Research Article

Linear Complexity of Generalized Cyclotomic Sequences of Order 4 over $\mathbb{F}_l$

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Generalized cyclotomic sequences of period pq have several desirable randomness properties if the two primes p and q are chosen properly. In particular, Ding deduced the exact formulas for the autocorrelation and the linear complexity of these sequences of order 2. In this paper, we consider the generalized sequences of order 4. Under certain conditions, the linear complexity of these sequences is developed over a finite field $\mathbb{F}_l$. The results show that, in many cases, they have high linear complexity.

1. Introduction

Let l be a prime number and $\mathbb{F}_l$ denote a finite field with l elements. A sequence $\lambda^\infty = \lambda_0\lambda_1\cdots\lambda_{n-1}\cdots$ is called to be n -periodic if $\lambda_i = \lambda_{i+n}$ for all $i \geq 0$. Periodic sequences with certain properties are widely used in software testing, radar systems, stream ciphers, and so on. For cryptography applications, the linear complexity is an important factor. It is defined to be the length of the shortest linear feedback shift register which generates this sequence. For cyclotomic sequences, many researchers are devoted to studying their random properties [1–8]. The generalized cyclotomic sequences have been described and studied [9–11, 12, 13] for the past decades. A number of periodic sequences with attractive random properties have been constructed [6, 14–20].

Let p and q be two distinct odd primes with $\gcd(p-1, q-1) = d$. Define $n = pq$ and $e = (p-1)(q-1)/d$. The Chinese remainder theorem guarantees that there exists a common primitive root g of both p and q . Let x be an integer satisfying

$$\begin{aligned} x &\equiv g \pmod{p}, \\ x &\equiv 1 \pmod{q}. \end{aligned} \quad (1)$$

Whiteman proved that [21]

$$\mathbb{Z}_n^* = \{g^i x^j : i = 0, 1, \dots, e-1; j = 0, 1, \dots, d-1\}, \quad (2)$$

where $\mathbb{Z}_n^*$ denotes the set of all invertible elements of the residue class ring $\mathbb{Z}_n$.

The generalized cyclotomic classes D_j ($0 \leq j \leq d-1$) of order d with respect to n are defined by [21]

$$D_j = \{g^i x^j : i = 0, 1, \dots, e-1\}, \quad 0 \leq j \leq d-1, \quad (3)$$

where the multiplication is that of $\mathbb{Z}_n$. Clearly, the cosets D_j depend on the choice of the common primitive root g if $d \geq 3$. It is not hard to prove that [21]

$$\mathbb{Z}_n^* = \bigcup_{j=0}^{d-1} D_j, D_i \cap D_j = \emptyset, \quad \text{for } i \neq j, \quad (4)$$

where $\emptyset$ denotes the empty set. Define

$$\begin{aligned} P &= \{p, 2p, \dots, (q-1)p\} = p\mathbb{Z}_q^*, \\ Q &= \{q, 2q, \dots, (p-1)q\} = q\mathbb{Z}_p^*. \end{aligned} \quad (5)$$

Then,

$$\mathbb{Z}_n = \bigcup_{i=0}^{d-1} D_i \cup \{0\} \cup P \cup Q. \quad (6)$$

Let S be a nonempty subset of $\{0, 1, \dots, d-1\}$ and

$$\sum_S = \bigcup_{i \in S} D_i. \quad (7)$$

We define the binary sequence $\lambda^\infty = \lambda_0 \lambda_1 \dots \lambda_{n-1} \dots$ of period n as follows:

$$\lambda_i = \begin{cases} 1, & \text{if } i \bmod n \in (P \cup \sum_S), \\ \rho, & \text{if } i \bmod n \in \{0\}, \\ 0, & \text{otherwise,} \end{cases} \quad (8)$$

where $i \geq 0$ and $\rho \in \{0, 1\}$. For $\rho = 0$ and $S = \{0\}$, the linear complexity of these sequences over $\mathbb{F}_2$ has been calculated by Ding [16] with $d = 2$ and Hu et al. [20] with $d = 4$. Furthermore, for $S = \{0, 2, \dots, d-2\}$ and $\rho = 0$, Ding [9] determined the linear complexity of the two-prime sequences over a finite field $\mathbb{F}_l$ and used these sequences to construct several classes of cyclic codes over $\mathbb{F}_l$ with optimal or almost optimal property, where $\gcd(l, n) = 1$. In this paper, we only consider the case $\rho = 1$, $d = 4$, and $S = \{0, 1\}$. Under the assumption that $(n-1)/4 \equiv 0 \pmod{l}$ or $l \nmid D_0$, we calculate the linear complexity of these sequences over the finite field $\mathbb{F}_l$. The results show that, in many cases, these sequences have high linear complexity.

This paper is organized as follows. Section 2 presents basic notations and results of periodic sequences and the generalized cyclotomy [21]. In Section 3, we give an expression for the linear complexity of the generalized cyclotomic sequences over $\mathbb{F}_l$. In the last section, we present concluding remarks of this paper.

2. Preliminaries

Firstly, we give the definition and formula of linear complexity of periodic sequences over a finite field. See [22] or [23] for more details.

Let l be a prime number and $\lambda^\infty = \lambda_0 \lambda_1 \dots \lambda_{n-1} \dots$ be a periodic sequence over $\mathbb{F}_l$ with period n , where $\lambda_i \in \mathbb{F}_l$ for $i \geq 0$. The sequence λ^∞ can be viewed as a power series

$$\Lambda^\infty(x) = \sum_{i=0}^{\infty} \lambda_i x^i = \frac{\Lambda(x)}{1-x^n}, \quad (9)$$

$$\Lambda(x) = \lambda_0 + \lambda_1 x + \dots + \lambda_{n-1} x^{n-1} \in \mathbb{F}_l[x],$$

in the power series ring $\mathbb{F}_l[[x]]$.

Let $h(x) = \gcd(\Lambda(x), 1-x^n)$, then

$$\Lambda^\infty(x) = \frac{w(x)}{v(x)},$$

$$v(x) = \frac{1-x^n}{h(x)}, \quad (10)$$

$$w(x) = \frac{\Lambda(x)}{h(x)},$$

where $w(x), v(x), h(x) \in \mathbb{F}_l[x]$.

Definition 1. The polynomial $v(x)$ is called the minimal polynomial of the periodic sequence λ^∞ over $\mathbb{F}_l$. The $\deg v(x) = n - \deg h(x)$ is called the linear complexity of the sequence λ^∞ over $\mathbb{F}_l$, which is denoted by $L_l(\lambda)$.

Indeed, $L_l(\lambda)$ is the length of the shortest linear feedback shift register which generates the sequence λ^∞ .

If $\gcd(n, l) = 1$, then $1-x^n$ has n distinct roots ζ_n^i ($0 \leq i \leq n-1$) in the algebraic closure Ω_l of $\mathbb{F}_l$, where ζ_n denotes the n -th primitive root of unity. It is easy to see that

$$L_l(\lambda) = n - \left| \{i: 0 \leq i \leq n-1, \Lambda(\zeta_n^i) = 0\} \right|. \quad (11)$$

In order to determine the linear complexity of generalized cyclotomic sequences, we introduce generalized cyclotomy.

Let symbols be the same as in the introduction and $d = \gcd(p-1, q-1) = 4$. The generalized cyclotomic numbers of order 4 with respect to n is defined by

$$(i, j) = |(D_i + 1) \cap D_j|, \quad \text{for } 0 \leq i, j \leq 3. \quad (12)$$

By the well-known theorem ([24], P.128), there are exactly two representations of n in the form $n = a^2 + 4b^2$ with $a \equiv 1 \pmod{4}$ and the sign of b indeterminate.

Let g_1 and g_2 be a fixed primitive root of p and q , respectively. For $i = 1, 2$, let x_i and y_i be the integers given uniquely by

$$\begin{aligned} p &= x_1^2 + 4y_1^2, \\ q &= x_2^2 + 4y_2^2, \\ x_1 &\equiv x_2 \equiv 1 \pmod{4}, \end{aligned} \quad (13)$$

$$\begin{aligned} 2y_1 &\equiv -(-g_1)^{(p-1)/4} x_1 \pmod{p}, \\ 2y_2 &\equiv -(-g_2)^{(q-1)/4} x_2 \pmod{q}. \end{aligned}$$

Define a and b to be integers satisfying

$$a = x_1 x_2 + 4 \left(\frac{2}{p} \right) \left(\frac{2}{q} \right) y_1 y_2, \quad (14)$$

$$b = x_2 y_1 - \left(\frac{2}{p} \right) \left(\frac{2}{q} \right) x_1 y_2,$$

where $(\div)$ denotes the Legendre symbol.

It is clear that $a \equiv 1 \pmod{4}$ and $n = a^2 + 4b^2$ is one of the two representations of n . The following lemma shows that the generalized cyclotomic numbers of order 4 with respect to n depend uniquely on this representation.

Lemma 1 (see [19], Theorem IV.1.). *Let $p \equiv q \equiv 1 \pmod{4}$ be two distinct primes with the fixed primitive roots g_1 and g_2 , respectively. $M = ((p-2)(q-2)-1)/4$, and a and b are the integers defined in (14).*

If $p \not\equiv q \pmod{8}$, then in Table 1, $8A = -a + 2M + 3$, $8B = -a - 4b + 2M - 1$, $8C = 3a + 2M - 1$, $8D = -a + 4b + 2M - 1$, and $8E = a + 2M + 1$.

If $p \equiv q \pmod{8}$, then in Table 2, $8A = 3a + 2M + 5$, $8B = -a + 4b + 2M + 1$, $8C = -a + 2M + 1$, $8D = -a - 4b + 2M + 1$, and $8E = a + 2M - 1$.

3. Generalized Cyclotomic Sequences of Order 4

Throughout this section, let p and q be two distinct odd primes with $\gcd(p-1, q-1) = 4$. Define $n = pq$ and $e = (p-1)(q-1)/4$. Let l be a prime and satisfy $\gcd(l, n) = 1$.

The generalized cyclotomic sequence λ^∞ of order 4 of period n is defined by

$$\lambda_i = \begin{cases} 1, & \text{if } i \bmod n \in \{0\} \cup P \cup D_0 \cup D_1, \\ 0, & \text{otherwise,} \end{cases} \quad (15)$$

where D_0 and D_1 are defined by (3) and $P = p\mathbb{Z}_q^*$. Here, in this paper, we treat it as a sequence over a finite field $\mathbb{F}_l$, where $\gcd(l, n) = 1$.

Denote $\text{ord}_n(l)$ the multiplicative order of l modulo n . Let ζ_n be an n -th primitive root of unity over $\mathbb{F}_{\text{ord}_n(l)}$. For the sequence λ^∞ defined by (15), we know

$$\Lambda(x) = \sum_{i=0}^{n-1} \lambda_i x^i = 1 + \sum_{i \in P} x^i + \sum_{i \in D_0} x^i + \sum_{i \in D_1} x^i, \quad (16)$$

$$\Lambda(1) = 1 + \frac{(p+1)(q-1)}{2} \bmod l. \quad (17)$$

Define δ as follows:

$$\delta = \begin{cases} 1, & \text{if } l \mid 1 + \frac{(p+1)(q-1)}{2}, \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

Note that the generalized cyclotomic classes of order 2 are given by

$$\begin{aligned} C_0 &= D_0 \cup D_2, \\ C_1 &= D_1 \cup D_3. \end{aligned} \quad (19)$$

Define $\eta_0 = \sum_{i \in C_0} \zeta_n^i$. The following lemma has been proven in [9].

Lemma 2 (see [9], Lemma 3.13). *If $n \equiv 1 \pmod{4}$, then we have*

$$\eta_0(1 - \eta_0) = -\frac{n-1}{4}. \quad (20)$$

Hence, $\eta_0 \in \{0, 1\}$ if and only if $(n-1)/4 \equiv 0 \pmod{l}$.

To compute the linear complexity of λ^∞ , we need to compute $\gcd(x^n - 1, \Lambda(x))$. For this purpose, we require a number of auxiliary results.

TABLE 1: $p \not\equiv q \pmod{8}$.

(i, j)	0	1	2	3
0	A	B	C	D
1	E	E	D	B
2	A	E	A	E
3	E	D	B	E

TABLE 2: $p \equiv q \pmod{8}$.

(i, j)	0	1	2	3
0	A	B	C	D
1	B	D	E	E
2	C	E	C	E
3	D	E	E	B

Lemma 3 (see [16], Lemma 5). *Let m be the least common multiple of two positive integers m_1 and m_2 . The system of congruences*

$$\begin{aligned} x &\equiv a_1 \pmod{m_1}, \\ x &\equiv a_2 \pmod{m_2}, \end{aligned} \quad (21)$$

has solutions if and only if

$$\gcd(m_1, m_2) \mid a_1 - a_2, \quad (22)$$

where $a \mid b$ means that a divides b . When condition (22) holds, the system of the congruences of (21) has only one solution modulo m .

Lemma 4. *For $a \in D_j$, then $aD_i = D_{(i+j) \bmod 4}$.*

Proof. To prove this lemma, we need to prove $x^4 \in D_0$ for the integer x defined by (1).

By the generalized Chinese remainder theorem, there exists an integer s with $0 \leq s \leq e-1$ such that

$$\begin{cases} s \equiv 4 \pmod{(p-1)}, \\ s \equiv 0 \pmod{(q-1)}. \end{cases} \quad (23)$$

That is, the integer s satisfies

$$\begin{cases} x^4 \equiv g^s \pmod{p}, \\ x^4 \equiv g^s \pmod{q}. \end{cases} \quad (24)$$

This s is unique. Hence, $x^4 \equiv g^s \pmod{pq}$ and $x^4 \in D_0$.

Because ζ_n is an n -th primitive root of unity over $\mathbb{F}_{\text{ord}_n(l)}$, we have

$$\begin{aligned} \sum_{i \in P} \zeta_n^i &= -1, \\ \sum_{i \in Q} \zeta_n^i &= -1. \end{aligned} \quad (25)$$

By the definition of ζ_n , we have

$$\begin{aligned} \zeta_n^n - 1 &= (\zeta_n - 1) \left(1 + \sum_{i \in P} \zeta_n^i + \sum_{i \in Q} \zeta_n^i + \sum_{j=0}^3 \sum_{i \in D_j} \zeta_n^i \right) \\ &= 0. \end{aligned} \quad (26)$$

Together with (25), we obtain

$$\sum_{j=0}^3 \sum_{i \in D_j} \zeta_n^i = 1. \quad (27)$$

Define $t(x) = \sum_{i \in D_1} x^i + \sum_{i \in D_2} x^i$. $\square$

Lemma 5. *Let symbols be the same as before. Then,*

$$\Lambda(\zeta_n^a) = \begin{cases} \Lambda(\zeta_n), & a \in D_0, \\ t(\zeta_n), & a \in D_1, \\ -(\Lambda(\zeta_n) - 1), & a \in D_2, \\ -(t(\zeta_n) - 1), & a \in D_3, \\ \frac{p-1}{2}, & a \in P, \\ \frac{q+1}{2}, & a \in Q. \end{cases} \quad (28)$$

Proof. By Lemma 4, $aD_0 = D_0$ and $aD_1 = D_1$ if $a \in D_0$. If $a \in D_0$, then $aP = P$ since $\gcd(a, q) = 1$. Hence, we obtain

$$\begin{aligned} \Lambda(\zeta_n^a) &= 1 + \sum_{i \in P} \zeta_n^{ai} + \sum_{i \in D_0 \cup D_1} \zeta_n^{ai} \\ &= 1 + \sum_{i \in P} \zeta_n^i + \sum_{i \in D_0 \cup D_1} \zeta_n^i \\ &= \Lambda(\zeta_n). \end{aligned} \quad (29)$$

If $a \in D_1$, by Lemma 4, $aD_0 = D_1$ and $aD_1 = D_2$. By (25), we have

$$\begin{aligned} \Lambda(\zeta_n^a) &= 1 + \sum_{i \in P} \zeta_n^{ai} + \sum_{i \in D_0 \cup D_1} \zeta_n^{ai} \\ &= \sum_{i \in D_1} \zeta_n^i + \sum_{D_2} \zeta_n^i \\ &= t(\zeta_n). \end{aligned} \quad (30)$$

If $a \in D_2$, by Lemma 4, $aD_0 = D_2$ and $aD_1 = D_3$. By (25) and (27), we obtain

$$\begin{aligned} \Lambda(\zeta_n^a) &= 1 + \sum_{i \in P} \zeta_n^{ai} + \sum_{i \in D_0 \cup D_1} \zeta_n^{ai} \\ &= \sum_{i \in D_2} \zeta_n^i + \sum_{i \in D_3} \zeta_n^i \\ &= 1 - \sum_{i \in D_0 \cup D_1} \zeta_n^i \\ &= 1 - \Lambda(\zeta_n). \end{aligned} \quad (31)$$

If $a \in D_3$, by Lemma 4, $aD_0 = D_3$ and $aD_1 = D_0$. By (27), we have

$$\begin{aligned} \Lambda(\zeta_n^a) &= 1 + \sum_{i \in P} \zeta_n^{ai} + \sum_{i \in D_0 \cup D_1} \zeta_n^{ai} \\ &= \sum_{i \in D_3} \zeta_n^i + \sum_{D_0} \zeta_n^i \\ &= 1 - t(\zeta_n). \end{aligned} \quad (32)$$

If $a \in P$, then $aP = P$. Then, by (25), we know

$$\begin{aligned} \Lambda(\zeta_n^a) &= 1 + \sum_{i \in P} \zeta_n^{ai} + \sum_{i \in (D_0 \cup D_1)} \zeta_n^{ai} \\ &= \sum_{i \in (D_0 \cup D_1)} \zeta_n^{ai}. \end{aligned} \quad (33)$$

When s ranges over $\{0, 1, \dots, e-1\}$, it can be checked that $(D_0 \cup D_1) \bmod q$ takes on each element of $\{1, 2, \dots, q-1\}$ exactly $(p-1)/2$ times. It follows from (25) that

$$\begin{aligned} \sum_{i \in (D_0 \cup D_1)} \zeta_n^{ai} &= \left(\frac{p-1}{2} \bmod l \right) \sum_{i \in P} \zeta_n^i \\ &= -\frac{p-1}{2} \bmod l. \end{aligned} \quad (34)$$

If $a \in Q$, then $aP = \{0\}$. Then,

$$\begin{aligned} \Lambda(\zeta_n^a) &= 1 + \sum_{i \in P} \zeta_n^{ai} + \sum_{i \in (D_0 \cup D_1)} \zeta_n^{ai} \\ &= 1 + (q-1) + \sum_{i \in (D_0 \cup D_1)} \zeta_n^{ai}. \end{aligned} \quad (35)$$

When s ranges over $\{0, 1, \dots, e-1\}$, $(D_0 \cup D_1) \bmod p$ takes on each element of $\{1, 2, \dots, p-1\}$ exactly $(q-1)/2$ times. It follows from (25) that

$$\begin{aligned} \Lambda(\zeta_n^a) &= 1 + (q-1) + \sum_{i \in (D_0 \cup D_1)} \zeta_n^{ai} \\ &= 1 + (q-1) + \frac{q-1}{2} \sum_{i \in Q} \zeta_n^i \\ &= \frac{q+1}{2} \bmod l. \end{aligned} \quad (36)$$

$\square$

Lemma 6 (see [20], Lemma 3.3). *Let notations be the same as before. Then,*

- (1) $-1 \in D_0$ if and only if $p \not\equiv q \pmod{8}$
- (2) $-1 \in D_2$ if and only if $p \equiv q \pmod{8}$

Lemma 7 (see [21], Lemmas 2 and 4). For each $w \in P \cup Q$,

$$|\{D_i \cap (D_j + w)\}| = \begin{cases} \frac{(p-1)(q-1)}{16}, & \text{if } i \neq j, \\ \frac{(p-1)(q-5)}{16}, & \text{if } i = j \text{ and } p \mid w, \\ \frac{(p-5)(q-1)}{16}, & \text{if } i = \text{ and } q \mid w. \end{cases} \quad (37)$$

Define

$$\begin{aligned} \Delta_1 &= \frac{p-1}{2} \bmod l, \\ \Delta_2 &= \frac{q+1}{2} \bmod l. \end{aligned} \quad (38)$$

Theorem 1. Let $(n-1/4) \equiv 0 \pmod{l}$. Then, the linear complexity of the sequence λ^∞ defined by (15) is given as follows:

(1) When $p \equiv q \pmod{8}$ and $b/2 \equiv 0 \pmod{l}$, we have

$$L_l(\lambda) = \begin{cases} \frac{pq + p + q - 1}{2} - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 \neq 0, \\ \frac{pq - p + q + 1}{2} - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \\ \frac{pq + p - q + 1}{2} - \delta, & \text{if } \Delta_1 = 0, \Delta_2 \neq 0, \\ \frac{pq - p - q + 1}{2}, & \text{if } \Delta_1 = 0, \Delta_2 = 0. \end{cases} \quad (39)$$

(2) When $q \not\equiv p \pmod{8}$ and $(a^2 + 3)/4 \equiv 0 \pmod{l}$, we have

$$L_l(\lambda) = \begin{cases} \frac{3pq + p + q - 1}{4} - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 \neq 0, \\ \frac{3pq - 3p + q + 3}{4} - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \\ \frac{3pq + p - 3q + 3}{4} - \delta, & \text{if } \Delta_1 = 0, \Delta_2 \neq 0, \\ \frac{3pq - 3p - 3q + 5}{4}, & \text{if } \Delta_1 = 0, \Delta_2 = 0. \end{cases} \quad (40)$$

(3) When $p \equiv q \pmod{8}$ and $b/2 \not\equiv 0 \pmod{l}$ or $q \not\equiv p \pmod{8}$ and $(a^2 + 3)/4 \not\equiv 0 \pmod{l}$, we have

$$L_l(\lambda) = \begin{cases} n - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 \neq 0, \\ n + 1 - p - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \\ n + 1 - q - \delta, & \text{if } \Delta_1 = 0, \Delta_2 \neq 0, \\ n + 2 - p - q, & \text{if } \Delta_1 = 0, \Delta_2 = 0, \end{cases} \quad (41)$$

where a and b are the integers defined in (14) and δ is defined in (18).

Proof. By definition, we have

$$\Lambda(\zeta_n)^2 = \left(\sum_{i \in D_0, j \in D_0} + \sum_{i \in D_0, j \in D_1} + \sum_{i \in D_1, j \in D_0} + \sum_{i \in D_1, j \in D_1} \right) \zeta_n^{i+j}. \quad (42)$$

We first prove the conclusions for the case that $q \equiv p \pmod{8}$. In this case, by Lemma 6, $-1 \in D_0$ and b must be even. By Table 2, in Lemma 1 and Lemma 7, we have

$$\begin{aligned} \Lambda(\zeta_n)^2 &= \left(\sum_{i \in D_0, j \in D_0} + \sum_{i \in D_0, j \in D_1} + \sum_{i \in D_1, j \in D_0} + \sum_{i \in D_1, j \in D_1} \right) \zeta_n^{i-j} \\ &= ((0, 0) + (3, 3) + (0, 1) + (3, 0)) \sum_{i \in D_0} \zeta_n^i \\ &\quad + ((01, 0) + (0, 3) + (1, 1) + (0, 0)) \sum_{i \in D_1} \zeta_n^i \\ &\quad + ((2, 0) + (1, 3) + (2, 1) + (1, 0)) \sum_{i \in D_2} \zeta_n^i \\ &\quad + ((3, 0) + (2, 3) + (3, 1) + (2, 0)) \sum_{i \in D_3} \zeta_n^i \\ &\quad + 2 \left(\frac{(p-1)(q-1)}{16} \sum_{i \in P} \zeta_n^i \right) + 2 \left(\frac{(p-1)(q-1)}{16} \sum_{i \in P} \zeta_n^i \right) \\ &\quad + 2 \left(\frac{(p-1)(q-5)}{16} \sum_{i \in P} \zeta_n^i \right) + 2 \left(\frac{(p-5)(q-1)}{16} \sum_{i \in Q} \zeta_n^i \right) \\ &\quad + |D_0| + |D_1| \\ &= \Lambda(\zeta_n) + b \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - \frac{b}{2} + M + \frac{p+q-2}{2}. \end{aligned} \quad (43)$$

From $(n-1)/4 \equiv 0 \pmod{l}$, we know $M + (p+q-2)/2 \equiv 0 \pmod{l}$. Hence,

$$\Lambda(\zeta_n)^2 = \Lambda(\zeta_n) + b \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - \frac{b}{2}. \quad (44)$$

Whence,

$$\Lambda(\zeta_n)(\Lambda(\zeta_n) - 1) = \frac{b}{2} \left(2 \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - 1 \right). \quad (45)$$

Note that $(n-1)/4 \equiv 0 \pmod{l}$. By Lemma 2, we have $\sum_{i \in (D_0 \cup D_2)} \zeta_n^i \in \{1, 0\}$. It follows that

$$2 \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - 1 \in \{1, -1\}. \quad (46)$$

Similarly, we have

$$t(\zeta_n)(t(\zeta_n) - 1) = -\frac{b}{2} \left(2 \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - 1 \right). \quad (47)$$

By Lemma 5, (45), and (47), we know when $b/2 \equiv 0 \pmod{l}$, there are exactly half of a 's with $a \in \mathbb{Z}_n^*$ such that $s(\zeta_n^a) = 0$. When $b/2 \not\equiv 0 \pmod{l}$, $s(\zeta_n^a) \neq 0$ for all $a \in \mathbb{Z}_n^*$. Then, the desirable results on the linear complexity of the sequence s follow from (11), (17), and Lemma 5.

Now, we prove the conclusions for the case that $q \not\equiv p \pmod{8}$. In this case, by Lemma 6, $-1 \in D_2$. It follows from Table 1 in Lemmas 1 and 7 that

$$\begin{aligned} \Lambda(\zeta_n)^2 &= \left(\sum_{i \in D_0, j \in D_2} + \sum_{i \in D_0, j \in D_3} + \sum_{i \in D_1, j \in D_2} + \sum_{i \in D_1, j \in D_3} \right) \zeta_n^{i-j} \\ &= ((2, 2) + (1, 1) + (2, 3) + (1, 2)) \sum_{i \in D_0} \zeta_n^i \\ &\quad + ((3, 2) + (2, 1) + (3, 3) + (2, 2)) \sum_{i \in D_1} \zeta_n^i \\ &\quad + ((0, 2) + (3, 1) + (0, 3) + (3, 2)) \sum_{i \in D_1} \zeta_n^i \\ &\quad + ((1, 2) + (0, 1) + (1, 3) + (0, 2)) \sum_{i \in D_1} \zeta_n^i \\ &\quad + 4 \left(\frac{(p-1)(q-1)}{16} \sum_{i \in P} \zeta_n^i \right) + 4 \left(\frac{(p-1)(q-1)}{16} \sum_{i \in Q} \zeta_n^i \right) \\ &= \Lambda(\zeta_n) + \frac{b \left(2 \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - 1 \right) - 1}{2} + M - \frac{(p-1)(q-1)}{2}. \end{aligned} \quad (48)$$

From $(n-1)/4 \equiv 0 \pmod{l}$, we have $M - ((p-1)(q-1))/2 \equiv 0 \pmod{l}$. Hence,

$$\Lambda(\zeta_n)^2 = \Lambda(\zeta_n) + \frac{b \left(2 \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - 1 \right) - 1}{2}. \quad (49)$$

Whence,

$$\Lambda(\zeta_n)(\Lambda(\zeta_n) - 1) = \frac{b \left(2 \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - 1 \right) - 1}{2}. \quad (50)$$

Similarly, we get

$$t(\zeta_n)(t(\zeta_n) - 1) = -\frac{b \left(2 \sum_{i \in (D_0 \cup D_2)} \zeta_n^i - 1 \right) + 1}{2}. \quad (51)$$

Since $\gcd(p-1, q-1) = 4$ and $p \equiv q+4 \pmod{8}$, we get $p \equiv 1$ or $5 \pmod{8}$. Hence, $n = pq \equiv 5 \pmod{8}$. Together with $(n-1)/4 \equiv 0 \pmod{l}$, we obtain l which is an odd prime. By the representation $n = a^2 + 4b^2$ with $a \equiv 1 \pmod{4}$, we have

$$\frac{n-1}{4} = \frac{a^2+3}{4} + (|b|-1)(|b|+1). \quad (52)$$

Therefore, $(a^2+3)/4 \equiv 0 \pmod{l}$ if and only if $(|b|-1)(|b|+1) \equiv 0 \pmod{l}$. Since l is odd, we obtain $(|b|-1)(|b|+1) \equiv 0 \pmod{l}$ if and only if l divides one of $|b|-1$ and $|b|+1$. By Lemma 5, if $(a^2+3)/4 \equiv 0 \pmod{l}$, there are exactly $(p-1)(q-1)/4$ a 's such that $s(\zeta_n^a) = 0$ for $a \in \mathbb{Z}_n^*$. If $(a^2+3)/4 \not\equiv 0 \pmod{l}$, $s(\zeta_n^a) \neq 0$ for all $a \in \mathbb{Z}_n^*$. Then, the desirable conclusions on the linear complexity of the sequence λ^∞ follow from (11), (17), (46), and Lemma 5.

For the case $l \notin D_0$, the linear complexity of the sequence λ^∞ defined by (15) can also be determined in the following theorem. $\square$

Theorem 2. *If $l \notin D_0$, then for the sequence λ^∞ defined in (15), we have*

$$L_l(\lambda) = \begin{cases} n - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 \neq 0, \\ n + 1 - p - \delta, & \text{if } \Delta_1 \neq 0, \Delta_2 = 0, \\ n + 1 - q - \delta, & \text{if } \Delta_1 = 0, \Delta_2 \neq 0, \\ n + 2 - p - q, & \text{if } \Delta_1 = 0, \Delta_2 = 0. \end{cases} \quad (53)$$

Proof. If $l \notin D_0$, then there exists $i \in \{1, 2, 3\}$ such that $i \in D_i$. No matter what i is, there exists $j \in \{1, 2, 3\}$ satisfying the congruence equation $ij \equiv 2 \pmod{4}$. Then, $l^j \in D_2$ and

$$\Lambda^{l^j}(\zeta_n) = \Lambda(\zeta_n^{l^j}) = 1 - \Lambda(\zeta_n). \quad (54)$$

This implies $\Lambda(\zeta_n) \notin \{0, 1\}$.

Similarly, $t^{l^j}(\zeta_n) = t(\zeta_n^{l^j}) = 1 - t(\zeta_n)$ and $t(\zeta_n) \notin \{0, 1\}$. Hence, $s(\zeta_n^a) \neq 0$ for all $a \in \mathbb{Z}_n^*$. Combining (11) and Lemma 5, we have the desirable results.

For $j \in \{0, 1, 2, 3\}$, define

$$d_j(x) = \prod_{i \in D_j} (x - \zeta_n^i). \quad (55)$$

If $l \in D_0$, it can be easily proved that $d_j(x) \in \mathbb{F}_l[x]$ for all j . Let $d(x) = \prod_{j=0}^3 d_j(x)$, then $d(x) \in \mathbb{F}_l[x]$. We have

$$x^n - 1 = \prod_{i=0}^{n-1} (x - \zeta_n^i) = \frac{(x^q - 1)(x^p - 1)d(x)}{x - 1}. \quad (56)$$

Lemma 8 (see [20], Lemma 3.3). *Let notations be the same as before. Then,*

- (1) $2 \in D_0 \cup D_2$ if and only if $p \equiv q \pmod{8}$
- (2) $2 \in D_1 \cup D_3$ if and only if $p \equiv q \pmod{8}$

Lemma 9 (see [20], Lemma 3.5). Let $p \equiv q \pmod{8}$. Then, there are exactly two representations over the integer ring $\mathbb{Z}$:

$$\begin{aligned} pq &= a^2 + 4b^2 = a'^2 + 4b'^2, \\ a &\equiv a' \equiv 1 \pmod{4}, \end{aligned} \quad (57)$$

where one of b and b' is divided by 4 and another is exactly divided by 2.

Lemma 10 (see [20], Corollary 3.9). Let $p \equiv q \pmod{8}$. Fix a common primitive root g of p and q . Then, $2 \in D_0$ if and only if the generalized cyclotomic numbers in Lemma 1 depend on the decomposition $n = a^2 + 4b^2$ with $4|b$; $2 \in D_2$ if and only if the generalized cyclotomic numbers depend on the

decomposition $n = a^2 + 4b^2$ with $2||b$, where $2||b$ means that $2|b$ and $2^2 \nmid b$.

After the preparations above, we are ready to compute the linear complexity and minimal polynomials of the sequence defined in (15) over $\mathbb{F}_2$.

Corollary 1. Let $l = 2$ and $n = a^2 + 4b^2 \equiv 1 \pmod{8}$. Then, $p \equiv q \pmod{8}$ and the linear complexity and minimal polynomials of the sequence λ^∞ defined by (15) are given as follows:

(1) When $4|b$, then

$$\begin{aligned} L_2(\lambda) &= \frac{pq - q + p + 1}{2}, \\ m(x) &= \begin{cases} \frac{(x^n - 1)(x - 1)}{(x^q - 1)d_2(x)d_1(x)}, & \text{if } \Lambda(\zeta_n) = 1 \text{ and } t(\zeta_n) = 0, \\ \frac{(x^n - 1)(x - 1)}{(x^q - 1)d_0(x)d_3(x)}, & \text{if } \Lambda(\zeta_n) = 0 \text{ and } t(\zeta_n) = 1, \\ \frac{(x^n - 1)(x - 1)}{(x^q - 1)d_0(x)d_1(x)}, & \text{if } \Lambda(\zeta_n) = 0 \text{ and } t(\zeta_n) = 0, \\ \frac{(x^n - 1)(x - 1)}{(x^q - 1)d_2(x)d_3(x)}, & \text{if } \Lambda(\zeta_n) = 1 \text{ and } t(\zeta_n) = 1. \end{cases} \end{aligned} \quad (58)$$

(2) When $2||b$, then

$$\begin{aligned} L_2(\lambda) &= n + 1 - q, \\ m(x) &= \frac{(x^n - 1)(x - 1)}{x^q - 1}. \end{aligned} \quad (59)$$

Proof. If $p \equiv q \pmod{8}$, then $p \equiv q + 4 \pmod{8}$ and $n = pq \equiv 5 \pmod{8}$. This contradicts with $n \equiv 1 \pmod{8}$. Hence, $p \not\equiv q \pmod{8}$. Also, in this case, by Lemma 8, $2 \in D_0 \cup D_2$.

By (45), (47), and Lemma 10, if $4|b$, i.e., $2 \in D_0$, we have

$$\begin{aligned} \Lambda(\zeta_n) &\in \{0, 1\}, \\ t(\zeta_n) &\in \{0, 1\}, \end{aligned} \quad (60)$$

and if $2||b$, i.e., $2 \in D_2$, we obtain

$$\begin{aligned} \Lambda(\zeta_n) &\notin \{0, 1\}, \\ t(\zeta_n) &\notin \{0, 1\}. \end{aligned} \quad (61)$$

Then, the desired conclusions on the linear complexity and the minimal polynomial of the sequence λ^∞ follow from (11) and Lemma 5.

If $p \not\equiv q \pmod{8}$, then $n = pq \equiv 5 \pmod{8}$, and by Lemma 8, $2 \in D_1 \cup D_3$. Hence, by Theorem 2, we have the following corollary.

Corollary 2. Let $l = 2$ and $n \equiv 5 \pmod{8}$. Then, for the sequence λ^∞ defined by (15), we have

$$\begin{aligned} L_2(\lambda) &= n + 1 - q, \\ m(x) &= \frac{(x^n - 1)(x - 1)}{x^q - 1}. \end{aligned} \quad (62)$$

Example 1. Let $(p, q) = (5, 13)$ and $(g_1, g_2) = (2, 2)$. Then, $a = 1$ and $b = 4$. If $l = 2$, Magma program shows that $2 \in D_0$ and $L_2(\lambda) = 29$. If $l = 3$, then $b/2 \not\equiv 0 \pmod{l}$ and $L_3(\lambda) = 65$.

Example 2. Let $(p, q) = (5, 17)$ and $(g_1, g_2) = (2, 3)$. Then, $a = -7$ and $b = 3$. If $l = 2$, it can be easily checked that $2 \in D_3$ and $L_2(\lambda) = 69$. If $l = 7$, then $(a^2 + 3)/4 \not\equiv 0 \pmod{l}$ and $L_7(\lambda) = 85$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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