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Research Article

Fractional Theoretical Model for Gravity Waves and Squall Line in Complex Atmospheric Motion

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The evolution of nonlinear gravity solitary waves in the atmosphere is related to the formation of severe weather. The nonlinear concentration of gravity solitary wave leads to energy accumulation, which further forms the disastrous weather phenomenon such as squall line. This paper theoretically proves that the formation of squall line in baroclinic nonstatic equilibrium atmosphere can be reduced to the fission process of algebraic gravity solitary waves described by the $(2+1)$-dimensional generalized Boussinesq-BO (B-BO) equation. Compared with previous models describing isolated waves, the Boussinesq-BO model can describe the propagation process of waves in two media, which is more suitable for actual atmospheric conditions. In order to explore more structural features of this solitary wave, the derived integer order model is transformed into the more practical time fractional-order model by using the variational method. By obtaining the exact solution and the conservation laws, the fission properties of algebraic gravity solitary waves are discussed. When the disturbance with limited width appears along the low-level jet stream, these solitary waves can be excited. When the disturbance intensity and width reach a certain value, solitary wave formation takes place, which is exactly the squall line or thunderstorm formation observed in the atmosphere.

1. Introduction

On June 4, 2016, a squall line weather occurred in northern Sichuan, China, lasting for 12 hours. On that day, from 09:00 to 20:00 Beijing time, a large area of catastrophic winds hit Aba, Guangyuan, Mianyang, Nanchong, and Bazhong in Sichuan province and the northern part of Chongqing province, with the maximum speed reaching $33.5\,\mathrm{m\cdot s^{-1}}$, causing serious loss of life and property. Among them, the squall line's strong convection cell generated local thunderstorm strong wind which capsized Guangyuan city Bailong Lake cruise ship, killing 15 people. So, the research of squall lines has important theoretical meaning and real value. But, the research on the formation and prediction of squall line phenomenon is very difficult, which is a problem.

The phenomenon of gravitational waves occurring in the upper atmosphere is the most common and important dynamic process. They propagate in the middle atmosphere and generate energy and momentum transfer between atmospheres at different altitudes, which leads to the process of energy coupling between atmospheres [1]. The pressure jump, strong wind, strong convergence, and upward movement of gravity waves are caused by nonlinear action in the dispersion process. Mesoscale strong convection and rainstorm systems often have the characteristics of gravity wave [2]. Therefore, the gravity wave is an important factor in the formation of many mesoscale weather phenomena.

Many previous studies have shown that the squall line phenomenon is closely related to gravity solitary waves. In the 1960s, Long [3] proposed that the amplitude of atmospheric fluctuation satisfied the Korteweg–de Vries (KdV) equation in the first time. In 1978, Li [4] obtained the periodic and elliptic cosine solutions of the nonlinear atmospheric gravity waves, nonlinear Rossby waves, and elliptic cosine waves from the nonlinear atmospheric motion equations. Liu and Liu [5] found that the amplitude of gravity solitary waves was proportional to the propagation speed in 1983. Then, Bu et al. [6] and Wang [7] studied gravity waves by using the nonlinear Korteweg–de Vries (KdV) equation from the nonlinear atmospheric dynamics equation. In 2002, Lott and Plougonven [8] calculated the EP flux using the WKB approximation method and parameterized the gravity wave source. In recent years, with the rapid development of numerical models, many experts at home and abroad began to use weather models, cloud models, and climate models to research the development of gravity waves. Jewett et al. [9] and Plougonven and Snyder [10] successfully simulated storm, frontal, and gravity waves with mesoscale models MM5 and WRF, respectively. In 2011, Lane and Zhang [11] used a two-dimensional convective cloud model to study the mechanism of convection-induced gravity waves. Regarding the formation of squall line, Li and Xue [12] and Li [13] pointed out that the nonlinear concentration of gravity solitary waves was the formation mechanism of squall line. Yao et al. [14] also analyzed the formation and maintenance of strong squall line in detail in 2005. Then, in 2014, Srinivasan et al. [15] studied the gravitational wave characteristics of squall line during its propagation. Stephan and Alexander [16] discussed the relationship between the formation of summer squall line and gravity solitary waves. Recently, in 2000, Scinocca and Ford [17] found that large gravity solitary waves forced unstable layered shear flows.

It can be seen from previous studies that the research on gravity waves and squall line mainly focuses on numerical simulation, while the research on theoretical analysis is very few. We know that the basic dynamic equations describing the motion of baroclinic atmosphere are more and more complex [18, 19], and the gravity waves of baroclinic atmosphere [20, 21] are seldom studied. However, because of the baroclinic characteristics of the atmosphere, the baroclinic problem in the actual atmosphere is inevitable. In addition, these studies describe classical gravity solitary waves [22, 23]. In fact, some classical gravity solitary wave models, such as the Korteweg–de Vries (KdV) equation, modified Korteweg–de Vries (mKdV) equation, and Boussinesq equation, describe the propagation of gravity solitary waves in a certain direction [24–27]. But, true gravity solitary waves travel in both directions [28]. Therefore, we need to establish a new model to replace the classical gravity solitary waves and explore the formation mechanism of squall line so as to better adapt to the actual atmospheric conditions.

In recent decades, fractional partial differential equations have been increasingly used to describe problems in optical and thermal systems, rheological and material and mechanical systems, signal processing and system identification, control and robotics, and other applications [29–31]. Fractional-order equations are divided into time fractional equations and space fractional equations [32–34] and have various definitions. Because fractional derivatives are historically dependent and nonlocal, they can more accurately describe complex physical and dynamic system processes in nature. The solution of fractional-order equation has attracted more and more attention of scholars, who have studied many effective solutions, such as the (G'/G)-expansion method [35], Khater method [36], sun-equation method [37], functional variable method [38], modified extended Tanh method [39], Godunov-type method [40], Lie group analysis method [41], and so on [42–45].

2. Perturbation Expansion of Atmospheric Dynamics Equation Set

In order to obtain a new integer-order model to describe the evolution of algebraic gravity solitary waves, we start from the classical dynamic equation set and carry out perturbation expansion and perturbation analysis. A general gas is not under extreme conditions, and the properties of a real gas are very close to that of an ideal gas. In addition, just to simplify our calculation, the gases that we are dealing with here are all in ideal gas states, so they all satisfy the ideal gas equation, the Clapeyron–Mendeleev equation. The basic dynamic equation set of baroclinic nonstatic equilibrium atmosphere is transformed to the dimensionless form (Appendix A) as follows:

$$\begin{cases} \dfrac{\partial U}{\partial T} + U\dfrac{\partial U}{\partial X} + V\dfrac{\partial U}{\partial Y} + W\dfrac{\partial U}{\partial Z} = -\dfrac{1}{\rho_s}\dfrac{\partial P}{\partial X} + V, \\ \dfrac{\partial V}{\partial T} + U\dfrac{\partial V}{\partial X} + V\dfrac{\partial V}{\partial Y} + W\dfrac{\partial V}{\partial Z} = -\dfrac{1}{\rho_s}\dfrac{\partial P}{\partial Y} - U, \\ \dfrac{\partial W}{\partial T} + U\dfrac{\partial W}{\partial X} + V\dfrac{\partial W}{\partial Y} + W\dfrac{\partial W}{\partial Z} = \varepsilon^{-1}\left(-\dfrac{1}{\rho_s}\dfrac{\partial P}{\partial Z} + \Theta\right), \\ \dfrac{\partial \Theta}{\partial T} + U\dfrac{\partial \Theta}{\partial X} + V\dfrac{\partial \Theta}{\partial Y} + W = 0, \\ \dfrac{\partial \rho_s U}{\partial X} + \dfrac{\partial \rho_s V}{\partial Y} + \dfrac{\partial \rho_s W}{\partial Z} = 0. \end{cases} \tag{1}$$

In the vicinity of the low-level jet, the basic flow tends to have strong horizontal shear, while when it is away from the low-level jet, the shear of the basic flow is small. Therefore, we can assume that the basic flow of the low-level jet is almost constant and the region is divided into two parts $[0, h_0]$ and $[h_0, \infty)$; the corresponding shear flow is written as

$$u = \begin{cases} u(Y, Z), & Y \in [0, h_0], \\ u_1, & Y \in [h_0, \infty), \end{cases} \tag{2}$$

where u_1 is a constant, and the boundary conditions can be written as

$$\begin{cases} P = 0, & \text{as } Y = 0, \\ P \longrightarrow 0, & \text{as } Y \longrightarrow \infty. \end{cases} \tag{3}$$

This is equivalent to assuming that the boundary of the atmosphere at $y = 0$ is a rigid boundary, and the disturbance in the region $y \geq h_0$ is only generated by the disturbance in region $0 \leq y \leq h_0$, so the disturbance disappears at infinity.

First of all, the situation of domain $[0, h_0]$ is studied. Introducing the multiscale slow time and space transformations [46, 47],

$$\begin{aligned}\frac{\partial}{\partial T} &= \varepsilon^{3/2}\frac{\partial}{\partial t} - c\varepsilon\frac{\partial}{\partial x},\\ \frac{\partial}{\partial X} &= \varepsilon\frac{\partial}{\partial x},\\ \frac{\partial}{\partial Y} &= \varepsilon\frac{\partial}{\partial y} + \frac{\partial}{\partial y_0},\\ \frac{\partial}{\partial Z} &= \frac{\partial}{\partial z},\end{aligned} \tag{4}$$

and supposing that U, V, W, P, and Θ, we have the following small parameter expansions:

$$\begin{aligned}U &= \left(u\left(y_0, z\right) + \varepsilon\alpha\right) + \varepsilon u_0 + \varepsilon^{3/2}u_1 + \varepsilon^2 u_2 + \cdots,\\ V &= \varepsilon^2 v_0 + \varepsilon^{5/2}v_1 + \varepsilon^3 v_2 + \cdots,\\ W &= \varepsilon^2 w_0 + \varepsilon^{5/2}w_1 + \varepsilon^3 w_2 + \cdots,\\ P &= p\left(y_0, z\right) + \varepsilon p_0 + \varepsilon^{3/2}p_1 + \varepsilon^2 p_2 + \cdots,\\ \Theta &= \theta\left(y_0, z\right) + \varepsilon\theta_0 + \varepsilon^{3/2}\theta_1 + \varepsilon^2\theta_2 + \cdots,\end{aligned} \tag{5}$$

where α is the order of magnitude parameter of 1.

Substituting equations (4) and (5) into equation (1) yields

$$\begin{aligned}&\varepsilon^{5/2}\frac{\partial u_0}{\partial t} + \varepsilon^3\frac{\partial u_1}{\partial t} + \varepsilon^{7/2}\frac{\partial u_2}{\partial t} - c\varepsilon^2\frac{\partial u_0}{\partial x} - c\varepsilon^{5/2}\frac{\partial u_1}{\partial x} - c\varepsilon^3\frac{\partial u_2}{\partial x} + \left[(u+\varepsilon\alpha) + \varepsilon u_0 + \varepsilon^{3/2}u_1 + \varepsilon^2 u_2\right]\\
&\quad\cdot\left(\varepsilon^2\frac{\partial u_0}{\partial x} + \varepsilon^{5/2}\frac{\partial u_1}{\partial x} + \varepsilon^3\frac{\partial u_2}{\partial x}\right) + \left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\left(\frac{\partial u}{\partial y_0} + \varepsilon\frac{\partial u_0}{\partial y_0} + \varepsilon^{3/2}\frac{\partial u_1}{\partial y_0} + \varepsilon^2\frac{\partial u_2}{\partial y_0}\right)\\
&\quad+\left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\left(\varepsilon^2\frac{\partial u_0}{\partial y} + \varepsilon^{5/2}\frac{\partial u_1}{\partial y} + \varepsilon^3\frac{\partial u_2}{\partial y}\right) + \left(\varepsilon^2 w_0 + \varepsilon^{2/5}w_1 + \varepsilon^3 w_2\right)\left(\frac{\partial u}{\partial z} + \varepsilon\frac{\partial u_0}{\partial z} + \varepsilon^{3/2}\frac{\partial u_1}{\partial z} + \varepsilon^2\frac{\partial u_2}{\partial z}\right)\\
&\quad+\varepsilon^2\frac{1}{\rho_s}\frac{\partial p_0}{\partial x} + \varepsilon^{5/2}\frac{1}{\rho_s}\frac{\partial p_1}{\partial x} + \varepsilon^3\frac{1}{\rho_s}\frac{\partial p_2}{\partial x} - \varepsilon^2 v_0 - \varepsilon^{5/2}v_1 - \varepsilon^3 v_2 = 0,\\
&\varepsilon^{5/2}\frac{\partial v_0}{\partial t} + \varepsilon^3\frac{\partial v_1}{\partial t} + \varepsilon^{7/2}\frac{\partial v_2}{\partial t} - c\varepsilon^2\frac{\partial v_0}{\partial x} - c\varepsilon^{5/2}\frac{\partial v_1}{\partial x} - c\varepsilon^3\frac{\partial v_2}{\partial x} + \left[(u+\varepsilon\alpha) + \varepsilon u_0 + \varepsilon^{3/2}u_1 + \varepsilon^2 u_2\right]\\
&\quad\cdot\left(\varepsilon^3\frac{\partial v_0}{\partial x} + \varepsilon^{7/2}\frac{\partial v_1}{\partial x} + \varepsilon^4\frac{\partial v_2}{\partial x}\right) + \left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\left(\varepsilon^2\frac{\partial v_0}{\partial y_0} + \varepsilon^{5/2}\frac{\partial v_1}{\partial y_0} + \varepsilon^3\frac{\partial v_2}{\partial y_0}\right) + \left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\\
&\quad\cdot\left(\varepsilon^3\frac{\partial v_0}{\partial y} + \varepsilon^{7/2}\frac{\partial v_1}{\partial y} + \varepsilon^4\frac{\partial v_2}{\partial y}\right) + \left(\varepsilon^2 w_0 + \varepsilon^{2/5}w_1 + \varepsilon^3 w_2\right)\left(\varepsilon^2\frac{\partial v_0}{\partial z} + \varepsilon^{5/2}\frac{\partial v_1}{\partial z} + \varepsilon^3\frac{\partial v_2}{\partial z}\right)\\
&\quad+\frac{1}{\rho_s}\frac{\partial p}{\partial y_0} + \varepsilon\frac{1}{\rho_s}\frac{\partial p_0}{\partial y_0} + \varepsilon^{3/2}\frac{1}{\rho_s}\frac{\partial p_1}{\partial y_0} + \varepsilon^2\frac{1}{\rho_s}\frac{\partial p_2}{\partial y_0} + \varepsilon^2\frac{1}{\rho_s}\frac{\partial p_0}{\partial y} + \varepsilon^{5/2}\frac{1}{\rho_s}\frac{\partial p_1}{\partial y} + \varepsilon^3\frac{1}{\rho_s}\frac{\partial p_2}{\partial y}\\
&\quad+(u+\varepsilon\alpha) + \varepsilon u_0 + \varepsilon^{3/2}u_1 + \varepsilon^2 u_2 = 0,\\
&\varepsilon^{5/2}\frac{\partial w_0}{\partial t} + \varepsilon^3\frac{\partial w_1}{\partial t} + \varepsilon^{7/2}\frac{\partial w_2}{\partial t} - c\varepsilon^2\frac{\partial w_0}{\partial x} - c\varepsilon^{5/2}\frac{\partial w_1}{\partial x} - c\varepsilon^3\frac{\partial w_2}{\partial x} + \left[(u+\varepsilon\alpha) + \varepsilon u_0 + \varepsilon^{3/2}u_1 + \varepsilon^2 u_2\right]\\
&\quad\cdot\left(\varepsilon^3\frac{\partial w_0}{\partial x} + \varepsilon^{7/2}\frac{\partial w_1}{\partial x} + \varepsilon^4\frac{\partial w_2}{\partial x}\right) + \left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\left(\varepsilon^2\frac{\partial w_0}{\partial y_0} + \varepsilon^{5/2}\frac{\partial w_1}{\partial y_0} + \varepsilon^3\frac{\partial w_2}{\partial y_0}\right) +\\
&\quad\cdot\left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\left(\varepsilon^3\frac{\partial w_0}{\partial y} + \varepsilon^{7/2}\frac{\partial w_1}{\partial y} + \varepsilon^4\frac{\partial w_2}{\partial y}\right) + \left(\varepsilon^2 w_0 + \varepsilon^{2/5}w_1 + \varepsilon^3 w_2\right)\left(\varepsilon^2\frac{\partial w_0}{\partial z} + \varepsilon^{5/2}\frac{\partial w_1}{\partial z} + \varepsilon^3\frac{\partial w_2}{\partial z}\right)\\
&\quad+\varepsilon^{-1}\frac{1}{\rho_s}\frac{\partial p}{\partial z} + \frac{1}{\rho_s}\frac{\partial p_0}{\partial z} + \varepsilon^{1/2}\frac{1}{\rho_s}\frac{\partial p_1}{\partial z} + \varepsilon^1\frac{1}{\rho_s}\frac{\partial p_2}{\partial z} - \varepsilon^{-1}\theta - \theta_0 - \varepsilon^{1/2}\theta_1 - \varepsilon\theta_2 = 0,\\
&\varepsilon^{5/2}\frac{\partial\theta_0}{\partial t} + \varepsilon^3\frac{\partial\theta_1}{\partial t} + \varepsilon^{7/2}\frac{\partial\theta_2}{\partial t} - c\varepsilon^2\frac{\partial\theta_0}{\partial x} - c\varepsilon^{5/2}\frac{\partial\theta_1}{\partial x} - c\varepsilon^3\frac{\partial\theta_2}{\partial x} + \left[(u+\varepsilon\alpha) + \varepsilon u_0 + \varepsilon^{3/2}u_1 + \varepsilon^2 u_2\right]\\
&\quad\cdot\left(\varepsilon^2\frac{\partial\theta_0}{\partial x} + \varepsilon^{5/2}\frac{\partial\theta_1}{\partial x} + \varepsilon^3\frac{\partial\theta_2}{\partial x}\right) + \left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\left(\frac{\partial\theta}{\partial y_0} + \varepsilon\frac{\partial\theta_0}{\partial y_0} + \varepsilon^{3/2}\frac{\partial\theta_1}{\partial y_0} + \varepsilon^2\frac{\partial\theta_2}{\partial y_0}\right)\\
&\quad+\left(\varepsilon^2 v_0 + \varepsilon^{2/5}v_1 + \varepsilon^3 v_2\right)\left(\varepsilon^2\frac{\partial\theta_0}{\partial y} + \varepsilon^{5/2}\frac{\partial\theta_1}{\partial y} + \varepsilon^3\frac{\partial\theta_2}{\partial y}\right) + \varepsilon^2 w_0 + \varepsilon^{5/2}w_1 + \varepsilon^3 w_2 = 0,\\
&\varepsilon^2\frac{\partial\rho_s u_0}{\partial x} + \varepsilon^{5/2}\frac{\partial\rho_s u_1}{\partial x} + \varepsilon^3\frac{\partial\rho_s u_2}{\partial x} + \varepsilon^2\frac{\partial\rho_s v_0}{\partial y_0} + \varepsilon^{5/2}\frac{\partial\rho_s v_1}{\partial y_0} + \varepsilon^3\frac{\partial\rho_s v_2}{\partial y_0} + \varepsilon^3\frac{\partial\rho_s v_0}{\partial y}\\
&\quad+\varepsilon^{7/2}\frac{\partial\rho_s v_1}{\partial y} + \varepsilon^3\frac{\partial\rho_s v_2}{\partial y} + \varepsilon^2\frac{\partial\rho_s w_0}{\partial z} + \varepsilon^{5/2}\frac{\partial\rho_s w_1}{\partial z} + \varepsilon^3\frac{\partial\rho_s w_2}{\partial z} = 0.\end{aligned} \tag{6}$$

Taking the zero-order approximation of ε, we have

$$\begin{cases} \dfrac{1}{\rho_s}\dfrac{\partial p}{\partial y_0} + u = 0, \\ \dfrac{1}{\rho_s}\dfrac{\partial p}{\partial z} - \theta = 0. \end{cases} \tag{7}$$

The above equation indicates that the basic flow is balanced. First of all, we take the first-order approximations of ε as

$$\begin{cases} (u-c)\dfrac{\partial u_0}{\partial x} + \left(\dfrac{\partial u}{\partial y_0} - 1\right)v_0 + \dfrac{\partial u}{\partial z}w_0 + \dfrac{1}{\rho_s}\dfrac{\partial p_0}{\partial x} = 0, \\ \dfrac{1}{\rho_s}\dfrac{\partial p_0}{\partial y_0} + \alpha + u_0 = 0, \\ \dfrac{1}{\rho_s}\dfrac{\partial p_0}{\partial z} - \theta_0 = 0, \\ (u-c)\dfrac{\partial \theta_0}{\partial x} + \dfrac{\partial \theta}{\partial y_0}v_0 + w_0 = 0, \\ \dfrac{\partial \rho_s u_0}{\partial x} + \dfrac{\partial \rho_s v_0}{\partial y_0} + \dfrac{\partial \rho_s w_0}{\partial z} = 0. \end{cases} \tag{8}$$

Suppose the equation has the following solutions in separate variable form:

$$\begin{cases} u_0 = \tilde{u}_0(y_0, z)n(t, x, y), \\ v_0 = \tilde{v}_0(y_0, z)n_x(t, x, y), \\ w_0 = \tilde{w}_0(y_0, z)n_x(t, x, y), \\ \theta_0 = \tilde{\theta}_0(y_0, z)n(t, x, y), \\ p_0 = \tilde{p}_0(y_0, z)n(t, x, y), \end{cases} \tag{9}$$

Then, we plug these solutions in equation (8); an equation that only depends on p_0 can be obtained:

$$\ell_{y_0,z}\left(\frac{\partial p_0}{\partial x}\right) = 0, \tag{10}$$

where the specific form of $\ell_{y_0,z}$ is shown in Appendix B. In addition to that, the second-order approximations of ε are given as

$$\begin{cases} (u-c)\dfrac{\partial u_1}{\partial x} + \left(\dfrac{\partial u}{\partial y_0} - 1\right)v_1 + \dfrac{\partial u}{\partial z}w_1 + \dfrac{1}{\rho_s}\dfrac{\partial p_1}{\partial x} = -\dfrac{\partial u_0}{\partial t}, \\ \dfrac{1}{\rho_s}\dfrac{\partial p_1}{\partial y_0} + u_1 = 0, \\ \dfrac{1}{\rho_s}\dfrac{\partial p_1}{\partial z} - \theta_1 = 0, \\ (u-c)\dfrac{\partial \theta_1}{\partial x} + \dfrac{\partial \theta}{\partial y_0}v_1 + w_1 = -\dfrac{\partial \theta_0}{\partial t}, \\ \dfrac{\partial \rho_s u_1}{\partial x} + \dfrac{\partial \rho_s v_1}{\partial y_0} + \dfrac{\partial \rho_s w_1}{\partial z} = 0. \end{cases} \tag{11}$$

In the same way, we assume that the separable solutions of equation (11) have the following form:

$$\begin{cases} u_{1x} = \tilde{u}_0(y_0, z)n_t(t, x, y), \\ v_1 = \tilde{v}_0(y_0, z)n_t(t, x, y), \\ w_1 = \tilde{w}_0(y_0, z)n_t(t, x, y), \\ \theta_{1x} = \tilde{\theta}_0(y_0, z)n_t(t, x, y), \\ p_{1x} = \tilde{p}_0(y_0, z)n_t(t, x, y), \end{cases} \tag{12}$$

Substituting the solutions into equation (11) yields

$$\begin{cases} (u-c)\tilde{u}_0 + \left(\dfrac{\partial u}{\partial y_0}\right)\tilde{v}_0 + \dfrac{\partial u}{\partial z}\tilde{w}_0 + \tilde{p}_0 = -\tilde{u}_0, \\ \dfrac{\partial \tilde{p}_0}{\partial y_0} + \tilde{u}_0 = 0, \\ \dfrac{\partial \tilde{p}_0}{\partial z} - \tilde{\theta}_0 = 0, \\ (u-c)\tilde{\theta}_0 + \dfrac{\partial \theta}{\partial y_0}\tilde{v}_0 + \tilde{w}_0 = -\tilde{\theta}_0, \\ \tilde{u}_0 + \dfrac{\partial \tilde{v}_0}{\partial y_0} + \dfrac{\partial \tilde{w}_0}{\partial z} = 0. \end{cases} \tag{13}$$

We find that the governing model of gravity solitary waves cannot be derived, so we need to proceed to the higher-order equation of ε:

$$\begin{cases} (u-c)\dfrac{\partial u_2}{\partial x}+\left(\dfrac{\partial u}{\partial y_0}-1\right)v_2+\dfrac{\partial u}{\partial z}w_2+\dfrac{1}{\rho_s}\dfrac{\partial p_2}{\partial x} \\ =-\dfrac{\partial u_1}{\partial t}-\alpha\dfrac{\partial u_0}{\partial x}-u_0\dfrac{\partial u_0}{\partial x}-v_0\dfrac{\partial u_0}{\partial y_0}-w_0\dfrac{\partial u_0}{\partial z}, \\ \dfrac{1}{\rho_s}\dfrac{\partial p_2}{\partial y_0}+u_2=-\dfrac{1}{\rho_s}\dfrac{\partial p_0}{\partial y}, \\ \dfrac{1}{\rho_s}\dfrac{\partial p_2}{\partial z}-\theta_2=0, \\ (u-c)\dfrac{\partial \theta_2}{\partial x}+\dfrac{\partial \theta}{\partial y_0}v_2+w_2 \\ =-\dfrac{\partial \theta_1}{\partial t}-\alpha\dfrac{\partial \theta_0}{\partial x}-u_0\dfrac{\partial \theta_0}{\partial x}-v_0\dfrac{\partial \theta_0}{\partial y_0}, \\ \dfrac{\partial \rho_s u_2}{\partial x}+\dfrac{\partial \rho_s v_2}{\partial y_0}+\dfrac{\partial \rho_s w_2}{\partial z}=-\dfrac{\partial \rho_s v_0}{\partial y}, \end{cases} \tag{14}$$

$$\begin{aligned} A_1&=\frac{\partial u_1}{\partial t}+\alpha\frac{\partial u_0}{\partial x}+u_0\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y_0}+w_0\frac{\partial u_0}{\partial z},\\ A_2&=\frac{\partial p_0}{\partial y},\\ A_3&=\frac{\partial \theta_1}{\partial t}+\alpha\frac{\partial \theta_0}{\partial x}+u_0\frac{\partial \theta_0}{\partial x}+v_0\frac{\partial \theta_0}{\partial y_0},\\ A_4&=\frac{\partial v_0}{\partial y}. \end{aligned} \tag{15}$$

Eliminating u_2, v_2, w_2, and θ_2 of equation (14), we can obtain an equation about p_2:

$$\ell_{y_0,z}\left(\frac{\partial p_2}{\partial x}\right)=\ell_{1y_{0,z}}(A_1)+\ell_{2y_{0,z}}(A_2)+\ell_{3y_{0,z}}(A_3)-A_4, \tag{16}$$

where the specific forms of $\ell_{1y_{0,z}}$, $\ell_{2y_{0,z}}$, and $\ell_{3y_{0,z}}$ are shown in Appendix B.

Substituting equations (9) and (12) into equation (15) yields

$$\begin{cases} A_{1x}=\tilde{u}_0 n_{tt}+\alpha\tilde{u}_0 n_{xx}+\dfrac{1}{2}\left(\tilde{u}_0^{\,2}+\tilde{v}_0\tilde{u}_{0y_0}+\tilde{w}_0\,\tilde{u}_{0z}\right)\left(n^2\right)_{xx}, \\ A_{2x}=\tilde{p}_0 n_{xy}, \\ A_{3x}=\tilde{\theta}_0 n_{tt}+\alpha\tilde{\theta}_0 n_{xx}+\dfrac{1}{2}\left(\tilde{u}_0\tilde{\theta}_0 t+n\tilde{v}_0 q\tilde{\theta}_{0y_0}\right)\left(n^2\right)_{xx}, \\ A_{4x}=\tilde{v}_0 n_{xxy}. \end{cases} \tag{17}$$

Then, we have

$$\begin{aligned} \ell_{y_0,z}\left(\frac{\partial^2 p_2}{\partial x^2}\right)&=\ell_{1y_{0,z}}(A_{1x})+\ell_{2y_{0,z}}(A_{2x})+\ell_{3y_{0,z}}(A_{3x})-A_{4x}\\ &=\left[\ell_{1y_{0,z}}(\tilde{u}_0)+\ell_{3y_{0,z}}(\tilde{\theta}_0)\right]n_{tt}+\alpha\Big[\ell_{1y_{0,z}}(\tilde{u}_0)\\ &\quad+\ell_{3y_{0,z}}(\tilde{\theta}_0)\Big]n_{xx}+\frac{1}{2}\Big[\ell_{1y_{0,z}}\big(\tilde{u}_0^{\,2}+\tilde{v}_0\tilde{u}_{0y_0}\\ &\quad+\tilde{w}_0\tilde{u}_{0z}\big)+\ell_{3y_{0,z}}\big(\tilde{u}_0\tilde{\theta}_0 t+n\tilde{v}_0 q\tilde{\theta}_{0y_0}\big)\Big]\left(n^2\right)_{xx}\\ &\quad+\ell_{2y_{0,z}}(\tilde{p}_0)n_{xy}-\tilde{v}_0 n_{xxy}. \end{aligned} \tag{18}$$

Because the homogeneous part of equation (10) is the same as that of equation (16), we can obtain

$$\begin{aligned} &\frac{\partial}{\partial x}\left\{\frac{d}{dy_0}\left[\frac{\partial p_2}{\partial x}\frac{d\tilde{p}_0}{dy_0}-\tilde{p}_0\frac{d}{dy_0}\left(\frac{\partial p_2}{\partial x}\right)\right]\right\}\\ &\quad+\left[\ell_{y_0z}{}^{*}(\tilde{p}_0)\frac{\partial^2 p_2}{\partial x^2}-\ell^{*}_{y_0z}\left(\frac{\partial^2 p_2}{\partial x^2}\right)\tilde{p}_0\right]\\ &=\tilde{p}_0\Big\{\left[\ell_{1y_{0,z}}(\tilde{u}_0)+\ell_{3y_{0,z}}(\tilde{\theta}_0)\right]n_{tt}+\alpha\Big[\ell_{1y_{0,z}}(\tilde{u}_0)\\ &\quad+\ell_{3y_{0,z}}(\tilde{\theta}_0)\Big]n_{xx}+\frac{1}{2}\Big[\ell_{1y_{0,z}}\big(\tilde{u}_0^{\,2}+\tilde{v}_0\tilde{u}_{0y_0}+\tilde{w}_0\tilde{u}_{0z}\big)\\ &\quad+\ell_{3y_{0,z}}\big(\tilde{u}_0\tilde{\theta}_0 t+n\tilde{v}_0 q\,\tilde{\theta}_{0y_0}\big)\Big]\left(n^2\right)_{xx}+\ell_{2y_{0,z}}(\tilde{p}_0)n_{xy}-\tilde{v}_0 n_{xxy}\Big\}, \end{aligned} \tag{19}$$

where the specific form of $\ell_{y_0,z}{}^{*}$ is shown in Appendix B.

Integrating equation (19) with respect to y_0 from 0 to h_0 yields

$$\begin{aligned} &\int_{-\infty}^{+\infty}\frac{\partial}{\partial x}\left\{\left[\tilde{p}_0(h_0,z)t\frac{\mathrm{d}}{\mathrm{d}y_0}n\left(\frac{\partial p_2}{\partial x}\right)q-h\frac{\mathrm{d}\tilde{p}_0(h_0,z)}{dy_0}\right.\right.\\ &\quad\left.\left.\cdot\left(\frac{\partial p_2}{\partial x}\right)\right]\right\}\mathrm{d}z+\int_{-\infty}^{+\infty}\int_0^{h_0}\left[\ell^{*}_{y_0z}(\tilde{p}_0(h_0,z))\frac{\partial^2 p_2}{\partial x^2}\right.\\ &\quad\left.-\ell^{*}_{y_0z}\left(\frac{\partial^2 p_2}{\partial x^2}\right)\tilde{p}_0(h_0,z)\right]\\ &\mathrm{d}y_0\mathrm{d}z=\int_{-\infty}^{+\infty}\int_0^{h_0}\tilde{p}_0\Big\{\left[\ell_{1y_{0,z}}(\tilde{u}_0)+\ell_{3y_{0,z}}(\tilde{\theta}_0)\right]n_{tt}\\ &\quad+\alpha\left[\ell_{1y_{0,z}}(\tilde{u}_0)+\ell_{3y_{0,z}}(\tilde{\theta}_0)\right]n_{xx}\\ &\quad+\frac{1}{2}\Big[\ell_{1y_{0,z}}\big(\tilde{u}_0^{\,2}+\tilde{v}_0\tilde{u}_{0y_0}+\tilde{w}_0\tilde{u}_{0y_0}\big)+\ell_{3y_{0,z}}\big(\tilde{u}_0\tilde{\theta}_0 t\\ &\quad+n\tilde{v}_0 q\tilde{\theta}_{0y_0}\big)\Big]\left(n^2\right)_{xx}+\ell_{2y_{0,z}}(\tilde{p}_0)n_{xy}-\tilde{v}_0 n_{xxy}\Big\}\mathrm{d}y_0\mathrm{d}z. \end{aligned} \tag{20}$$

Secondly, we introduce the following transformation in the domain $y\geq h_0$:

$$\frac{\partial}{\partial T} = \varepsilon^2 \frac{\partial}{\partial t} - c\varepsilon \frac{\partial}{\partial \widetilde{X}},$$
$$\frac{\partial}{\partial X} = \varepsilon \frac{\partial}{\partial \widetilde{X}},$$
$$\frac{\partial}{\partial Y} = \varepsilon \frac{\partial}{\partial y},\tag{21}$$
$$\frac{\partial}{\partial Z} = \frac{\partial}{\partial z},$$

and U, V, W, Θ, and P are rewritten as

$$\begin{aligned} U &= \varepsilon^{3/2} U^* (X, y, z, t), \\ V &= \varepsilon^{3/2} V^* (X, y, z, t), \\ W &= \varepsilon^{3/2} W^* (X, y, z, t), \\ \Theta &= \varepsilon \Theta^* (X, y, z, t), \\ P &= \varepsilon^{3/2} P^* (X, y, z, t). \end{aligned}\tag{22}$$

Substituting (21) and (22) into (1), the following can be obtained:

$$\begin{cases} c\dfrac{\partial U^*}{\partial \widetilde{X}} = \dfrac{1}{\rho_s} \dfrac{\partial P}{\partial \widetilde{X}}, \\ \dfrac{\partial V^*}{\partial \widetilde{X}} = \dfrac{1}{\rho_s} \dfrac{\partial P}{\partial y}, \\ \dfrac{\partial U^*}{\partial \widetilde{X}} + \dfrac{\partial V^*}{\partial y} = 0. \end{cases}\tag{23}$$

When$c \neq 0$, from equation (23), we obtain a Laplace equation for P^*:

$$\frac{\partial^2 P}{\partial \widetilde{X}^2} + \frac{\partial^2 P}{\partial y^2} = 0,\tag{24}$$

Also, the boundary conditions are

$$\begin{cases} P^* (\widetilde{X}, y, z, t) \longrightarrow 0, & y \longrightarrow \infty, \\ P^* (\widetilde{X}, y, z, t) = P_0^* (\widetilde{X}, y, z, t), & y = h_0. \end{cases}\tag{25}$$

Therefore, the solution of equation (24) under boundary conditions can be obtained:

$$P^* (\widetilde{X}, y, z, t) = \frac{P_r}{\pi} \int_{-\infty}^{+\infty} P_0^* (X', y, z, t) \cdot \frac{(y - h_0) \mathrm{d}X'}{(y - h_0)^2 + (X - X')^2},\tag{26}$$

where P_r is the principal value of Cauchy integral equation (26). Taking the derivative with respect to y for both sides of equation (26) yields

$$\frac{\partial P^* (\widetilde{X}, y, z, t)}{\partial y} = \frac{P_r}{\pi} \int_{-\infty}^{+\infty} P_0^* \cdot (X', y, z, t) \frac{\left[(X - X')^2 - (y - h_0)^2 \right] \mathrm{d}X'}{\left[(y - h_0)^2 + (X - X')^2 \right]^2}.\tag{27}$$

At the point $y = h_0$, matching the value of zone $[0, h_0]$ and zone $[h_0, \infty)$, we obtain

$$\begin{aligned} &\varepsilon^{1/2} \frac{\partial p_0 (x, h_0, z, t)}{\partial x} + \varepsilon \frac{\partial p_1 (x, h_0, z, t)}{\partial x} + \varepsilon^{3/2} \frac{\partial p_2 (x, h_0, z, t)}{\partial x} \\ &\quad = \varepsilon^{1/2} \frac{\partial P^* (\widetilde{X}, th_0 n, qzh,_t)}{\partial x} + O(\varepsilon), \\ &\varepsilon^{1/2} \frac{\partial^2 p_0 (x, h_0, z, t)}{\partial x \partial y} + \varepsilon \frac{\partial^2 p_1 (x, h_0, z, t)}{\partial x \partial y} + \varepsilon^{3/2} \frac{\partial^2 p_2 (x, h_0, z, t)}{\partial x \partial y} \\ &\quad = \varepsilon^{1/2} \frac{\partial^2 P^* (\widetilde{X}, th_0 n, qzh,_t)}{\partial x \partial y} + O\left(\varepsilon^{2/3}\right). \end{aligned}\tag{28}$$

Applying equation (27), we obtain the following:

$$\begin{aligned} n(t, x, y) \widetilde{p}_0 (h_0, z) &= P^* (x, h_0, z, t), \\ p_2 (x, h_0, z, t) &= 0, \end{aligned}\tag{29}$$

$$\frac{\partial^2 P^*}{\partial x \partial y} = \varepsilon \widetilde{p}_0 (h_0) \frac{\partial^3 \mathscr{H} (n(t, x, y))}{\partial x^3},\tag{30}$$

where $\mathscr{H} (n(t, x, y)) = P_r / \pi \int_{-\infty}^{+\infty} n(t, x', y) \ln|x - x'| \mathrm{d}x'$. Furthermore, we have

$$\begin{aligned} \frac{\partial \widetilde{p}_0}{\partial y} &= 0, \\ \frac{\partial^2 p_2 (x, h_0, z, t)}{\partial x \partial y} &= \widetilde{p}_0 (h_0) \frac{\partial^3 \mathscr{L} (n(t, x, y))}{\partial x^3}, \end{aligned}\tag{31}$$

and the boundary conditions

$$\frac{\partial p_0}{\partial x} (x, 0, z, t) = \frac{\partial p_2}{\partial x} (x, 0, z, t) = 0.\tag{32}$$

In conclusion, substituting (31) and (32) into (20), a new equation is obtained as follows:

$$\begin{aligned} &n_{tt} + a_1 n_{xx} + a_2 \left(n^2\right)_{xx} + a_3 n_{xy} + a_4 n_{xxy} \\ &\quad + a_5 \frac{\partial^3}{\partial x^3} \mathscr{H} (n(x, y, t)) = 0, \end{aligned}\tag{33}$$

where $\mathscr{H} (n(x, y, t)) = (P_r / \pi) \int_{-\infty}^{+\infty} (n(x', y, t) / (x - x')) \mathrm{d}x'$ is the Hilbert transformation and the coefficients are seen in Appendix B.

The new equation is the $(2 + 1)$-dimensional generalized Boussinesq-BO equation. Previous researchers have studied the $(1 + 1)$-dimensional Boussinesq equation, which can only describe the propagation of algebraic gravity solitary waves on a line, and the results described are quite different from the actual environment. Our new model can describe the propagation of algebraic gravity solitary waves on a surface that is more in line with the actual natural atmospheric environment.

In equation (33), $\partial^3/\partial x^3\mathscr{H}(n)$ denotes the dispersion effect and $(n^2)_{xx}$ and n_{xxy} denote the nonlinear effects, which shows that the generalized Boussinesq-BO equation contains two kinds of processes of gravity solitary waves, namely, dispersion process and nonlinear process. The dispersion and nonlinear processes in barotropic atmosphere are the main causes of squall line formation. Therefore, the specific formation mechanism of squall line can be obtained by researching this new model.

3. The (2 + 1)-Dimensional Time-Fractional Generalized Boussinesq-BO Equation

In this section, we obtain the (2 + 1)-dimensional time-fractional generalized Boussinesq-BO equation by using the Agrawal method, semi-inverse method, and fractional variational principle. At the beginning, we introduce some definitions as follows.

Definition 1 (see [48]). The Riemann–Liouville fractional derivation of a function $f(x,y,t)$ is defined by

$$D_t^{\omega} f = \frac{\partial^{\omega} f}{\partial t^{\omega}} = \begin{cases} \dfrac{1}{\Gamma(k-\omega)}\dfrac{\partial^k}{\partial t^k}\displaystyle\int_0^T (t-s)^{k-\omega-1} f\mathrm{d}s, & k-1\le\omega<k, \\ \dfrac{\partial^k f}{\partial t^k}, & \omega = k. \end{cases} \tag{34}$$

Definition 2 (see [48]). The fractional integration by parts rule is defined as follows:

$$\int_a^b (\mathrm{d}\tau)^{\gamma} f(z) D_z^{\gamma} g(z) = \int_a^b \mathrm{d}\tau g(z) D_z^{\gamma} f(z), \quad f(z), g(z) \in [a,b], \tag{35}$$

where $D_z^{\gamma} g(z)$ is the Riemann–Liouville fractional derivation.

According to the (2 + 1)-dimensional generalized Boussinesq-BO equation,

$$n_{tt} + a_1 n_{xx} + a_2\left(n^2\right)_{xx} + a_3 n_{xy} + a_4 n_{xxy} + a_5\frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t)) = 0. \tag{36}$$

It is worth noting that we cannot obtain the ideal form of variation by integral transformation of the (2 + 1)-dimensional time-fractional generalized Boussinesq-BO equation. Suppose $n(x,y,t) = m_x(x,y,t)$, where $m(x,y,t)$ refers to a potential function. The potential equation of the generalized Boussinesq-BO equation is

$$m_{xtt} + a_1 m_{xxx} + a_2\left(n^2\right)_{xx} + a_3 m_{xxy} + a_4 m_{xxxy} + a_5\frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t)) = 0, \tag{37}$$

where $(n^2)_{xx}$ and $\partial^3/\partial x^3\mathscr{H}(n(x,y,t))$ are considered as fixed functions because of their specificity. Then, we give the function of the potential equation as follows:

$$J(m) = \int_R \mathrm{d}x\int_R \mathrm{d}y\int_T \mathrm{d}t\Big\{m_x\Big[c_1 m_{xtt} + c_2 a_1 m_{xxx} + c_3 a_2\left(n^2\right)_{xx} + c_4 a_3 m_{xxy} + c_5 a_4 m_{xxxy} + c_6 a_5\frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t))\Big]\Big\}, \tag{38}$$

where $c_i\ (i = 1,2,3,4,5)$ are Lagrangian multipliers and can be determined later, and R and T are the boundary of space and the limit of time, respectively.

Using integration by parts for equation (38) and taking $m_x|_x = m_t|_R = m_{xx}|_R = 0$, equation (38) is rewritten as

$$J(m) = \int_R \mathrm{d}x\int_R \mathrm{d}y\int_T \mathrm{d}t I\left(m_x, m_{tx}, m_{xx}, m_{xy}\right), \tag{39}$$

where

$$I\left(m_x, m_{tx}, m_{xx}, m_{xy}\right) = -c_1 m_{tx}^2 - c_2 a_1 m_{xx}^2 + c_3 a_2\left(n^2\right)_{xx} m_x - c_4 a_3 m_{xy}^2 - c_5 a_4 m_{xx} m_{xxy} + c_6 a_5\frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t)) m_x, \tag{40}$$

Then, taking the variation of this function, we have

$$\delta J(m) = \int_R \mathrm{d}x\int_R \mathrm{d}y\int_T \mathrm{d}t\left[\left(\frac{\partial U}{\partial m_x}\right)\delta m - \left(\frac{\partial U}{\partial m_{tx}}\right)\delta m_t - \left(\frac{\partial U}{\partial m_{xx}}\right)\delta m_x - \left(\frac{\partial U}{\partial m_{xy}}\right)\delta m_x\right]. \tag{41}$$

Applying the variation optimum condition, i.e., $\delta J(m) = 0$, we can obtain the following:

$$2c_1 m_{xtt} + 2c_2 a_1 m_{xxx} + c_3 a_2\left(n^2\right)_{xx} + 2c_4 a_3 m_{xxy} + c_5 a_4 m_{xxxy} + c_6 a_5\frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t)) = 0. \tag{42}$$

Because equation (42) is equivalent to equation (37), the unknown constants become

$$c_1 = c_2 = c_4 = \frac{1}{2},\quad c_3 = c_5 = c_6 = 1. \tag{43}$$

Therefore, the Lagrangian form of the integer-order generalized Boussinesq-BO equation is as follows:

$$\begin{aligned} I\left(m_x, m_{tx}, m_{xx}, m_{xy}\right) = &-\frac{1}{2}m_{tx}^2 - \frac{1}{2}a_1 m_{xx}^2 + a_2\left(n^2\right)_{xx} m_x \\ &- \frac{1}{2}a_3 m_{xy}^2 \\ &- a_4 m_{xx} m_{xxy} \\ &+ a_5 \frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t)) m_x. \end{aligned} \tag{44}$$

Similarly, the Lagrangian form of the $(2+1)$-dimensional time fractional generalized Boussinesq-BO equation is expressed as

$$\begin{aligned} F\left(m_x, D_t^\alpha m_x, m_{xx}, m_{xy}\right) = &-\frac{1}{2}\left(D_t^\alpha m_x\right)^2 - \frac{1}{2}a_1 m_{xx}^2 - a_2\left(n^2\right)_{xx} m_x \\ &- \frac{1}{2}a_3 m_{xy}^2 - a_4 m_{xx} m_{xxy} \\ &+ a_5 \frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t)) m_x, \end{aligned} \tag{45}$$

where $D_t^\alpha f = \partial^\alpha f/\partial t^\alpha$ is Riemann–Liouville fractional partial derivative of α with respect to t. Also, the function of the time-fractional generalized Boussinesq-BO equation can be given as

$$J_F(m) = \int_R \mathrm{d}x \int_R \mathrm{d}y \int_T (\mathrm{d}t)^\alpha F\left(m_x, D_t^\alpha m_x, m_{xx}, m_{xy}\right). \tag{46}$$

Using Agrawal's method [49, 50], the variation of functional equation (46) yields

$$\begin{aligned} \delta J_F(m) = \int_R \mathrm{d}x \int_R \mathrm{d}y \int_T (\mathrm{d}t)^\alpha \Bigg[&-\left(\frac{\partial F}{\partial D_t^\alpha m_x}\right)\delta D_t^\alpha m \\ &+ \left(\frac{\partial F}{\partial m_x}\right)\delta m - \left(\frac{\partial F}{\partial m_{xx}}\right)\delta m_x - \left(\frac{\partial F}{\partial m_{xy}}\right)\delta m_x \Bigg]. \end{aligned} \tag{47}$$

Applying the fractional integration by parts (Definition 2) and considering $\delta I|_T = \delta I|_R = \delta I_x|_R = 0$, we can obtain

$$\begin{aligned} \delta J_F(m) = \int_R \mathrm{d}x \int_R \mathrm{d}y \int_T (\mathrm{d}t)^\alpha \Bigg[&-D_t^\alpha\left(\frac{\partial F}{\partial D_t^\alpha m_x}\right) + \left(\frac{\partial F}{\partial m_x}\right) \\ &- \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial m_{xx}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial m_{xy}}\right)\Bigg]\delta m. \end{aligned} \tag{48}$$

Obviously, optimizing the variation of the functional $J_F(m)$, i.e., $\delta J_F(m) = 0$, the Euler–Lagrange equation for the $(2+1)$-dimensional time-fractional generalized Boussinesq-BO equation can be expressed as

$$-D_t^\alpha\left(\frac{\partial F}{\partial D_t^\alpha m_x}\right) + \left(\frac{\partial F}{\partial m_x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial m_{xx}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial m_{xy}}\right) = 0. \tag{49}$$

Substituting equation (45) into the Euler–Lagrange equation, we have

$$\begin{aligned} &D_t^{\alpha\alpha} m_x + a_1 m_{xxx} + a_2\left(n^2\right)_{xx} + a_3 m_{xxy} + a_4 m_{xxxy} \\ &\quad + a_5 \frac{\partial^3}{\partial x^3}\mathscr{H}(n(x,y,t)) = 0, \end{aligned} \tag{50}$$

where $D_t^{\alpha\alpha} f = D_t^\alpha [D_t^\alpha f]$. Again, substituting the potential function $m_x(x,y,t) = u(x,y,t)$ leads to

$$\begin{aligned} &D_t^{\alpha\alpha} u + a_1 u_{xx} + a_2\left(u^2\right)_{xx} + a_3 u_{xy} + a_4 u_{xxy} \\ &\quad + a_5 \frac{\partial^3}{\partial x^3}\mathscr{H}(u(x,y,t)) = 0. \end{aligned} \tag{51}$$

This is the $(2+1)$-dimensional time-fractional generalized Boussinesq-BO equation, which is an extension of the $(2+1)$-dimensional integer-order Boussinesq-BO equation. The time-fractional model is nonlocal, which makes it more suitable than the integer-order model to describe the propagation process of algebraic gravity solitary waves in baroclinic atmosphere.

4. A Solution of the Fractional Generalized Boussinesq-BO Equation

In this section, the exact solution of the $(2+1)$-dimensional time fractional generalized Boussinesq-BO equation is obtained by applying the trial function method. We can explore some propagation characteristics of gravity solitary waves by calculation.

First, we transform the partial differential equation into an ordinary differential equation by assuming that u is a function of $\eta = x + y - ct^\alpha/\Gamma(1+\alpha)$, equation (51) can be rewritten as

$$c^2 u_{\eta\eta} + a_1 u_{\eta\eta} + a_2\left(u^2\right)_{\eta\eta} + a_3 u_{\eta\eta} + a_4 u_{\eta\eta\eta} + a_5 \mathscr{H}_{\eta\eta\eta}(u) = 0. \tag{52}$$

Integrating twice with respect to η and assuming the integral constant to be zero, we have

$$c^2 u + a_1 u + a_2 u^2 + a_3 u + a_4 u_\eta + a_5 \mathscr{H}_\eta(u) = 0. \tag{53}$$

Then, the solution of equation (53) has the following form:

$$u(\eta, t) = \frac{\lambda\sigma^2}{\eta^2 + \sigma^2}. \tag{54}$$

Substituting the solution into equation (53) yields

$$\begin{aligned}&\frac{c^2\lambda\sigma^2}{\eta^2+\sigma^2} + \frac{a_1\lambda\sigma^2}{\eta^2+\sigma^2} + \frac{a_2\lambda^2\sigma^4}{\left(\eta^2+\sigma^2\right)^2} + a_4\frac{\partial}{\partial\eta}\left(\frac{\lambda\sigma^2}{\eta^2+\sigma^2}\right)\\ &\quad + a_5\frac{\partial}{\partial\eta}\left[\frac{\lambda\sigma^2\eta}{|\sigma|\left(\eta^2+\sigma^2\right)}\right] = 0,\end{aligned} \tag{55}$$

where the relation used is

$$\mathscr{H}\left(\frac{1}{\eta^2+\sigma^2}\right) = \frac{1}{|\sigma|}\frac{\eta}{\eta^2+\sigma^2}. \tag{56}$$

And then, finally, by calculating, we can figure out

$$\begin{cases} |\sigma| = \dfrac{2\left(a_3+a_4+a_5\right)}{\lambda a_2},\\[2ex] c = \pm\sqrt{a_1 + \dfrac{1}{2}\lambda a_2 + a_3}\,. \end{cases} \tag{57}$$

It is well known that the real gravity solitary waves are spread along the two directions. Thus, when c is positive, the gravity solitary waves propagate towards the right; when c is negative, the gravity solitary waves propagate in the opposite direction. This is a special feature of the Boussinesq-Bo equation compared with the BO equation. Compared with the solution of Camassa–Holm equation [28], the solution form of the Boussinesq-Bo equation is very different not only because of the difference in the nonlinear terms of the two equations but also because of the steps involved in solving them. The solution of the Boussinesq-Bo equation is simpler than the solution of the Camassa–Holme equation.

5. Fission of Waves and Formation of Squall Lines

In Section 4, a solution of the fractional generalized Boussinesq-BO equation is obtained. In addition to solving equations, the conservation law of equation is something that a dynamic equation must be explored because it plays an important role [51]. In this section, conservation laws of the $(2+1)$-dimensional time fractional generalized Boussinesq-BO equation are derived from the variational principle. Then, most importantly, we will combine the conservation laws and the exact solution to study the fission process of gravitational solitary waves and the formation mechanism of squall line.

The first step is to obtain the conservation law of the equation. Because the Lagrangian density of equation (51) does not exist, we introduce a new variable v to construct an equal formula as follows:

$$\begin{cases} D_t^\alpha u = v_x,\\ D_t^\alpha v = -\left[a_1u_x + a_2\left(u^2\right)_x + a_3u_y + a_4u_{xy} + a_5\mathscr{H}_{xx}(u)\right]. \end{cases} \tag{58}$$

Letting $\psi = (u, v)$ be a section, the first prolongation of ψ be denoted as

$$P_1(\psi) = \left(x, y, t, u, v, u_x, v_x, u_y, D_t^\alpha u, D_t^\alpha v\right). \tag{59}$$

The Lagrange density of equation (58) is as follows:

$$\mathscr{D}\left(P_1(\psi)\right) = L\left(P_1(\psi)\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}t^\alpha, \tag{60}$$

where

$$\begin{aligned}L\left(P_1(\psi)\right) = {}& D_t^\alpha v u_x - D_t^\alpha u v_x - \frac{1}{2}v_x^2 - \frac{a_1}{2}u_x^2 - a_2\left(u^2\right)_x\\ &- \frac{a_3}{2}u_y^2 + \frac{a_4}{2}u_{xy}u_x - a_5\mathscr{H}_{xx}(u)u_x.\end{aligned} \tag{61}$$

Combining the Lagrangian density, the action functional is given as

$$A(\psi) = \int_{\mathscr{S}}\mathscr{D}\left(P_1(\psi)\right), \tag{62}$$

where $\mathscr{S}$ is an open set of X, $X = (x, y, t)$ represents the space of independent variables, and $U = (u, v)$ represents the space of dependent variables. Introducing a vector J yields

$$\begin{aligned}J = {}& \tau(x, y, t)\frac{\partial^\alpha}{\partial t} + \xi(x, y, t)\frac{\partial}{\partial x} + \eta_1(x, y, t, u, v)\frac{\partial}{\partial u}\\ &+ \eta_2(x, y, t, u, v)\frac{\partial}{\partial v}.\end{aligned} \tag{63}$$

Transform a section $\psi\colon \mathscr{S}\longrightarrow U$ to a family section $\tilde{\psi}\colon \tilde{\mathscr{S}}\longrightarrow U$ that depends on the parameter κ [52]. Then, we calculate the variation for the action functional as follows:

$$\begin{aligned}\delta A = \frac{d}{d\kappa}_{\kappa=0}A(\tilde{\psi}) &= \int_{\tilde{\mathscr{S}}}\left(L\left(\tilde{P}_1(\psi)\right)\right)\mathrm{d}\tilde{x}\wedge\mathrm{d}\tilde{y}\wedge\mathrm{d}\tilde{t^\alpha}\\ &= \int_{\mathscr{S}} N\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}t^\alpha + M,\end{aligned} \tag{64}$$

where

$$
\begin{aligned}
N &= \tau\Big[D_t^{\alpha}\Big(\frac{1}{2}v_x^2+\frac{a_1}{2}u_x^2+a_2\big(u^2\big)_x+a_5\mathscr{H}_{xx}(u)\Big)+D_x\left(-D_t^{\alpha}vu_x-D_t^{\alpha}uv_x\right)+\frac{1}{2}D_y\big(a_3u_y^2+a_4u_{xy}u_x\big)\Big]\\
&\quad+\xi\Big[\frac{1}{2}D_t^{\alpha}\left(uv_x-vu_x\right)+D_x\Big(D_t^{\alpha}vu_x-D_t^{\alpha}uv_x-\frac{1}{2}v_x^2-\frac{a_1}{2}u_x^2+a_2\big(u^2\big)_x+a_5\mathscr{H}_{xx}(u)\Big)+\frac{1}{2}D_y\big(a_3u_y^2+a_4u_{xy}u_x\big)\Big]+\eta_1\\
M &= \int_{\widetilde{\mathscr{S}}}\Big\{\tau\Big[\Big(\frac{1}{2}v_x^2+\frac{a_1}{2}u_x^2+a_2\big(u^2\big)_x+a_5\mathscr{H}_{xx}(u)\Big)\mathrm{d}x+\left(D_t^{\alpha}vu_x+D_t^{\alpha}uv_x\right)\mathrm{d}t^{\alpha}+\frac{1}{2}\big(a_3u_y^2+a_4u_{xy}u_x\big)\mathrm{d}y\Big]\\
&\quad+\xi\frac{1}{2}\left(uv_x-vu_x\right)\mathrm{d}x-\Big(D_t^{\alpha}vu_x-D_t^{\alpha}uv_x-\frac{1}{2}v_x^2-\frac{a_1}{2}u_x^2+a_2\big(u^2\big)_x+a_5\mathscr{H}_{xx}(u)\Big)\mathrm{d}t^{\alpha}\\
&\quad+\frac{1}{2}\big(a_3u_y^2+a_4u_{xy}u_x\big)\mathrm{d}y+\eta_1\Big(\frac{1}{2}v\mathrm{d}x+u_{xy}\mathrm{d}y-u_x\mathrm{d}t^{\alpha}\Big)+\eta_2\Big(-\frac{1}{2}u\mathrm{d}x-u_y\mathrm{d}y-v_x\mathrm{d}t^{\alpha}\Big).\big(D_t^{\alpha}v+a_1u_x+a_2\big(u^2\big)_x\\
&\quad+a_3u_y+a_4u_{xy}+a_5\mathscr{H}_{xx}(u)\big)+\eta_2\left(D_t^{\alpha}u-v_x\right).
\end{aligned}
\tag{65}
$$

If $\tau, \xi_1, \xi_2, \eta_1$, and η_2 are supported by $\mathscr{S}$ compactness, then $M = 0$.

Assuming $|x| \longrightarrow \infty$ yields the first conservation law:

$$
\begin{aligned}
E_1 &= \int_{-\infty}^{+\infty} u\mathrm{d}x,\\
\frac{\partial E_1}{\partial t} &= 0,
\end{aligned}
\tag{66}
$$

where E_1 means the mass of the algebraic gravity solitary waves, and equation (66) shows that the mass of the algebraic gravity solitary waves is conserved.

Then, from $\delta A = 0$, we can see from equation (64) that the variation τ gives the local energy conservation law:

$$
\begin{aligned}
&D_t^{\alpha}\Big(\frac{1}{2}v_x^2+\frac{a_1}{2}u_x^2+a_2\big(u^2\big)_x+a_5\mathscr{H}_{xx}(u)\Big)+D_x\big(-D_t^{\alpha}vu_x\\
&\quad-D_t^{\alpha}uv_x\big)+\frac{1}{2}D_y\big(a_3u_y^2+a_4u_{xy}u_x\big)=0,
\end{aligned}
\tag{67}
$$

where

$$
\begin{aligned}
E_2 &= \int_{-\infty}^{+\infty}\frac{1}{2}v_x^2+\frac{a_1}{2}u_x^2+a_2\big(u^2\big)_x+a_5\mathscr{H}_{xx}(u)\mathrm{d}x,\\
\frac{\partial E_2}{\partial t} &= 0,
\end{aligned}
\tag{68}
$$

where the equations indicate that the energy of the algebraic gravity solitary waves is conserved.

Also, the variation ξ gives the local momentum conservation law:

$$
\begin{aligned}
&\frac{1}{2}D_t^{\alpha}\left(uv_x-vu_x\right)+D_x\Big(D_t^{\alpha}vu_x-D_t^{\alpha}uv_x-\frac{1}{2}v_x^2-\frac{a_1}{2}u_x^2\\
&\quad+a_2\big(u^2\big)_x+a_5\mathscr{H}_{xx}(u)\Big)+\frac{1}{2}D_y\big(a_3u_y^2+a_4u_{xy}u_x\big)=0,
\end{aligned}
\tag{69}
$$

where

$$
\begin{aligned}
E_3 &= \frac{1}{2}\int_{-\infty}^{+\infty}\left(uv_x-vu_x\right)\mathrm{d}x,\\
\frac{\partial E_3}{\partial t} &= 0,
\end{aligned}
\tag{70}
$$

where the equations indicate that the momentum of the algebraic gravity solitary waves is conserved.

Second, combining the conservation laws and the exact solution that we have, we forecast the amplitude of gravity solitary waves emerging from a special type of initial value. The initial value is given as

$$
u(\eta,0)=\frac{\lambda_0\sigma_0^2}{\eta^2+\sigma_0^2}. \tag{71}
$$

Without loss of generality, assuming that the initial gravitational solitary waves gradually change to n solitary waves with different amplitudes and phases as follows:

$$
u(\eta,t)=\sum_{n_0=1}^{n}\frac{\lambda_n\sigma_n^2}{\left(\eta-c_nt-\eta_n\right)+\sigma_n^2}, \tag{72}
$$

where

$$
\begin{aligned}
c_n^2 &= a_1+\frac{1}{2}\lambda_na_2+a_3,\\
\left|\sigma_n\right| &= \frac{2\left(a_3+a_4+a_5\right)}{\lambda_na_2},\\
\sum_{n_0=1}^{n}\eta_n &= 0.
\end{aligned}
\tag{73}
$$

According to equations (52) and (71), we obtain

$$
\begin{aligned}
u_t &= \sum_{n_0=1}^{n}\frac{2\lambda_n\sigma_n^2c_n\left(\eta-c_nt-\eta_n\right)}{\left[\left(\eta-c_nt-\eta_n\right)^2+\sigma_n^2\right]^2},\\
v_x &= \sum_{n_0=1}^{n}\frac{\lambda_n\sigma_n^2c_n}{\left(\eta-c_nt-\eta_n\right)^2+\sigma_n^2},\\
v &= \sum_{n_0=1}^{n}\lambda_n\sigma_nc_n\pi.
\end{aligned}
\tag{74}
$$

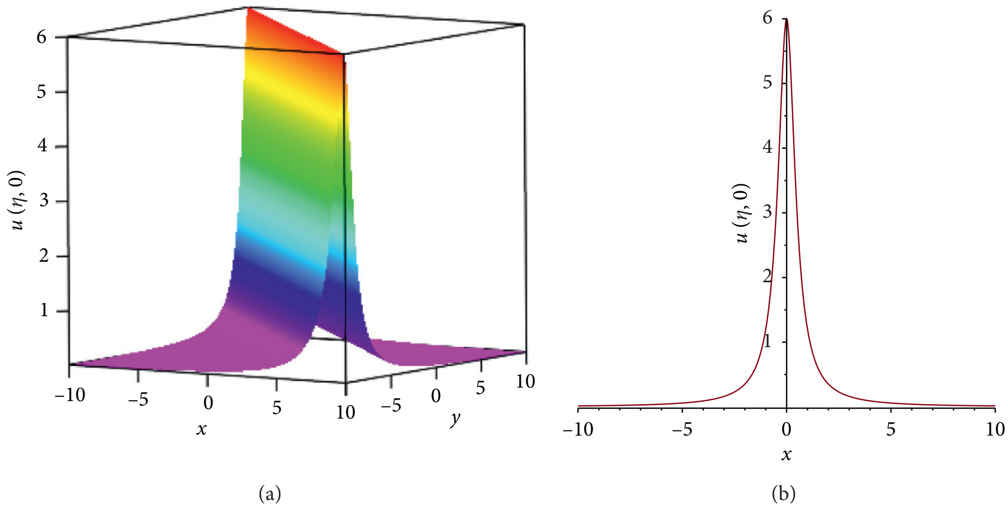

FIGURE 1: A single algebraic gravity solitary wave at $t = 0$. (a) Space diagram. (b) Planar graph.

Then, at $t = 0$, applying the conservation laws, the conserved quantities E_1, E_2, and E_3 can be calculated from equations (71) and (74) as

$$\begin{aligned} E_1 &= \pi\lambda_0\sigma_0, \\ E_2 &= \left(\pi^2 + 2\pi\right)\lambda_0^2\sigma_0, \\ E_3 &= \frac{(1 + a_1 - a_3 - a_4 - a_5 + \lambda_0\sigma_0)\lambda_0^2\pi}{8}. \end{aligned} \tag{75}$$

In a given period of time, the conserved quantities can be thought of as the sum of the conserved quantities related to each emerging gravity solitary wave. According to equations (58)–(62) and (72) and (74), we have

$$\begin{aligned} E_1 &= \frac{2(a_3 + a_4 + a_5)\pi n}{a_2}, \\ E_2 &= \frac{2(\pi^2 + 2\pi)(a_3 + a_4 + a_5)}{a_2 n} \sum_{n_0=1}^{n} \lambda_n, \\ E_3 &= \frac{(a_3 + 2a_4 + 2a_5)\pi}{8} \sum_{n_0=1}^{n} (\lambda_n)^2 + \frac{(1 + a_1)\pi}{8} \sum_{n_0=1}^{n} \frac{(\lambda_n)^2}{\sigma_n}. \end{aligned} \tag{76}$$

Equating corresponding quantities in equations (75) and (76), as well as taking the three gravity solitary waves as the object of study and assuming that $\lambda_1 > \lambda_2 > \lambda_3, \sigma_0 = 0$, and $a_i = 1, \quad i = 1 \cdots 5$, we obtain

$$\begin{aligned} \lambda_0 &= 6, \\ \lambda_1 &= 8.3, \\ \lambda_2 &= 3.5, \\ \lambda_3 &= 2.1. \end{aligned} \tag{77}$$

Finally, according to the above calculation results, we can see that the two algebraic gravity solitary waves which are generated due to fission can be described clearly. We will demonstrate the fission process of algebraic gravity solitary waves by figures with $N = 3$ (Figures 1–3).

As can be seen from Figure 1, a solitary wave described by the equation is propagating forward at a fixed wave speed. In this way, the solitary wave is characterized by a pulsing nature and stable propagation in the atmosphere. In fact, many of these waves have been observed on the low jet stream in recent years.

In the atmosphere, the velocity of algebraic gravity solitary waves depends on the amplitude of the waves. Under certain conditions, the larger the amplitude of the wave, the higher the propagation speed of the gravity isolated wave, which reflects the characteristics of the fast propagation of the gravity solitary waves with large amplitude. According to the conservation laws, the energy of a gravity solitary wave is conserved during its propagation. When the energy dispersion velocity of the gravity solitary waves is slower than the propagation velocity of the waves, the propagation velocity of the gravity solitary waves is faster for the long waves. For short waves, gravity waves travel more slowly or even backwards. When the nonlinear energy and wave energy dispersion reach a balance, the stable algebraic gravity solitary wave will be formed in the atmosphere.

As can be seen from Figure 2, with dispersion and nonlinear changes, when a disturbance occurs within a certain range, a single gravitational solitary wave will be excited and split into a number of stable solitary waves and oscillatory wave train. Because of the amplitude of the solitary wave moving block, after the interaction of the initial time, at some time T, the solitary wave of amplitude a will appear at the position of $2aT$. Therefore, the gravity solitary waves are arranged into an array according to their intensity, that is, the strong waves are in front and the weak waves are behind, as shown in Figure 3.

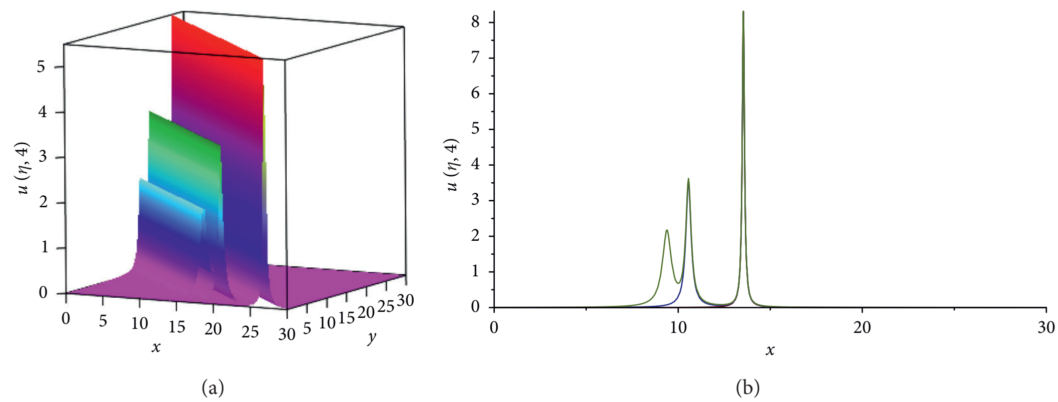

Figure 2: The fission of an algebraic gravity solitary wave at $t = 4$. (a) Space diagram. (b) Planar graph.

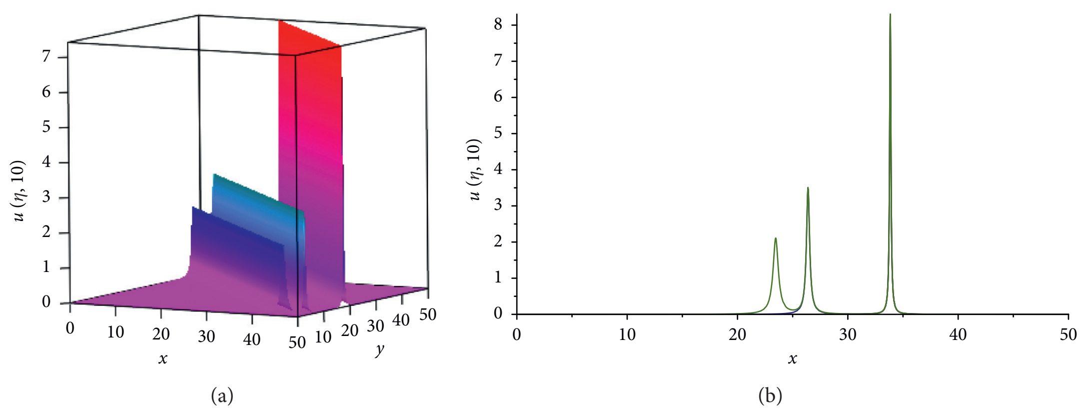

Figure 3: The fission of an algebraic gravity solitary wave at $t = 10$. (a) Space diagram. (b) Planar graph.

The above discussion shows that the disturbance in a limited range in the atmosphere can form a solitary wave array composed of several solitary waves, and the increase of both the width and intensity of the initial disturbance can increase the number, intensity, and velocity of solitary waves in the solitary wave array. This is consistent with the squall line observed in the low-altitude air and the formation of thunderstorm group caused by the squall line.

All in all, the fission of algebraic gravity solitary waves will produce the squall line phenomenon, that is to say, the newly derived $(2+1)$-dimensional fractional generalized Boussinesq-BO equation can reasonably describe the generation and evolution of squall line.

6. Conclusions

In barotropic nonstatic equilibrium atmosphere, squall line evolution is essentially a kind of algebraic gravity solitary wave under the interaction of nonlinear process and dispersion process, which can be described by the Boussinesq-BO equation. For the first time, we derived the $(2+1)$-dimensional generalized Boussinesq-BO equation from the basic equation of atmospheric dynamics by using the multiscale analysis and perturbation method. Compared with the previous equations, the generalized Boussinesq-BO equation described the properties of solitary waves similar to those observed in the atmosphere.

In addition, we extended the generalized Boussinesq-BO equation to the fractional order. By solving the fractional generalized Boussinesq-BO equation and studying the conservation laws, we find that the squall line and thunderstorm formation observed in the atmosphere can be explained as a solitary wave formation occurred by the fission of gravity solitary waves excited by a disturbance source.

Appendix

A. Dimensionless Process of the Basic Dynamic Equation Set of Baroclinic Nonstatic Equilibrium Atmosphere

We expressed the pressure gradient force and gravity in the vertical direction as the sum of the disturbance pressure gradient and buoyancy and used the Boussinesq approximation. The dimensionless process of the basic dynamic equation set in the baroclinic nonstatic equilibrium atmosphere is

$$\begin{cases} \dfrac{\partial U}{\partial T} + U\dfrac{\partial U}{\partial X} + V\dfrac{\partial U}{\partial Y} + W\dfrac{\partial W}{\partial Z} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial X} + fV, \\ \dfrac{\partial V}{\partial T} + U\dfrac{\partial V}{\partial X} + V\dfrac{\partial V}{\partial Y} + W\dfrac{\partial V}{\partial Z} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial Y} - fU, \\ \dfrac{\partial W}{\partial T} + U\dfrac{\partial W}{\partial X} + V\dfrac{\partial W}{\partial Y} + W\dfrac{\partial W}{\partial Z} = -\dfrac{1}{\rho_0}\dfrac{\partial P}{\partial Z} + \dfrac{g\Theta}{\Theta_0}, \\ \dfrac{\partial \Theta}{\partial T} + u\dfrac{\partial \Theta}{\partial X} + V\dfrac{\partial \Theta}{\partial Y} + \sigma W = 0, \\ \dfrac{(\partial \rho_0 U)}{\partial X} + \dfrac{(\partial \rho_0 V)}{\partial Y} + \dfrac{(\partial \rho_0 W)}{\partial Z} = 0, \end{cases} \tag{A.1}$$

where t is the time variable and u, v, and w are the three fluid velocity components in the x-, y-, and z-directions, respectively. P denotes the atmospheric pressure, f denotes the Coriolis parameter, g denotes the gravitational acceleration, Θ_0 and ρ_0 are the potential and density of environmental flow field, respectively, and $\sigma = \mathrm{d}\Theta_0/\mathrm{d}Z$.

The characteristic scales and dimensionless quantities related to variables are introduced as follows:

$$\begin{aligned} &(X, Y) \\ &Z = D'(\widehat{Z}), \\ &T = f'^{-1}(\widehat{T}), \\ &(U, V) = U'(\widehat{U} \\ &W = \frac{U'}{L'}D'(\widehat{W}), \\ &\Theta = \delta\Theta(\widehat{\Theta}), \\ &\delta P_{X,Y} = \frac{P'}{H}f'L'U'(\widehat{P}), \\ &\delta P_Z = \frac{P'}{\Theta'}\delta\Theta(\widehat{P}), \\ &\rho_0 = \frac{P'}{gH}(\rho_s), \end{aligned} \tag{A.2}$$

where P' is the characteristic pressure on the ground, H is the height of homogeneous atmosphere, and $\delta P_{X,Y}$ and δP_Z represent changes in pressure in the horizontal and vertical directions, respectively. The ones with a prime at the top right are dimensionless variables.

By substituting equation (A.2) into equation (A.1), dimensionless equations can be obtained as follows:

$$\begin{cases} \dfrac{\partial \widehat{U}}{\partial \widehat{T}} + \dfrac{U'}{f'L'}\left(\widehat{U}\dfrac{\partial \widehat{U}}{\partial \widehat{X}}t + n\widehat{V}q\dfrac{\partial \widehat{U}}{\partial \widehat{Y}}h +_{\widehat{W}x}\dfrac{\partial \widehat{U}}{\partial \widehat{Z}}\right) = -\dfrac{1}{\rho_s}\dfrac{\partial \widehat{P}}{\partial \widehat{X}} + \widehat{V}, \\ \dfrac{\partial \widehat{V}}{\partial \widehat{T}} + \dfrac{U'}{f'L'}\left(\widehat{U}\dfrac{\partial \widehat{V}}{\partial \widehat{X}}t + n\widehat{V}q\dfrac{\partial \widehat{V}}{\partial \widehat{Y}}h +_{\widehat{W}x}\dfrac{\partial \widehat{V}}{\partial \widehat{Z}}\right) = -\dfrac{1}{\rho_s}\dfrac{\partial \widehat{P}}{\partial \widehat{Y}} - \widehat{U}, \\ \dfrac{\partial \widehat{W}}{\partial \widehat{T}} + \dfrac{U'}{f'L'}\left(\widehat{U}\dfrac{\partial \widehat{W}}{\partial \widehat{X}}t + n\widehat{V}q\dfrac{\partial \widehat{W}}{\partial \widehat{Y}}h +_{\widehat{W}x}\dfrac{\partial \widehat{W}}{\partial \widehat{Z}}\right) \\ \quad = \dfrac{gL'\delta\Theta}{D'f'U'\Theta'}\left(-\dfrac{1}{\rho_s}\dfrac{\partial \widehat{P}}{\partial \widehat{Z}} + \widehat{\Theta}\right), \\ \dfrac{\partial \widehat{\Theta}}{\partial \widehat{T}} + \dfrac{U'}{f'L'}\left(\widehat{U}\dfrac{\partial \widehat{\Theta}}{\partial \widehat{X}}t + n\widehat{V}q\dfrac{\partial \widehat{\Theta}}{\partial \widehat{Y}}\right) = \dfrac{\sigma U'D'}{f'L'\delta\Theta}\widehat{W}, \\ \dfrac{\partial \rho_s \widehat{U}}{\partial \widehat{X}} + \dfrac{\partial \rho_s \widehat{V}}{\partial \widehat{Y}} + \dfrac{\partial \rho_s \widehat{W}}{\partial \widehat{Z}} = 0. \end{cases} \tag{A.3}$$

We assumed that $D' \sim H$, $\delta\Theta \sim (\sigma U'D')/(f'L')$, and $U'/(f'L') \sim o(1)$ introduced a small parameter $\varepsilon = f\prime^2/N^2$, where N is the Brunt–Vaisala frequency and $N^2 = g\Theta/\Theta\prime$. If the tip sign is omitted, equation (A.3) is changed into

$$\begin{cases} \dfrac{\partial U}{\partial T} + U\dfrac{\partial U}{\partial X} + V\dfrac{\partial U}{\partial Y} + W\dfrac{\partial U}{\partial Z} = -\dfrac{1}{\rho_s}\dfrac{\partial P}{\partial X} + V, \\ \dfrac{\partial V}{\partial T} + U\dfrac{\partial V}{\partial X} + V\dfrac{\partial V}{\partial Y} + W\dfrac{\partial V}{\partial Z} = -\dfrac{1}{\rho_s}\dfrac{\partial P}{\partial Y} - U, \\ \dfrac{\partial W}{\partial T} + U\dfrac{\partial W}{\partial X} + V\dfrac{\partial W}{\partial Y} + W\dfrac{\partial W}{\partial Z} = \varepsilon^{-1}\left(-\dfrac{1}{\rho_s}\dfrac{\partial P}{\partial Z} + \Theta\right), \\ \dfrac{\partial \Theta}{\partial T} + U\dfrac{\partial \Theta}{\partial X} + V\dfrac{\partial \Theta}{\partial Y} + W = 0, \\ \dfrac{\partial \rho_s U}{\partial X} + \dfrac{\partial \rho_s V}{\partial Y} + \dfrac{\partial \rho_s W}{\partial Z} = 0. \end{cases} \tag{A.4}$$

B. The Concrete Forms of Complex Parameters

The specific form of $\ell_{y_0,z}$ is

$$\begin{cases} \ell_{y_0,z} = -\frac{1}{\rho_s}\frac{\partial}{\partial y_0} + C\left[\left(\frac{1}{(u-c)^2} + u_{zz}\right)\frac{\partial}{\partial y_0} + \frac{\partial^2}{\partial y_0^2} + 2u_z\frac{\partial^2}{\partial z\partial y_0} + \left(u_{y_{0z}} + + \frac{u_z}{(u-c)}\frac{\left(1-u_{y_0}\right)u_z}{(u-c)^2}\right)\frac{\partial}{\partial z} + \frac{1-u_{y_0}}{(u-c)}\frac{\partial^2}{\partial z^2}\right. \\ \left. + \frac{u_{zz} - u_{y_0 z}}{(u-c)} - \frac{u_{y_0} + u_z^2}{(u-c)^2}\right] + C_{y_0}\left[\frac{1}{(u-c)} + \frac{\partial}{\partial y_0} + u_z\frac{\partial}{\partial z}\right] + C_z\left[\frac{u_z}{u-c} + (u-c)\frac{\partial}{\partial y_0} + \frac{1-u_{y_0}}{(u-c)}\frac{\partial}{\partial z}\right], \\ C = \frac{1}{\rho_s\left(u_{y_0} - 1 + u_z^2\right)}. \end{cases} \tag{B.1}$$

The specific forms of $\ell_{1y_{0,z}}$, $\ell_{2y_{0,z}}$, and $\ell_{3y_{0,z}}$ are

$$\begin{cases} \ell_{1y_{0,z}} = \rho_s\left[\frac{C}{(u-c)}\frac{\partial}{\partial y_0} + \frac{Cu_z}{(u-c)}\frac{\partial}{\partial z} + \frac{Cu_{zz}}{(u-c)}\frac{C\left(u_{y_0} + u_z^2\right)}{(u-c)^2} + \frac{C_{y_0}}{(u-c)} + \frac{C_z u_z}{(u-c)}\right], \\ \ell_{2y_{0,z}} = C\left(\frac{\partial}{\partial y_0} + u_{zz} + u_Z\frac{\partial}{\partial Z}\right) + C_{y_0} + C_z u_z, \\ \ell_{3y_{0,z}} = \rho_s\left[\frac{Cu_{zy_0}}{(u-c)} + \frac{u_z}{(u-c)}\frac{\partial}{\partial y_0} - \frac{Cu_{y_0}u_z}{(u-c)} - \frac{Cu_{y_0 z}}{(u-c)} + \frac{C\left(1-u_{y_0}\right)}{(u-c)}\frac{\partial}{\partial z} - \frac{C\left(1-u_y\right)u_z}{(u-c)^2} + \frac{C_y u_z}{(u-c)} + \frac{C_z\left(1-u_{y_0}\right)}{(u-c)}\right]. \end{cases} \tag{B.2}$$

The specific form of $\ell^*_{y_0,z}$ is

$$\begin{cases} \ell^*_{y_0,z} = -\frac{1}{\rho_s}\frac{\partial}{\partial y_0} + C\left[\left(\frac{1}{(u-c)^2} + u_{zz}\right)\frac{\partial}{\partial y_0} + 2u_z\frac{\partial^2}{\partial z\partial y_0} + \left(u_{y_{0z}} + \frac{u_z}{(u-c)} + \frac{\left(1-u_{y_0}\right)u_z}{(u-c)^2}\right) + \frac{1-u_{y_0}}{(u-c)}\frac{\partial^2}{\partial z^2} + \frac{u_{zz} - u_{y_0 z}}{(u-c)} - \frac{u_{y_0} + u_z^2}{(u-c)^2}\right] \\ + C_{y_0}\left[\frac{1}{(u-c)} + \frac{\partial}{\partial y_0} + u_z\frac{\partial}{\partial z}\right] \\ + C_z\left[(u-c)\frac{\partial}{\partial y_0} + \frac{1-u_{y_0}}{(u-c)}\frac{\partial}{\partial z}\right]. \end{cases} \tag{B.3}$$

The coefficients of equation (33) are

$$\begin{cases} a_0 = \int_{-\infty}^{+\infty}\int_0^{h_0} \widetilde{p}_0\left[\ell_{1y_{0,z}}\left(\widetilde{u}_0\right) + \ell_{3y_{0,z}}\left(\widetilde{\theta}_0\right)\right]\mathrm{d}y_0\mathrm{d}z, \\ a_1 = \int_{-\infty}^{+\infty}\int_0^{h_0} \widetilde{p}_0\alpha\left[\ell_{1y_{0,z}}\left(\widetilde{u}_0\right) + \ell_{3y_{0,z}}\left(\widetilde{\theta}_0\right)\right]\frac{\mathrm{d}y_0\mathrm{d}z}{a_0}, \\ a_2 = \int_{-\infty}^{+\infty}\int_0^{h_0} \widetilde{p}_0\begin{bmatrix} \ell_{1y_{0,z}}\left(\widetilde{u}_0^{\,2} + \widetilde{v}_0\widetilde{u_{0y_0}} + \widetilde{w}_0\widetilde{u_{0z}}\right) \\ +\ell_{3y_{0,z}}\left(\widetilde{u}_0\widetilde{\theta}_0 t + n\widetilde{v}_0 q\widetilde{\theta_{0y_0}}\right)\end{bmatrix}\frac{\mathrm{d}y_0\mathrm{d}z}{2a_0}, \\ a_3 = \int_{-\infty}^{+\infty}\int_0^{h_0} \widetilde{p}_0\ell_{2y_{0,z}}\left(\widetilde{p}_0\right)\frac{\mathrm{d}y_0\mathrm{d}z}{a_0}, \\ a_4 = \int_{-\infty}^{+\infty}\int_0^{h_0} \widetilde{p}_0\widetilde{v}_0\frac{\mathrm{d}y_0\mathrm{d}z}{a_0}, \\ a_5 = \int_{-\infty}^{+\infty} \widetilde{p}_0\frac{\mathrm{d}z}{a_0}. \end{cases} \tag{B.4}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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