Hindawi
Complexity
Volume 2020, Article ID 7967328, 13 pages
https://doi.org/10.1155/2020/7967328

Research Article

An Extension of the Double $(G'/G, 1/G)$-Expansion Method for Conformable Fractional Differential Equations

Altaf A. Al-Shawba,[1] Farah A. Abdullah,[1] Amirah Azmi,[1] and M. Ali Akbar [2]

[1] *School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia*
[2] *Department of Applied Mathematics, University of Rajshahi, Rajshahi, Bangladesh*

Correspondence should be addressed to M. Ali Akbar; ali_math74@yahoo.com

Received 21 June 2020; Revised 25 September 2020; Accepted 19 October 2020; Published 6 November 2020

Academic Editor: Toshikazu Kuniya

The phenomena, molecular path in a liquid or a gas, fluctuating price stoke, fission and fusion, quantum field theory, relativistic wave motion, etc., are modeled through the nonlinear time fractional clannish random Walker's parabolic (CRWP) equation, nonlinear time fractional SharmaTassoOlver (STO) equation, and the nonlinear space-time fractional KleinGordon equation. The fractional derivative is described in the sense of conformable derivative. From there, the $(G'/G, 1/G)$-expansion method is found to be ensuing, effective, and capable to provide functional solutions to nonlinear models concerning physical and engineering problems. In this study, an extension of the $(G'/G, 1/G)$-expansion method has been introduced. This enhancement establishes broad-ranging and adequate fresh solutions. In addition, some existing solutions attainable in the literature also confirm the validity of the suggested extension. We believe that the extension might be added to the literature as a reliable and efficient technique to examine a wide variety of nonlinear fractional systems with parameters including solitary and periodic wave solutions to nonlinear FDEs.

1. Introduction

The subject FDEs can be considered as a generalization of the typical ordinary differential equations (ODEs). The advantages of the FDEs become apparent for us to understand real world problems. In fact, the FDEs have attracted significant attention to the past couple of decades due to their relatively much effectiveness than ODEs. There are several definitions of fractional derivatives, as for instance, the RiemannLiouville derivative, the Caputo derivative, and the conformal fractional derivative [1–3]. Recently, Khalil et al. [4] proposed a compatible definition of fractional derivative called the conformable fractional derivative. Therefore, several properties related to this new definition have been studied. Jarad et al. [5] used conformable type derivatives defined in [6] to generate new type of generalized fractional derivatives with memory effect. In [7, 8], the authors used conformable derivatives and integrals to formulate new generalized Liapunov-type inequality. The exact solutions of nonlinear FDEs play fundamental role in describing different qualitative and quantitative features of nonlinear complex physical phenomena. For this reason, researchers have proposed different methods to obtain exact solutions to nonlinear FDEs such as $\exp(-\varphi(\xi))$-expansion method [9–12], the (G'/G)-expansion method [13–19], auxiliary equation method [20, 21], $(G'/G, 1/G)$-expansion method [22–27] the trial equation method [28, 29], fractional sub-equation method [30, 31], modified simple equation method [32], generalized Kudrayshov method [33, 34], and others [35–38].

In this study, we introduce an extension of the $(G'/G, 1/G)$-expansion method for analyzing nonlinear FDEs in mathematical physics, engineering, and applied mathematics. To demonstrate the reliability and advantages of the suggested extension, the time-fractional CRWP equation, the time fractional STO equation, and the space-time fractional KleinGordon equation are examined and further broad-ranging and new families of exact wave solutions are established. This new extension can be applied to further nonlinear FDEs which can be done in forthcoming work.

2. Conformable Fractional Derivative and Its Important Properties

In the last years, Khalil et al. [4] introduced a simple, interesting, and compatible with typical definition of derivative named conformable fractional derivative, which can rectify the deficiencies of the other definitions. One can also find several useful studies related to this new definition in [5–8]. In [39], the geometrical and physical interpretations of this definition are investigated and the potential applications in science and engineering are pointer out. The conformable fractional derivative of a function g of order β is defined as

$$T_{\beta}(g)(t) = \lim_{x \longrightarrow 0} \frac{g\left(t + x t^{1-\beta}\right) - g(t)}{x}, \tag{1}$$

where g: $[0, \infty) \longrightarrow R$, $t > 0$ and $\beta \in (0, 1)$. Some important properties of above definition are given below:

(i) $T_{\beta}(ag + bf) = aT_{\beta}(g) + bT_{\beta}(f)$, $\forall\, a, b \in R$
(ii) $T_{\beta}(t^{\mu}) = \mu t^{\mu-\beta}$, $\forall\, \mu \in R$
(iii) $T_{\beta}(\lambda) = 0$, for each constant function $g(t) = \lambda$
(iv) $T_{\beta}(\mathrm{gof})(t) = t^{1-\beta} f'(t) g'(f(t))$

3. Methodology

In this section, we will suggest an extension of the $(G'/G, 1/G)$-expansion method to ascertain the analytic solutions to nonlinear FDEs. We begin with considering the second-order linear ODE:

$$G''(\xi) + \lambda G(\xi) - \mu = 0. \tag{2}$$

We choose

$$\phi = \frac{G'}{G}, \qquad \psi = \frac{1}{G'}. \tag{3}$$

Thus, from equation (2) and (3), it is found

$$(1/\phi)' = \lambda\left(\frac{1}{\phi}\right)^2 - \mu\psi\left(\frac{1}{\phi}\right) + 1, \qquad \psi' = \lambda\psi\left(\frac{1}{\phi}\right) - \mu\psi^2. \tag{4}$$

From the different general solutions of the linear ODE (2), we attain the following:

Case 1: when $\lambda < 0$, the hyperbolic function solution is

$$G(\xi) = A_1 \sinh\sqrt{-\lambda}\xi + A_2 \cosh\sqrt{-\lambda}\xi + \frac{\mu}{\lambda}, \tag{5}$$

and thus we obtain

$$\psi' = \frac{-\lambda}{\lambda^2\sigma + \mu^2}\left[\lambda\left(\frac{1}{\phi}\right)^2 - 2\mu\psi\left(\frac{1}{\phi}\right) + 1\right], \tag{6}$$

where $\sigma = A_1^2 - A_2^2$ and A_1, A_2 are arbitrary constants.
Case 2: when $\lambda > 0$, the trigonometric function solution is

$$G(\xi) = A_1 \sin\sqrt{\lambda}\xi + A_2 \cos\sqrt{\lambda}\xi + \frac{\mu}{\lambda}, \tag{7}$$

and corresponding relation is

$$\psi^2 = \frac{\lambda}{\lambda^2\sigma - \mu^2}\left[\lambda\,(1/\phi)^2 - 2\mu\psi\,(1/\phi) + 1\right], \tag{8}$$

where $\sigma = A_1^2 + A_2^2$ and A_1 and A_2 are arbitrary constants.
Case 3: when $\lambda = 0$, the rational function solution is

$$G(\xi) = \frac{\mu}{2}\xi^2 + A_1\xi + A_2, \tag{9}$$

and thus it is found

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left[-2\mu\psi\left(\frac{1}{\phi}\right) + 1\right], \tag{10}$$

where A_1 and A_2 are arbitrary constants.

Let us consider a general nonlinear FDE:

$$P\left(u, D_t^{\alpha}u, D_x^{\alpha}u, D_y^{\alpha}u, D_t^{\alpha}D_t^{\alpha}u, \right) = 0, \quad 0 < \alpha < 1, \tag{11}$$

where $D_t^{\alpha}u$, $D_x^{\alpha}u$ and $D_y^{\alpha}u$ are the conformable fractional derivatives of the wave function u with respect to spatial variables x and y and the temporal variable t, and P is a polynomial of $u = u(x, y, t, \ldots)$ and its various partial derivatives.

The main steps of the new extension of the $(G'/G, 1/G)$-expansion method to seek exact solutions of nonlinear FDEs are as follows:

Step 1: we estimate the new form of the fractional wave variable:

$$u(x, y, t) = U(\xi), \qquad \xi = k\frac{x^{\alpha}}{\alpha} + l\frac{y^{\alpha}}{\alpha} \pm v\frac{t^{\alpha}}{\alpha}, \tag{12}$$

where k and l are nonzero constants and v is the wave velocity to be determined later. The complex wave transformation (12) translates equation (11) into an ODE as follows:

$$P\left(U, U', U'', \ldots\right) = 0. \tag{13}$$

Step 2: suppose that the general solution of nonlinear ODE (13) can be expressed by a polynomial in $(1/\phi)$ and ψ as

$$U(\xi) = \sum_{i=0}^{N} a_i \left(\frac{1}{\phi}\right)^i + \sum_{i=0}^{N} b_i \left(\frac{1}{\phi}\right)^i \psi, \tag{14}$$

where $G = G(\xi)$ satisfies the auxiliary linear ODE (2) and a_i and b_i are arbitrary constants to be determined later, and balancing the highest order derivative with the nonlinear terms in equation (13), we can find the positive integer N.

Step 3: inserting solution (14) into equation (13) and utilizing (4) and (6) (here case 1 is selected as an example), the left-hand side of equation (13) can be converted into a polynomial in $(1/\phi)$ and ψ, where the degree of ψ is not more than one. Equating all coefficients of this polynomial to zero yield a set of algebraic equations for a_i, b_i, k, l, v, A_1, A_2, λ $(\lambda < 0)$, and μ.

Step 4: the solution of the algebraic equations found in the step 3 can be found with the aid of Maple software package. Making use of the values of a_i, b_i, k, l, v, A_1, A_2, λ, and μ into (14), we might determine exact solutions expressed by hyperbolic functions of equation (13).

Step 5: similar to step 3 and step 4, substituting (14) into equation (13) and utilizing equation (4) and (6) (or equation (4) and (10)), we can get the exact solutions of equation (13) expressed by the trigonometric functions (or expressed by the rational functions).

4. Determination of Solutions

In this paragraph, we will search out solutions to three conformable FDEs as appertain of the new extension of the $(G'/G, 1/G)$-expansion method.

4.1. Time Fractional CRWP Equation. First, we consider the time fractional CRWP equation [22, 40]:

$$D_t^{\alpha} u - u_x + 2uu_x + u_{xx} = 0, \quad 0 < \alpha \leq 1. \tag{15}$$

The ensuing wave transformation is

$$\begin{aligned} u(x, t) &= U(\xi), \\ \xi &= kx - c\frac{t^{\alpha}}{\alpha}, \end{aligned} \tag{16}$$

where k is the wave number and c is the velocity; reducing the equation (15) into the subsequent ODE, we get

$$-(c + k)U' + 2kUU' + k^2 U'' = 0. \tag{17}$$

Integrating (17) once, we find

$$-(c + k)U + kU^2 + k^2 U' + \xi_0 = 0, \tag{18}$$

where ξ_0 is a constant of integration. It is clear that the homogeneous balance between U^2 and U' present in equation (18) gives $N = 1$. Therefore, the shape of the exact solution of equation (18) is

$$U(\xi) = a_0 + a_1 \left(\frac{1}{\phi}\right) + b_1 \psi. \tag{19}$$

There are three cases to be considered:

Case 1: when $\lambda < 0$, substituting solution (19) into equation (18) and utilizing (4) and (6), equation (18) can be transmuted to a polynomial in $(1/\phi)$ and ψ. Equalizing its coefficients to zero yields a set of algebraic equations in a_0, a_1, b_1, k, c, σ, λ, μ, and ξ_0:

$$\begin{aligned} &\left(\frac{1}{\phi}\right)^2: \frac{b_1 k^2 \mu \lambda^2}{\lambda^2 \sigma + \mu^2} - \frac{b_1^2 k \lambda^2}{\lambda^2 \sigma + \mu^2} + a_1^2 k + a_1 k^2 \lambda = 0, \\ &\left(\frac{1}{\phi}\right)\psi: \frac{2 b_1^2 k \lambda \mu}{\lambda^2 \sigma + \mu^2} - \frac{2 b_1 k^2 \mu^2 \lambda}{\lambda^2 \sigma + \mu^2} - a_1 k^2 \mu + b_1 k^2 \lambda + 2 a_1 b_1 k = 0, \\ &\left(\frac{1}{\phi}\right): a_1 k - a_1 c + 2 a_0 a_1 k = 0, \\ &\psi: b_1 k + 2 a_0 b_1 k - b_1 c = 0, \\ &\psi^0: \frac{b_1 k^2 \mu \lambda}{\lambda^2 \sigma + \mu^2} - \frac{b_1^2 k \lambda}{\lambda^2 \sigma + \mu^2} + a_0^2 k - a_0 c + a_0 k + a_1 k^2 + \xi_0 = 0. \end{aligned} \tag{20}$$

Resolving these algebraic equations by Maple software package, we attain the following values:

$$\begin{aligned} a_0 &= \frac{c + k}{2k}, \\ a_1 &= -k\lambda, \\ b_1 &= \mu k, \\ \xi_0 &= \frac{4k^4 \lambda + c^2 + 2ck + k^2}{4k}, \end{aligned} \tag{21}$$

$$\begin{aligned} a_0 &= \frac{c + k}{2k}, \\ a_1 &= -\frac{1}{2} k\lambda, \\ b_1 &= \frac{1}{2} k \left(\mu \mp \lambda \sqrt{-\sigma}\right), \\ \xi_0 &= \frac{k^4 \lambda + c^2 + 2ck + k^2}{4k}, \end{aligned} \tag{22}$$

where k and c are free constants.

Inserting the values of the parameters assembled in (21) into solution (19) along with (6) and (16), the following wide-ranging hyperbolic function solution is ascertained:

$$u_1(x,t) = \frac{c+k}{2k} + k\sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{\mu\kappa}{\sqrt{-\lambda}}\left(\frac{1}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \quad (23)$$

In particular, if we set $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 \neq 0$ and $\mu = 0$) into the solution (23), we obtain the successive singular kink and kink soliton solutions to the CRWP equation (15), respectively:

$$u_{1_1} = \frac{c+k}{2k} + k\sqrt{-\lambda}\coth\sqrt{-\lambda}\xi, \quad (24)$$

$$u_{1_2} = \frac{c+k}{2k} + k\sqrt{-\lambda}\tanh\sqrt{-\lambda}\xi, \quad (25)$$

where $\xi = kx - c(t^{\alpha}/\alpha)$.

Furthermore, by means of the parameter values gathered in (22), we derive

$$u_2(x,t) = \frac{c+k}{2k} + \frac{1}{2}k\sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{\kappa}{2\sqrt{-\lambda}}\left(\frac{\mu \mp \lambda\sqrt{-\sigma}}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \quad (26)$$

Since A_1 and A_2 are integral constants, one might select their values spontaneously. Thus, if we select $A_1 = 0$, $A_2 > 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 > 0$, and $\mu = 0$) into solution (26), we gain the following solitary wave solutions to the CRWP equation, respectively:

$$u_{2_1} = \frac{c+k}{2k} + \frac{1}{2}k\sqrt{-\lambda}\left(\coth\sqrt{-\lambda}\,\xi \pm \operatorname{csc}h\sqrt{-\lambda}\,\xi\right), \quad (27)$$

$$u_{2_2} = \frac{c+k}{2k} + \frac{1}{2}k\sqrt{-\lambda}\left(\tanh\sqrt{-\lambda}\,\xi \pm i\operatorname{sec}h\sqrt{-\lambda}\,\xi\right), \quad (28)$$

where $\xi = kx - c(t^{\alpha}/\alpha)$.

Case 2: when $\lambda > 0$, embedding solution (19) int$(1/\phi)$o (18) and putting in use (4) and (8), equation (18) becomes polynomial in and ψ. Computing the action similar to case 1 and after resolving the algebraic equations, we ascertain the following values:

$$\begin{aligned} a_0 &= \frac{c+k}{2k}, \\ a_1 &= -k\lambda, \\ b_1 &= \mu k, \\ \xi_0 &= \frac{4k^4\lambda + c^2 + 2ck + k^2}{4k}, \end{aligned} \quad (29)$$

$$\begin{aligned} a_0 &= \frac{c+k}{2k}, \\ a_1 &= -\frac{1}{2}k\lambda, \\ b_1 &= \frac{1}{2}k(\mu \pm \lambda\sqrt{\sigma}), \\ \xi_0 &= \frac{k^4\lambda + c^2 + 2ck + k^2}{4k}, \end{aligned} \quad (30)$$

where k and c are arbitrary constants.

Substituting the values scheduled in (29) into solution (19) along with (8) and (16), the succeeding trigonometric solution to the CRWP equation (15) is established:

$$u_3(x,t) = \frac{c+k}{2k} - k\sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{\mu k}{\sqrt{\lambda}}\left(\frac{1}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \quad (31)$$

Since A_1 and A_2 are free constants, we may set $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$) into solution (31), we derive the under mentioned periodic wave solutions to the CRWP equation:

$$u_{3_1} = \frac{c+k}{2k} + k\sqrt{\lambda}\cot\sqrt{\lambda}\xi, \quad (32)$$

$$u_{3_2} = \frac{c+k}{2k} - k\sqrt{\lambda}\tan\sqrt{\lambda}\xi, \quad (33)$$

where $\xi = kx - c(t^\alpha/\alpha)$.

By means of the values organized in (30), we extract the following solution:

$$u_4(x,t) = \frac{c+k}{2k} - \frac{1}{2}k\sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + \mu/\lambda}\right)^{-1} + \frac{k}{2\sqrt{\lambda}}\left(\frac{\mu \pm \lambda\sqrt{\sigma}}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \quad (34)$$

Now, if we set $A_1 = 0$, $A_2 > 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 > 0$, and $\mu = 0$) into solution (34), we obtain other periodic wave solutions to the CRWP equation:

$$u_{4_1} = \frac{c+k}{2k} + \frac{1}{2}k\sqrt{\lambda}\left(\cot\sqrt{\lambda}\,\xi \mp i\csc\sqrt{\lambda}\,\xi\right), \quad (35)$$

$$u_{4_2} = \frac{c+k}{2k} - \frac{1}{2}k\sqrt{\lambda}\left(\tan\sqrt{\lambda}\,\xi \mp \sec\sqrt{\lambda}\,\xi\right), \quad (36)$$

where $\xi = kx - c(t^\alpha/\alpha)$.

Case 3: when $\lambda = 0$, introducing (19) into equation (18) and executing (4) and (10), equation (18) becomes a polynomial in $(1/\phi)$ and ψ. Vanishing the coefficients yields a set of algebraic equations, and solving this system the following results are obtained:

$$a_0 = \frac{c+k}{2k}, \quad a_1 = 0, \quad b_1 = \mu k, \quad \xi_0 = \frac{(c+k)^2}{4k}. \quad (37)$$

Setting the values gathered in (37) into solution (19) along with (10) and (16), we carry out the next rational function solution to the CRWP equation:

$$u_5 = \frac{c+k}{2k} + \frac{k\mu}{\mu\xi + A_1}, \quad (38)$$

where $\xi = kx - c(t^\alpha/\alpha)$.

The obtained solutions can be compared with the solutions accessible in the literature. We detect that, by setting $c = -k^2\lambda + 2a_0k - k$ and $\mu = -(\lambda^2/4)$, the obtained solutions (24) and (25) fully agree with the corresponding solutions (3.19) and (3.20) established in [22], while the other solutions are different.

4.2. General Time Fractional STO Equation. The conformable general time fractional STO equation [41–43] is

$$D_t^\alpha u + 3\beta u_x^2 + 3\beta u^2 u_x + 3\beta u u_{xx} + \beta u_{xxx} = 0, \quad t > 0,\ 0 < \alpha \le 1, \quad (39)$$

where β is nonzero constant. The wave transformation is as follows:

$$u(x,t) = U(\xi), \quad \xi = x + \omega\frac{t^\alpha}{\alpha}, \quad (40)$$

where ω is the velocity of the travelling wave; converting equation (39) into an ODE, we get

$$\omega U' + 3\beta(U')^2 + 3\beta U^2U' + 3\beta UU'' + \beta U''' = 0. \quad (41)$$

Integrating equation (41) once, we find

$$\omega U + 3\beta UU' + \beta U^3 + \beta U' = 0. \quad (42)$$

Balancing U' and U^3 obtained from equation (42), we get $N = 1$ Thus, the solution structure of equation (42) is same of solution (19). Three cases as described in Section 3 will be discussed further:

Case 1: when $\lambda < 0$, placing the solution (19) into (42) and utilizing (4) and (6), equation (42) modifies to a polynomial in $(1/\phi)$ and ψ. Setting each coefficient of the polynomial to zero yields a set of algebraic equations in $a_0, a_1, b_1, \sigma, \lambda, \mu, \beta$, and ω. From these equations with the help of Maple algebra software, we find the ensuing results of constants:

$$a_0 = \pm\frac{1}{2}\sqrt{-\lambda}, \quad a_1 = -\frac{1}{2}\lambda, \quad b_1 = \frac{1}{2}\left(\mu \mp \lambda\sqrt{-\sigma}\right), \quad \omega = \beta\lambda, \quad (43)$$

$$a_0 = 0, \quad a_1 = -\frac{1}{2}\lambda, \quad b_1 = \frac{1}{2}\left(\mu \mp \lambda\sqrt{-\sigma}\right), \quad \omega = \frac{1}{4}\beta\lambda, \quad (44)$$

$$a_0 = 0, \quad a_1 = -\lambda, \quad b_1 = \mu \mp \lambda\sqrt{-\sigma}, \quad \omega = \beta\lambda, \quad (45)$$

$$\begin{aligned} a_0 &= \pm\sqrt{-\lambda}, \\ a_1 &= -\lambda, \\ b_1 &= \mu, \\ \omega &= 4\beta\lambda, \end{aligned} \quad (46)$$

$$\begin{aligned} a_0 &= 0, \\ a_1 &= -\lambda, \\ b_1 &= \mu, \\ \omega &= \beta\lambda, \end{aligned} \quad (47)$$

$$\begin{aligned} a_0 &= 0, \\ a_1 &= -2\lambda, \\ b_1 &= 2\mu, \\ \omega &= 4\beta\lambda. \end{aligned} \quad (48)$$

By means of parameter values sorted out in (43), along with (6) and (40), the following hyperbolic function solutions of the general time fractional STO equation are obtained:

$$u_1(x,t) = \pm\frac{1}{2}\sqrt{-\lambda} + \frac{1}{2}\sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{1}{2\sqrt{-\lambda}}\left(\frac{\mu\mp\lambda\sqrt{-\sigma}}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \quad (49)$$

In solution (49), A_1 and A_2 are free parameters; therefore, we freely can choose our values. Thus, if we choose $A_1 = 0$, $A_2 > 0$, and $\mu = 0$ or $A_2 = 0$, $A_1 > 0$, and $\mu = 0$, from solution (49), we obtain the under mentioned soliton solutions to the general time fractional STO equation:

$$u_{1_1} = \pm\frac{1}{2}\sqrt{-\lambda}\left(1 \pm \coth\sqrt{-\lambda}\,\xi + \csc h\sqrt{-\lambda}\,\xi\right), \quad (50)$$

$$u_{1_2} = \pm\frac{1}{2}\sqrt{-\lambda}\left(1 \pm \tanh\sqrt{-\lambda}\,\xi\right) \pm \frac{1}{2}\sqrt{\lambda}\ \sec h\sqrt{-\lambda}\xi, \quad (51)$$

where $\xi = x + \beta\lambda(t^\alpha/\alpha)$.

Furthermore, by means of the values scheduled in (44), we establish

$$u_2(x,t) = \frac{1}{2}\sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{1}{2\sqrt{-\lambda}}\left(\frac{\mu\mp\lambda\sqrt{-\sigma}}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \quad (52)$$

Now, if we use $A_1 = 0$, $A_2 > 0$, and $\mu = 0$ or $A_2 = 0$, $A_1 > 0$, and $\mu = 0$, in (52), we attain the next soliton solutions to the general time fractional STO equation:

$$u_{2_1} = \frac{1}{2}\sqrt{-\lambda}\left(\coth\sqrt{-\lambda}\,\xi \pm \csc h\sqrt{-\lambda}\,\xi\right), \quad (53)$$

$$u_{2_2} = \frac{1}{2}\sqrt{-\lambda}\ \tanh\sqrt{-\lambda}\xi \pm \frac{1}{2}\sqrt{\lambda}\sec h\sqrt{-\lambda}\xi, \quad (54)$$

where $\xi = x + (1/4)\beta\lambda(t^\alpha/\alpha)$.

Moreover, by means of the values organized in (45), we carry out

$$u_3(x,t) = \sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{1}{\sqrt{-\lambda}}\left(\frac{\mu\mp\lambda\sqrt{-\sigma}}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \quad (55)$$

If we consider $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$, from solution (55), we derive the solitary wave solutions given underneath:

$$u_{3_1} = \sqrt{-\lambda}\left(\coth\sqrt{-\lambda}\,\xi \pm \csc h\sqrt{-\lambda}\,\xi\right), \quad (56)$$

$$u_{3_2} = \sqrt{-\lambda}\ \tanh\left(\sqrt{-\lambda}\ \xi\right) \pm \sqrt{\lambda}\ \sec h\sqrt{-\lambda}\xi, \quad (57)$$

where $\xi = x + \beta\lambda(t^\alpha/\alpha)$.

Similarly, making use of the values accumulated in (46), we determine

$$u_4(x,t) = \pm\sqrt{-\lambda} + \sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{\mu}{\sqrt{-\lambda}}\left(\frac{1}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \quad (58)$$

For $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$, the solution (58) generates the kink and singular kink solutions given underneath:

$$u_{4_1} = \pm\sqrt{-\lambda}\left(1 \pm \coth\sqrt{-\lambda}\,\xi\right), \quad (59)$$

$$u_{4_2} = \pm\sqrt{-\lambda}\left(1 \pm \tanh\sqrt{-\lambda}\,\xi\right), \quad (60)$$

where $\xi = x + 4\beta\lambda(t^\alpha/\alpha)$.

Likewise, with the help of the parametric values amassed in (47), we achieve

$$u_5(x,t) = \sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{\mu}{\sqrt{-\lambda}}\left(\frac{1}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \tag{61}$$

Choosing $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$, solution (61) turns into

$$u_{5_1} = \sqrt{-\lambda}\coth\sqrt{-\lambda}\xi, \tag{62}$$

$$u_{5_2} = \sqrt{-\lambda}\tanh\sqrt{-\lambda}\xi, \tag{63}$$

where $\xi = x + \beta\lambda(t^\alpha/\alpha)$.

For the values gathered in (48), we found

$$u_6(x,t) = 2\sqrt{-\lambda}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{2\mu}{\sqrt{-\lambda}}\left(\frac{1}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right). \tag{64}$$

In particular, if we set $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$), the solution (64) reduces to

$$u_{6_1} = 2\sqrt{-\lambda}\coth\sqrt{-\lambda}\xi, \tag{65}$$

$$u_{6_2} = 2\sqrt{-\lambda}\tanh\sqrt{-\lambda}\xi, \tag{66}$$

where $\xi = x + 4\beta\lambda(t^\alpha/\alpha)$.

Case 2: when $\lambda > 0$, we established by completing the parallel course of algorithms to case 1 and the following values for the constants:

$$a_0 = \pm\frac{1}{2}i\sqrt{\lambda},\quad a_1 = -\frac{1}{2}\lambda,\quad b_1 = \frac{1}{2}(\mu \pm \lambda\sqrt{\sigma}),\quad \omega = \beta\lambda, \tag{67}$$

$$a_0 = 0,\quad a_1 = -\frac{1}{2}\lambda,\quad b_1 = \frac{1}{2}(\mu \pm \lambda\sqrt{\sigma}),\quad \omega = \frac{1}{4}\beta\lambda, \tag{68}$$

$$a_0 = 0,\quad a_1 = -\lambda,\quad b_1 = \mu \pm \lambda\sqrt{\sigma},\quad \omega = \beta\lambda, \tag{69}$$

$$a_0 = \pm i\sqrt{\lambda},\quad a_1 = \lambda,\quad b_1 = \mu,\quad \omega = 4\beta\lambda, \tag{70}$$

$$a_0 = 0,\quad a_1 = -\lambda,\quad b_1 = \mu,\quad \omega = \beta\lambda, \tag{71}$$

$$a_0 = 0,\quad a_1 = -2\lambda,\quad b_1 = 2\mu,\quad \omega = 4\beta\lambda. \tag{72}$$

Substituting the sets of values ((67)–(72)) into solution (19) along with (8) and (40), we get the trigonometric function solutions for equation (39) as follows:

By means of (67), we obtain

$$u_7(x,t) = \pm\frac{1}{2}i\sqrt{\lambda} - \frac{1}{2}\sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{1}{2\sqrt{\lambda}}\left(\frac{\mu \pm \lambda\sqrt{\sigma}}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \tag{73}$$

If we consider $A_1 = 0$, $A_2 > 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 > 0$, and $\mu = 0$) in (73), the periodic wave solutions of equation (39) are found as follows:

$$u_{7_1} = \pm\frac{1}{2}\sqrt{\lambda}\left(i \pm \cot\sqrt{\lambda}\,\xi - \csc\sqrt{\lambda}\,\xi\right), \tag{74}$$

$$u_{7_2} = \pm\frac{1}{2}\sqrt{\lambda}\left(i \mp \tan\sqrt{\lambda}\,\xi + \sec\sqrt{\lambda}\,\xi\right), \tag{75}$$

where $\xi = x + \beta\lambda(t^\alpha/\alpha)$.

Through the values of the parameters gathered in (68), we derive

$$u_8(x,t) = -\frac{1}{2}\sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{1}{2\sqrt{\lambda}}\left(\frac{\mu \pm \lambda\sqrt{\sigma}}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \tag{76}$$

Since A_1 and A_2 are subjective constants, one may pick their values freely. Thus, if we pick $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$), from (76), we found the periodic wave solutions of equation (39) as follows:

$$u_{8_1} = \frac{1}{2}\sqrt{\lambda}\left(\cot\sqrt{\lambda}\,\xi \mp \csc\sqrt{\lambda}\,\xi\right), \tag{77}$$

$$u_{8_2} = -\frac{1}{2}\sqrt{\lambda}\left(\tan\sqrt{\lambda}\,\xi \mp \sec\sqrt{\lambda}\,\xi\right), \tag{78}$$

where $\xi = x + (1/4)\beta\lambda(t^\alpha/\alpha)$.

By virtue of the values scheduled in (69), we obtain

$$u_9(x,t) = -\sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{1}{\sqrt{\lambda}}\left(\frac{\mu \pm \lambda\sqrt{\sigma}}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \tag{79}$$

If we set $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$) into solution (79), we ascertain the following periodic wave solutions to the general time fractional STO equation:

$$u_{9_1} = \sqrt{\lambda}\left(\cot\sqrt{\lambda}\,\xi \mp \csc\sqrt{\lambda}\,\xi\right), \tag{80}$$

$$u_{9_2} = -\sqrt{\lambda}\left(\tan\sqrt{\lambda}\,\xi \mp \sec\sqrt{\lambda}\,\xi\right), \tag{81}$$

where $\xi = x + \beta\lambda(t^\alpha/\alpha)$.

For the values accumulated in (70), we establish

$$u_{10}(x,t) = \pm i\sqrt{\lambda} - \sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{\mu}{\sqrt{\lambda}}\left(\frac{1}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \tag{82}$$

Setting $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$), the solution (82) transformed to

$$u_{10_1} = \pm\sqrt{\lambda}\left(i \pm \cot\sqrt{\lambda}\,\xi\right), \tag{83}$$

$$u_{10_2} = \pm\sqrt{\lambda}\left(i \mp \tan\sqrt{\lambda}\,\xi\right), \tag{84}$$

where $\xi = x + 4\beta\lambda(t^\alpha/\alpha)$.

Utilizing the values of the parameters arranged in (71), we obtain

$$u_{11}(x,t) = -\sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{\mu}{\lambda}\left(\frac{1}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \tag{85}$$

Now setting $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$ and $\mu = 0$), the solution (85) becomes

$$u_{11_1} = \sqrt{\lambda}\cot\sqrt{\lambda}\xi, \tag{86}$$

$$u_{11_2} = -\sqrt{\lambda}\tan\sqrt{\lambda}\xi, \tag{87}$$

where $\xi = x + \beta\lambda(t^\alpha/\alpha)$.

Embedding the parametric values compiled in (72), we find out

$$u_{12}(x,t) = -2\sqrt{\lambda}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} + \frac{2\mu}{\sqrt{\lambda}}\left(\frac{1}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right). \tag{88}$$

Suppose that $A_1 = 0$, $A_2 \neq 0$, and $\mu = 0$ (or $A_2 = 0$, $A_1 \neq 0$, and $\mu = 0$), then the solution (88) developed into

$$u_{12_1} = 2\sqrt{\lambda}\cot\sqrt{\lambda}\xi, \tag{89}$$

$$u_{12_2} = -2\sqrt{\lambda}\tan\sqrt{\lambda}\xi, \tag{90}$$

where $\xi = x + 4\beta\lambda(t^\alpha/\alpha)$.

From the obtained broad-ranging solutions, it is observed that setting definite values of the associated parameters, we manage to determine some particular solutions which coincide with those accessible in the literature and some fresh solutions are established. It is seen that, by putting $\beta = d$, the obtained solutions (53), (54), (77), and (78) completely agree with the solutions (4.8), (4.7), (4.11), and (4.10), respectively, found in [43]. In addition to these solutions, we found many more solutions that were not found in any other studies.

4.3. Space-Time Fractional KleinGordon Equation. In this subsection, we extract the closed form solutions to the space-time fractional KleinGordon equation. Let us consider the KleinGordon equation with space-time fractional order [44, 45]:

$$D_t^\alpha (D_t^\alpha u) - D_x^\alpha (D_x^\alpha u) - \beta u - \gamma u^3 = 0, \quad t > 0,\ 0 < \alpha \le 1, \tag{91}$$

where β and γ are nonzero constants. We apply the following transformation for reducing equation (91) to an ODE:

$$\begin{aligned} u(x, t) &= U(\xi), \\ \xi &= k\frac{x^\alpha}{\alpha} - c\frac{t^\alpha}{\alpha}, \end{aligned} \tag{92}$$

where k and c are nonzero constants. Thus, the space-time fractional KleinGordon equation turns out as follows:

$$\left(k^2 - c^2\right)U'' + \beta U + \gamma U^3 = 0. \tag{93}$$

Balancing U'' and U^3 in equation (93), we found $N = 1$. On account of this, the structure of the solution of equation (93) is identical to the shape of the solution of equation (19) and therefore has not been repeated. There are three cases should be discussed as described in Section 3:

Case 1: when $\lambda < 0$, plugging in (19) into equation (93) and utilizing (4) and (6), equation (93) will be converted into a polynomial in $(1/\phi)$ and ψ. Equalizing the coefficients of this polynomial to zero yields a set of algebraic equations for a_0, a_1, b_1, c, k, σ, λ, μ, β, and γ. Resolving these equations with the assistance of computer algebra, like Maple software package, we found the following values for the constants:

$$\begin{aligned} a_0 &= 0, \\ a_1 &= \pm\sqrt{\frac{\lambda\beta}{\gamma}}, \\ b_1 &= \pm\mu\sqrt{\frac{\beta}{\lambda\gamma}}, \\ c &= \pm\sqrt{\frac{2\lambda k^2 + \beta}{2\lambda}}, \end{aligned} \tag{94}$$

$$\begin{aligned} a_0 &= 0, \\ a_1 &= 0, \\ b_1 &= \pm\sqrt{\frac{2\lambda\beta\sigma}{\gamma}}, \\ c &= \pm\sqrt{\frac{\lambda k^2 - \beta}{\lambda}}, \end{aligned} \tag{95}$$

$$\begin{aligned} a_0 &= 0, \\ a_1 &= \pm\sqrt{\frac{\lambda\beta}{\gamma}}, \\ b_1 &= \pm\sqrt{\frac{\beta}{\lambda\gamma}}\left(\mu \pm \lambda\sqrt{-\sigma}\right), \\ c &= \mp\sqrt{\frac{\lambda k^2 + 2\beta}{\lambda}}, \end{aligned} \tag{96}$$

where k, β, γ, and μ are free parameters.

Inserting the sets of constant values from (94) to (96) into solution (19) along with (6) and (92), the following hyperbolic function solutions to the space-time fractional KleinGordon are obtained:

By means of the values assembled in (94), we attain

$$\begin{aligned} u_1(x, t) = &\pm\sqrt{\frac{-\beta}{\gamma}}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} \\ &\mp\frac{\mu}{\lambda}\sqrt{\frac{-\beta}{\gamma}}\left(\frac{1}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right), \end{aligned} \tag{97}$$

where $\xi = k(x^\alpha/\alpha) \mp \sqrt{(2\lambda k^2 + \beta/2\lambda)}\,(t^\alpha/\alpha)$.

With the help of the values arranged in (95), we carry out

$$u_2(x, t) = \pm\sqrt{\frac{-2\beta\sigma}{\gamma}}\left(\frac{1}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right), \tag{98}$$

where $\xi = k(x^\alpha/\alpha) \mp \sqrt{((\lambda k^2 - \beta)/\lambda)}\,(t^\alpha/\alpha)$.

On the other hand, by means of (96), we derive

$$\begin{aligned} u_3(x, t) = &\pm\sqrt{\frac{-\beta}{\gamma}}\left(\frac{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}{A_1\sinh\sqrt{-\lambda}\,\xi + A_2\cosh\sqrt{-\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} \\ &\mp\frac{1}{\lambda}\sqrt{\frac{-\beta}{\gamma}}\left(\frac{\mu \pm \lambda\sqrt{-\sigma}}{A_1\cosh\sqrt{-\lambda}\,\xi + A_2\sinh\sqrt{-\lambda}\,\xi}\right), \end{aligned} \tag{99}$$

where $\xi = k(x^\alpha/\alpha) \pm \sqrt{(\lambda k^2 + 2\beta/\lambda)}\,(t^\alpha/\alpha)$.

Case 2: when $\lambda > 0$, executing the analogous steps as case 1, a system of algebraic equations can be found and after resolving this system of equations, we obtain three sets of values of the constants as follows:

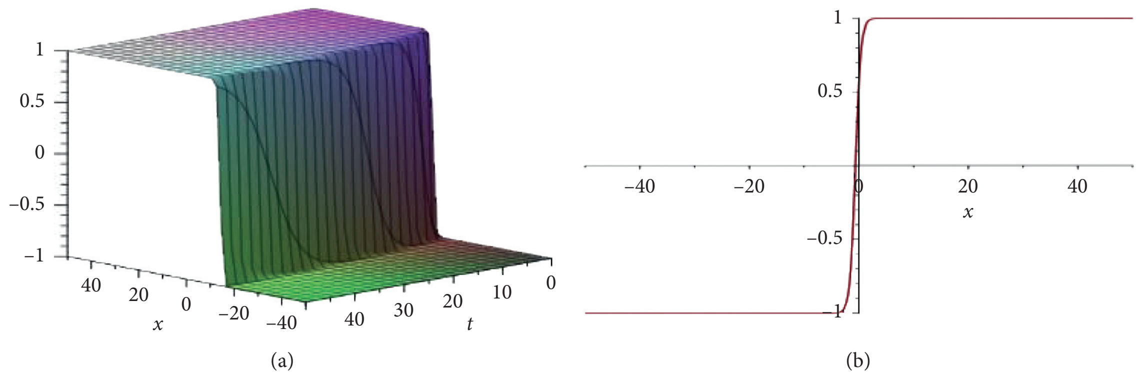


Figure 1: Kink-shape wave of (63) when $\lambda = -1$, $\beta = -1$, $\alpha = 0.5$, $-50 \le x \le 50$, and $0 \le t \le 50$.

$$a_0 = 0, \quad a_1 = \mp\sqrt{\frac{\lambda\beta}{\gamma}}, \quad b_1 = \pm\mu\sqrt{\frac{\beta}{\lambda\gamma}}, \quad c = \pm\sqrt{\frac{2\lambda k^2 + \beta}{2\lambda}}, \tag{100}$$

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = \pm\sqrt{-\frac{2\lambda\beta\sigma}{\gamma}}, \quad c = \pm\sqrt{\frac{\lambda k^2 - \beta}{\lambda}}, \tag{101}$$

$$a_0 = 0, \quad a_1 = \pm\sqrt{\frac{\lambda\beta}{\gamma}}, \quad b_1 = \mp\sqrt{\frac{\beta}{\lambda\gamma}}(\mu \mp \lambda\sqrt{\sigma}), \quad c = \pm\sqrt{\frac{\lambda k^2 + 2\beta}{\lambda}}, \tag{102}$$

where k, β, γ, and μ are arbitrary constants.

By the use of the constant values scheduled in (100)–(102) into solution (19) alongside with (8) and (92) function solutions to the space-time, the following trigonometric function solutions to the space-time fractional KleinGordon equation are obtained.

For the values arranged in (100), we obtain the general solution:

$$u_4(x,t) = \mp\sqrt{\frac{\beta}{\gamma}}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} \pm \frac{\mu}{\lambda}\sqrt{\frac{\beta}{\gamma}}\left(\frac{1}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right), \tag{103}$$

where $\xi = k(x^\alpha/\alpha) \mp \sqrt{((2\lambda k^2 + \beta)/2\lambda)}\,(t^\alpha/\alpha)$.

Similarly, for the values organized in (101), we accomplish

$$u_5(x,t) = \pm\sqrt{\frac{-2\beta\sigma}{\gamma}}\left(\frac{1}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right), \tag{104}$$

where $\xi = k(x^\alpha/\alpha) \mp \sqrt{((\lambda k^2 - \beta)/\lambda)}\,(t^\alpha/\alpha)$.

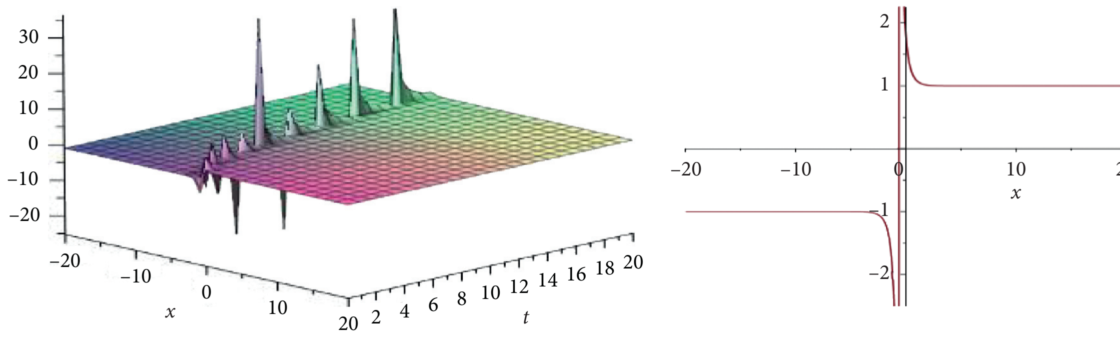
And, for the values laid out in (102), we ascertain

$$u_6(x,t) = \pm\sqrt{\frac{\beta}{\gamma}}\left(\frac{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}{A_1\sin\sqrt{\lambda}\,\xi + A_2\cos\sqrt{\lambda}\,\xi + (\mu/\lambda)}\right)^{-1} \mp \frac{1}{\lambda}\sqrt{\frac{\beta}{\gamma}}\left(\frac{\mu \mp \lambda\sqrt{\sigma}}{A_1\cos\sqrt{\lambda}\,\xi - A_2\sin\sqrt{\lambda}\,\xi}\right), \tag{105}$$

where $\xi = k(x^\alpha/\alpha) \mp \sqrt{(\lambda k^2 + 2\beta/\lambda)}\,(t^\alpha/\alpha)$.

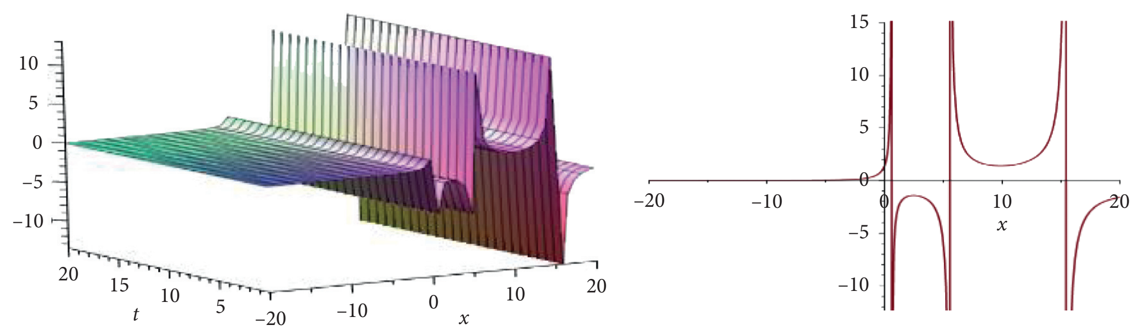
In this section, we have established the general solutions (97)–(99) and (103)–(105) to the space-time fractional KleinGordon equation from where scores of periodic solitary wave solutions can be extracted selecting special values for parameters. But, for simplicity, particular solutions are omitted.

The obtained results can be compared with the exact solutions accessible in the literature. In [45], the exact solutions of the space-time fractional KleinGordon equation are established by using the $(G'/G, 1/G)$-expansion method. It is seen that the solutions established in this study are different than the solutions found in [45].



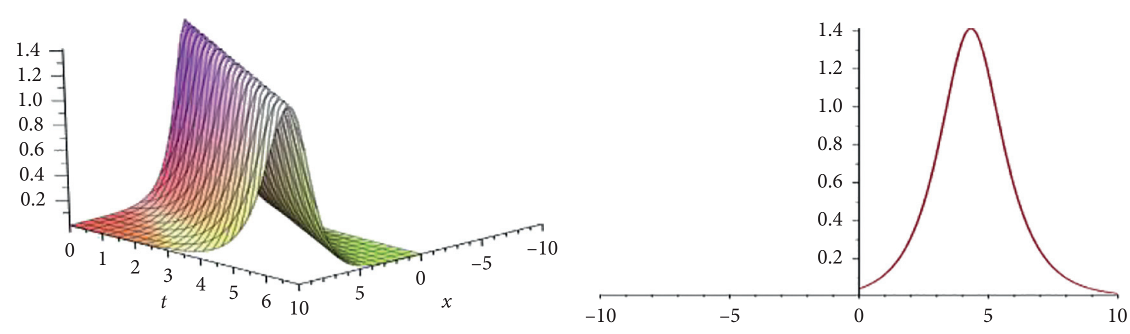
(a) (b)

Figure 2: Singular kink-shape wave of (24) when $c = 1$, $k = -1$, $\lambda = -1$, $\alpha = 0.5$, $-20 \le x \le 20$, and $0.1 \le t \le 20$.



(a) (b)

Figure 3: Singular periodic wave shape of (104) when $\beta = 1$, $k = 1$, $\gamma = -1$, $\lambda = 1$, $\alpha = 0.5$, $-20 \le x \le 20$, and $0 \le t \le 20$.



(a) (b)

Figure 4: Soliton wave shape of (98) when $\beta = 1$, $k = 1$, $\gamma = -1$, $\lambda = -1$, $\alpha = 0.95$, $-10 \le x \le 10$, and $0 \le t \le 7$.

5. Physical Explanations

In this section, we put forth the physical explanation and the 2D and 3D graphical representation of the solutions obtained for the time fractional CRWP equation, the general time fractional STO equation, and the space-time fractional KleinGordon equation as follows:

Solutions (25), (28), (51), (54), (57), (60), (63), (66), and (97) are the kink-shape soliton. Kink waves are travelling waves which arise from one asymptotic position to another. Figure 1 shows the shape of the kink solution (63). Other figures are omitted for convenience.

Moreover, solutions (24), (27), (50), (53), (56), (59), (62), (65), and (99) are singular kink solitons. Figure 2 shows the shape of the singular kink solution (24). The residual figures are left for simplicity.

On the other hand, solutions (32), (33), (35), (36), (74), (75), (77), (78), (80), (81), (83), (84), (86), (87), (89), 90), and (103)–(105) are the exact periodic wave solutions. Periodic solutions are travelling wave solutions that are periodic like $\sin(x-t)$. Figure 3 shows the singular periodic solution of (104). The remaining graphs are left for minimalism.

Solution (98) is the soliton solution. Figure 4 shows the shape of the exact soliton solution of (98) of the space-time fractional KleinGordon equation.

6. Conclusion

In this article, we have introduced an extension of the $(G'/G, 1/G)$-expansion method to look into nonlinear fractional differential equations in the sense of conformable derivative. Taking the advantage of this extension, the time fractional CRWP equation, the general time fractional STO equation, and the space-time fractional Klein-Gordon equation have been investigated. Scores of broad-ranging exact solutions have successfully been found as a linear combination of hyperbolic, trigonometric, and rational function associated with free parameters. For definite values of these parameters, some known periodic, kink, and solitary wave solutions accessible in the literature are derived from the general solutions and some fresh solutions are originated. This study shows that the proposed extension is quite efficient, useful, direct, and easily computable with the aid of Maple software package and practically well suited to be used in finding analytical exact solutions to many other nonlinear FDEs, and this is our scheme in the future.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

This work was done in coordinated effort among the authors. All the authors have a good contribution to plan the study and to complete the investigation of this work. All authors read and endorsed the final version of the manuscript.

Acknowledgments

The authors would like to thank the School of Mathematical Sciences, University of Science Malaysia, Penang, Malaysia, for providing computing facilities. This work was supported by the Publication Fee Funding Grant and Bridging Grant Scheme (304/PMATHS/6316285) by Research Creativity and Management Office (RCMO), University Sains Malaysia.

References

[1] K. S. Miller and B. Ross, "An introduction to the fractional calculus and fractional differential equations," *Mathematics in Science and Engineering*, 1993.

[2] I. Podlubny, "Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications," *Mathematics in Science and Engineering*, vol. 198, 1999.

[3] A. Kilbas, H. Srivastava, and J. Trujillo, "New Book: Theory and applications of fractional differential equations," *North-Holland Mathematics Studies*, Elsevier, vol. 9, p. 71, Amsterdam, Netherlands, 2006.

[4] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[5] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, "On a new class of fractional operators," *Advances in Difference Equations*, vol. 2017, no. 1, p. 247, 2017.

[6] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015.

[7] T. Abdeljawad, J. Alzabut, and F. Jarad, "A generalized Lyapunov-type inequality in the frame of conformable derivatives," *Advances in Difference Equations*, vol. 2017, no. 1, p. 321, 2017.

[8] M. Al-Refai and T. Abdeljawad, "Fundamental results of conformable Sturm-Liouville eigenvalue problems," *Complexity*, vol. 2017, pp. 1–7, Article ID 3720471, 2017.

[9] M. Kaplan and A. Bekir, "A novel analytical method for time-fractional differential equations," *Optik*, vol. 127, no. 20, pp. 8209–8214, 2016.

[10] M. Kaplan and A. Bekir, "Construction of exact solutions to the space-time fractional differential equations via new approach," *Optik*, vol. 132, pp. 1–8, 2017.

[11] K. Hosseini, A. Bekir, and R. Ansari, "Exact solutions of nonlinear conformable time-fractional Boussinesq equations using the exp - $\phi\varepsilon$ -expansion method," *Optical and Quantum Electronics*, vol. 49, no. 4, p. 131, 2017.

[12] K. Hosseini, P. Mayeli, A. Bekir, and O. Guner, "Density-dependent conformable space-time fractional diffusion-reaction equation and its exact solutions," *Communications in Theoretical Physics*, vol. 69, no. 1, p. 1, 2018.

[13] A. A. Al-Shawba, K. A. Gepreel, F. A. Abdullah, and A. Azmi, "Abundant closed form solutions of the conformable time fractional Sawada-Kotera-Ito equation using (G'/G)-expansion method," *Results in Physics*, vol. 9, pp. 337–343, 2018.

[14] M. Younis and A. Zafar, "Exact solution to nonlinear differential equations of fractional order via (G'/G)-expansion method," *Applied Mathematics*, vol. 5, no. 1, pp. 1–6, 2014.

[15] M. Wang, X. Li, and J. Zhang, "The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.

[16] A. A. Al-Shawba, F. A. Abdullah, and A. Azmi, "Travelling wave solutions for fractional Boussinesq equation using modified (G'/G) expansion method," in *Proceedings of the AIP Conference*, Article ID 020036, Pahang, Malaysia, July 2018.

[17] A. Jabbari, H. Kheiri, and A. Bekir, "Exact solutions of the coupled Higgs equation and the Maccari system using He's semi-inverse method and (G'/G)-expansion method,"

Computers & Mathematics with Applications, vol. 62, no. 5, pp. 2177–2186, 2011.

[18] O. Guner, A. Bekir, and K. Hosseini, "Solitary wave solutions of nonlinear conformable time-fractional Boussinesq equations," *Acta Physica Polonica A*, vol. 136, no. 1, p. 135, 2019.

[19] A. A. Al-Shawba, F. A. Abdullah, K. A. Gepreel, and A. Azmi, "The (G'/G)-expansion method for solving the conformable time fractional Hirota-Satsuma coupled KdV system," in *Proceedings of the AIP Conference*, Article ID 060027, 2019.

[20] A. Akbulut, M. Kaplan, and A. Bekir, "Auxiliary equation method for fractional differential equations with modified Riemann-Liouville derivative," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 17, no. 7-8, pp. 413–420, 2016.

[21] A. Akbulut and M. Kaplan, "Auxiliary equation method for time-fractional differential equations with conformable derivative," *Computers & Mathematics with Applications*, vol. 75, no. 3, pp. 876–882, 2018.

[22] O. Guner, A. Bekir, and Ö. Ünsal, "Two reliable methods for solving the time fractional Clannish Random Walker's parabolic equation," *Optik*, vol. 127, no. 20, pp. 9571–9577, 2016.

[23] M. H. Uddin, M. A. Akbar, M. A. Khan, and M. A. Haque, "Close form solutions of the fractional generalized reaction duffing model and the density dependent fractional diffusion reaction equation," *Applied and Computational Mathematics*, vol. 6, no. 4, pp. 177–184, 2017.

[24] A. A. Al-Shawba, F. A. Abdullah, K. A. Gepreel, and A. Azmi, "Solitary and periodic wave solutions of higher-dimensional conformable time-fractional differential equations using the -$((G'/G)(1/G))$expansion method," *Advances in Difference Equations*, vol. 2018, no. 1, p. 362, 2018.

[25] S. Demiray, Ö. Ünsal, and A. Bekir, "New exact solutions for Boussinesq type equations by using $((G'/G)(1/G))$ and $(1/G')$-expansion methods," *Acta Physica Polonica A*, vol. 125, no. 5, p. 1093, 2014.

[26] S. Demiray, Ö. Ünsal, and A. Bekir, "Exact solutions of nonlinear wave equations using $((G'/G)(1/G))$-expansion method," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 1, pp. 78–84, 2015.

[27] A. A. Al-Shawba, F. A. Abdullah, A. Azmi, and M. A. Akbar, "Reliable methods to study some nonlinear conformable systems in shallow water," *Advances in Difference Equations*, vol. 2020, pp. 1–27, 2020.

[28] M. Ekici, M. Mirzazadeh, Q. Zhou, S. P. Moshokoa, A. Biswas, and M. Belic, "Solitons in optical metamaterials with fractional temporal evolution," *Optik*, vol. 127, no. 22, pp. 10879–10897, 2016.

[29] Y. Pandir, Y. Gurefe, and E. Misirli, "The extended trial equation method for some time fractional differential equations," *Discrete Dynamics in Nature and Society*, vol. 2013, pp. 1–13, Article ID 491359, 2013.

[30] S. Zhang and H.-Q. Zhang, "Fractional sub-equation method and its applications to nonlinear fractional PDEs," *Physics Letters A*, vol. 375, no. 7, pp. 1069–1073, 2011.

[31] S. Guo, L. Mei, Y. Li, and Y. Sun, "The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics," *Physics Letters A*, vol. 376, no. 4, pp. 407–411, 2012.

[32] M. Kaplan, A. Bekir, A. Akbulut, and E. Aksoy, "The modified simple equation method for nonlinear fractional differential equations," *Romanian Journal of Physics*, vol. 60, pp. 1374–1383, 2015.

[33] S. Tuluce Demiray, Y. Pandir, and H. Bulut, "Generalized Kudryashov method for time-fractional differential equations," *Abstract and Applied Analysis*, vol. 2014, pp. 1–13, Article ID 901540, 2014.

[34] M. Eslami, "Exact traveling wave solutions to the fractional coupled nonlinear Schrodinger equations," *Applied Mathematics and Computation*, vol. 285, pp. 141–148, 2016.

[35] M. Eslami, "Solitary wave solutions for perturbed nonlinear Schrodinger's equation with Kerr law nonlinearity under the DAM," *Optik*, vol. 126, no. 13, pp. 1312–1317, 2015.

[36] B. Lu, "Bäcklund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations," *Physics Letters A*, vol. 376, no. 28-29, pp. 2045–2048, 2012.

[37] C. Liu, "The relation between the kink-type solution and the kink-bell-type solution of nonlinear evolution equations," *Physics Letters A*, vol. 312, no. 1-2, pp. 41–48, 2003.

[38] J.-H. He and L.-N. Zhang, "Generalized solitary solution and compacton-like solution of the Jaulent-Miodek equations using the exp-function method," *Physics Letters A*, vol. 372, no. 7, pp. 1044–1047, 2008.

[39] D. Zhao and M. Luo, "General conformable fractional derivative and its physical interpretation," *Calcolo*, vol. 54, no. 3, pp. 903–917, 2017.

[40] H. Bulut, "Exact solutions for some fractional nonlinear partial differential equations via Kudryashov method," *Physical Sciences*, vol. 8, pp. 24–63, 2013.

[41] L. Song, Q. Wang, and H. Zhang, "Rational approximation solution of the fractional Sharma-Tasso-Olever equation," *Journal of Computational and Applied Mathematics*, vol. 224, no. 1, pp. 210–218, 2009.

[42] M. S. Rawashdeh, "An efficient approach for time-fractional damped Burger and time-sharma-tasso-olver equations using the FRDTM," *Applied Mathematics & Information Sciences*, vol. 9, no. 3, p. 1239, 2015.

[43] M. H. Uddin, M. A. Khan, M. A. Akbar, and M. A. Haque, "Multi-solitary wave solutions to the general time fractional Sharma-Tasso-Olver equation and the time fractional Cahn-Allen equation," *Arab Journal of Basic and Applied Sciences*, vol. 26, no. 1, pp. 193–201, 2019.

[44] M. A. Shallal, H. N. Jabbar, and K. K. Ali, "Analytic solution for the space-time fractional Klein-Gordon and coupled conformable Boussinesq equations," *Results in Physics*, vol. 8, pp. 372–378, 2018.

[45] E. Yaşar and I. B. Giresunlu, "The $((G'/G)(1/G))$-expansion method for solving nonlinear space–time fractional differential equations," *Pramana-Journal of Physics*, vol. 87, no. 2, p. 17, 2016.