Review Article

Boundary Value Problem for a Second-Order Difference Equation with Resonance

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In this paper, we study the existence and multiplicity of nontrivial solutions of a second-order discrete boundary value problem with resonance and sublinear or superlinear nonlinearity. The main methods are based on the Morse theory and the minimax methods. In addition, some examples are given to illustrate our results.

1. Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, $\mathbb{Z}(a, b)$ denotes the discrete interval $\{a, a+1, \dots, b\}$ if $a \leq b$.

In this paper, we consider the existence and multiplicity of nontrivial solutions for the following discrete Dirichlet boundary value problem:

$$\begin{cases} \Delta[p(k)\Delta u(k-1)] + q(k)u(k) + f(k, u(k)) = 0, & k \in \mathbb{Z}(1, T), \\ u(0) = u(T+1) = 0, \end{cases} \quad (1)$$

where the forward difference operator is defined by $\Delta u(k) = u(k+1) - u(k)$; $p(k)$ and $q(k)$ are real valued on $\mathbb{Z}$, and $p(k)$ is nonzero; and $f(k, \cdot) \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $f(k, 0) = 0$ for each $k \in \mathbb{Z}(1, T)$. Clearly, problem (1) has the trivial solution $u = 0$.

In the recent years, the existence of solutions for nonlinear difference equations has been widely studied by many authors. We note that these results were usually obtained by means of critical point theory, for example, the existence of ground state solutions [1], homoclinic orbits [2–5], the boundary value problem, and periodic solutions [6–8].

The boundary value problem (1) may be regarded as a discrete analogue of the following boundary value problem:

$$\begin{aligned} (p(t)u'(t))' + q(t)u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u(0) &= u(1) = 0, \end{aligned} \quad (2)$$

which has been successfully applied to the modeling of astrophysics, gas dynamics, and chemically reacting system, see [9–11].

By various methods and techniques, many authors studied the similar second-order difference equation under various boundary value conditions. For example, Agarwal in [10] considered the existence of solutions for the boundary value problem (1) by the contraction mapping principle and Brouwer fixed point theorems in Euclidean space. Yu and Guo in [12] first used the critical point theory to study the following discrete boundary value problem:

$$\begin{cases} \Delta[p(k)\Delta u(k-1)] + q(k)u(k) = f(k, u(k)), & k \in \mathbb{Z}(1, T), \\ u(0) = A, \\ u(T+1) = B, \end{cases} \quad (3)$$

where A and B are the constants, and they proved the existence of solutions of problem (3) when the nonlinear term f is sublinear or superlinear. We refer the readers to [13–16] and the reference therein for more information.

It is well known that the resonance exists in many real-world applications, and the equations with resonance have

been extensively studied in various fields [17–21]. In fact, it is more difficult to study the boundary value problem under the case of resonance because the resonance case can change the local geometric properties of the critical points [17]. In this paper, we study the existence and multiplicity of nontrivial solutions for problem (1) at resonance by means of the interaction between the nonlinearity and the spectrum of the symmetric matrix $P + Q$, where $P + Q$ denotes a matrix whose elements are given by $p(k)$ and $q(k)$ for $k \in \mathbb{Z}(1, T)$.

Note that in [12], the authors obtained the existence of one solution for (3) via the variational methods or the saddle point theorem. In the case where $A = B$, the solution could be trivial, which was not considered in [12]. In this paper, we will show that the boundary value problem (1) has at least one nontrivial solution. In fact, our Theorem 1 and Corollary 1 extend and complement the existence results given in [12] and establish the existence of multiple solutions.

Our results are obtained by combining the Morse theory [22–24], critical group computation, and the minimax methods including the local linking [17, 25]. We prove the existence of nontrivial solutions for problem (1) at resonance by the relationship between these nonlinear techniques and methods. In order to apply these ideas to identify the unknown critical point, we compute the corresponding critical groups of functional J . Obviously, when we consider the existence of nontrivial solutions of problem (1) at resonance, we need to overcome much more difficulties than those in the literature [12].

We consider the T -dimensional Banach space:

$$S = \{u: \mathbb{Z}(0, T+1) \longrightarrow \mathbb{R} \text{ such that } u(0) = u(T+1) = 0\}. \quad (4)$$

The space S is a Hilbert space with the inner product

$$\langle u, v \rangle = \sum_{k=1}^T u(k)v(k), \quad \forall u, v \in S, \quad (5)$$

endowed with the norm

$$\|u\| = \sqrt{\langle u, u \rangle} = \left(\sum_{k=1}^T |u(k)|^2 \right)^{(1/2)}. \quad (6)$$

Now, we define the C^1 -functional J on S as follows:

$$J(u) = \sum_{k=1}^{T+1} \frac{1}{2} (p(k)(\Delta u(k-1))^2 - q(k)u^2(k)) - \sum_{k=1}^T F(k, u(k)), \quad (7)$$

for every $u \in S$, where $F(k, t) = \int_0^t f(k, s)ds$, $(k, t) \in \mathbb{Z}(1, T) \times \mathbb{R}$. We can compute the Frechet derivative as

$$\begin{aligned} \langle J'(u), v \rangle &= \sum_{k=1}^{T+1} (p(k)\Delta u(k-1)\Delta v(k-1) - q(k)u(k)v(k)) \\ &\quad - f(k, u(k))v(k) \\ &= - \sum_{k=1}^T (\Delta(p(k)\Delta u(k-1)) + q(k)u(k) \\ &\quad + f(k, u(k)))v(k), \end{aligned} \quad (8)$$

for all $u, v \in S$. It is clear that the critical points of J are the solutions of problem (1).

For convenience, we identify $u \in S$ with $u = (u(1), u(2), \dots, u(T)) \in \mathbb{R}^T$. Thus, we rewrite $J(u)$ and $\langle J'(u), v \rangle$ as

$$J(u) = \frac{1}{2} u^T (P + Q) u - \sum_{k=1}^T F(k, u(k)), \quad (9)$$

$$\langle J'(u), v \rangle = u^T (P + Q) v - \sum_{k=1}^T f(k, u(k))v(k),$$

further,

$$\langle J''(u)v, w \rangle = w^T (P + Q) v - \sum_{k=1}^T f'(k, u(k))v(k)w(k), \quad (10)$$

where u^T denotes the transpose of u and P and Q are the $T \times T$ symmetric matrices given by

$$P = \begin{pmatrix} p(1) + p(2) & -p(2) & 0 & \cdots & 0 & 0 \\ -p(2) & p(2) + p(3) & -p(3) & \cdots & 0 & 0 \\ 0 & -p(3) & p(3) + p(4) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & p(T-1) + p(T) & -p(T) \\ 0 & 0 & 0 & \cdots & -p(T) & p(T) + p(T+1) \end{pmatrix}, \quad (11)$$

$$Q = \begin{pmatrix} -q(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & -q(2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & -q(3) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -q(T-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & -q(T) \end{pmatrix}.$$

Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. For $c \in \mathbb{R}$, we write $J^c = \{u \in E: J(u) \leq c\}$ and $K = \{u \in E: J'(u) = 0\}$.

In order to obtain the critical points of the functional J on E , we recall some basic concepts and results of the Morse theory.

Definition 1 (see [22]). J is said to satisfy the Palais–Smale condition (PS condition for short), if any sequence $\{u_n\} \subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ possesses a convergent subsequence in E .

In the Morse theory, functional J is always required to satisfy the deformation condition (D), which was introduced by Bartsch and Li [26]. Bartolo et al. [27] proved that J satisfies the deformation condition (D) if J satisfies the PS condition.

Assume that $J \in C^1(E, \mathbb{R})$, and let $u_0 \in K$ be an isolated critical point of J with $J(u_0) = c \in \mathbb{R}$, and U is a neighborhood of u_0 , containing the unique critical point, then we call

$$C_q(J, u_0) = H_q(J^c \cap U, J^c \cap U \setminus \{u_0\}), \quad q \in \mathbb{Z}, \quad (12)$$

the q -th critical group of J at u_0 , where $H_q(\cdot, \cdot)$ stands for the q -th singular relative homology group with integer coefficients [22, 23].

We say that u_0 is a homological nontrivial critical point of J , if at least one of its critical groups is nontrivial.

Let $J \in C^1(E, \mathbb{R})$ satisfy the PS condition and all critical values of J be greater than some $\alpha \in \mathbb{R}$, then the group

$$C_q(J, \infty) = H_q(E, J^\alpha), \quad q \in \mathbb{Z}, \quad (13)$$

is called the q -th critical group of J at infinity [26].

Assume $\#K < \infty$ and J satisfies the PS condition. The Morse-type numbers of the pair (E, J^α) are defined by $M_q = M_q(E, J^\alpha) = \sum_{u \in K} \dim C_q(J, u)$, and the Betti numbers of the pair (E, J^α) are $\beta_q = \dim C_q(J, \infty)$. By the Morse theory [22, 23], the following relations hold:

$$\sum_{j=0}^q (-1)^{q-j} M_j \geq \sum_{j=0}^q (-1)^{q-j} \beta_j, \quad \forall q \in \mathbb{Z}, \quad (14)$$

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q. \quad (15)$$

By (14), it follows that $M_q \geq \beta_q$ for all $q \in \mathbb{Z}$. Hence, when $\beta_q \neq 0$ for some $q \in \mathbb{Z}$ and J satisfies the PS condition, then J must have a critical point u_0 satisfying $C_q(J, u_0) \neq 0$. If J has only one critical point u_0 , then $C_q(J, \infty) \cong C_q(J, u_0)$, $\forall q \in \mathbb{Z}$. If $C_q(J, \infty) \neq C_q(J, u_0)$ for some q , then J must have another critical point $u_1 \neq u_0$. Further, if u and v are two critical points of J and $C_q(J, u) \neq C_q(J, v)$ for some q , then $u \neq v$.

We need the following lemmas to prove the main results.

Lemma 1 (see [17]). *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the (D) condition and be bounded from below. If J has a critical point that is homological nontrivial and is not the minimizer of J , then J has at least three critical points.*

Obviously, we need to show that J has a homological nontrivial critical point, which is not the global minimizer in this lemma.

Lemma 2 (see [17, 25]). *Assume that J has a critical point $u = 0$ with $J(0) = 0$. If J has a local linking at 0 with respect to $E = V \oplus W$ and $h = \dim V < \infty$, i.e., there exists $\gamma > 0$ such that*

$$\begin{cases} J(u) \leq 0, & u \in V, \|u\| \leq \gamma, \\ J(u) > 0, & u \in W, 0 < \|u\| \leq \gamma. \end{cases} \quad (16)$$

Then, $C_h(J, 0) \neq 0$. Moreover, if $h = \mu(0)$ or $h = \mu(0) + \nu(0)$, then $C_q(J, 0) \cong \delta_{q,h} \mathbb{Z}$, where $\mu(0)$ and $\nu(0)$ denote the Morse index and the nullity of J at 0, respectively.

From Lemma 2, if $u = 0$ is a trivial critical point and J has a local linking structure at 0, then 0 is a homological nontrivial critical point of J .

Lemma 3 (see [26]). *Let E be a Banach space and $J \in C^1(E, \mathbb{R})$ satisfy the PS condition. Suppose E splits as $E^+ \oplus E^-$ such that J is bounded from below on E^+ and $J(u) \rightarrow -\infty$ for $u \in E^-$ as $\|u\| \rightarrow \infty$. Then, $C_h(J, \infty) \neq 0$ for $h = \dim E^-$.*

Lemma 4 (see [28]). *Let $Y \subset B \subset A \subset X$ be topological spaces and $q \in \mathbb{Z}$. If*

$$\begin{aligned} H_q(A, B) &\neq 0, \\ H_q(X, Y) &\neq 0, \end{aligned} \quad (17)$$

then

$$H_{q+1}(X, A) \neq 0 \text{ or } H_{q-1}(B, Y) \neq 0. \quad (18)$$

2. Sublinear Case

In this section, we will consider the case where $f(k, t)$ is sublinear in t for each $k \in \mathbb{Z}(1, T)$. Let the spectrum of the symmetric matrix $P + Q$ consist of isolated eigenvalues of finite multiplicity denoted by

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_l \leq \lambda_{l+1} \leq \dots \leq \lambda_T. \quad (19)$$

It follows that

$$\lambda_1 \|u\|^2 \leq u^T (P + Q) u \leq \lambda_T \|u\|^2, \quad (20)$$

for every $u \in S$.

We assume the following conditions (G_1) and (G_2) are satisfied:

(i) (G_1) There exist $1 < \theta < 2$ and $M > 0$ such that

$$0 < t f(k, t) \leq \theta F(k, t), \quad |t| \geq M, \quad \forall k \in \mathbb{Z}(1, T). \quad (21)$$

(ii) (G_2) There exist $\delta > 0$, $\lambda_l \neq \lambda_{l+1}$ for some $l < T$, and $\bar{\lambda} \in (\lambda_l, \lambda_{l+1})$ such that

$$\lambda_l |t|^2 \leq 2F(k, t) \leq \bar{\lambda} |t|^2, |t| \leq \delta, \forall k \in \mathbb{Z}(1, T). \quad (22)$$

Theorem 1. Assume that the matrix $P + Q$ is positive definite and $f(k, t)$ satisfies (G_1) and (G_2) . Then, problem (1) has at least two nontrivial solutions in S .

To prove Theorem 1, we need a series of lemmas.

Lemma 5. If the matrix $P + Q$ is positive definite and (G_1) holds. Then, J satisfies the PS condition.

Proof. For any sequence $\{u_n\} \subset S$, with $\{J(u_n)\}$ is bounded and $J'(u_n) \rightarrow 0$ as $n \rightarrow +\infty$, there exists a positive constant $C \in \mathbb{R}$ such that $|J(u_n)| \leq C$. By (G_1) , we have

$$\begin{aligned} -C\theta - \|u_n\| &\leq \theta J(u_n) - J'(u_n)(u_n) = \left(\frac{\theta}{2} - 1\right) u_n^\tau (P + Q) u_n - \sum_{k=1}^T (\theta F(k, u_n(k)) - f(k, u_n(k)) u_n(k)) \\ &= \left(\frac{\theta}{2} - 1\right) u_n^\tau (P + Q) u_n - \sum_{|u_n(k)| < M} (\theta F(k, u_n(k)) - f(k, u_n(k)) u_n(k)) \\ &\quad - \sum_{|u_n(k)| \geq M} (\theta F(k, u_n(k)) - f(k, u_n(k)) u_n(k)) \\ &\leq \left(\frac{\theta}{2} - 1\right) \lambda_1 \|u_n\|^2 - L, \end{aligned} \quad (23)$$

where $L = \min_{(k,t) \in \mathbb{Z}(1,T) \times [-M,M]} (\theta F(k, t) - f(k, t)t)$. Then, for any $n \in \mathbb{Z}$,

$$\left(1 - \frac{\theta}{2}\right) \lambda_1 \|u_n\|^2 - \|u_n\| \leq C\theta - L. \quad (24)$$

Since $1 < \theta < 2$, thus $\{u_n\}$ is bounded in S and the Bolzano–Weierstrass theorem implies that $\{u_n\}$ has a convergent subsequence. $\square$

Lemma 6. If the matrix $P + Q$ is positive definite and (G_1) holds. Then, J is coercive on S , that is, $J(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$.

Proof. If the matrix $P + Q$ is positive definite, then $\lambda_i > 0$, $\forall i \in \mathbb{Z}(1, T)$. Since (G_1) is satisfied, it implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$F(k, t) \leq a_1 |t|^\theta + a_2, \quad \text{for all } k \in \mathbb{Z}(1, T), t \in \mathbb{R}, \quad (25)$$

then

$$\begin{aligned} J(u) &= \frac{1}{2} u^\tau (P + Q) u - \sum_{k=1}^T F(k, u(k)) \geq \frac{1}{2} \lambda_1 \|u\|^2 \\ &\quad - a_1 \sum_{k=1}^T |u(k)|^\theta - a_2 T. \end{aligned} \quad (26)$$

Since

$$\sum_{k=1}^T |u(k)|^\theta \leq T^{(1-\theta/2)} |u|^\theta, \quad (27)$$

we have

$$J(u) \geq \frac{1}{2} \lambda_1 |u|^2 - a_1 T^{(1-\theta/2)} |u|^\theta - a_2 T \rightarrow +\infty, \quad \text{as } |u| \rightarrow \infty. \quad (28)$$

The proof is complete.

Let $S(\lambda_l)$ denote the eigenspace of λ_l for some $l \in \mathbb{Z}(1, T-1)$, V denote the subspace of S spanned by the eigenfunctions corresponding to the eigenvalues $\lambda_1, \dots, \lambda_{l-1}$, and W denote the subspace of S spanned by the eigenfunctions corresponding to the eigenvalues $\lambda_{l+1}, \dots, \lambda_T$, then we are given an orthogonal decomposition

$$S = V \oplus S(\lambda_l) \oplus W. \quad (29) \quad \square$$

Lemma 7. Assume that (G_2) holds. Then, J has a local linking at the origin with respect to $S = (V \oplus S(\lambda_l)) \oplus W$ and $h = \dim(V \oplus S(\lambda_l)) < T$.

Proof. Let $\gamma = \delta$, for each $u \in V \oplus S(\lambda_l)$ and $\|u\| \leq \delta$ implies $|u(k)| \leq \delta \forall k \in \mathbb{Z}(1, T)$. By (G_2) , we have

$$\begin{aligned} J(u) &= \frac{1}{2} u^\tau (P + Q) u - \sum_{k=1}^T F(k, u(k)) \\ &\leq \frac{\lambda_l}{2} \|u\|^2 - \sum_{k=1}^T F(k, u(k)) \end{aligned} \quad (30)$$

$$= \sum_{k=1}^T \left(\frac{\lambda_l}{2} |u(k)|^2 - F(k, u(k)) \right) \leq 0.$$

For each $u \in W$, $0 < \|u\| \leq \delta$ implies $|u(k)| \leq \delta \forall k \in \mathbb{Z}(1, T)$. We have the following estimates:

$$\begin{aligned}
J(u) &= \frac{1}{2}u^\tau(P+Q)u - \sum_{k=1}^T F(k, u(k)) \geq \frac{\lambda_{l+1}}{2}\|u\|^2 - \sum_{k=1}^T F(k, u(k)) \\
&= \frac{\lambda_{l+1}}{2}\|u\|^2 - \frac{\bar{\lambda}}{2}\|u\|^2 - \sum_{k=1}^T \left(F(k, u(k)) - \frac{\bar{\lambda}}{2}|u(k)|^2 \right) \geq \frac{\lambda_{l+1} - \bar{\lambda}}{2}\|u\|^2 > 0.
\end{aligned} \tag{31}$$

This completes the proof. $\square$

Proof. Proof of Theorem 1. By Lemmas 5 and 6, J satisfies the PS condition and is coercive, hence J is bounded from below. Combining Lemma 7 with Lemma 2, the trivial solution $u = 0$ is homological nontrivial and is not a minimizer. We apply Lemma 1 to conclude that problem (1) has at least two nontrivial solutions in S . $\square$

Remark 1

- (i) The conclusion of Theorem 1 holds, if the condition (G_2) is replaced by the following condition:

(G'_2) There exist $\delta > 0$, $\lambda_l \neq \lambda_{l+1}$ for some $l < T$, and $\bar{\lambda} \in (\lambda_l, \lambda_{l+1})$ such that

$$\lambda_l |t|^2 \leq t f(k, t) \leq \bar{\lambda} |t|^2, \quad \text{for } |t| \leq \delta, \forall k \in \mathbb{Z}(1, T). \tag{32}$$

- (ii) The condition (G_2) means that problem (1) is resonant near 0 at the eigenvalue λ_l from the right side, and (G_2) contains the situation $\limsup_{|t| \rightarrow 0} (2F(k, t)/|t|^2) = \lambda \in [\lambda_l, \lambda_{l+1}]$.

- (iii) If $f(k, 0) = 0$ for each $k \in \mathbb{Z}(1, T)$, then problem (1) admits the trivial solution $u = 0$. In Theorem 1, we find two nontrivial solutions for (1). From (G_1) and (G_2) , we see that the existence of nontrivial solutions for problem (1) depends on the behaviors of the term $f(k, t)$ or $F(k, t)$ at 0 and at infinity $\forall k \in \mathbb{Z}(1, T)$.

Example 1. Let $T = 2$, and we consider the boundary value problem (1) with

$$f(k, t) = f(t) = \begin{cases} t, & \text{if } |t| < 1, \\ \frac{t}{|t|}, & \text{if } |t| \geq 1, \end{cases} \tag{33}$$

for all $k \in \mathbb{Z}(1, 2)$. Then,

$$F(k, t) = F(t) = \begin{cases} \frac{t^2}{2}, & \text{if } |t| < 1, \\ |t| - \frac{1}{2}, & \text{if } |t| \geq 1. \end{cases} \tag{34}$$

Fix $\theta = 3/2$, $p(k) = 1$, and $q(k) = 0 \forall k \in \mathbb{Z}(1, 2)$, then the matrix $P + Q$,

$$P + Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{35}$$

is positive definite and admits two distinct eigenvalues given by $\lambda_1 = 1$ and $\lambda_2 = 3$.

For each $k \in \mathbb{Z}(1, 2)$, we have

$$\frac{3}{2}F(k, t) - t f(k, t) = \frac{1}{2}|t| - \frac{3}{4} \longrightarrow +\infty, \quad \text{as } |t| \longrightarrow +\infty, \tag{36}$$

thus the condition (G_1) holds.

Let $\delta = 1$, we see that

$$|t|^2 = 2F(k, t) < 3|t|^2, \quad |t| \leq 1, \forall k \in \mathbb{Z}(1, 2), \tag{37}$$

then, the condition (G_2) holds. Clearly, the conditions of Theorem 1 are satisfied, and problem (1) admits at least two nontrivial solutions in S .

In fact, if $|u(k)| \geq 1$ for $k \in \mathbb{Z}(1, 2)$, then problem (1) is

$$\begin{cases} -2u(1) + u(2) + \frac{u(1)}{|u(1)|} = 0, \\ u(1) - 2u(2) + \frac{u(2)}{|u(2)|} = 0, \\ u(0) = u(3) = 0. \end{cases} \tag{38}$$

We can show that $\{u(0) = 0, u(1) = 1, u(2) = 1, u(3) = 0\}$ and $\{u(0) = 0, u(1) = -1, u(2) = -1, u(3) = 0\}$ are the only two nontrivial solutions of problem (38) in S .

Theorem 2. Assume that the matrix $P + Q$ is negative definite, and the conditions (G_1) and (G_2) hold. Then, problem (1) possesses at least one nontrivial solution.

Proof. We now show that we can find at least one nontrivial critical point of functional J .

Since the matrix $P + Q$ is negative definite, in this case, $\lambda_i < 0$ for every $i \in \mathbb{Z}(1, T)$ and the inequality (20) holds. If $u \in S$ satisfies $\|u\| \geq M$, by (G_1) , we have

$$J(u) \leq \frac{\lambda_T}{2}\|u\|^2 \longrightarrow -\infty, \quad \text{as } \|u\| \longrightarrow \infty. \tag{39}$$

We notice that if $-J$ is continuity and coercive, then J has at least a local maximum $u_1 \in S$ by the above inequality. This means that $C_q(J, u_1) \cong \delta_{q, T}\mathbb{Z}$. If $u_1 = 0$, then $C_q(J, 0) \cong \delta_{q, T}\mathbb{Z}$, noticing Lemma 7, this contradicts Lemma 2. Hence, 0 is different from u_1 . $\square$

Theorem 3. Assume that matrix $P + Q$ is negative semi-definite, and

$$(G_3) t f(k, t) > 0, \quad \text{for } t \neq 0, k \in \mathbb{Z}(1, T). \tag{40}$$

Then, problem (1) has no nontrivial solution.

Proof. If we assume that (1) has a nontrivial solution, then $J(u)$ has a nonzero critical point $\bar{u}$. Since $J'(\bar{u}) = 0$ and the matrix $P + Q$ is negative semidefinite, one has

$$\bar{u}^T (P + Q) \bar{u} = \sum_{k=1}^T f(k, \bar{u}(k)) \bar{u}(k) \leq 0. \quad (41)$$

This contradicts with (G_3) . The proof is complete.

If the matrix $P + Q$ is just nonsingular, we may suppose that its first positive eigenvalue is $\lambda_l > 0$. We denote its eigenvalues as follows:

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{l-1} < \lambda_l \leq \lambda_{l+1} \leq \dots \leq \lambda_T. \quad (42)$$

Then, we have

$$\begin{aligned} \lambda_1 \|u\|^2 &\leq u^T (P + Q) u \leq \lambda_l \|u\|^2, \quad \forall u \in V \oplus S(\lambda_l), \\ \lambda_{l+1} \|u\|^2 &\leq u^T (P + Q) u \leq \lambda_T \|u\|^2, \quad \forall u \in W. \end{aligned} \quad (43)$$

□

Theorem 4. *If the matrix $P + Q$ is just nonsingular and $2 \leq l < T$, we assume that (G_1) and the following conditions hold:*

- (i) (G_4) $\lim_{t \rightarrow 0} (f(k, t)/t) = \lambda_l, \quad \forall k \in \mathbb{Z}(1, T)$
- (ii) (G_5) *There exists $\delta > 0$ such that $2F(k, t) \geq \lambda_l |t|^2$ for $|t| \leq \delta, \forall k \in \mathbb{Z}(1, T)$*

Then, problem (1) has at least one nontrivial solution.

Proof. We show that J has a local linking at 0. By (G_4) , let $\varepsilon \in (0, \lambda_{l+1} - \lambda_l)$, then there exists $\gamma \in (0, \delta)$ such that

$$|2F(k, t) - \lambda_l |t|^2| < \varepsilon |t|^2, |t| \leq \gamma, \forall k \in \mathbb{Z}(1, T). \quad (44)$$

Since (G_5) holds, then we obtain

$$0 \leq 2F(k, t) - \lambda_l |t|^2 < \varepsilon |t|^2, |t| \leq \gamma, \forall k \in \mathbb{Z}(1, T). \quad (45)$$

For every $u \in W$ and $0 < \|u\| \leq \gamma$, one has $|u(k)| \leq \gamma \forall k \in \mathbb{Z}(1, T)$. Then,

$$\begin{aligned} J(u) &= \frac{1}{2} u^T (P + Q) u - \sum_{k=1}^T F(k, u(k)) \geq \frac{\lambda_{l+1}}{2} \|u\|^2 - \frac{\lambda_l + \varepsilon}{2} \sum_{k=1}^T |u(k)|^2 \\ &= \frac{\lambda_{l+1} - \lambda_l - \varepsilon}{2} \|u\|^2 > 0. \end{aligned} \quad (46)$$

For every $u \in V \oplus S(\lambda_l)$, $\|u\| \leq \gamma$ implies $|u(k)| \leq \gamma \forall k \in \mathbb{Z}(1, T)$. Thus, we get

$$J(u) = \frac{1}{2} u^T (P + Q) u - \sum_{k=1}^T F(k, u(k)) \leq \frac{\lambda_l}{2} \|u\|^2 - \frac{\lambda_l}{2} \sum_{k=1}^T |u(k)|^2 = 0. \quad (47)$$

We can apply Lemma 2 to conclude that J has a local linking at 0, then $C_h(J, 0) \neq 0$, $h = \dim(V \oplus S(\lambda_l))$, and 0 is not local maximum in S .

Let ζ be the element of $S(\lambda_l)$, and owing to (G_4) , we have

$$\langle J''(0)\zeta, \zeta \rangle = \sum_{k=1}^T (\lambda_l - f'(k, 0)) |\zeta(k)|^2 = 0, \quad (48)$$

therefore, $\nu(0) = \dim S(\lambda_l) \geq 1$ and $C_q(J, 0) \cong \delta_{q, l-1+\nu(0)} \mathbb{Z}$.

We see that J satisfies the PS condition from (G_1) , and we have

$$J(u) \leq \frac{\lambda_{l-1}}{2} \|u\|^2 \longrightarrow -\infty, \quad \text{as } \|u\| \longrightarrow \infty, \quad (49)$$

for every $u \in V$.

On the other hand, we have

$$J(u) \geq \frac{1}{2} \lambda_l |u|^2 - a_1 T^{(1-\theta/2)} |u|^\theta - a_2 T \longrightarrow +\infty, \quad \text{as } |u| \longrightarrow \infty, \quad (50)$$

for every $u \in S(\lambda_l) \oplus W$. Hence, J is bounded from below on $S(\lambda_l) \oplus W$.

It can be easily seen that the functional J satisfies all the assumptions of Lemma 3; hence, we have

$$C_{l-1}(J, \infty) \neq 0. \quad (51)$$

We note that $l-1 \neq l-1+\nu(0)$, and then J has a critical point $u_1 \neq 0$ such that $C_{l-1}(J, u_1) \neq 0$. Thus, J has at least one nontrivial solution u_1 . □

3. Superlinear Case

In this section, we will study the case where $f(k, t)$ is superlinear in t at infinity for each $k \in \mathbb{Z}(1, T)$. We give the following assumption:

- (i) (G_6) There exist $\beta \in (2, \infty)$ and $M_1 > 0$ such that

$$tf(k, t) \geq \beta F(k, t) > 0, \quad \text{for } |t| \geq M_1, \forall k \in \mathbb{Z}(1, T). \quad (52)$$

We apply a similar method in [18] to look for nontrivial critical points of J in S .

In order to prove the main results of this section, we need to make use of the following lemmas.

Lemma 8. *If (G_6) holds, then J satisfies the PS condition.*

Proof. Suppose that the eigenvalues of the matrix $P + Q$ are $\lambda_1, \lambda_2, \dots, \lambda_T$, let $\lambda_{\max} = \max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_T|\} > 0$. Using (G_6) , there exist constants $a_3 > 0$ and $a_4 > 0$ such that

$$F(k, t) \geq a_3 |t|^\beta - a_4, \quad \text{for all } k \in \mathbb{Z}(1, T), t \in \mathbb{R}. \quad (53)$$

For any sequence $\{u_n\} \subset E$, $|J(u_n)| \leq C$ for some $C \in \mathbb{R}$ and $J'(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$, then we have

$$-C \leq J(u_n) = \frac{1}{2} u_n^T (P + Q) u_n - \sum_{k=1}^T F(k, u_n(k)) \quad (54)$$

$$\leq \frac{1}{2} \lambda_{\max} \|u_n\|^2 - a_3 \sum_{k=1}^T |u_n(k)|^\beta + a_4 T.$$

Since

$$\sum_{k=1}^T |u_n(k)|^2 \leq T^{(1-2/\beta)} \left(\sum_{k=1}^T |u_n(k)|^\beta \right)^{(2/\beta)}, \quad (55)$$

then,

$$\sum_{k=1}^T |u_n(k)|^\beta \geq T^{(1-\beta/2)} \|u_n\|^\beta. \quad (56)$$

We have

$$-C \leq J(u_n) \leq \frac{1}{2} \lambda_{\max} |u_n|^2 - a_3 T^{(1-\beta/2)} |u_n|^\beta + a_4 T. \quad (57)$$

For any $n \in \mathbb{Z}$,

$$a_3 T^{(1-\beta/2)} |u_n|^\beta - \frac{1}{2} \lambda_{\max} |u_n|^2 \leq a_4 T + C. \quad (58)$$

Note that $\beta > 2$, thus $\{u_n\}$ is bounded in S , and the Bolzano–Weierstrass theorem implies that $\{u_n\}$ has a convergent subsequence. $\square$

Lemma 9. Assume that (G_6) holds. Then, there exists a constant $D > 0$ such that

$$\hat{J}^{\hat{a}} \cong S^{T-1}, \quad \text{for } \hat{a} - \frac{3TM_1}{2} D, \quad (59)$$

where S^{T-1} is the unit sphere in S .

Proof. From (G_6) , we obtain

$$F(k, t) \geq a_3 |t|^\beta, \quad \text{for } |t| \geq M_1. \quad (60)$$

Let $u \in S^{T-1}$, then

$$\begin{aligned} J(su) &= \frac{|s|^2}{2} u^\tau (P + Q) u - \sum_{k=1}^T F(k, su(k)) \\ &\leq \frac{\lambda_{\max}}{2} |s|^2 - a_3 |s|^\beta \sum_{k=1}^T |u(k)|^\beta \leq \frac{\lambda_{\max}}{2} |s|^2 - a_3 |s|^\beta T^{(1-\beta/2)}, \end{aligned} \quad (61)$$

we have $J(su) \rightarrow -\infty$ as $|s| \rightarrow +\infty$.

Let $D = \max |f(k, t)|$ for $(k, t) \in \mathbb{Z}(1, T) \times [-M_1, M_1]$.

In view of (G_6) , we have

$$\begin{aligned} \sum_{k=1}^T F(k, u(k)) - \frac{1}{2} \sum_{k=1}^T u(k) f(k, u(k)) &= \sum_{|u(k)| > M_1} F(k, u(k)) + \sum_{|u(k)| \leq M_1} F(k, u(k)) \\ &\quad - \frac{1}{2} \sum_{|u(k)| > M_1} u(k) f(k, u(k)) - \frac{1}{2} \sum_{|u(k)| \leq M_1} u(k) f(k, u(k)) \\ &\leq \left(\frac{1}{\beta} - \frac{1}{2} \right) \sum_{|u(k)| > M_1} u(k) f(k, u(k)) + \sum_{|u(k)| \leq M_1} |F(k, u(k))| \\ &\quad + \frac{1}{2} \sum_{|u(k)| < M_1} |u(k) f(k, u(k))| \\ &\leq \left(\frac{1}{\beta} - \frac{1}{2} \right) \sum_{|u(k)| > M_1} u(k) f(k, u(k)) + \frac{3TM_1}{2} D. \end{aligned} \quad (62)$$

Let $\bar{a} < (-3TM_1/2)D$, $s > 0$, and

$$J(su) = \frac{s^2}{2} u^\tau (P + Q) u - \sum_{k=1}^T F(k, su(k)) \leq \bar{a}, \quad \text{for } u \in S^{T-1}. \quad (63)$$

We can compute the derivative of $J(su)$ with respect to s and obtain

$$\begin{aligned} \frac{dJ(su)}{ds} &= s(u^\tau (P + Q) u) - \sum_{k=1}^T u(k) f(k, su(k)) \\ &\leq \frac{2}{s} \left(\sum_{k=1}^T F(k, su(k)) - \frac{1}{2} \sum_{k=1}^T su(k) f(k, su(k)) + \bar{a} \right) \\ &\leq \frac{2}{s} \left(\left(\frac{1}{\beta} - \frac{1}{2} \right) \sum_{|su(k)| > M_1} su(k) f(k, su(k)) + \frac{3TM_1}{2} D + \bar{a} \right) \\ &\leq \frac{2}{s} \left(\frac{1}{\beta} - \frac{1}{2} \right) \sum_{|su(k)| > M_1} su(k) f(k, su(k)) < 0. \end{aligned} \quad (64)$$

By the implicit function theorem, there exists a unique $\varphi \in C(S^{T-1}, \mathbb{R})$ such that $s = \varphi(u)$ and $J(\varphi(u)u) = \hat{a} \leq \bar{a}$ for $u \in S^{T-1}$. If $u \neq 0$, let $\bar{\varphi}(u) = ((1/\|u\|)\varphi(u/\|u\|))$, then $\bar{\varphi} \in C(S \setminus 0, \mathbb{R})$ and $J(\bar{\varphi}(u)u) = \hat{a}$, for $u \in S \setminus 0$. Further, if $J(u) = \hat{a}$, then $\bar{\varphi}(u) = 1$.

Now, we define a function

$$\Phi(u) = \begin{cases} \bar{\varphi}(u), & \text{if } J(u) \geq \hat{a}, \\ 1, & \text{if } J(u) \leq \hat{a}. \end{cases} \quad (65)$$

Obviously, $\Phi: S \setminus 0 \rightarrow \mathbb{R}$ is a continuous function.

We set a map $\mu: [0, 1] \times (S \setminus 0) \rightarrow S \setminus 0$ as $\mu(s, u) = (1 - s)u + s\Phi(u)u$ since J^a is a strong deformation retract of $S \setminus 0$, that is, $J^a \cong S \setminus 0 \cong S^{T-1}$. $\square$

Theorem 5. Assume that the conditions (G_2) and (G_6) hold. Then, problem (1) has at least one nontrivial solution.

Proof. Since the conditions (G_2) and (G_6) hold, it follows from Lemma 2, Lemma 7, and ([23], Theorem 4.2) that there exists a $\varepsilon > 0$ such that

$$H_h(J^\varepsilon, J^{-\varepsilon}) = C_h(J, 0) \neq 0, \quad (66)$$

where $h = \dim(V \oplus S(\lambda_l))$.

By Lemma 9, we have

$$H_h(S, J^{\hat{a}}) \cong H_h(S, S^{T-1}) = 0. \quad (67)$$

Using Lemma 4,

$$H_{h+1}(S, J^\varepsilon) \neq 0 \text{ or } H_{h-1}(J^{-\varepsilon}, J^{\hat{a}}) \neq 0. \quad (68)$$

Hence, J has a critical point $u \neq 0$ for which

$$J(u) > \varepsilon \text{ or } \hat{a} < J(u) < -\varepsilon. \quad (69)$$

The proof is complete.

Note that the conclusion of Theorem 5 holds if the symmetric matrix $P + Q$ is positive definite or negative definite or just nonsingular. $\square$

Example 2. Let $T = 3$, and we study the boundary value problem (1) with

$$f(k, t) = f(t) = \begin{cases} 2t, & \text{if } |t| < \frac{1}{4}, \\ \frac{t}{\sqrt{|t|}}, & \text{if } \frac{1}{4} \leq |t| \leq 1, \\ |t|^2 t, & \text{if } |t| > 1, \end{cases} \quad (70)$$

for $k \in \mathbb{Z}(1, 3)$.

Then,

$$F(k, t) = F(t) = \begin{cases} t^2, & \text{if } |t| < \frac{1}{4}, \\ \frac{2}{3}|t|^{(3/2)} - \frac{1}{48}, & \text{if } \frac{1}{4} \leq |t| \leq 1, \\ \frac{|t|^4}{4} + \frac{19}{48}, & \text{if } |t| > 1. \end{cases} \quad (71)$$

Let $\beta = 3$, $p(k) = 1$, and $q(k) = 0$ for each $k \in \mathbb{Z}(1, 3)$. The matrix $P + Q$,

$$P + Q = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad (72)$$

admits three distinct eigenvalues given by

$$\lambda_1 = 2 - \sqrt{2}, \lambda_2 = 2, \lambda_3 = 2 + \sqrt{2}. \quad (73)$$

Clearly, we have

$$2|t|^2 = 2F(k, t) < (2 + \sqrt{2})|t|^2, \quad |t| \leq \frac{1}{4}, \forall k \in \mathbb{Z}(1, 3), \quad (74)$$

and

$$tf(k, t) - 3F(k, t) = \frac{|t|^4}{4} - \frac{19}{16} \rightarrow +\infty, \quad (75)$$

as $|t| \rightarrow +\infty, \forall k \in \mathbb{Z}(1, 3)$.

This means that the conditions (G_2) and (G_6) hold.

All conditions of Theorem 5 are verified, and then problem (1) admits one nontrivial solution in S .

Corollary 1. Let (G_2) and (G_6) hold and $\dim(V \oplus S(\lambda_l)) \pm 1 \neq T$. Then, problem (1) possesses at least two nontrivial solutions.

Proof. We notice that condition (G_6) implies (53). Combining with (56), we have

$$\begin{aligned} J(u) &= \frac{1}{2}u^T(P + Q)u - \sum_{k=1}^T F(k, u(k)) \\ &\leq \frac{\lambda_T}{2}\|u\|^2 - a_3 \sum_{k=1}^T |u(k)|^\beta + a_4 T \\ &\leq \frac{\lambda_T}{2}\|u\|^2 - a_3 T^{(1-\beta/2)}\|u\|^\beta \\ &\quad + a_4 T \rightarrow -\infty \text{ as } \|u\| \rightarrow \infty. \end{aligned} \quad (76)$$

Hence, $-J$ is coercive, and J has a local maximum at some $u_1 \in S$. This means $C_q(J, u_1) \cong \delta_{q,T}\mathbb{Z}$. Note that the condition (G_2) holds, so $u_1 \neq 0$ by Lemma 7. Now, $\dim(V \oplus S(\lambda_l)) \pm 1 \neq T$, combining with (68) and (69), there exists another nontrivial critical point $u_2 \neq u_1$ of J . $\square$

Example 3. Let $T = 3$, and we consider the following boundary value problem (1) with

$$f(k, t) = f(t) = \begin{cases} -2t, & \text{if } |t| < \frac{1}{4}, \\ -\frac{t}{\sqrt{|t|}}, & \text{if } \frac{1}{4} \leq |t| \leq 1, \\ -|t|^2 t, & \text{if } |t| > 1, \end{cases} \quad (77)$$

for $k \in \mathbb{Z}(1, 3)$.

Then,

$$F(k, t) = F(t) = \begin{cases} -t^2, & \text{if } |t| < \frac{1}{4}, \\ -\frac{2}{3}|t|^{(3/2)} + \frac{1}{48}, & \text{if } \frac{1}{4} \leq |t| \leq 1, \\ -\frac{|t|^4}{4} - \frac{19}{48}, & \text{if } |t| > 1. \end{cases} \quad (78)$$

Let $p(k) = 1$ and $q(k) = 7/2$ for each $k \in \mathbb{Z}(1, 3)$. Then, the matrix

$$P + Q = \begin{pmatrix} -\frac{3}{2} & -1 & 0 \\ -1 & -\frac{3}{2} & -1 \\ 0 & -1 & -\frac{3}{2} \end{pmatrix}, \quad (79)$$

is negative definite, and the eigenvalues are given by

$$\lambda_1 = -\frac{3+2\sqrt{2}}{2}, \lambda_2 = -\frac{3}{2}, \lambda_3 = \frac{2\sqrt{2}-3}{2}. \quad (80)$$

Let $\delta = 1/4$ and $\beta = 8$, then we have

$$-\frac{3+2\sqrt{2}}{2}|t|^2 < 2F(k, t) < -\frac{3}{2}|t|^2, |t| \leq \frac{1}{4}, \forall k \in \mathbb{Z}(1, 3), \quad (81)$$

and

$$tf(k, t) - 8F(k, t) = |t|^4 + \frac{19}{6} \longrightarrow +\infty, \quad (82)$$

as $|t| \longrightarrow +\infty, \forall k \in \mathbb{Z}(1, 3)$.

Then, the conditions (G_2) and (G_6) hold. Moreover, $\dim S(\lambda_1) = 1$, and obviously, $\dim S(\lambda_1) \pm 1 \neq 3$. Then, problem (1) admits at least two nontrivial solutions in S by Corollary 1.

4. Conclusion

In this paper, we have studied the existence of one or multiple nontrivial solutions of the boundary value problem (1) for two cases: (i) f is sublinear and (ii) f is superlinear. Examples are provided to illustrate the applicability of our results.

We point out that our work is different from the previous works and the results of this paper improve, extend, and complement some related results in the literature [12]. Since our solutions are obtained by the variational approach combining with the Morse theory, functional J is required to satisfy the PS conditions, and some computations on the critical group of a local linking-type critical point are used to deal with the resonance at zero.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Both the authors contributed equally to the writing of this paper and they read and approved the final manuscript.

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