

## Supplementary information

### Quantifying the effect of the intraespecific term

To understand the effect of the intraspecific term in the number of free-equilibrium solutions, we use the technique proposed by<sup>11</sup>. In this appendix, we calculate the number of free equilibrium points in the mutualistic model<sup>5</sup> and in our general model with the intraespecific interaction.

The mutualistic model is depicted, for two species, as,

$$\begin{aligned}\dot{X}_1 &= [(r_1 + b_{12}X_2) - (a_1 + c_1b_{12}X_2)X_1]X_1, \\ \dot{X}_2 &= [(r_2 + b_{21}X_1) - (a_2 + c_2b_{21}X_1)X_2]X_2,\end{aligned}\tag{19}$$

Following the steps proposed in<sup>11</sup>, first we obtain the set of exponents of the monomials of the free-isoclines functions  $S_1 \{(0,0), (0,1), (1,0), (1,1)\}$  and  $S_2 \{(0,0), (1,0), (0,1), (1,1)\}$ . These set of points allow to plot the Newton polytopes,  $P_1$  and  $P_2$ . Then we need to calculate the Minkowski sum  $P_1 \oplus P_2$ . And finally we obtain the mixed volume of the Newton polytopes,  $M(P_1, P_2)$

$$M(P_1, P_2) = vol_2(P_1 \oplus P_2) - vol_2(P_1) - vol_2(P_2) = 4 - 1 - 1 = 2\tag{20}$$

consequently, the number of free solution is 2.

Now, we perform the same procedure in our model with the intraespecific term, for two species.

$$\begin{aligned}\dot{X}_1 &= [(r_1 + b_{11}X_1 + b_{12}X_2) - (a_1 + c_1b_{11}X_1^2 + c_1b_{12}X_2X_1)]X_1, \\ \dot{X}_2 &= [(r_2 + b_{21}X_1 + b_{22}X_2) - (a_2 + c_2b_{21}X_1X_2 + c_2b_{22}X_2^2)]X_2,\end{aligned}\tag{21}$$

Now we obtain the new set of exponents of the monomials of the free-isoclines functions  $S_3 \{(0,0), (0,1), (1,0), (2,0), (1,1)\}$  and

$S_4 \{(0,0), (1,0), (0,1), (1,1), (0,2)\}$ . These points allow to plot the Newton polytopes,  $P_3$  and  $P_4$  (see figure 13 ). Then we calculate the Minkowski sum  $P_3 \oplus P_4$  and consequently the mixed volume,  $M(P_3, P_4)$

$$M(P_3, P_4) = vol_2(P_3 \oplus P_4) - vol_2(P_3) - vol_2(P_4) = 7 - 1.5 - 1.5 = 4\tag{22}$$

We can stand out that the equation with the intraspecific term, with the same order, introduce 2 new free solutions, with a full ecological meaning, as we can see in Section ??.

### The Jacobian matrix for N species

For N species, making use of [13], the diagonal terms of the Jacobian matrix at the free-equilibrium points can be written as:

$$J_{ii} = -r_i - \sum_{j \neq i} b_{ij}X_j^* - c_i b_{ii}X_i^{*2}\tag{23}$$

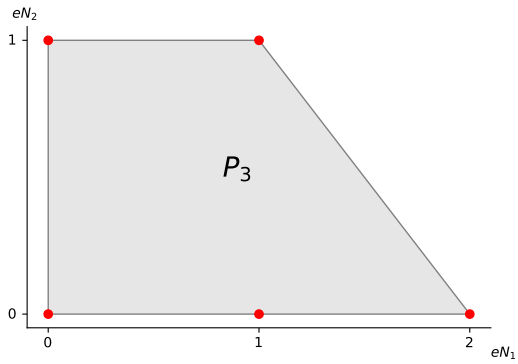
and the non-diagonal terms:

$$J_{ij} = b_{ij}X_i^* [1 - c_i X_i^*]\tag{24}$$

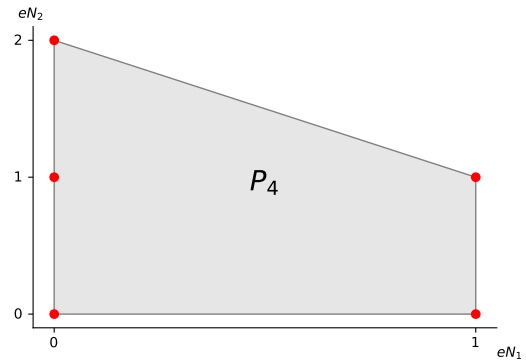
So, the Jacobian matrix can be written as:

$$J_{\{X_1^*, X_2^*\}} = \begin{pmatrix} \ddots & \dots & \dots & \dots & \dots \\ \dots & J_{ii} & \dots & J_{ij} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dots & J_{ji} & \dots & J_{jj} & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

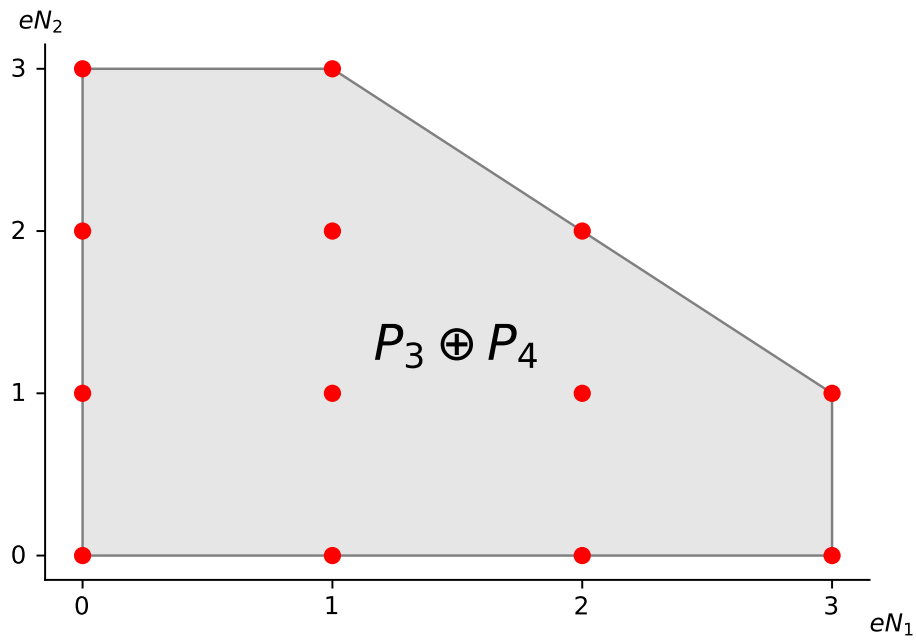
Note that the intraespecific coefficient appears in the Jacobian matrix as a new negative quadratic term.



(a) Polytope  $P_3$



(b) Polytope  $P_4$



(c) Polytope  $P_3 \oplus P_4$

**Figure 13.** Mixed volumes of Newton polytopes of the new model with the intraspecific terms. Plots (a) and (b) represent the Newton polytopes of sets  $S_3$  and  $S_4$  (red points), respectively. Plot (c) represents the mixed volume of the Newton polytope of the Minkowski sum  $P_3 \oplus P_4$ .