter $\tilde{\alpha}$ does not satisfy $0 < \tilde{\alpha} < 1$, it is obvious that t^* located in inequality (36) cannot be derived by Lemma 4. In addition, according to Theorem 1, the control parameters of controller (6) can be chosen and designed. For instance, for the given network (2), by choosing the control parameters $\iota_i^{k,\tilde{r}} > 0$, $\hat{\eta}_i^{k,\tilde{r}} > 0$, $\hat{\theta}^{\tilde{r}} > 0$, $\hat{Q} = \mathrm{diag}(\hat{q}_1,\hat{q}_2,\ldots,\hat{q}_n) > 0$, $\hat{F}_i^k \in R^{n\times n} > 0$, $0 \le \rho < 1$, and $\tilde{r} = 1,2,\ldots,\hat{s}$, it is easy to realize conditions (I) and (II) of Theorem 1. Furthermore, it can also be seen that for network (2), there exist many solutions of controller (6) designed by Theorem 1 and these solutions can make conditions (I)–(III) hold.

Remark 9. In controller (6), if Theorem 1 holds, there must be $\mathscr{L}V^{(1)}(\tilde{e}(t), t, p) \leq 0$ and

$$\mathscr{L}V(\tilde{e}(t), t, p) \leq \mathscr{L}V^{(1)}(\tilde{e}(t), t, p) + \mathscr{L}V^{(2)}(\tilde{e}(t), t, p) \leq -2\tilde{\beta}(\chi(\tilde{h}))m\hat{\eta}\{V(\tilde{e}(t), t, p)\}^{(\tilde{\alpha}+1/2)}, \quad (40)$$

where $p \in \{1, 2, \ldots, \hat{s}\}$. Thus, it is derived that

$$\mathbb{E}\left[\mathscr{L}V^{(1)}(\tilde{e}(t), t, \tilde{r})\right] \leq 0$$
$$\mathbb{E}[\mathscr{L}V(\tilde{e}(t), t, p)] \leq -2\tilde{\beta}(\chi(\tilde{h}))m\hat{\eta}\{\mathbb{E}[V(\tilde{e}(t), t, p)]\}^{(\tilde{\alpha}+1/2)}. \quad (41)$$

Considering Theorem 1 together with equalities (29)–(32) and (36), we can draw the following results:

(i) The term $-\sum_{k=1}^{m} \iota_i^{k,\tilde{r}}\tilde{e}_i(t)$ is a global synchronization negative feedback control term, where $\tilde{r} = 1, 2, \ldots, \hat{s}$.

(ii) The terms

$$-\sum_{k=1}^{m}\tilde{\beta}(\chi(\tilde{h}))\hat{\eta}_i^{k,\tilde{r}}\hat{Q}^{(\tilde{\alpha}-1/2)}\operatorname{diag}(\operatorname{sgn}(\tilde{e}_i(t)))\left|\tilde{e}_i(t)\right|^{\tilde{\alpha}}$$
$$-m\hat{Q}^{-1}\sum_{k=1}^{m}\tilde{\beta}(\chi(\tilde{h}))\hat{\eta}_i^{k,\tilde{r}}\left(\frac{\hat{a}_k\tilde{\hat{\theta}}^{\tilde{r}}}{1-\rho}\int_{t-\tau(t)}^{t}\tilde{e}_i^T(s)\hat{F}_i^k\tilde{e}_i(s)\mathrm{d}s\right)^{(\tilde{\alpha}+1/2)}\frac{\tilde{e}_i(t)}{\left\|\tilde{e}_i(t)\right\|_1^2}, \quad (42)$$

represent finite-time synchronization negative feedback control term, where $0 \leq \dot{\tau}(t) \leq \rho < 1$ and the matrix $\hat{Q}$ is a positive definite matrix of network (2).

(iii) Parameters $\iota_i^{k,\tilde{r}} > 0$ and $\hat{\theta}^{\tilde{r}} > 0$ and matrix $\hat{F}_i^k \in \mathbb{R}^{n\times n} > 0$ are global synchronization negative feedback control parameter and matrix.

(iv) Parameter $\tilde{\hat{\theta}}^{\tilde{r}} > 0$ and matrix $\hat{F}_i^k \in \mathbb{R}^{n\times n} > 0$ are the global synchronization positive feedback control parameter. From (14) and (15), we can obtain that the designed parameter $\tilde{\hat{\theta}}^{\tilde{r}}$ and matrix $\hat{F}_i^k$ should first make inequality (15) hold. Then, some feasible values of parameter $\iota_i^{k,\tilde{r}}$ are chosen to make (14) hold.

(v) Parameters $\tilde{\beta}(\chi(\tilde{h})) > 0$, $\hat{\eta}_i^{k,\tilde{r}} > 0$, and $0 < \tilde{\alpha} < 1$ are finite-time synchronization negative feedback control parameters.

Remark 10. Now we give the steps of designing controller (6) as follows:

(i) Step 1: according to Markov chain with the known transition rate and network (2), one can obtain the values of $\hat{s}$, δ_{pq}, m, $\tilde{a}_k$, $\hat{a}_k$, $\tilde{g}_{ij}^{k,r}\tau(t)$, $\tilde{A}$, $\tilde{\Gamma}_k$, and $\hat{\Gamma}_k$

(ii) Step 2: the values of $\tilde{v}_p \geq 1$ and $\hat{Q} > 0$ are chosen

(iii) Step 3: combining Assumptions 1 and 2 and network (2), one has the values of L, $\tilde{L}$, ψ, $\tilde{\psi}_p$, τ_M, and $0 \leq \dot{\tau}(t) \leq \rho < 1$

(iv) Step 4: by inequality (15) of condition (II) in Theorem 1, $\tilde{\hat{\theta}}^p > 0$ and $\hat{F}_i^k > 0$ are designed

(v) Step 5: from inequality (14) of condition (I) in Theorem 1, one gets the value of $\iota_i^{k,\tilde{r}} > 0$

(vi) Step 6: combining steps 2–4 and condition (I) of Theorem 1, one can obtain the value of $\hat{\varrho}_p$

(vii) Step 7: combined with condition (III) of Theorem 1, t^* is estimated, where the values of $0 < \tilde{\alpha} < 1$ and $\hat{\eta}_i^{k,p} > 0$ are chosen. Thus, the design of controller (6) is completed

In the steps above, there is $p = 1, 2, \ldots, \hat{s}$.

Remark 11. The nonlinearity of nonlinear coupling function may impact synchronization dynamics and synchronization convergence time of the considered nonlinear coupled complex network [2, 48, 55]. As discussed in Section 1, the nonlinearity of nonlinear coupled function $\tilde{h}(\cdot)$ in network (2) will make $\chi(\tilde{h})$ become more complex and lead to the following two questions:

(i) If the nonlinearity of $\tilde{h}(\cdot)$ is more serious, does synchronization dynamics in finite time for network (2) become poorer or does synchronization convergence time of network (2) become longer?

(ii) How can we use the settling finite time t^* to reflect the impact of the nonlinearity of $\tilde{h}(\cdot)$ on finite-time synchronization dynamics and synchronization convergence time of network (2)?

It is a pity that controller (6) designed by Theorem 1 cannot answer the two questions above. Next, we give the answer to the two questions above in Corollaries 1 and 2, respectively.

Corollary 1. *Let $\tilde{\beta}(\chi(\tilde{h})) > 0$ be a decreasing function. Under Theorem 1, before network (2) with controller (6) achieves global synchronization in finite time t_{C1}^* if*

$$\tilde{e}^T(t)\left(\tilde{G}_k^{\tilde{r}} \otimes \widehat{Q}\widehat{\Gamma}_k\right)\widetilde{H}(\tilde{e}(t)) > 0, \quad (43)$$

$$\tilde{e}^T(t)\left(\tilde{G}_k^{\tilde{r}} \otimes \widehat{Q}\widehat{\Gamma}_k\right)\widetilde{H}(\tilde{e}(t-\tau(t))) > 0, \quad (44)$$

$$V(\tilde{e}(0), 0, \tilde{r}(0)) = \tilde{v}_{\tilde{r}(0)} \sum_{i=1}^{N} \tilde{e}_i^T(0)\widehat{Q}\tilde{e}_i(0) > 0, \quad (45)$$

where $\tilde{v}_{\tilde{r}(0)} \geq 1$, $\widetilde{H}(\tilde{e}_j(t)) = \tilde{h}(\tilde{y}_j(t)) - \tilde{h}(s(t))$, $\widetilde{H}(\tilde{e}_j(t-\tau(t))) = \tilde{h}\tilde{y}_j(t-\tau(t)) - \tilde{h}(s(t-\tau(t)))$, *and* $\tilde{G}_k^{\tilde{r}} > 0$ *or* $\tilde{G}_k^{\tilde{r}} < 0$, *with the increase of* $\chi(\tilde{h})$ *of nonlinear coupled function* $\tilde{h}(\cdot)$ *in network (2), synchronization dynamics of network (2) with controller (6) within finite time* t_{C1}^* *becomes poorer and synchronization convergence time of network (2) with controller (6) becomes longer.*

Proof. If network (2) with controller (6) satisfies Theorem 1, it must achieve finite-time synchronization. Thus, under Theorem 1, we can analyze the impact of $\chi(\tilde{h})$ of $\tilde{h}(\cdot)$ on finite-time synchronization dynamics and synchronization convergence time of network (2) with controller (6).

Assume that inequalities (43) and (44) of Corollary 1 hold; then, combining inequalities (21) and (22) in the proof of Theorem 1, we can get

$$\begin{aligned} V_{\tilde{e}(t)}(\tilde{e}(t), t, p) \sum_{k=1}^{m}\sum_{j=1}^{N} \tilde{a}_k \tilde{g}_{ij}^{k,p} \widetilde{\Gamma}_k \widetilde{H}(\tilde{e}(t)) &= 2\tilde{v}_p \sum_{i=1}^{N} \tilde{e}_i^T(t)\widehat{Q} \sum_{k=1}^{m}\sum_{j=1}^{N} \tilde{a}_k \tilde{g}_{ij}^{k,p} \widetilde{\Gamma}_k \widetilde{H}(\tilde{e}(t)) \\ &= 2\tilde{v}_p \sum_{k=1}^{m} \tilde{a}_k \tilde{e}^T(t)\left(\tilde{G}_k^p \otimes \widehat{Q}\widetilde{\Gamma}_k\right)\widetilde{H}(\tilde{e}(t)) > 0, \end{aligned} \quad (46)$$

$$\begin{aligned} V_{\tilde{e}(t)}(\tilde{e}(t), t, p) \sum_{k=1}^{m}\sum_{j=1}^{N} \widehat{a}_k \tilde{g}_{ij}^{k,p} \widehat{\Gamma}_k \widetilde{H}\left(\tilde{e}_j(t-\tau(t))\right) &= 2\tilde{v}_p \sum_{i=1}^{N} \tilde{e}_i^T(t)\widehat{Q} \sum_{k=1}^{m}\sum_{j=1}^{N} \widehat{a}_k \tilde{g}_{ij}^{k,p} \widehat{\Gamma}_k \widetilde{H}\left(\tilde{e}_j(t-\tau(t))\right) \\ &= 2\tilde{v}_p \sum_{k=1}^{m} \widehat{a}_k \tilde{e}^T(t)\left(\tilde{G}_k^p \otimes \widehat{Q}\widehat{\Gamma}_k\right)\widetilde{H}(\tilde{e}(t-\tau(t))) > 0, \end{aligned} \quad (47)$$

where $\widetilde{\Gamma}_k > 0$, $\widehat{Q} > 0$, $\tilde{v}_p \geq 1$, $\widehat{a}_k > 0$, $e_i(t) = \tilde{y}_i(t) - s(t)$, $\widetilde{H}(\tilde{e}_j(t)) = \tilde{h}(\tilde{y}_j(t)) - \tilde{h}(s(t))$, $\widetilde{H}(\tilde{e}_j(t-\tau(t))) = \tilde{h}(\tilde{y}_j(t-\tau(t))) - \tilde{h}(s(t-\tau(t)))$, $\widetilde{H}(\tilde{e}(t)) = [\widetilde{H}^T(\tilde{e}_j(t)), \ldots, \widetilde{H}^T(\tilde{e}_j(t))]^T$, $\widetilde{H}(\tilde{e}(t-\tau(t))) = [\widetilde{H}^T(\tilde{e}_j(t-\tau(t))), \ldots, \widetilde{H}^T(\tilde{e}_j(t-\tau(t)))]^T$, and $i, j = 1, 2, \ldots, N$ $V(\tilde{e}(0), 0, \tilde{r}(0)) = \tilde{v}_{\tilde{r}(0)} \sum_{i=1}^{N} \tilde{e}_i^T(0)\widehat{Q}\tilde{e}_i(0) > 0$.

If $\tilde{v}_{\tilde{r}(0)} \geq 1$ holds, where $\widehat{Q} > 0$ and $\tilde{e}(t) \neq 0$, there must be $\tilde{e}(t) \neq 0$. It is obvious that if $\tilde{e}(t-\tau(t)) \neq 0$, there is $t \geq \tau(t) \geq 0$, where $\tilde{G}_k^{\tilde{r}} > 0$.

Thus, under $\tilde{G}_k^{\tilde{r}} < 0$ or $\tilde{G}_k^{\tilde{r}} > 0$, in order to verify inequalities (46) and (47), we have the following four cases:

(i) Case 1: $e(t) > 0$, $(\tilde{e}(t-\tau(t))) > 0$, $\widetilde{H}(\tilde{e}(t)) > 0$, $\widetilde{H}(\tilde{e}(t-\tau(t))) > 0$, and $\tilde{G}_k^{\tilde{r}} > 0$

(ii) Case 2: $\tilde{e}(t) < 0$, $\tilde{e}(t-\tau(t)) < 0$, $\widetilde{H}(\tilde{e}(t)) < 0$, and $\widetilde{H}(\tilde{e}(t-\tau(t))) < 0$

(iii) Case 3: $\tilde{G}_k^{\tilde{r}} < 0$, $e(t) > 0$, $(\tilde{e}(t-\tau(t))) > 0$, $\widetilde{H}(\tilde{e}(t)) < 0$, and $\widetilde{H}(\tilde{e}(t-\tau(t))) < 0$

(iv) Case 4: $\tilde{G}_k^{\tilde{r}} < 0$, $\tilde{e}(t) < 0$, $\tilde{e}(t-\tau(t)) < 0$, $\widetilde{H}(\tilde{e}(t)) > 0$, and $\widetilde{H}(\tilde{e}(t-\tau(t))) > 0$

Combined with the results in Remark 11, it can be seen that if the nonlinearity $\chi(\tilde{h})$ of $\tilde{h}(\cdot)$ increases, $\|\widetilde{H}(\tilde{e}(t))\|_1$ and $\|\widetilde{H}(\tilde{e}(t-\tau(t)))\|_1$ increase. Therefore, it is not hard to derive that if $\chi(\tilde{h})$ increases, $2\tilde{v}_p \sum_{k=1}^{m} \widehat{a}_k \tilde{e}^T(t)(\tilde{G}_k^p \otimes \widehat{Q}\widehat{\Gamma}_k)\widetilde{H}(\tilde{e}(t-\tau(t))) > 0$ and $2\tilde{v}_p \sum_{k=1}^{m} \widehat{a}_k \tilde{e}^T(t)(\tilde{G}_k^p \otimes \widehat{Q}\widehat{\Gamma}_k)\widetilde{H}(\tilde{e}(t-\tau(t))) > 0$ become larger. For example, in Case 1, there are $\tilde{G}_k^{\tilde{r}} > 0$, $\tilde{e}(t) > 0$, $\tilde{e}(t-\tau(t)) > 0$, $\widetilde{H}(\tilde{e}(t)) > 0$, and $\widetilde{H}(\tilde{e}(t-\tau(t))) > 0$. Therefore, we have

$$\begin{aligned} 2\tilde{v}_p \sum_{k=1}^{m} \widehat{a}_k \tilde{e}^T(t)\left(\tilde{G}_k^p \otimes \widehat{Q}\widetilde{\Gamma}_k\right)\widetilde{H}(\tilde{e}(t)) &\geq 2\tilde{v}_p \lambda_{\min} \\ \cdot \left\{\left(\tilde{G}_k^p \otimes \widehat{Q}\widetilde{\Gamma}_k\right)\right\} &\sum_{k=1}^{m} \tilde{a}_k \tilde{e}^T(t)\widetilde{H}(\tilde{e}(t)) > 0, \end{aligned} \quad (48)$$

$$\begin{aligned} &2\tilde{v}_p \sum_{k=1}^{m} \widehat{a}_k \tilde{e}^T(t)\left(\tilde{G}_k^p \otimes \widehat{Q}\widehat{\Gamma}_k\right)\widetilde{H}(\tilde{e}(t \\ &- \tau(t))) \geq 2\tilde{v}_p \lambda_{\min}\left\{\left(\tilde{G}_k^p \otimes \widehat{Q}\widetilde{\Gamma}_k\right)\right\} \sum_{k=1}^{m} \tilde{a}_k \tilde{e}^T(t)\widetilde{H}(\tilde{e}(t \\ &- \tau(t))) > 0, \end{aligned} \quad (49)$$

where $\tilde{a}_k > 0$, $\widehat{Q} > 0$, and $\widetilde{\Gamma}_k > 0$.

Because $\tilde{G}_k^p > 0$, $\widehat{Q} > 0$, and $\widetilde{\Gamma}_k > 0$, $\lambda_{\min}\left\{(\tilde{G}_k^p \otimes \widehat{Q}\widetilde{\Gamma}_k)\right\} > 0$. Thus, combining (45) and (46), we derive that

$$\tilde{e}^T(t)\widetilde{H}(\tilde{e}(t)) = \sum_{i=1}^{N}\sum_{j=1}^{n} \tilde{e}_{ij}(t) H\left(\tilde{e}_{ij}(t)\right) > 0, \quad (50)$$

$$\tilde{e}^T(t)\widetilde{H}(\tilde{e}(t-\tau(t))) = \sum_{i=1}^{N}\sum_{j=1}^{n} \tilde{e}_{ij}(t) H\left(\tilde{e}_{ij}(t-\tau(t))\right) > 0. \quad (51)$$

It is clear that if $H(\tilde{e}_{ij}(t)) > 0$ increases, $\tilde{e}^T(t)\tilde{H}(\tilde{e}(t))$ and $\tilde{e}^T(t)\tilde{H}(\tilde{e}(t-\tau(t))) > 0$ also increase. The analyses of Cases 2–4 are similar to that of case 1. Therefore, before network (2) with controller (6) achieves global synchronization, if conditions (43)–(45) of Corollary 1 hold, with the increase of the nonlinearity $\chi(\tilde{h})$ of $\tilde{h}(\cdot)$, synchronization dynamics of network (2) with controller (6) in finite time t^* given by condition (III) of Theorem 1 becomes poorer. This makes synchronization convergence time of network (2) with controller (6) become longer. In order to more accurately describe synchronization convergence time of network (2) with controller (6), let $\tilde{\beta}(\chi(\tilde{h})) > 0$ be a decreasing function. That is to say, if $\chi(\tilde{h})$ becomes larger, $\tilde{\beta}(\chi(\tilde{h})) > 0$ also becomes smaller. Thus, by using condition (III) of Theorem 1, we can obtain t^*_{C1}. And if $\tilde{\beta}(\chi(\tilde{h})) > 0$ becomes smaller, t^*_{C1} becomes longer. This completes the proof. □

Corollary 2. *Let $\tilde{\beta}(\chi(\tilde{h})) > 0$ be an increasing function. Under Theorem 1, before network (2) with controller (6) achieve global synchronization in finite time t^*_{C2}, if*

$$\begin{aligned}
&\tilde{e}^T(t)\left(\tilde{G}_k^{\tilde{r}} \otimes \hat{Q}\hat{\Gamma}_k\right)\tilde{H}(\tilde{e}(t)) < 0,\\
&\tilde{e}^T(t)\left(\tilde{G}_k^{\tilde{r}} \otimes \hat{Q}\hat{\Gamma}_k\right)\tilde{H}(\tilde{e}(t-\tau(t))) < 0,\\
&V(\tilde{e}(0), 0, \tilde{r}(0)) = \tilde{\upsilon}_{\tilde{r}(0)} \sum_{i=1}^{N} \tilde{e}_i^T(0)\hat{Q}\tilde{e}_i(0) > 0,
\end{aligned} \tag{52}$$

where $\tilde{\upsilon}_{\tilde{r}(0)} \geq 1$, $\tilde{H}(\tilde{e}_j(t)) = \tilde{h}(\tilde{y}_j(t)) - \tilde{h}(s(t))$, $\tilde{H}(\tilde{e}_j(t-\tau(t))) = \tilde{h}(\tilde{y}_j(t-\tau(t))) - \tilde{h}(s(t-\tau(t)))$, and $\tilde{G}_k^{\tilde{r}} > 0$ or $\tilde{G}_k^{\tilde{r}} < 0$, with the increase of $\chi(\tilde{h})$ of nonlinear coupled function $\tilde{h}(\cdot)$ in network (2), synchronization dynamics of network (2) with controller (6) within finite time t^*_{C2} becomes better and synchronization convergence time of network (2) with controller (6) becomes shorter.

Proof. The proof is similar to that of Corollary 1, and hence, we omit it here. □

Remark 12. From Corollaries 1 and 2, we can observe that with the increase of the nonlinearity $\chi(\tilde{h})$ of nonlinear coupled function $\tilde{h}(\cdot)$ in network (2), synchronization dynamic of network (2) in Corollaries 1 and 2 become poorer and better, respectively. Meanwhile, if $\chi(\tilde{h})$ increases, synchronization convergence time and the derived settling finite time t^*_{C1} and t^*_{C2} of network (2) with controller (6) in Corollaries 1 and 2, respectively, become longer and shorter. It is obvious that compared with t^*, $t^*_{C1}$and t^*_{C2} can more effectively reflect and accurately describe the impact of $\chi(\tilde{h})$ of $\tilde{h}(\cdot)$ on finite-time synchronization dynamics and synchronization convergence time of network (2) with controller (6).

Besides this, from the proofs of Corollaries 1 and 2, it can also be obtained that the impact of the nonlinearity of nonlinear coupled function $\tilde{h}(\cdot)$ on synchronization dynamics and synchronization convergence time of network (2) with controller (6) is not only related to $\chi(\tilde{h})$ of $\tilde{h}(\cdot)$ but also closely connected with the synchronization state $s(t)$ and the initial conditions.

Remark 13. Due to some factors such as limited communications and environmental changes, parameter switching in a dynamic system is usually inevitable [63, 64]. Parameter switching may add some interesting dynamic behaviors to a dynamic system, which reveals that it is essential to investigate dynamic problems of systems with switching parameters. During the past decades, many researchers began to explore dynamic problems of some classes of systems with switching parameters [10, 65–73]. For instance, based on the Lyapunov function method and inequality technology, Mao et al. [69] studied the stability problem of switched continuous-time systems with all subsystems unstable. In [70], Xu et al. derived a sufficient condition to ensure global synchronization of a class of complex networks with switched adjacent matrices. Regrettably, up to now, although a great deal of valuable and meaningful results on dynamic problems for some classes of systems with switching parameters have been obtained [10, 63–73], to the best of our knowledge, there are still no literature studies to discuss the following two problems:

(i) If the overall system consists of a subsystem achieving synchronization within finite time t_1 and subsystems not achieving synchronization, can the overall system achieve synchronization?

(ii) If the overall system consists of a subsystem achieving synchronization within finite time t_1 and subsystems achieving synchronization within finite time t_2, can the overall system achieve finite-time synchronization? If yes, what is the finite time?

In the next two remarks, we will focus on the two questions above.

Remark 14. In network (2), the switching signal $\tilde{r}(t)$ satisfying the rule given by formula (3) has $\hat{s}$ finite state, which shows that overall network (2) has $\hat{s}$ subsystems. Assume that the $\hat{p}$th subsystem with controller (6) can achieve synchronization within finite time t^* and the $\hat{q}$th subsystem without controller (6) cannot achieve synchronization, where $\hat{p} = 1$ and $\hat{q} = 2, \ldots, \hat{s}$. In this situation, can overall network (2) achieve the synchronization?

First, combining formula (3) and finite state space $\hat{S}$, we suppose the following switching sequence:

$$S = \left\{(t_1, 1), (t_2, 2), \ldots, \left(t_{\hat{s}}, \hat{s}\right), \left(t_{\hat{s}+1}, 1\right), \left(t_{\hat{s}+2}, 2\right), \ldots\right\}, \tag{53}$$

where $\{t_1 \in [0, \Delta t]$, $t_2 \in [\Delta t, 2\Delta t], \ldots, t_{\hat{s}} \in [(\hat{s}-1)\Delta t, \hat{s}\Delta t], t_{\hat{s}+1} \in [\hat{s}\Delta t, (\hat{s}+1)\Delta t], \ldots\}$. In error system (5) of network (2), $\tilde{e}_i(t) \in R^n$, $\tilde{F}(\tilde{y}_i(t)) \in R^n$, $\tilde{A} \in R^{n\times n}$, $\tilde{a}_k > 0$ and

$\widehat{a}_k > 0$, and $\widetilde{\Gamma}_k \in R^{n\times n} > 0$ and $\widehat{\Gamma}_k \in R^{n\times n} > 0$, which reveal that there must exist the solutions $\widetilde{e}_i^{(p)}(t_{\widehat{k}})$ and $\widetilde{e}_i^{(q)}(t_{\widehat{m}})$ in the error system of the $\widehat{p}$ subsystem with controller (6) and the error system of the $\widehat{q}$ subsystem without controller (6), where $\widehat{k} = 1, \widehat{s}+1, \ldots$ and $\widehat{m} = 2, 3, \ldots, \widehat{s}, \widehat{s}+2, \ldots$;

Second, according to formula (17), one can obtain that $\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(p)}(t_{\widehat{k}}), t_{\widehat{k}}, \widehat{p})$, $\mathscr{L}_{\widehat{q}}V(\widetilde{e}^{(\widehat{q})}(t_{\widehat{m}}), t_{\widehat{m}}, \widehat{q})$, and $\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})$, where $\widehat{r}$ stands for the switching sequence S, and $\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(p)}(t_{\widehat{k}}), t_{\widehat{k}}, \widehat{p})$, $\mathscr{L}_{\widehat{q}}V(\widetilde{e}^{(\widehat{q})}(t_{\widehat{m}}), t_{\widehat{m}}, \widehat{q})$, and $\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})$, respectively, represent the derivative of the Lyapunov functional (16) for the $\widehat{p}$ subsystem with controller (6), the $\widehat{q}$th subsystem without controller (6), and overall network (2) which consists of a $\widehat{p}$ subsystem with controller (6) and $\widehat{s}-1$ subsystems without controller (6).

In fact, it can be derived that $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] = \mathbb{E}[\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(p)}(t_{\widehat{r}_1}), t_{\widehat{r}_1}, \widehat{r}_1) + \mathscr{L}_{\widehat{q}}V(\widetilde{e}^{(\widehat{q})}(t_{r_2}), t_{\widehat{r}_2}, \widehat{r}_2)]$ and $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})]$ is a smooth function, where $\widehat{r} = \widehat{r}_1 \cup \widehat{r}_2$, $\widehat{r}_1 = \{(t_1, 1), (t_{\widehat{s}+1}, 1), \ldots\}$ and $\widehat{r}_2 = \{(t_2, 2), (t_3, 3), \ldots, (t_{\widehat{s}}, \widehat{s}), (t_{\widehat{s}+2}, 2), \ldots\}$. This can be obtained as follows.

Markovian switching usually models random abrupt variations which are often caused by random failures and repairs of the components [59]. This means that network (2) with Markovian switching parameter $\widetilde{g}_{ij}^k(\widetilde{r}(t))$ which consists of $\widehat{s}$ subsystems actually originates from a subsystem of network (2). For example, due to noise perturbations and some other elements [59], the parameter $\widetilde{g}_{ij}^k(\widetilde{r}(t_1))$ of the 1st subsystem is randomly switched to $\widetilde{g}_{ij}^k(\widetilde{r}(t_2))$. Thus, the 2nd subsystem is produced. By the similar rule, the 3rd... and the $\widehat{s}$th subsystem are also produced. Therefore, the overall network (2) with a Markovian switching parameter actually is one dynamical system and the number of the solution $\widetilde{y}_{i\widetilde{j}}(t)$ of the overall network (2) is $N \times n$, instead of $\widehat{s} \times N \times n$, where $i = 1, 2, \ldots, N$ and $\widetilde{j} = 1, 2, \ldots, n$.

Moreover, the solution $\widetilde{y}_{i\widetilde{j}}(t)$ of the overall network (2) with or without controller (6) is a smooth function.

In fact, let $\widetilde{y}_i(t) = (\widetilde{y}_{i1}(t), \widetilde{y}_{i2}(t), \ldots, \widetilde{y}_{in}(t))^T$ and assume that $\widetilde{y}_i(t)$ is not a smooth function; then, there exists $\|\lim_{\triangle\longrightarrow 0}(\widetilde{y}_i(t+\triangle) - \widetilde{y}_i(t))\|_1 = \widetilde{\eta}$, where $\widetilde{\eta} > 0$ is a constant. This leads to $\|\dot{\widetilde{y}}_i(t)\|_1 = \|\lim_{\triangle\longrightarrow 0}((\widetilde{y}_i(t+\triangle) - \widetilde{y}_i(t))/\triangle)\|_1 = \|\lim_{\triangle\longrightarrow 0}(\widetilde{\eta}/\triangle)\|_1 \longrightarrow +\infty$. Combining network (2) with or without controller (6), one can obtain that if $\|\dot{\widetilde{y}}_i(t)\|_1 \longrightarrow +\infty$, there must be $\|\widetilde{y}_i(t)\|_1 \longrightarrow +\infty$. Obviously, this is wrong. Therefore, there must be $\|\lim_{\triangle\longrightarrow 0}(\widetilde{y}_i(t+\triangle) - \widetilde{y}_i(t))\|_1 = 0$, which means that $\widetilde{y}_i(t)$ is a smooth function.

From (4)–(6), one can obtain that $\widetilde{e}_i(t)$ is also a smooth function. Combining (16) with (17), one can obtain that $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})]$ is also a smooth function.

Therefore, though $\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})$ is divided into

$$\begin{gathered}\mathscr{L}_{\widehat{p}}V\left(\widetilde{e}^{(\widehat{p})}\left(t_{\widehat{r}_1}\right), t_{\widehat{r}_1}, \widehat{r}_1\right),\\ \mathscr{L}_{\widehat{q}}V\left(\widetilde{e}^{(\widehat{q})}\left(t_{\widehat{r}_2}\right), t_{\widehat{r}_2}, \widehat{r}_2\right),\end{gathered} \tag{54}$$

by the switching sequence S, because $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})]$ is a smooth function, then

$$\mathbb{E}\left[\mathscr{L}_{\widehat{p}}V\left(\widetilde{e}^{(\widehat{p})}\left(t_{\widehat{r}_1}\right), t_{\widehat{r}_1}, \widehat{r}_1\right) + \mathscr{L}_{\widehat{q}}V\left(\widetilde{e}^{(\widehat{q})}\left(t_{\widehat{r}_2}\right), t_{\widehat{r}_2}, \widehat{r}_2\right)\right], \tag{55}$$

must be a smooth function. It is clear that

$$\begin{aligned}\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] = \mathbb{E}\Big[&\mathscr{L}_{\widehat{p}}V\left(\widetilde{e}^{(\widehat{p})}\left(t_{\widehat{r}_1}\right), t_{\widehat{r}_1}, \widehat{r}_1\right)\\ &+ \mathscr{L}_{\widehat{q}}V\left(\widetilde{e}^{(\widehat{q})}\left(t_{\widehat{r}_2}\right), t_{\widehat{r}_2}, \widehat{r}_2\right)\Big].\end{aligned} \tag{56}$$

Then, we can conclude that although network (2) is composed by $\widehat{s}$ subsystems which are coupled by Markovian switching parameter $\widetilde{g}_{ij}^k(\widetilde{r}(t))$, the overall network (2) is actually a dynamical system. This reflects that in network (2), there does not exist independent subsystems.

And if subsystems in network (2) are independent, every subsystem may exhibit different synchronization dynamical behaviors. All these testify that if every subsystem of network (2) is independent, it may be realized that the $\widehat{p}$th subsystem with controller (6) can achieve synchronization within finite time t^* and the $\widehat{q}$th subsystem without controller (6) cannot achieve synchronization, where $\widehat{p} = 1$ and $\widehat{q} = 2, \ldots, \widehat{s}$. Otherwise, if every subsystem of network (2) is coupled by Markovian switching parameter $\widetilde{g}_{ij}^k(\widetilde{r}(t))$, the above addressed synchronization dynamic behaviors of subsystems in network (2) cannot be realized. Therefore, every subsystem of overall network (2) is independent.

Similar to the proof of Theorem 1, we can get $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t) < 0$ and $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t) > 0$, where $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t)$ and $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t)$ are with respect to the $\widehat{p}$th independent subsystem with controller (6) and the $\widehat{q}$th independent subsystem without controller (6), respectively. Here, $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t)$ and $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t)$ replace $\mathscr{L}V_{\widehat{p}}(\widetilde{e}(t), t, \widehat{p})$ and $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t)$. Because in the independent $\widehat{p}$th subsystem with controller (6) and the independent $\widehat{q}$th subsystem without controller (6), there is no Markovian switching phenomenon; thus, if $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t) \le 0$ and $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t) > 0$, then $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] \le 0$, the overall network (2) can achieve global synchronization within infinite time interval or finite time interval.

Remark 15. If the $\widehat{p}$th subsystem with controller (6) can achieve synchronization within finite time $t^*_{(1)}$ and the $\widehat{q}$th subsystem with controller (6) can achieve synchronization within finite time $t^*_{(2)}$, can overall network (2) achieve synchronization? If yes, what type of finite-time synchronization is the obtained finite-time synchronization? By the similar analysis of Remark 14, one can get the following conclusion: if $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t) \le -\widetilde{\vartheta}_1 V_{\widehat{q}}^{\varsigma_1}(\widetilde{e}(t), t)$ and $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t), t) \le -\widetilde{\vartheta}_2 V_{\widehat{q}}^{\varsigma_2}(\widetilde{e}(t), t)$ can make $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] \le -\widetilde{\vartheta}_3\{\mathbb{E}[V(\widetilde{e}(t), t, \widehat{r})]\}^{\varsigma_3}$ hold, overall network (2) can achieve synchronization, where $\widetilde{\vartheta}_1 > 0$, $\widetilde{\vartheta}_2 > 0$, $\widetilde{\vartheta}_3 > 0$, $0 < \varsigma_1 < 1$, $0 < \varsigma_2 < 1$, $0 < \varsigma_3 < 1$, $t_1^* \le V_{\widehat{p}}^{1-\varsigma_1}(0)/\widetilde{\vartheta}_1(1-\varsigma_1)$, $t_2^* \le V_{\widehat{q}}^{1-\varsigma_2}(0)/\widetilde{\vartheta}_2(1-\varsigma_2)$, and $T^* \le [V(\widetilde{e}(0), 0, \widetilde{r}(0))]^{1-\varsigma_3}/\widetilde{\vartheta}_3(1-\varsigma_3)$. For overall network (2), the obtained finite-time synchronization is global synchronization within finite time interval.

4. Numerical Examples

In order to illustrate the effectiveness of the obtained results, this section gives four numerical examples. And synchronization total error of the addressed network is defined as $e(t)=\sum_{i=1}^{N}\sum_{j=1}^{n}e_{ij}(t)$, where $e_{ij}(t)$ is synchronization error of the addressed network. For a given rate transition matrix, the right-continuous Markov chain $\widetilde{r}(t)$ can be generated. As an example, we adopt $\widehat{S}=\{1,2,3\}$ and the rate transition matrix as follows:

$$\Upsilon=\begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}. \tag{57}$$

Example 1. Consider the following network with sixty coupling nodes:

$$\begin{aligned}\dot{\widetilde{y}}_i(t)&=\widetilde{A}\widetilde{f}(\widetilde{y}_i(t))+\sum_{k=1}^{2}\sum_{j=1}^{60}\widetilde{a}_k\widetilde{g}_{ij}^k(\widetilde{r}(t))\widetilde{\Gamma}_k\widetilde{h}\big(\widetilde{y}_j(t)\big)\\&\quad+\sum_{k=1}^{2}\sum_{j=1}^{60}\widehat{a}_k\widetilde{g}_{ij}^k(\widetilde{r}(t))\widehat{\Gamma}_k\widetilde{h}\big(\widetilde{y}_j(t-\tau(t))\big),\end{aligned} \tag{58}$$

where $i=1,2,\ldots,60$, $\widetilde{A}=\mathrm{diag}\{1,1\}$, $\widetilde{\Gamma}_k=\widehat{\Gamma}_k=\mathrm{diag}\{1,1\}$, $\widetilde{f}(\widetilde{y}_i)=[-\widetilde{y}_{i1},-\widetilde{y}_{i2}]^T$, $\widetilde{a}_k=\widehat{a}_k=1$, $\tau(t)=0.1-0.1e^{-t}$, $\widetilde{h}(\widetilde{y}_j(t))=[\tanh(\widetilde{y}_{j1}(t)),\tanh(\widetilde{y}_{j2}(t))]^T$, and $\widetilde{h}(\widetilde{y}_j(t-\tau(t)))=[\tanh(\widetilde{y}_{j1}(t-\tau(t))),\tanh(\widetilde{y}_{j2}(t-\tau(t)))]^T$.

The coupling matrix $\widetilde{G}_k^{\widetilde{r}}$ is as follows:

$$\begin{aligned}\widetilde{G}_k^1&=I_{20}\otimes\begin{bmatrix}1.5 & 0.3 & 0.6\\ 0.3 & 1.6 & 0.5\\ 0.6 & 0.5 & 1.4\end{bmatrix},\\ \widetilde{G}_k^2&=I_{20}\otimes\begin{bmatrix}1.4 & 0.2 & 0.5\\ 0.2 & 1.5 & 0.3\\ 0.5 & 0.3 & 1.6\end{bmatrix},\\ \widetilde{G}_k^3&=I_{20}\otimes\begin{bmatrix}1.3 & 0.2 & 0.4\\ 0.2 & 1.5 & 0.5\\ 0.4 & 0.5 & 1.4\end{bmatrix}.\end{aligned} \tag{59}$$

By using the steps in Remark 14, we can design controller (6):

(i) Let $\widetilde{v}_p=1$ and $\widehat{Q}=I_2$, where $p=1,2,3$.

(ii) Combining Assumptions 1 and 2, $\widetilde{f}(\cdot)$, and $\widetilde{h}(\cdot)$, we have $L=\widetilde{L}=1.2$, $\psi=I_2$, $\widetilde{\psi}_p=I_{120}$, and $\tau_M\doteq\rho=0.1$.

(iii) From inequality (15) of condition (II) of Theorem 1, we obtain that $\widehat{\theta}_p=1$ and $\widehat{F}^k=1.5\otimes I_{120}$.

(iv) According to inequality (14) of condition (II) of Theorem 1, we get $\Xi_p^k=4.5I_{60}$. Thus, combining condition (I) of Theorem 1, there is $\widehat{\varrho}_p\doteq 1.11$.

(v) Let the initial state vector value $\widetilde{y}(0)=2(1,1.1,+\ldots,1+0.1\widetilde{k},\ldots,12.9)^T$, $\widetilde{k}=0,1,\ldots,119$, $\widetilde{\beta}(\chi(\widetilde{h}))=1$, $\widetilde{\alpha}=0.5$, and $\widehat{\eta}_i^{k,p}=4$, then by condition (III) of Theorem 1, t^* satisfies $t^*\le 2.74$, where $V(e(0),0,\widetilde{r}(0))=14472$, $e(0)=1668$, and $m=2$.

Thus, the design of controller (6) is completed and the finite time t^* is successfully estimated.

Actually, from Definition 1 and the derived t^* in condition (III) of Theorem 1, it can be seen that finite time t^* heavily depends on the initial state value $\widetilde{y}(0)$ of network (58). This shows that the initial value of the addressed system can affect the simulation result.

In order to further test the result, let $\widetilde{y}^{(1)}(0)=3(1,1.1,+\ldots,1+0.1\widetilde{k},\ldots,12.9)^T$. We still choose the designed controller (6). Thus, one can obtain that $t_{(1)}^*\le 3.04$, where $V^{(1)}(e^{(1)}(0),0,\widetilde{r}(0))=21709$ and $e^{(1)}(0)=2502$.

In the simulation results of Figure 1, the trajectories marked by red and blue are with respect to the initial conditions $\widetilde{y}(0)$ and $\widetilde{y}^{(1)}(0)$, respectively. From Figure 1, we can see that within finite time $t^*=2.74$ and $t_{(1)}^*=3.04$, synchronization errors $e_{ij}(t)$ and $e_{ij}^{(1)}(t)$ and synchronization total errors $e(t)$ and $e^{(1)}(t)$ gradually tend to zero, where $i=1,2,\ldots,60$ and $j=1,2$. This means that network (58) with designed controller (6) can achieve global synchronization in finite time t^* and $t_{(1)}^*$, respectively.

Moreover, compared with the simulation results in Figure 1, we can find that under the same control rule, if the initial state value of network (58) becomes larger, synchronization convergence time of network (58) becomes longer. The obtained t^* and $t_{(1)}^*$ can also reflect the results.

Example 2. In this example, we consider network (2) with eighteen coupling nodes presented by

$$\begin{aligned}\dot{\widetilde{y}}_i(t)&=\widetilde{A}\widetilde{f}(\widetilde{y}_i(t))+\sum_{k=1}^{2}\sum_{j=1}^{18}\widetilde{a}_k\widetilde{g}_{ij}^k(\widetilde{r}(t))\widetilde{\Gamma}_k\widetilde{h}\big(\widetilde{y}_j(t)\big)\\&\quad+\sum_{k=1}^{2}\sum_{j=1}^{18}\widehat{a}_k\widetilde{g}_{ij}^k(\widetilde{r}(t))\widehat{\Gamma}_k\widetilde{h}\big(\widetilde{y}_j(t-\tau(t))\big).\end{aligned} \tag{60}$$

From Corollary 1, it can be seen that there are

$$\begin{aligned}&\widetilde{e}^T(t)\Big(\widetilde{G}_k^{\widetilde{r}}\otimes\widehat{Q}\widetilde{\Gamma}_k\Big)\widetilde{H}(\widetilde{e}(t))>0,\\&\widetilde{e}^T(t)\Big(\widetilde{G}_k^{\widetilde{r}}\otimes\widehat{Q}\widehat{\Gamma}_k\Big)\widetilde{H}(\widetilde{e}(t-\tau(t)))>0,\end{aligned} \tag{61}$$

where

$$\begin{aligned}&\widetilde{H}(\widetilde{e}(t))=\widetilde{h}\big(\widetilde{y}_j(t)\big)-\widetilde{h}(s(t)),\quad \widetilde{H}(\widetilde{e}(t-\tau(t)))=\widetilde{h}\big(\widetilde{y}_j(t-\tau(t))\big)-\widetilde{h}(s(t-\tau(t)))\\&\widetilde{G}_k^{\widetilde{r}}>0 \text{ or } \widetilde{G}_k^{\widetilde{r}}<0,\ \widehat{Q}>0,\ \widetilde{\Gamma}_k>0.\end{aligned} \tag{62}$$

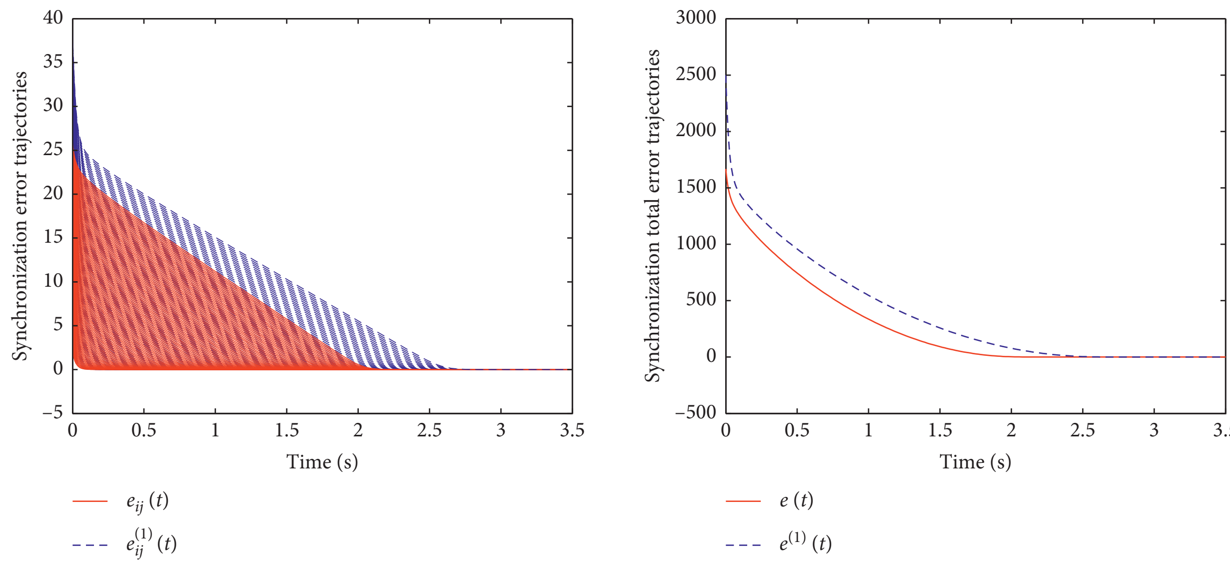

FIGURE 1: Synchronization error trajectories and synchronization total error trajectories of network (58) with controller (6) for Example 1.

Thus, there exist the following four cases if $\tilde{e}^T(t)(\tilde{G}_k^r \otimes \hat{Q}\hat{\Gamma}_k)H(\tilde{e}(t)) > 0$ and $\tilde{e}^T(t)(\tilde{G}_k^r \otimes \hat{Q}\hat{\Gamma}_k)\tilde{H}(\tilde{e}(t-\tau(t))) > 0$ hold.

Case 1: $\tilde{G}_k^{\tilde{r}} > 0$, $\tilde{e}(t) > 0$, $(\tilde{e}(t-\tau(t))) > 0$, $\tilde{H}(\tilde{e}(t)) > 0$, and $\tilde{H}(\tilde{e}(t-\tau(t))) > 0$. We choose

$$\begin{aligned} \tilde{G}_k^p &= I_6 \otimes [1.3, 0.3, 0.6; 0.3, 1.5, 0.5; 0.6, 0.5, 1.4], \\ \tilde{y}(0) &= 2(1, 1.1, \ldots, 1 + 0.1\tilde{k}, \ldots, 4.5)^T, \end{aligned} \tag{63}$$

where $\tilde{k} = 0, 1, \ldots, 35$.

First, let $\tilde{h}_{(1)}(\tilde{y}_j(t)) = [\tanh(\tilde{y}_{j1}(t)), \tanh(\tilde{y}_{j2}(t))]^T$ and $\tilde{h}_{(1)}(\tilde{y}_j(t-\tau(t))) = [\tanh(\tilde{y}_{j1}(t-\tau(t))), \tanh(\tilde{y}_{j2}(t-\tau(t)))]^T$. The other parameters of network (60) are the same as those of network (58). Using the similar steps of Example 1, we can design the parameters of controller (6) and these parameters are $\Xi_p^k = 4.5 I_{18}$, $\hat{Q} = I_2$, $\hat{\theta}_p = 1$ and $\hat{F}^k = 1.5 \otimes I_{36}$, $\tau_M = \rho = 0.1$, $\tilde{\beta}(\chi(\tilde{h}_{(1)})) = 1$, $\tilde{\alpha} = 0.5$, and $\hat{\eta}_i^{k,p} = 4$. Besides these, from Example 1, we have $\hat{\varrho}_p \doteq 1.11$. From the simulation results in Figure 2, we can derive that before network (60) with controller (6) achieves finite-time synchronization, there must be $\tilde{e}(t) > 0$, $\tilde{e}(t-\tau(t)) > 0$, $\tilde{H}(\tilde{e}(t)) > 0$, and $\tilde{H}(\tilde{e}(t-\tau(t))) > 0$, where $\tilde{e}(t)$ is with respect to $e(t)$ of Figure 2. This means that condition (III) of Corollary 1 holds. From condition (III) of Theorem 1, we have $t_{(1)}^* \le 1.25$, where $V(e(0), 0, \tilde{r}(0)) = 622.2$, $e(0) = 198$, and $\tilde{v}_p = 1$.

Second, in order to test the impact of the nonlinearity $\chi(\tilde{h})$ of nonlinear coupled function $\tilde{h}(\cdot)$ on synchronization dynamics and synchronization convergence time of network (60) with controller (6), let nonlinear coupled function $\tilde{h}(\cdot)$ be

$$\begin{aligned} \tilde{h}_{(2)}(\tilde{y}_j(t)) &= \begin{bmatrix} 0.7\tanh(0.7\tilde{y}_{j1}(t)) \\ 0.7\tanh(0.7\tilde{y}_{j2}(t)) \end{bmatrix}, \\ \tilde{h}_{(2)}(\tilde{y}_j(t-\tau(t))) &= \begin{bmatrix} 0.7\tanh(0.7\tilde{y}_{j1}(t-\tau(t))) \\ 0.7\tanh(0.7\tilde{y}_{j2}(t-\tau(t))) \end{bmatrix}, \\ \tilde{h}_{(3)}(\tilde{y}_j(t)) &= \begin{bmatrix} 0.2\tanh(0.2\tilde{y}_{j1}(t)) \\ 0.2\tanh(0.2\tilde{y}_{j2}(t)) \end{bmatrix}, \\ \tilde{h}_{(3)}(\tilde{y}_j(t-\tau(t))) &= \begin{bmatrix} 0.2\tanh(0.2\tilde{y}_{j1}(t-\tau(t))) \\ 0.2\tanh(0.2\tilde{y}_{j2}(t-\tau(t))) \end{bmatrix}. \end{aligned} \tag{64}$$

It is clear that $\chi(\tilde{h}_{(1)}) > \chi(\tilde{h}_{(2)}) > \chi(\tilde{h}_{(3)}) > 0$. We still choose $\Xi_p^k = 4.5 * I_{18}$, $\hat{Q} = I_2$, $\hat{\theta}_p = 1$, $\hat{F}^k = 1.5 \otimes I_{36}$, $\tau_M = \rho = 0.1$, $\tilde{\beta}(\chi(\tilde{h}_{(2)})) = \tilde{\beta}(\chi(\tilde{h}_{(2)})) = \tilde{\beta}(\chi(\tilde{h}_{(1)})) = 1$, $\tilde{\alpha} = 0.5$, and $\hat{\eta}_i^{k,p} = 4$. Combining $\tilde{h}_{(2)}(\cdot)$ and $\tilde{h}_{(1)}(\cdot)$, we easily testify that condition (II) of Theorem 1 holds. Besides this, it can also be derived that there are $t_{(2)}^* \le 1.25$ and $t_{(3)}^* \le 1.25$. In Figure 2, the trajectories marked by green, black, and red are with respect to $\tilde{h}_{(1)}(\cdot)$, $\tilde{h}_{(2)}(\cdot)$, and $\tilde{h}_{(3)}(\cdot)$, respectively.

From Figure 2, we can observe that with the decrease of $\chi(\tilde{h})$, synchronization dynamics of network (60) with controller (6) in finite time $t_{(\hat{v})}^* = 1.25$ becomes better and synchronization convergence of network (60) with controller (6) becomes shorter, where $\hat{v} = 1, 2, 3$. In Corollary 1, because $\tilde{\beta}(\chi(\tilde{h}))$ is a decreasing function; let $\tilde{\beta}_{\text{new}}(\chi(\tilde{h}_{(2)})) = 1.1$ and $\tilde{\beta}_{\text{new}}(\chi(\tilde{h}_{(3)})) = 1.2$. Thus, combining condition (III) of Theorem 1, we can obtain that $t_{\text{new}(2)}^* \le 1.14$ and $t_{\text{new}(3)}^* \le 1.04$. It is obvious that

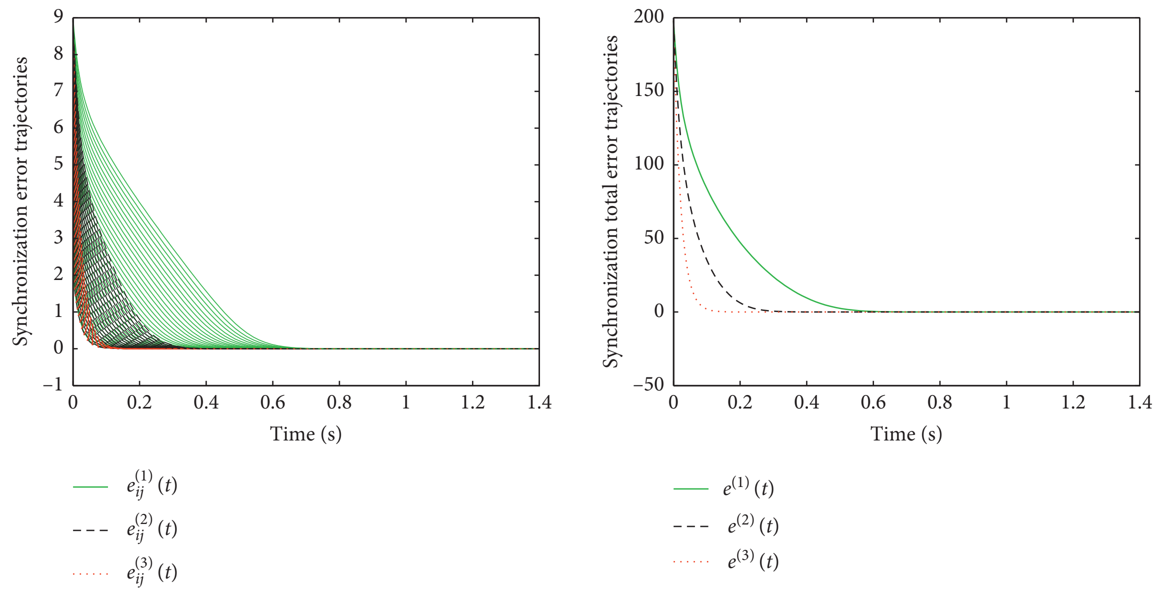

Figure 2: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case 1 of Example 2.

compared with $t^*_{(2)} \leq 1.58$ and $t^*_{(3)} \leq 1.58$, $t^*_{\text{new}(2)} \leq 1.14$ and $t^*_{\text{new}(3)} \leq 1.04$ can more accurately reflect the impact of nonlinear coupled function $\widetilde{h}(\cdot)$ on synchronization dynamics and synchronization convergence time of network (60) with the same designed controller (6).

Case 2: $\widetilde{G}_k^{\widetilde{r}} > 0, \widetilde{e}(t) < 0, \widetilde{e}(t-\tau(t)) < 0, \widetilde{H}(\widetilde{e}(t)) < 0$, and $\widetilde{H}(\widetilde{e}(t-\tau(t))) < 0$. Let

$$\widetilde{G}_k^p = I_6 \otimes [1.3, 0.3, 0.6; 0.3, 1.5, 0.5; 0.6, 0.5, 1.4],$$
$$\widetilde{h}^{II}_{(\widetilde{l})}(\cdot) = \widetilde{h}_{(\widetilde{l})}(\cdot),$$
$$\widetilde{y}(0) = 2 * (-1, -1.1, \ldots, -(1 + 0.1 * \widetilde{k}), \ldots, -4.5)^T, \tag{65}$$

where $\widetilde{l} = 1, 2, 3$, $\widetilde{k} = 0, 1, \ldots, 35$ and $\widetilde{h}_{(\widetilde{l})}(\cdot)$ is $\widetilde{h}_{(1)}(\cdot)$, $\widetilde{h}_{(2)}(\cdot)$, and $\widetilde{h}_{(3)}(\cdot)$ in Case 1, respectively.

Case 3: $\widetilde{G}_k^{\widetilde{r}} < 0, \widetilde{e}(t) > 0, \widetilde{e}(t-\tau(t)) > 0, \widetilde{H}(\widetilde{e}(t)) < 0$, and $\widetilde{H}(\widetilde{e}(t-\tau(t))) < 0$. Let

$$\widetilde{G}_k^p = I_6 \otimes [-1.3, -0.3, -0.6; -0.3, -1.5, -0.5; -0.6, -0.5, -1.4],$$
$$\widetilde{h}^{III}_{(\widetilde{l})}(\cdot) = -\widetilde{h}^{II}_{(\widetilde{l})}(\cdot),$$
$$\widetilde{y}(0) = 2 * (1, 1.1, \ldots, 1 + 0.1 * \widetilde{k}, \ldots, 4.5)^T, \tag{66}$$

where $\widetilde{l} = 1, 2, 3$ and $\widetilde{k} = 0, 1, \ldots, 35$.

Case 4: $\widetilde{G}_k^{\widetilde{r}} < 0, \widetilde{e}(t) < 0, \widetilde{e}(t-\tau(t)) < 0, \widetilde{H}(\widetilde{e}(t)) > 0$, and $\widetilde{H}(\widetilde{e}(t-\tau(t))) > 0$. Let

$$\widetilde{G}_k^p = I_6 \otimes [-1.3, -0.3, -0.6; -0.3, -1.5, -0.5; -0.6, -0.5, -1.4],$$
$$\widetilde{h}^{IV}_{(\widetilde{l})}(\cdot) = \widetilde{h}^{III}_{(\widetilde{l})}(\cdot),$$
$$\widetilde{y}(0) = 2 * (-1, -1.1, \ldots, -(1 + 0.1 * \widetilde{k}), \ldots, -4.5)^T, \tag{67}$$

where $\widetilde{l} = 1, 2, 3$ and $\widetilde{k} = 0, 1, \ldots, 35$.

The other parameters of Cases 2–4 are same as that of Case 1. Similar to Case 1, we can obtain that in Cases 2–4, finite time $t^*_{(1)} \leq 1.58$, $t^*_{\text{new}(2)} \leq 1.14$, and $t^*_{\text{new}(3)} \leq 1.04$. The simulation trajectories are shown in Figures 2–5, which further testify that the obtained results of Corollary 1 are reasonable.

Example 3. Network (60) is still chosen in this example. According to Corollary 2, there exist the following four cases:

Case I: $\widetilde{G}_k^{\widetilde{r}} > 0, \widetilde{e}(t) > 0, \widetilde{e}(t-\tau(t)) > 0, \widetilde{H}(\widetilde{e}(t)) < 0$, and $\widetilde{H}(\widetilde{e}(t-\tau(t))) < 0$. Let

$$\widetilde{G}_k^p = I_6 \otimes [1.3, 0.3, 0.6; 0.3, 1.5, 0.5; 0.6, 0.5, 1.4],$$
$$\widetilde{y}(0) = 2(1, 1.1, \ldots, 1 + 0.1\widetilde{k}, \ldots, 4.5)^T,$$
$$\widetilde{h}^{I}_{(\widetilde{l})}(\cdot) = -\widetilde{h}_{(\widetilde{l})}(\cdot), \tag{68}$$

where $\widetilde{k} = 0, 1, \ldots, 35$, $\widetilde{h}_{(\widetilde{l})}(\cdot)$ is a nonlinear coupled function in Case I of Example 2 and $\widetilde{l} = 1, 2, 3$.

Case II: $\widetilde{G}_k^{\widetilde{r}} > 0$, $\widetilde{e}(t) < 0$, $\widetilde{e}(t-\tau(t)) < 0$, $\widetilde{H}(\widetilde{e}(t)) > 0$, and $\widetilde{H}(\widetilde{e}(t-\tau(t))) > 0$. Let

$$\widetilde{G}_k^p = I_6 \otimes [1.3, 0.3, 0.6; 0.3, 1.5, 0.5; 0.6, 0.5, 1.4],$$
$$\widetilde{y}(0) = 2(-1, -1.1, \ldots, -(1 + 0.1 * \widetilde{k}), \ldots, -4.5)^T,$$
$$\widetilde{h}^{II}_{(\widetilde{l})}(\cdot) = -\widetilde{h}_{(\widetilde{l})}(\cdot). \tag{69}$$

Case III: $\widetilde{G}_k^{\widetilde{r}} < 0$, $\widetilde{e}(t) > 0$, $\widetilde{e}(t-\tau(t)) > 0$, $\widetilde{H}(\widetilde{e}(t)) > 0$, and $\widetilde{H}(\widetilde{e}(t-\tau(t))) > 0$. Let

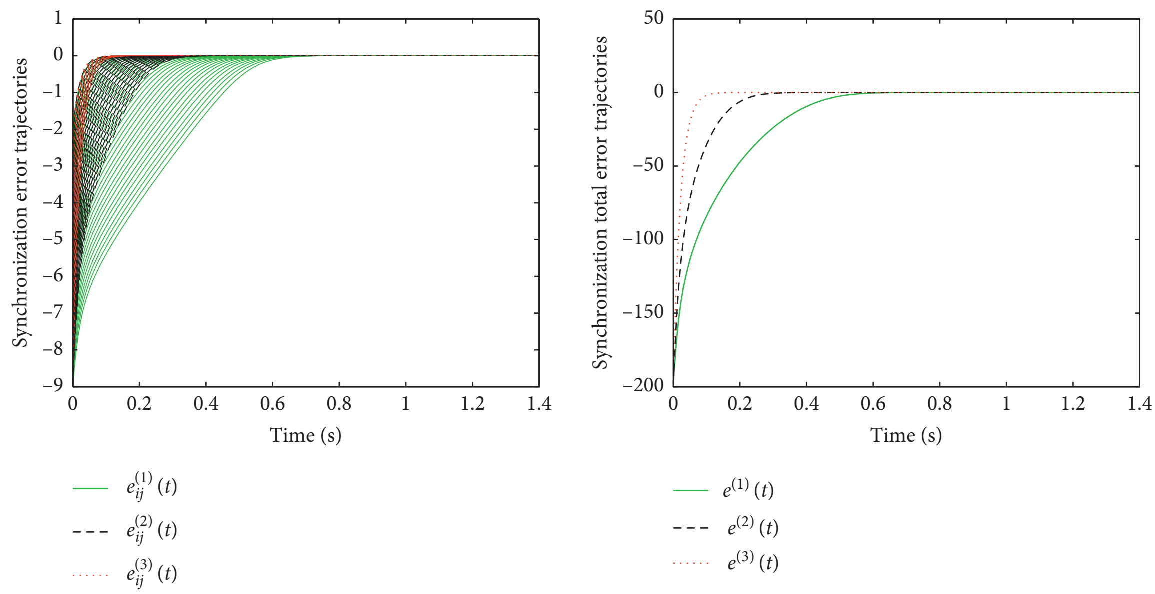

Figure 3: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case 2 of Example 2.

$$\widetilde{G}_k^p = I_6 \otimes [-1.3, -0.3, -0.6; -0.3, -1.5, -0.5; -0.6, -0.5, -1.4],$$
$$\widetilde{y}(0) = 2(1, 1.1, \ldots, 1 + 0.1\widetilde{k}, \ldots, 4.5)^T,$$
$$\widetilde{h}^{III}_{(\widetilde{l})}(\cdot) = \widetilde{h}_{(\widetilde{l})}(\cdot). \tag{70}$$

Case IV: $\widetilde{G}_k^{\widetilde{r}} < 0$, $\widetilde{e}(t) < 0$, $\widetilde{e}(t - \tau(t)) < 0$, $\widetilde{H}(\widetilde{e}(t)) < 0$, and $\widetilde{H}(\widetilde{e}(t - \tau(t))) < 0$. Let

$$\widetilde{G}_k^p = I_6 \otimes [-1.3, -0.3, -0.6; -0.3, -1.5, -0.5; -0.6, -0.5, -1.4],$$
$$\widetilde{y}(0) = 2(-1, -1.1, \ldots, -(1 + 0.1\widetilde{k}), \ldots, -4.5)^T,$$
$$\widetilde{h}^{IV}_{(\widetilde{l})}(\cdot) = \widetilde{h}_{(\widetilde{l})}(\cdot). \tag{71}$$

In Cases I–IV, according to Theorem 1, let $\widetilde{\beta}(\chi(\widetilde{h}_{(1)})) = \widetilde{\beta}(\chi(\widetilde{h}_{(2)})) = \widetilde{\beta}(\chi(\widetilde{h}_{(2)})) = 1$; then, we have $\Xi_p^k = 4.5 * I_{18}$, $\widehat{Q} = I_2$, $\widehat{\theta}_p = 1$, $\widehat{F}^k = 1.5 \otimes I_{36}$, $\tau_M = \rho = 0.1$, $\widetilde{\alpha} = 0.5$, $\widehat{\eta}_i^{k,p} = 2$, $\widetilde{v}_p = 1$, $V(\widetilde{e}(0), 0, \widetilde{r}(0)) = 99$, $t^*_{(1)} \leq 1.58$, $t^*_{(2)} \leq 1.58$, and $t^*_{(3)} \leq 1.58$. The simulation trajectories are shown in Figures 6–9, and we can observe that with the increase of $\chi(\widetilde{h})$, synchronization dynamics of network (60) with controller (6) becomes better and synchronization convergence time becomes shorter.

It is a pity that $t^*_{(1)}$ and $t^*_{(2)}$ above cannot reflect the fact. In order to more accurately estimate synchronization convergence time, according to Corollary 2, let $\widetilde{\beta}_{\text{new}}(\chi(\widetilde{h}_{(1)})) = 1.2$ and $\widetilde{\beta}_{\text{new}}(\chi(\widetilde{h}_{(2)})) = 1.1$, where $\chi(\widetilde{h}_{(1)}) > \chi(\widetilde{h}_{(2)}) > 0$. Thus, $t^*_{\text{new}(1)} \leq 1.32$, $t^*_{\text{new}(2)} \leq 1.44$, and $t^*_{\text{new}(3)} \leq 1.58$. It is clear that $t^*_{\text{new}(1)} < t^*_{\text{new}(2)} < t^*_{\text{new}(3)}$, which shows that for addressed network (60) with controller (6), the settling finite time t^*_{C2} given by Corollary 2 can more accurately estimate synchronization convergence time.

Example 4. In network (60) above, $\widehat{s} = 3$. Therefore, network (52) includes three subsystems. Assume that the 1st subsystem with controller (6) can achieve synchronization in finite time $t^*_{(1)}$ and the other two subsystems without controller (6) cannot achieve synchronization, where $\widetilde{r} = 1$. That is, in controller (6), $\widetilde{r} = 1$.

Next, we design controller (6) to make the overall network (60) realize synchronization as follows. Let

$$\widetilde{G}_k^1 = I_6 \otimes [1.5, 0.3, 0.6; 0.3, 1.6, 0.5; 0.6, 0.5, 1.4],$$
$$\widetilde{G}_k^2 = I_6 \otimes [1.4, 0.2, 0.5; 0.2, 1.5, 0.3; 0.5, 0.3, 1.6],$$
$$\widetilde{G}_k^3 = I_6 \otimes [1.3, 0.2, 0.4; 0.2, 1.5, 0.5; 0.4, 0.5, 1.4],$$
$$\widetilde{h}\left(\widetilde{y}_j(t)\right) = \left[\tanh\left(\widetilde{y}_{j1}(t)\right), \tanh\left(\widetilde{y}_{j2}(t)\right)\right]^T,$$
$$\widetilde{h}\left(\widetilde{y}_j(t - \tau(t))\right) = \left[\tanh\left(\widetilde{y}_{j1}(t - \tau(t))\right), \tanh\left(\widetilde{y}_{j2}(t - \tau(t))\right)\right]^T,$$
$$\widehat{Q} = I_2,$$
$$\widehat{\theta}^p = 1,$$
$$\widehat{F}^k = 0.8 \otimes I_{36}. \tag{72}$$

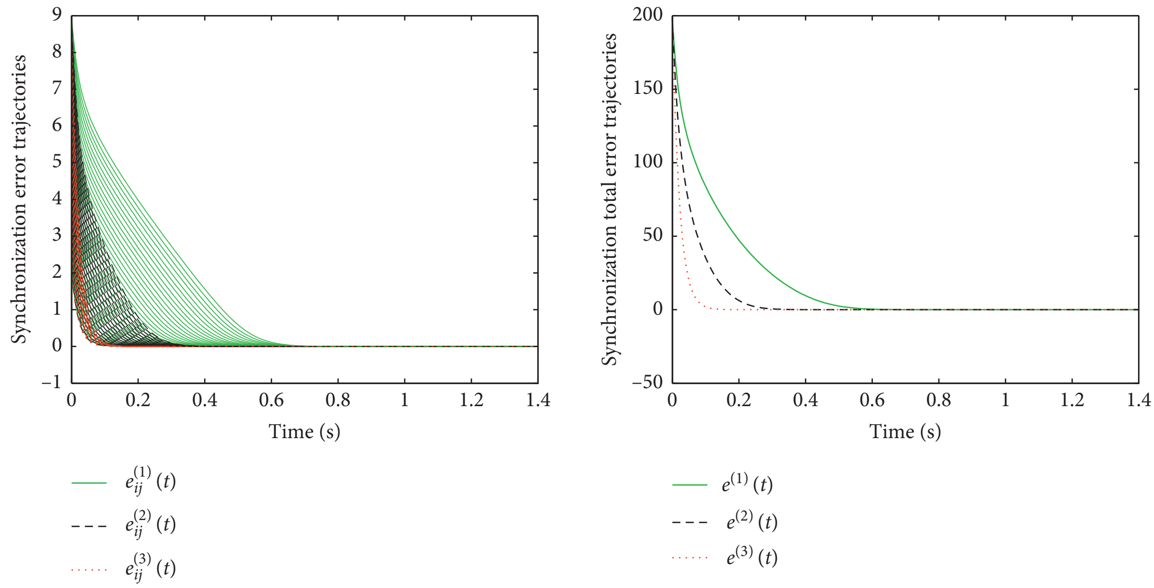

Figure 4: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case 3 of Example 2.

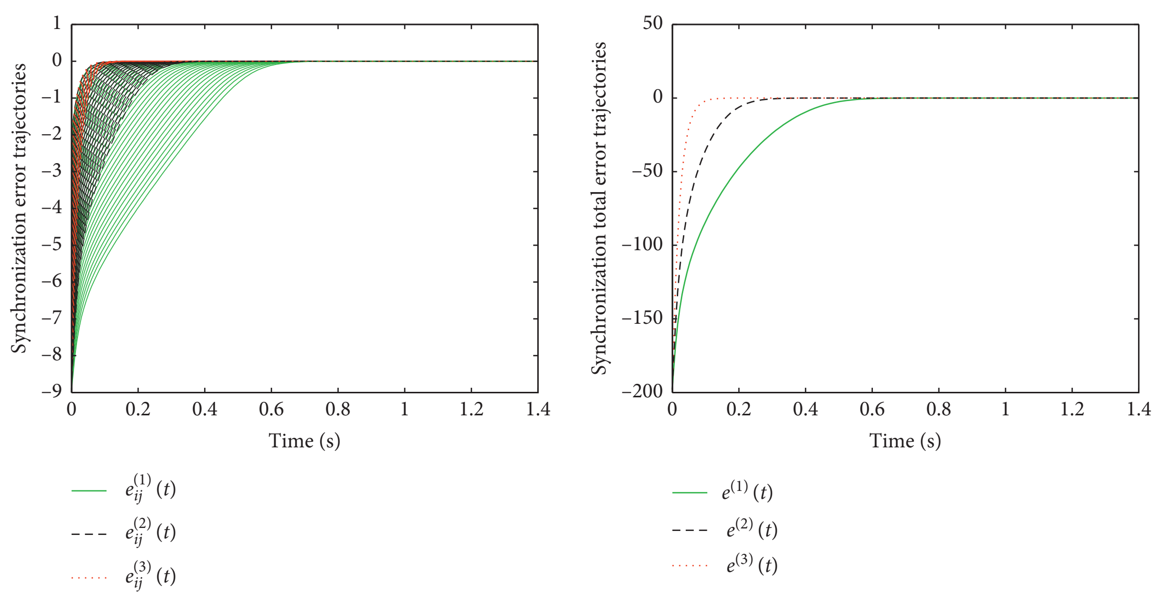

Figure 5: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case 4 of Example 2.

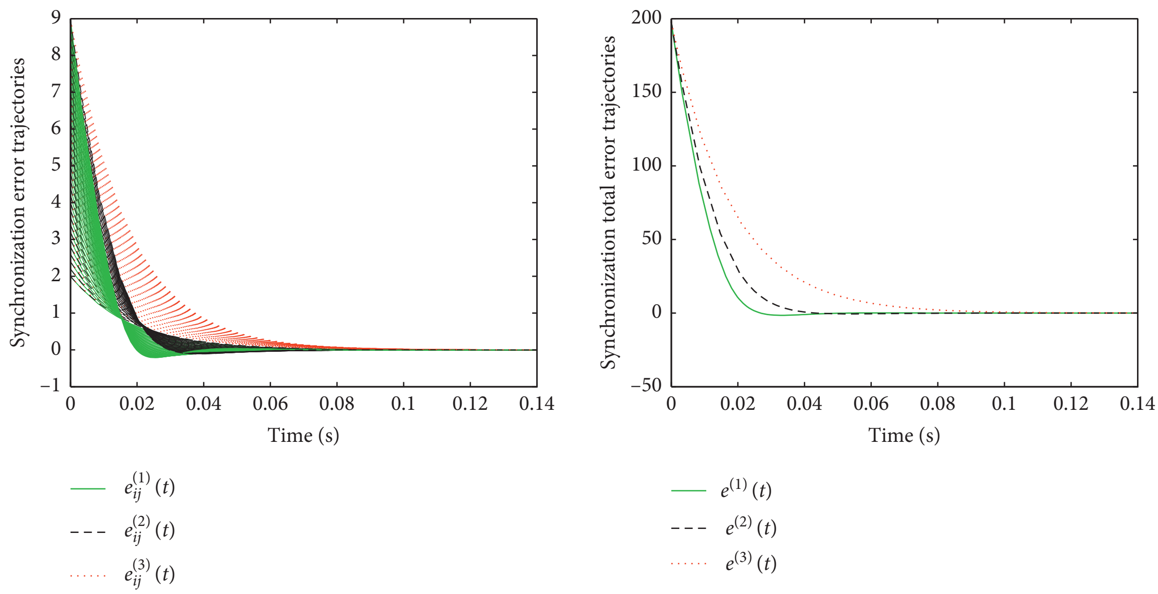

Figure 6: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case I of Example 3.

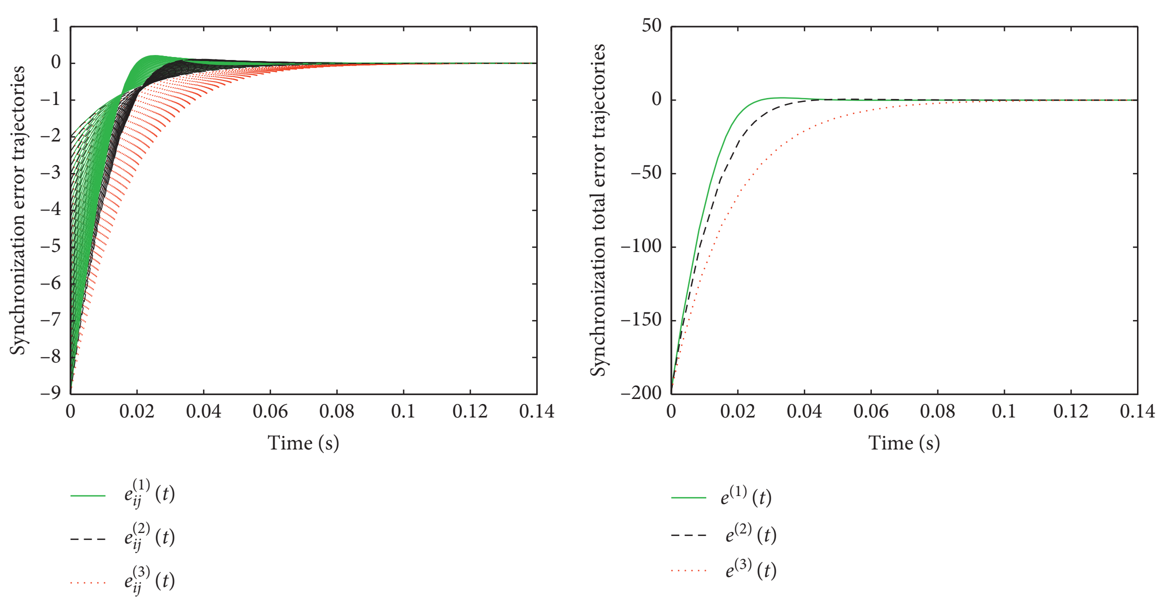

Figure 7: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case II of Example 3.

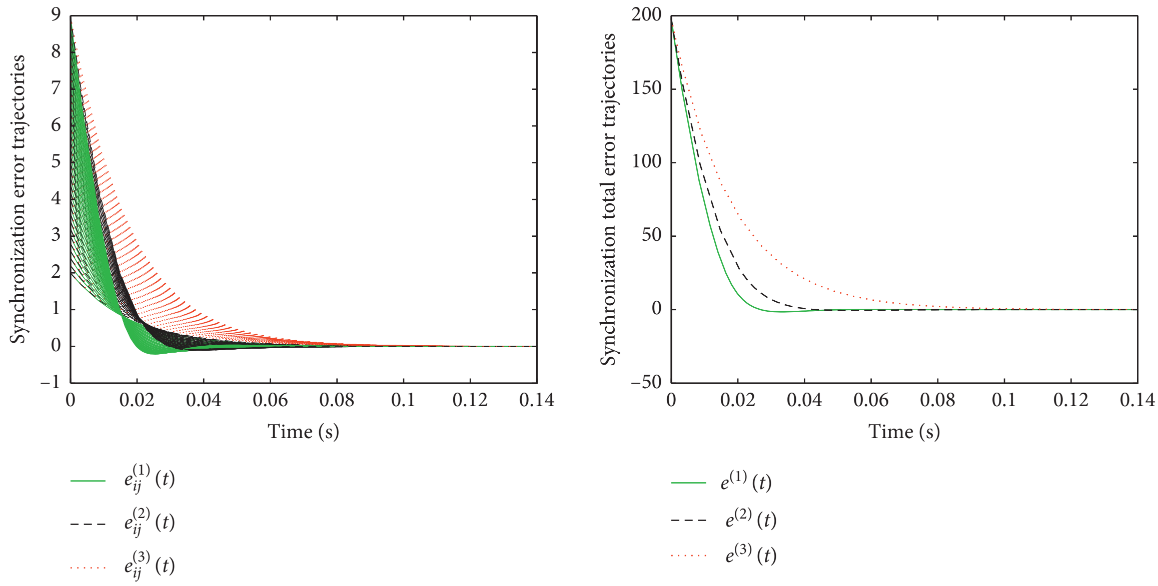

Figure 8: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case III of Example 3.

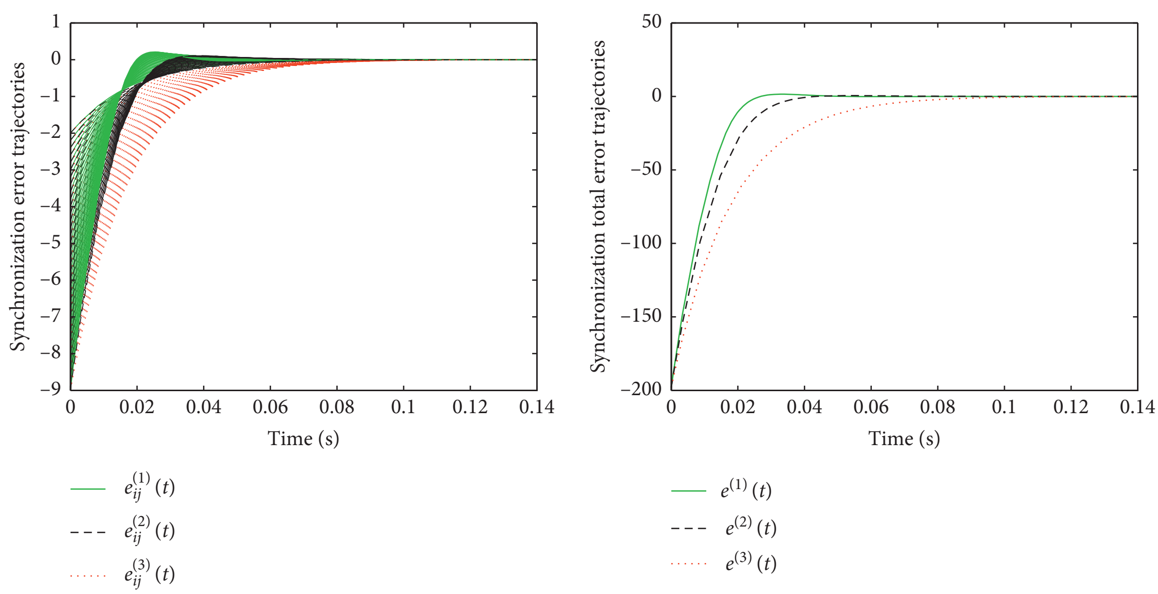

Figure 9: Synchronization error trajectories and synchronization total error trajectories of network (60) with controller (6) for Case IV of Example 3.

According to the analysis of Remark 14, we know the following:

(a) If $\mathscr{L}V_{\widehat{p}}(\widetilde{e}(t),t)\leq 0$ and $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t),t)>0$ can make $\mathbb{E}[\mathscr{L}V(\widetilde{e}((t)),t,\widehat{r})]\leq 0$ hold, where $\widehat{p}=1$, $\widehat{q}=2,3,$ and

$$
\begin{aligned}
&\mathbb{E}[\mathscr{L}V(\widetilde{e}(t),t,\widehat{r})]=\mathbb{E}\left[\mathscr{L}_{\widehat{p}}V\left(\widetilde{e}^{(\widehat{p})}\left(t_{\widehat{r}_1}\right),t_{\widehat{r}_1},\widehat{r}_1\right)+\mathscr{L}_{\widehat{q}}V\left(\widetilde{e}^{(\widehat{q})}\left(t_{\widehat{r}_2}\right),t_{\widehat{r}_2},\widehat{r}_2\right)\right],\\
&\widehat{r}=\widehat{r}_1\cup\widehat{r}_2,\widehat{r}_1=\left\{(t_1,1),\left(t_{\widehat{s}+1},1\right),\ldots\right\},\\
&\widehat{r}_2=\left\{(t_2,2),(t_3,3),\ldots,\left(t_{\widehat{s}},\widehat{s}\right),\left(t_{\widehat{s}+2},2\right),\ldots\right\},\\
&\left\{t_1\in[0,\Delta t],t_2\in[\Delta t,2\Delta t],\ldots,t_{\widehat{s}}\in[(\widehat{s}-1)\Delta t,\widehat{s}\Delta t],t_{\widehat{s}+1}\in[\widehat{s}\Delta t,(\widehat{s}+1)\Delta t],\ldots\right\},\\
&S=\left\{(t_1,1),(t_2,2),\ldots,\left(t_{\widehat{s}},\widehat{s}\right),\left(t_{\widehat{s}+1},1\right),\left(t_{\widehat{s}+2},2\right),\ldots\right\},
\end{aligned}
\tag{73}
$$

overall network (60) can achieve synchronization

(b) In order to make every subsystem of network (60) have different synchronization dynamic behaviors, every subsystem is an independent subsystem

(c) In every subsystem, there is no Markovian switching phenomenon

(d) If $\widehat{s}$ subsystems are coupled by Markovian switching parameter $\widetilde{g}_{ij}^{k}(r(t))$, the overall network (60) is actually one dynamical system

(e) Although there exists Markovian switching parameter $\widetilde{g}_{ij}^{k}(r(t))$, synchronization error $\widetilde{e}_{ij}(t)$ of the error system of the overall network (60) is a smooth function, where $i=1,2,\ldots,18$ and $j=1,2$

Actually, according to the principles above, it is very difficult to design controller (6) to make addressed overall network (60) achieve synchronization. Next, we will focus on the synchronization of the overall network (60) by the following steps:

Step 1: combining $\mathscr{L}V_{\widehat{p}}(\widetilde{e}(t),t)\leq 0$ and the proof of Theorem 1, we have

$$
\mathscr{L}V_{\widehat{p}}(\widetilde{e}(t),t)\leq\widetilde{e}^{T}(t)\widehat{\Theta}_1^{(1)}\widetilde{e}(t)+\widetilde{e}^{T}(t-\tau(t))\widehat{\Theta}_1^{(2)}\widetilde{e}(t-\tau(t))\leq 0,
\tag{74}
$$

where $\widehat{p}=1$ and

$$
\begin{aligned}
\widehat{\Theta}_1^{(1)}=&\left[\left(L^2\|\psi\|_2+\widetilde{L}^2\left\|\widetilde{\psi}_1\right\|_2\sum_{k=1}^{m}\widetilde{a}_k\right)I_N\otimes I_N+\frac{\widehat{\theta}^1}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\widehat{F}^k+I_N\otimes(\widehat{Q}\widetilde{A})\psi^{-1}(\widehat{Q}\widetilde{A})^T\right.\\
&\left.+\sum_{k=1}^{m}\widetilde{a}_k\left(\widetilde{G}_k^1\otimes\widehat{Q}\widetilde{\Gamma}_k\right)\widetilde{\psi}_1^{-1}\left(\widetilde{G}_k^1\otimes\widehat{Q}\widetilde{\Gamma}_k\right)^T+\sum_{k=1}^{m}\widehat{a}_k\left(\widetilde{G}_k^1\otimes\widehat{Q}\widehat{\Gamma}_k\right)\widehat{\psi}_1^{-1}\left(\widetilde{G}_k^1\otimes\widehat{Q}\widehat{\Gamma}_k\right)^T-2\sum_{k=1}^{m}\Xi_1^k\otimes\widehat{Q}\right]\leq 0,\\
\widehat{\Theta}_1^{(2)}=&\left[\sum_{k=1}^{m}\widehat{a}_k\left(\widetilde{L}^2\left\|\widehat{\psi}_1\right\|_2(I_N\otimes I_N)-\widehat{\theta}^1\widehat{F}^k\right)\right]\leq 0.
\end{aligned}
\tag{75}
$$

Thus, we can design Ξ_1^k, $\widehat{\theta}^1$, and $\widehat{F}^k$ to make the 1st subsystem of overall network (60) achieve global synchronization.

Step 2: if the $\widehat{q}$ subsystem of overall network (60) cannot achieve synchronization, then $\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t),t)>0$. Thus, similar to the proof of Theorem 1, it can be derived that

$$
\begin{aligned}
\mathscr{L}V_{\widehat{q}}(\widetilde{e}(t),t) &= V_t(\widetilde{e}(t),t) + V_{\widetilde{e}(t)}(\widetilde{e}(t),t)\left[\widetilde{A}\widetilde{F}(\widetilde{e}_i(t)) + \sum_{k=1}^{m}\sum_{j=1}^{N}\widetilde{a}_k\widetilde{g}_{ij}^{k,p}\widetilde{\Gamma}_k\widetilde{H}(\widetilde{e}_i(t)) + \sum_{k=1}^{m}\sum_{j=1}^{N}\widehat{a}_k\widetilde{g}_{ij}^{k,p}\widehat{\Gamma}_k\widetilde{H}\left(\widetilde{e}_j(t-\tau(t))\right)\right],\\
&= \frac{\widetilde{v}_p\widehat{\theta}^p}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\left[\widetilde{e}^T(t)\widehat{F}^k\widetilde{e}(t) - (1-\dot{\tau}(t))\widetilde{e}^T(t-\tau(t))\widehat{F}^k\widetilde{e}(t-\tau(t))\right] + 2\widetilde{v}_p\sum_{i=1}^{N}\widetilde{e}_i^T(t)\widehat{Q}\widetilde{A}\widetilde{F}(\widetilde{e}_i(t))\\
&\quad + 2\widetilde{v}_p\sum_{i=1}^{N}\widetilde{e}_i^T(t)\widehat{Q}\sum_{k=1}^{m}\sum_{j=1}^{N}\widetilde{a}_k\widetilde{g}_{ij}^{k,p}\widetilde{\Gamma}_k\widetilde{H}\left(\widetilde{e}_j(t)\right) + 2\widetilde{v}_p\sum_{i=1}^{N}\widetilde{e}_i^T(t)\widehat{Q}\sum_{k=1}^{m}\sum_{j=1}^{N}\widehat{a}_k\widetilde{g}_{ij}^{k,p}\widehat{\Gamma}_k\widetilde{H}\left(\widetilde{e}_j(t-\tau(t))\right),\\
&= \frac{\widetilde{v}_p\widehat{\theta}^p}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\left[\widetilde{e}^T(t)\widetilde{A}\widetilde{e}(t) - (1-\dot{\tau}(t))\widetilde{e}^T(t-\tau(t))\widehat{F}^k\widetilde{e}(t-\tau(t))\right] + 2\widetilde{v}_p\sum_{k=1}^{m}\left(\widetilde{e}^T(t)\left(I_N\otimes\widehat{Q}\widetilde{A}\right)\widetilde{F}(\widetilde{e}(t))\right.\\
&\quad \left. + \widetilde{a}_k\widetilde{e}^T(t)\left(\widetilde{G}_k^p\otimes\widehat{Q}\widetilde{\Gamma}_k\right)\widetilde{H}(\widetilde{e}(t)) + \widehat{a}_k\widetilde{e}^T(t)\left(\widetilde{G}_k^p\otimes\widehat{Q}\widehat{\Gamma}_k\right)\widetilde{H}(\widetilde{e}(t-\tau(t)))\right) > 0,
\end{aligned}
\tag{76}
$$

where $p = 1$.

Step 3: according to the proof of Theorem 1, we can obtain that

$$
\mathscr{L}_{\widehat{p}}V\left(\widetilde{e}^{(\widehat{p})}\left(t_{\widehat{r}_1}\right), t_{\widehat{r}_1}, \widehat{p}\right)\right) \le \widetilde{v}_{\widehat{p}}\left[\widetilde{e}^{(\widehat{p})T}(t)\widetilde{\Theta}_{\widehat{p}}^{(1)}\widetilde{e}^{(\widehat{p})}(t) + \widetilde{e}^{(\widehat{p})T}(t-\tau(t))\widetilde{\Theta}_{\widehat{p}}^{(2)}\widetilde{e}^{(\widehat{p})}(t-\tau(t))\right],
\tag{77}
$$

where $\widehat{p} = 1$, $\widehat{q} = 2, 3$, and

$$
\begin{aligned}
\widetilde{\Theta}_{\widehat{p}}^{(1)} &= \left[\left(L^2\|\psi\|_2 + \widetilde{L}^2\left\|\widetilde{\psi}_{\widehat{p}}\right\|_2\sum_{k=1}^{m}\widetilde{a}_k\right)I_N\otimes I_n + \frac{\widehat{\theta}^{\widehat{p}}}{1-\rho}\sum_{k=1}^{m}\widehat{a}_k\widehat{F}^k + I_N\otimes(\widehat{Q}\widetilde{A})\psi^{-1}(\widehat{Q}\widetilde{A})^T\right.\\
&\quad + \sum_{\widehat{q}=2}^{\widehat{s}}\frac{\delta_{\widehat{p}\widehat{q}}\widetilde{v}_{\widehat{q}}}{\widetilde{v}_{\widehat{p}}}I_N\otimes\widehat{Q} + \sum_{k=1}^{m}\widetilde{a}_k\left(\widetilde{G}_k^{\widehat{p}}\otimes\widehat{Q}\widetilde{\Gamma}_k\right)\widetilde{\psi}_{\widehat{p}}^{-1}\left(\widetilde{G}_k^{\widehat{p}}\otimes\widehat{Q}\widetilde{\Gamma}_k\right)^T\\
&\quad \left. + \sum_{k=1}^{m}\widehat{a}_k\left(\widetilde{G}_k^{\widehat{p}}\otimes\widehat{Q}\widehat{\Gamma}_k\right)\widehat{\psi}_{\widehat{p}}^{-1}\left(\widetilde{G}_k^{\widehat{p}}\otimes\widehat{Q}\widehat{\Gamma}_k\right)^T - 2\sum_{k=1}^{m}\Xi_{\widehat{p}}^k\otimes\widehat{Q}\right],
\end{aligned}
\tag{78}
$$

$$
\widetilde{\Theta}_{\widehat{p}}^{(2)} = \left[\sum_{k=1}^{m}\widehat{a}_k\left(\widetilde{L}^2\left\|\widehat{\psi}_{\widehat{p}}\right\|_2(I_N\otimes I_n) - \widehat{\theta}^{\widehat{p}}\widehat{F}^k\right)\right].
\tag{79}
$$

Substituting the above designed Ξ_1^k, $\widehat{\theta}^1$, and $\widehat{F}^k$ located in Step 1 into (78) and (79), we can make $\widetilde{\Theta}_{\widehat{p}}^{(1)} \le 0$ and $\widetilde{\Theta}_{\widehat{p}}^{(2)} \le 0$ hold. Thus $\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(\widehat{p})}(t_{\widehat{r}_1}), t_{\widehat{r}_1}, \widehat{p}) \le 0$.

Step 4: by the analysis of Remark 14, it can be obtained that if overall network (52) can achieve synchronization, there must be

$$
\begin{aligned}
\mathbb{E}[\mathscr{L}V((\widetilde{e}(t), t, \widehat{r}))] &= \mathbb{E}\left[\mathscr{L}_{\widehat{p}}V\left(\widetilde{e}^{(\widehat{p})}\left(t_{\widehat{r}_1}\right), t_{\widehat{r}_1}, \widehat{r}_1\right)\right.\\
&\quad \left. + \mathscr{L}_{\widehat{q}}V\left(\widetilde{e}^{(\widehat{q})}\left(t_{\widehat{r}_2}\right), t_{\widehat{r}_2}, \widehat{r}_2\right)\right)\right] \le 0.
\end{aligned}
\tag{80}
$$

From Step 3, it can be derived that if $\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(\widehat{p})}(t_{\widehat{r}_1}), t_{\widehat{r}_1}, \widehat{p}) \leq 0$, then $\mathbb{E}[\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(\widehat{p})}(t_{\widehat{r}_1}), t_{\widehat{r}_1}, \widehat{r}_1)] \leq 0$ holds. Combined with formula (17), it cannot be derived that under $\mathbb{E}[\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(\widehat{p})}(t_{\widehat{r}_1}), t_{\widehat{r}_1}, \widehat{r}_1)] \leq 0$, then $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] \leq 0$. This means that if $\mathbb{E}[\mathscr{L}_{\widehat{p}}V(\widetilde{e}^{(\widehat{p})}(t_{\widehat{r}_1}), t_{\widehat{r}_1}, \widehat{r}_1)] \leq 0$, there is $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] \leq 0$ or $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] > 0$. In the case of $\mathbb{E}[\mathscr{L}V(\widetilde{e}(t), t, \widehat{r})] \leq 0$, overall network (52) can achieve synchronization. Otherwise, overall network (52) cannot achieve synchronization.

5. Concluding Remarks

In this paper, we mainly focus on the impact of the nonlinearity of nonlinear coupled function on finite-time synchronization dynamics and synchronization convergence time for a class of NCMWCNs with Markovian switching and time-varying delay. In order to make the addressed network achieve global synchronization in finite time, we design a kind of finite-time synchronization controller. And based on the finite-time synchronization controller, we derive two kinds of finite-time estimation approaches and find that the impact of synchronization dynamics on finite time and synchronization convergence time can be reflected by the obtained settling finite time t^*. The proposed finite-time estimation methods can reflect how the nonlinearity of nonlinear coupled function impacts synchronization dynamics and synchronization convergence time of the addressed network. Furthermore, we investigate the relationship between Markovian switching parameters and synchronization problems of subsystems and the overall system.

It is worthy to note that the obtained finite-time estimation methods heavily depend on the initial conditions of the NCMWCNs with Markovian switching and time-varying delay. This shows that if the initial conditions of the addressed system are not accurately obtained, the proposed finite-time synchronization control methods are invalid.

Moreover, in NCMWCNs (2), nonlinear coupled function $\widetilde{h}(\cdot)$ must satisfy the Lipschitz condition. Compared with sector-bounded nonlinearity condition, Lipschitz condition is a special case [74]. If $\widetilde{h}(\cdot)$ satisfies sector-bounded nonlinearity condition, how can we design finite-time and fixed-time synchronization controllers of NCMWCNs with Markovian switching and time-varying delay? Furthermore, the fixed-time control can effectively overcome the faultiness [75, 76]. Therefore, it is necessary to explore fixed-time synchronization control and fixed-time synchronization dynamic for some classes of NCMWCNs with Markovian switching and time-varying delay. How can we analyze the impact of nonlinearity of nonlinear coupled function on finite-time and fixed-time synchronization dynamics and synchronization convergence time? In addition, if $\widetilde{h}(\cdot)$ is a discontinuous right-hand function, how can we investigate the related finite-time and fixed-time problems of nonlinear coupled delayed multiweighted complex networks with Markovian switching? These are desirable in future studies.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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