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Research Article

Treatment a New Approximation Method and Its Justification for Sturm–Liouville Problems

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In this paper, we propose a new approximation method (we shall call this method as α-parameterized differential transform method), which differs from the traditional differential transform method in calculating the coefficients of Taylor polynomials. Numerical examples are presented to illustrate the efficiency and reliability of our own method. Namely, two Sturm–Liouville problems are solved by the present α-parameterized differential transform method, and the obtained results are compared with those obtained by the classical DTM and by the analytical method. The result reveals that α-parameterized differential transform method is a simple and effective numerical algorithm.

1. Introduction

Many problems in mathematical physics, theoretical physics, and chemical physics are modelled by the so-called initial value and boundary value problems in the second-order ordinary differential equations. In most cases, these problems may be too complicated to solve analytically. Alternatively, the numerical methods can provide approximate solutions rather than the analytic solutions of problems. There are various approximation methods for solving a system of differential equations, e.g., Adomian decomposition method (ADM), Galerkin method, rationalized Haar functions method, homotopy perturbation method (HPM), variational iteration method (VIM), and the differential transform method (DTM).

The DTM is one of the numerical methods which enables to find an approximate solution in case of linear and nonlinear systems of differential equations. The main advantage of this method is that it can be applied directly to nonlinear ODEs without requiring linearization. The well-known advantage of DTM is its simplicity and accuracy in calculations and also wide range of applications. Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with a fast convergence rate. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. The concept of the differential transform method was first proposed by Zhou [1], who solved linear and nonlinear initial value problems in electric circuit analysis. Afterwards, Chiou and Tzeng [2] applied the Taylor transform to solve nonlinear vibration problems, Chen and Ho [3] developed this method to various linear and nonlinear problems such as two-point boundary value problems, and Ayaz [4] applied it to the system of differential equations. Abbasov and Bahadir [5] used the method of differential transform to obtain approximate solutions of the linear and nonlinear equations related to engineering problems and observed that the numerical results are in good agreement with the analytical solutions. In recent years, many authors have used different methods for solving various types of equations (see, for example, [6–9]). For example, DTM has been used for differential algebraic equations [10], partial differential equations [3, 11–13], fractional differential equations [14], difference equations [15], etc. In [16–18], this method has been utilized for Telegraph, Kuramoto–Sivashinsky, and Kawahara equations. Shahmorad et al. developed DTM to fractional-order integrodifferential equations with nonlocal boundary conditions [19] and class of two-dimensional Volterra integral equations [20]. Borhanifar and Abazari applied this method for Schrödinger equations [21]. Different applications of DTM can be found in [22, 23]. Although the differential transform method (DTM) is an effective numerical method for solving many initial value problems, there are also some disadvantages since this method is designed for problems that have analytic solutions (i.e., solutions that can be expanded in Taylor series).

In this paper, we suggest a new version of DTM which we shall call α-parameterized differential transform method (α-P DTM) to solve initial value and boundary value problems, as well as eigenvalue problems. Note that, in the special cases, the α-P DTM reduces to the standard DTM, so our method is the extension and generalization of the classical DTM.

2. Outline of the Classical DTM

In this section, we describe the definition and some basic properties of the classical DTM. Recall that an arbitrary analytic function $f(x)$ can be expanded in Taylor series about a point $x = x_0$ as

$$f(x) = \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=x_0}. \quad (1)$$

The classical differential transformation of $f(x)$ is defined as

$$F(k) = \frac{1}{k!} \left[\frac{d^k f(x)}{dx^k} \right]_{x=x_0}, \quad (2)$$

and the inverse differential transform is defined as

$$f(x) = \sum_{k=0}^{\infty} (x - x_0)^k F(k), \quad (3)$$

(see [4]).

Let $F(k)$, $G(k)$, and $H(k)$ be the differential transformation of $f(x)$, $g(x)$, and $h(x)$, respectively. The basic mathematical operations performed by the differential transform method are listed in following:

(i) If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$

(ii) If $f(x) = \alpha g(x)$, $\alpha \in \mathbb{R}$, then $F(k) = \alpha G(k)$

(iii) If $f(x) = (d^m g/dx^m)$, then $F(k) = (k+1)(k+2)\ldots(k+m)G(k+m)$

(iv) If $f(x) = x^m$, then $F(k) = \delta(k-m) = \begin{cases} 1, & \text{for } k = m \\ 0, & \text{for } k \neq m \end{cases}$

(v) If $f(x) = g(x)h(x)$, then $F(k) = \sum_{m=0}^{k} H(m)G(k-m)$

3. α-Parameterized Differential Transform Method (α-P DTM)

In this section, we suggest a new version of the classical differential transform method by following.

Let $I = [a, b] \subset \mathbb{R}$ be an arbitrary real interval, $f: I \longrightarrow \mathbb{R}$ be an infinitely differentiable function (in real applications, it is enough to require that $f(x)$ is sufficiently a large-order differentiable function), $\alpha \in [0, 1]$ be any real parameter, and N be any integer (large enough).

Let us define a parameterized sequence $D(f, \alpha; k)$, $k = 0, 1, 2, \ldots$ by

$$D(f, \alpha; k) \coloneqq \alpha D_a(f; k) + (1 - \alpha) D_b(f; k), \quad (4)$$

where $D_a(f; k)$ and $D_b(f; k)$ are Taylor's coefficients, that is,

$$D_a(f; k) \coloneqq \frac{f^{(k)}(a)}{k!}, \quad D_b(f; k) \coloneqq \frac{f^{(k)}(b)}{k!}. \quad (5)$$

Definition 1. The sequence

$$(D_\alpha(f)) \coloneqq (D(f, \alpha; 1), D(f, \alpha; 2), \ldots), \quad (6)$$

is called the α-P transformation of the original function $f(x)$. The differential inverse transformation of $D_\alpha(f)$ is defined as the series

$$E_\alpha(D_\alpha(f)) \coloneqq \sum_{k=0}^{\infty} D(f, \alpha; k)(x - x_\alpha)^k, \quad (7)$$

if the series is convergent, where $x_\alpha = \alpha a + (1 - \alpha) b$.

The function $\tilde{f}_\alpha(x)$ defined by equality

$$\tilde{f}_\alpha(x) \coloneqq E_\alpha(D_\alpha(f)), \quad (8)$$

is called the α-parameterized approximation of the original function $f(x)$.

Remark 1. In the cases of $\alpha = 1$ and $\alpha = 0$, the α-P differential transform (4) reduces to the classical differential transform (2) at the points $x = a$ and $x = b$, respectively. Namely, for $\alpha = 0$ and $\alpha = 1$, the equality $\tilde{f}_\alpha(x) = f(x)$ holds.

Remark 2. For practical application, instead of $\tilde{f}_\alpha(x)$, it is convenient to introduce the N-term α-parameterized approximation of the function $\tilde{f}_\alpha(x)$ which we shall define as

$$\tilde{f}_{\alpha,N}(x) \coloneqq E_{\alpha,N}(D_\alpha(f)) \coloneqq \sum_{k=0}^{N} D(f, \alpha; k)(x - x_\alpha)^k. \quad (9)$$

Theorem 1. *If $f(x)$ is a constant function, then $\tilde{f}_\alpha(x) = f(x)$ and $\tilde{f}_{\alpha,N}(x) = f(x)$ for each N.*

Proof. The proof is immediate from Definition 1 and Remark 2. □

Theorem 2. *If $f(x) = cg(x)$, $c \in \mathbb{R}$, then $D_\alpha(f) = cD_\alpha(g)$ and $\widetilde{f}_\alpha(x) = c\widetilde{g}_\alpha(x)$.*

Proof. By applying the well-known properties of classical DTM, we get

$$\begin{aligned} D(f, \alpha; k) &= \alpha D_a(f; k) + (1-\alpha) D_b(f; k) \\ &= \alpha c D_a(g; k) + (1-\alpha) c D_b(g; k) \\ &= c(\alpha D_a(g; k) + (1-\alpha) D_b(g; k)) \\ &= c D(g, \alpha; k). \end{aligned} \tag{10}$$

Consequently, $D_\alpha(f) = cD_\alpha(g)$, from which immediately follows that $\widetilde{f}_\alpha(x) = c\widetilde{g}_\alpha(x)$. □

Theorem 3. *If $f(x) = g(x) \pm h(x)$, then $D_\alpha(f) = D_\alpha(g) \pm D_\alpha(h)$ and $\widetilde{f}_\alpha(x) = \widetilde{g}_\alpha(x) \pm \widetilde{h}_\alpha(x)$.*

Proof. By using the definition of transform (4)

$$\begin{aligned} D(f, \alpha; k) &= \alpha D_a(f; k) \pm (1-\alpha) D_b(f; k) \\ &= \alpha D_a(g+h; k) \pm (1-\alpha) D_b(g+h; k) \\ &= D(g, \alpha; k) \pm D(h, \alpha; k). \end{aligned} \tag{11}$$

Consequently, $D_\alpha(f) = D_\alpha(g) \pm D_\alpha(h)$, from which immediately follows that $\widetilde{f}_\alpha(x) = \widetilde{g}_\alpha(x) \pm \widetilde{h}_\alpha(x)$. □

Theorem 4. *Let $f(x) = (d^m g/dx^m)$ and $m \in \mathbb{N}$. Then,*

$$\begin{aligned} D\left(f^{(m)}, \alpha; k\right) &= \frac{(k+m)!}{k!} D(f, \alpha; k+m), \\ \widetilde{f}_\alpha^{(m)}(x) &= \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} D(f, \alpha; k+m)(x - x_\alpha)^k, \end{aligned} \tag{12}$$

where $x_\alpha = \alpha a + (1-\alpha) b$.

Proof. We have from definition (4)

$$\begin{aligned} D\left(f^{(m)}, \alpha; k\right) &= \alpha D_a\left(f^{(m)}; k\right) + (1-\alpha) D_b\left(f^{(m)}; k\right) \\ &= \alpha(k+1)(k+2)\dots(k+m) D_a(f; k+m) \\ &\quad + (1-\alpha)(k+1)(k+2)\dots(k+m) D_b(f; k+m) \\ &= (k+1)(k+2)\dots(k+m)(\alpha D_a(f; k) \\ &\quad + (1-\alpha) D_b(f; k+m)) \\ &= \frac{(k+m)!}{k!} D(f, \alpha; k+m). \end{aligned} \tag{13}$$

Thus, we get $D(f^{(m)}, \alpha; k) = ((k+m)!/k!) D(f, \alpha; k+m)$. Using this, we find $\widetilde{f}_\alpha^{(m)}(x) = \sum_{k=0}^{\infty} ((k+m)!/k!) D(f, \alpha; k+m)(x - x_\alpha)^k$. □

Theorem 5. *Let $f(x) = x^m$, $m \in \mathbb{N}$. Then,*

$$D(f, \alpha; k) = \begin{cases} \begin{pmatrix} m \\ k \end{pmatrix} (\alpha a^{m-k} + (1-\alpha) b^{m-k}), & \text{for } k < m, \\ 1, & \text{for } k = m, \\ 0, & \text{for } k > m. \end{cases} \tag{14}$$

Proof. Let $k < m$. By using the definition of the transform (4), we have

$$\begin{aligned} D(f, \alpha; k) &= \alpha D_a(f; k) + (1-\alpha) D_b(f; k) \\ &= \alpha D_a\left(x^{(m)}; k\right) + (1-\alpha) D_b\left(x^{(m)}; k\right) \\ &= \begin{pmatrix} m \\ k \end{pmatrix} \left(\alpha a^{(m-k)} + (1-\alpha) b^{(m-k)}\right). \end{aligned} \tag{15}$$

The equalities $D(x^m, \alpha; m) = 1$ and $D(x^m, \alpha; m+s) = 0$ for $s \geq 1$ are obvious. □

Theorem 6. *If $f(x) = g(x)h(x)$, then $D(f, \alpha; k) = \sum_{m=0}^{k} [\alpha D_a(g; m) D_a(h; k-m) + (1-\alpha) D_b(g; m) D_b(h; k-m)]$*

Proof. By using the definition of transform given in equation (4), we have

$$\begin{aligned} D(f, \alpha; k) &= \alpha D_a(f; k) + m(1-\alpha) D_b(f; k) \\ &= \alpha D_a(gh; k) + (1-\alpha) D_b(gh; k) \\ &= \frac{\alpha}{k!} \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} g^{(m)}(a) h^{(k-m)}(a) \\ &\quad + \frac{(1-\alpha)}{k!} \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} g^{(m)}(b) h^{(k-m)}(b) \\ &= \sum_{m=0}^{k} [\alpha D_a(g; m) D_a(h; k-m) \\ &\quad + (1-\alpha) D_b(g; m) D_b(h; k-m)]. \end{aligned} \tag{16}$$

□

4. Justification of the α-P DTM

In order to show the effectiveness of α-P DTM for solving boundary value problems, we shall consider the following Sturm–Liouville problems.

Example 1 (application to the boundary value problem). Let us consider the Sturm–Liouville equation

$$\ell y \coloneqq y''(x) + \mu^2 y(x) = 0, \quad x \in [0, 1], \mu \in \mathbb{R}, \tag{17}$$

with the nonhomogeneous boundary conditions

$$\begin{aligned} y(0) &= 0, \\ y(1) &= 1. \end{aligned} \tag{18}$$

The exact solution for this problem is

$$y(x) = \frac{\sin \mu x}{\sin \mu}. \tag{19}$$

Applying the N-term α-P differential transform to both the sides of (17) and (18), we obtain the following α-parameterized boundary value problem as

$$\begin{aligned} (\widetilde{\ell y})_{\alpha,N} &= \widetilde{0}_{\alpha,N}, \\ \widetilde{y}_{\alpha,N}(0) &= \widetilde{0}_{\alpha,N}, \\ \widetilde{y}_{\alpha;N}(1) &= \widetilde{1}_{\alpha,N}. \end{aligned} \tag{20}$$

By using the fundamental operations of α-P DTM, we have

$$D(y,\alpha;k+2) = -\frac{\mu^2 D(y,\alpha;k)}{(k+1)(k+2)}. \tag{21}$$

The boundary conditions given in (18) can be transformed as follows:

$$\begin{aligned} \widetilde{y}_{\alpha,N}(0) &= \sum_{k=0}^{N} D(y,\alpha;k)(\alpha-1)^k = 0, \\ \widetilde{y}_{\alpha,N}(1) &= \sum_{k=0}^{N} D(y,\alpha;k)\alpha^k = 1. \end{aligned} \tag{22}$$

Using (21) and (22) and by taking $N = 5$, the following α-P approximate solution is obtained:

$$\begin{aligned} \widetilde{y}_\alpha(x) = {} & A + (x - x_\alpha)B \\ & - \frac{\mu^2 (x-x_\alpha)^2 A}{2} - \frac{\mu^2 (x-x_\alpha)^3 B}{6} + \frac{\mu^4 (x-x_\alpha)^4 A}{24} \\ & + \frac{\mu^4 (x-x_\alpha)^5 B}{120} + O(x^6), \end{aligned} \tag{23}$$

where $x_\alpha = (1-\alpha)$, according to (7), $D(y,\alpha;0) = A$, and $D(y,\alpha;1) = B$. The constants A and B evaluated from equations in (21) are as follows:

$$\begin{aligned} A = {} & \Big(2880\big(x_\alpha^4\mu^8 - 4x_\alpha^5\mu^8 + 6x_\alpha^6\mu^8 + 24\mu^4 - 4x_\alpha^7\mu^8 \\ & - 480\mu^2 + x_\alpha^8\mu^8 + 2880 \\ & - 12x_\alpha^2\mu^6 + 40x_\alpha^3\mu^6 - 60x_\alpha^4\mu^6 + 48x_\alpha^5\mu^6 - 16x_\alpha^6\mu^6\big)^{-1}\Big) \\ & \times \left(x_\alpha - \frac{\mu^2 x_\alpha^3}{6} + \frac{\mu^4 x_\alpha^5}{120}\right), \end{aligned} \tag{24}$$

$$\begin{aligned} B = {} & \Big(2880\big(x_\alpha^4\mu^8 - 4x_\alpha^5\mu^8 + 6x_\alpha^6\mu^8 + 24\mu^4 - 4x_\alpha^7\mu^8 \\ & - 480\mu^2 + x_\alpha^8\mu^8 + 2880 \\ & - 12x_\alpha^2\mu^6 + 40x_\alpha^3\mu^6 - 60x_\alpha^4\mu^6 + 48x_\alpha^5\mu^6 - 16x_\alpha^6\mu^6\big)^{-1}\Big) \\ & \times \left(1 - \frac{\mu^2 x_\alpha^2}{2} + \frac{\mu^4 x_\alpha^4}{24}\right). \end{aligned} \tag{25}$$

Remark 3. Putting $\alpha = 0$ in (23), we have the classical DTM solution $\widetilde{y}_0(x)$, given by

$$\begin{aligned} \widetilde{y}_0(x) = {} & A + (x-1)B - \mu^2 (x-1)^2 \frac{A}{2} - \mu^2 (x-1)^3 \frac{B}{6} \\ & + \mu^4 (x-1)^4 \frac{A}{24} + \mu^4 (x-1)^5 \frac{B}{120} + O(x^6). \end{aligned} \tag{26}$$

Now, we will show that the α-P DTM can be applied not only to find the solutions of Sturm–Liouville problems, but also to find the eigenvalues of this type boundary value problems.

Example 2 (application to eigenvalue problems). We consider the following eigenvalue problem:

$$y'' + \lambda y = 0, \quad x \in [0,1], \tag{27}$$

$$A_{11} y(0) + A_{12} y'(0) = 0, \tag{28}$$

$$A_{21} y(1) + A_{22} y'(1) = 0. \tag{29}$$

Taking the α-P differential transform of both sides of (27), we find

$$\begin{aligned} D(y'' + \lambda y, \alpha; k) = {} & (k+1)(k+2) D(y,\alpha;k+2) \\ & + \lambda D(y,\alpha;k) = 0. \end{aligned} \tag{30}$$

Then, the following recurrence relation is obtained:

$$D(y,\alpha;k+2) = -\frac{\lambda D(y,\alpha;k)}{(k+1)(k+2)}. \tag{31}$$

Using definition of the α-P differential transform, we get

$$\begin{aligned} \widetilde{y}_\alpha(x) &= \sum_{k=0}^{\infty} D(y,\alpha;k)(x-x_\alpha)^k, \\ \widetilde{y}'_\alpha(x) &= \sum_{k=0}^{\infty} k D(y,\alpha;k)(x-x_\alpha)^{k-1}. \end{aligned} \tag{32}$$

Consequently,

$$\begin{aligned}\tilde{y}_\alpha(0) &= \sum_{k=0}^{\infty} D(y,\alpha;k)(\alpha-1)^k = \sum_{k=0}^{\infty}(-1)^k D(y,\alpha;k)(1-\alpha)^k,\\ \tilde{y}'_\alpha(0) &= \sum_{k=0}^{\infty} kD(y,\alpha;k)(\alpha-1)^{k-1}\\ &= \sum_{k=0}^{\infty}(-1)^k kD(y,\alpha;k)(1-\alpha)^{k-1}.\end{aligned} \tag{33}$$

Thus, the boundary condition (28) can be transformed as follows:

$$A_{11}\tilde{y}_\alpha(0) + A_{12}\tilde{y}'_\alpha(0) = \sum_{k=0}^{\infty}\left(A_{11}(\alpha-1)^k + kA_{12}(\alpha-1)^{k-1}\right)D(y,\alpha;k) = 0. \tag{34}$$

Similarly, we have

$$\begin{aligned}\tilde{y}_\alpha(1) &= \sum_{k=0}^{\infty} D(y,\alpha;k)\alpha^k,\\ \tilde{y}'_\alpha(1) &= \sum_{k=0}^{\infty} kD(y,\alpha;k)\alpha^{k-1}.\end{aligned} \tag{35}$$

In this case, the boundary condition (29) can be written as follows:

$$A_{21}\tilde{y}_\alpha(1) + A_{22}\tilde{y}'_\alpha(1) = \sum_{k=0}^{\infty}\left(A_{21}\alpha^k + kA_{22}\alpha^{k-1}\right)D(y,\alpha;k) = 0. \tag{36}$$

Let $D(y,\alpha;0) = A$ and $D(y,\alpha;1) = B$. Substituting these values in (31), we have the following recursive procedure:

$$D(y,\alpha;k) = \begin{cases}\dfrac{A(-\lambda)^\ell}{(2\ell!)}, & \text{for } k = 2\ell,\\[2ex] \dfrac{B(-\lambda)^\ell}{(2\ell+1)!}, & \text{for } k = 2\ell+1.\end{cases} \tag{37}$$

Substituting (37) in (34) and (36), we find

$$\begin{aligned}&A\left\{\sum_{\ell=0}^{\infty}\left(A_{11}(\alpha-1)2^\ell + 2\ell A_{12}(\alpha-1)^{2\ell-1}\right)\frac{(-\lambda)^\ell}{(2\ell)!}\right\}\\ &\quad + B\left\{\sum_{\ell=0}^{\infty}\left(A_{11}\left((\alpha-1)^{2\ell+1} + (2\ell+1)A_{12}(\alpha-1)^{2\ell}\right)\right)\frac{(-\lambda)^\ell}{(2\ell+1)!}\right\} = 0,\\ &A\left\{\sum_{\ell=0}^{\infty}\left(A_{21}\alpha^{2\ell} + 2\ell A_{22}\alpha^{2\ell-1}\right)\frac{(-\lambda)^\ell}{(2\ell)!}\right\}\\ &\quad + B\left\{\sum_{\ell=0}^{\infty}\left(A_{21}\alpha^{2\ell+1} + (2\ell+1)A_{22}\alpha^{2\ell}\right)\frac{(-\lambda)^\ell}{(2\ell+1)!}\right\} = 0,\end{aligned} \tag{38}$$

respectively. In this case, we have a linear system of the equations with respect to the variables A and B as

$$AP_{11}(\lambda) + BP_{12}(\lambda) = 0, \tag{39}$$

$$AP_{21}(\lambda) + BP_{22}(\lambda) = 0, \tag{40}$$

where

$$\begin{aligned}P_{11}(\lambda) &:= \sum_{\ell=0}^{\infty}\left(A_{11}(\alpha-1)^{2\ell} + 2\ell A_{12}(\alpha-1)^{2\ell-1}\right)\frac{(-\lambda)^\ell}{(2\ell)!},\\ P_{12}(\lambda) &:= \sum_{\ell=0}^{\infty}\left(A_{11}(\alpha-1)^{2\ell+1} + (2\ell+1)A_{12}(\alpha-1)^{2\ell}\right)\frac{(-\lambda)^\ell}{(2\ell+1)!},\\ P_{21}(\lambda) &:= \sum_{\ell=0}^{\infty}\left(A_{21}\alpha^{2\ell} + 2\ell A_{22}\alpha^{2\ell-1}\right)\frac{(-\lambda)^\ell}{(2\ell)!},\\ P_{22}(\lambda) &:= \sum_{\ell=0}^{\infty}\left(A_{21}\alpha^{2\ell+1} + (2\ell+1)A_{22}\alpha^{2\ell}\right)\frac{(-\lambda)^\ell}{(2\ell+1)!}.\end{aligned} \tag{41}$$

Since systems (39) and (40) have a nontrivial solution for A and B, the characteristic determinant is zero, i.e.,

$$P(\lambda) = \begin{vmatrix}P_{11}(\lambda) & P_{12}(\lambda)\\ P_{21}(\lambda) & P_{22}(\lambda)\end{vmatrix} = 0. \tag{42}$$

The zeros of the characteristic equation $P(\lambda) = 0$ coincide with the α-parametrized eigenvalues of the Sturm–Liouville problem (27)–(29).

Now, let us find the exact eigenvalues and eigenfunctions of the Sturm–Liouville problem (27)–(29). The general solution of equation (29) has the form

$$y(x) = C\cos\mu x + D\sin\mu x, \tag{43}$$

where $\lambda = \mu^2$ and C and D are the arbitrary constants. Applying the boundary conditions (27) and (28), we get

$$\begin{aligned}A_{11}C + \mu A_{12}D &= 0,\\ \left(A_{21}\cos - \mu A_{22}\sin\mu\right)C + \left(A_{21}\cos - \mu A_{22}\sin\mu\right)D &= 0.\end{aligned} \tag{44}$$

Because we cannot have $C = D = 0$, this implies

$$\left(A_{11}A_{21} + \mu^2 A_{12}A_{22}\right)\sin\mu - \mu\left(A_{12}A_{21} - A_{11}A_{22}\right)\cos\mu = 0. \tag{45}$$

This is a transcendental equation which is solved graphically. Let $\mu = \mu_n, n \in \mathbb{N}$ are points of intersection of the graphs of the functions:

$$\begin{aligned}y &= \left(A_{11}A_{21} + \mu^2 A_{12}A_{22}\right)\sin\mu,\\ y &= \mu\left(A_{12}A_{21} - A_{11}A_{22}\right)\cos\mu.\end{aligned} \tag{46}$$

The eigenvalues and corresponding eigenfunctions are therefore given by

$$\lambda_n = \mu_n^2 \text{ and } y_n(x) = C_n\cos\mu_n x + D_n\sin\mu_n x,\ n \in \mathbb{N}. \tag{47}$$

Now, we consider a special case of the Sturm–Liouville problem (27)–(29), given by

$$y'' + \lambda y = 0, \tag{48}$$

$$y(0) + y'(0) = 0, \tag{49}$$

$$y(1) - y'(1) = 0. \tag{50}$$

The eigenvalues of this problem are determined by the following equation:

$$\tan \quad \mu = \frac{2\mu}{1-\mu^2}. \tag{51}$$

This equation can be solved graphically by the points of intersections of the graphs of functions:

$$y = \tan \mu, \qquad y = \frac{2\mu}{1-\mu^2}, \tag{52}$$

as shown by the sequence (μ_n) in Figure 1.

The eigenvalues of the considered problem are given by $\lambda_n = \mu_n^2$ and corresponding eigenfunctions are given by

$$y_n(x) = C_n \cos \mu_n x + D_n \sin \mu_n x, \quad n \in \mathbb{N}. \tag{53}$$

Taking the α-P differential transform of both sides of equation (48), the following recurrence relation is obtained:

$$D(y,\alpha;k+2) = -\frac{\lambda D(y,\alpha;k)}{(k+1)(k+2)}. \tag{54}$$

Applying the N-term α-P differential transform to the boundary conditions (49) and (50), we have

$$\tilde{y}_\alpha(0) + \tilde{y}'_\alpha(0) = \sum_{k=0}^{N}\left((\alpha-1)^k + k(\alpha-1)^{k-1}\right)D(y,\alpha;k) = 0, \tag{55}$$

$$\tilde{y}_\alpha(1) - \tilde{y}'_\alpha(1) = \sum_{k=0}^{N}\left(\alpha^k - k\alpha^{k-1}\right)D(y,\alpha;k) = 0. \tag{56}$$

By using (54), (55), and (56), we obtain the following equalities (for $N=6$):

$$\begin{aligned}A\Big[&1 + \left((\alpha-1)^2 + 2(\alpha-1)\right)\frac{(-\lambda)}{2!} + \left((\alpha-1)^4 + 4(\alpha-1)^3\right)\frac{\lambda^2}{4!} \\ &+ \left((\alpha-1)^6 + 6(\alpha-1)^5\right)\frac{(-\lambda^3)}{6!}\Big] \\ &+ B\Big[\alpha + \left((\alpha-1)^3 + 3(\alpha-1)^2\right)\frac{(-\lambda)}{3!} \\ &+ \left((\alpha-1)^5 + 5(\alpha-1)^4\right)\frac{\lambda^2}{5!}\Big] = 0,\end{aligned} \tag{57}$$

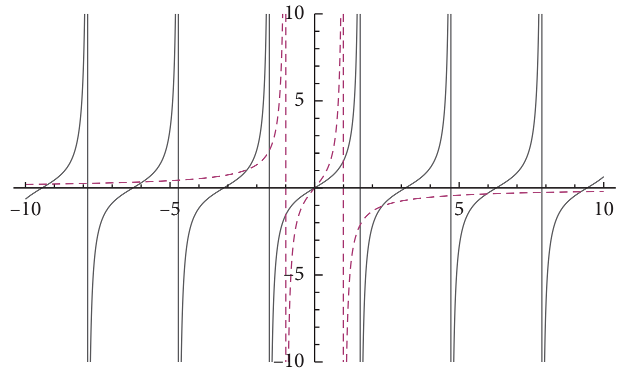

Figure 1: The exact eigenvalues of problems (48)–(50) (abscissas of intersection points of the graphs of the functions $y = \tan \mu$(black line) and $y = (2\mu/1-\mu^2)$ (red line)).

$$\begin{aligned}&A\left[1 + \left(\alpha^2 - 2\alpha\right)\frac{(-\lambda)}{2!} + \left(\alpha^4 - 4\alpha^3\right)\frac{\lambda^2}{4!} + \left(\alpha^6 - 6\alpha^5\right)\frac{(-\lambda^3)}{6!}\right] \\ &+ B\left[(\alpha-1) + \left(\alpha^3 - 3\alpha^2\right)\frac{(-\lambda)}{3!} + \left(\alpha^5 - 5\alpha^4\right)\frac{\lambda^2}{5!}\right] = 0.\end{aligned} \tag{58}$$

Since systems (57) and (58) have a nontrivial solution for A and B, the characteristic determinant is zero, i.e.,

$$a(\lambda) = \begin{vmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda)\end{vmatrix} = 0, \tag{59}$$

where

$$\begin{aligned} a_{11} &= 1 + \left((\alpha-1)^2 + 2(\alpha-1)\right)\frac{(-\lambda)}{2!} + \left((\alpha-1)^4 + 4(\alpha-1)^3\right)\frac{\lambda^2}{4!} + \left((\alpha-1)^6 + 6(\alpha-1)^5\right)\frac{(-\lambda^3)}{6!}, \\ a_{12} &= \alpha + \left((\alpha-1)^3 + 3(\alpha-1)^2\right)\frac{(-\lambda)}{3!} + \left((\alpha-1)^5 + 5(\alpha-1)^4\right)\frac{\lambda^2}{5!}, \\ a_{21} &= 1 + \left(\alpha^2 - 2\alpha\right)\frac{(-\lambda)}{2!} + \left(\alpha^4 - 4\alpha^3\right)\frac{\lambda^2}{4!} + \left(\alpha^6 - 6\alpha^5\right)\frac{(-\lambda^3)}{6!}, \\ a_{22} &= (\alpha-1) + \left(\alpha^3 - 3\alpha^2\right)\frac{(-\lambda)}{3!} + \left(\alpha^5 - 5\alpha^4\right)\frac{\lambda^2}{5!}. \end{aligned} \tag{60}$$

Taking $\alpha = (1/2)$, we have the following algebraic equation for approximate eigenvalues:

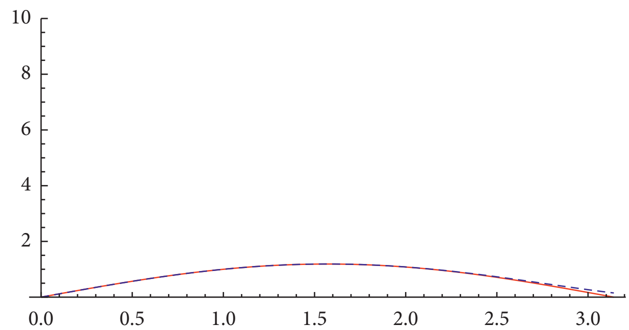

FIGURE 2: Comparison of the exact solution for $\mu = 1$ (red line) and the classical DTM solution for $\mu = 1$ (blue line).

$$-1 - \frac{\lambda}{6} + \frac{11\lambda^2}{120} - \frac{89\lambda^3}{15360} + \frac{299\lambda^4}{2211840} - \frac{11\lambda^5}{9830400} = 0. \quad (61)$$

This equation can be solved by various numerical methods.

5. Comparison Results and Discussion

It is important to note that the α-parametrized DTM is an extension and generalization of the classical DTM since in the special cases $\alpha = 0$ and $\alpha = 1$, our method reduced to the classical DTM.

To illustrate the accuracy of the α-parametrized DTM, solution (23) obtained using this method is compared with solution (26) obtained using the classical DTM and with exact solution (19) obtained using the analytical method in Figures 2–5.

Remark 4. As seen from Figures 2–5, in order to increase the accuracy of the approximate solutions, it is necessary to increase the number of terms $D(t, \alpha, k)$, and the convergence of α-parametrized DTM is quite obvious.

6. Analysis of the Method

In this study, we have introduced a new version of classical DTM that will extend the application of the method to spectral analysis of various types, initial and boundary value problems, which arise from problems of mathematical physics. Numerical results reveal that the α-P DTM is a powerful tool for solving many initial value and boundary value problems. It is concluded that comparing with the standard DTM, the α-P DTM reduces computational cost in obtaining approximated solutions. This method unlike most numerical techniques provides a closed-form solution. It may be concluded that α-P DTM is very powerful and efficient in finding approximate solutions and approximate eigenvalues for wide classes of boundary value problems. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, and in a rapidly convergent sequence with elegantly computed terms.

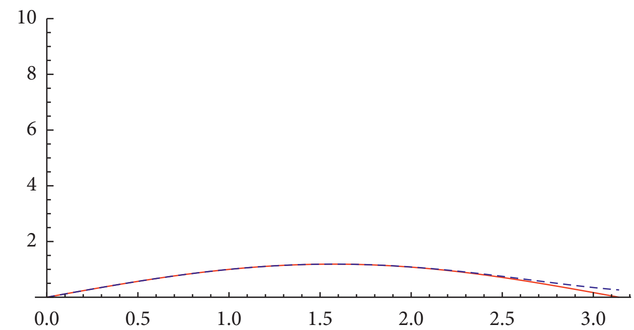

FIGURE 3: Comparison of the exact solution for $\mu = 1$ (red line) and the numerical α-parametrized solution for $\alpha = (1/4)$ (blue line), with $\mu = 1$.

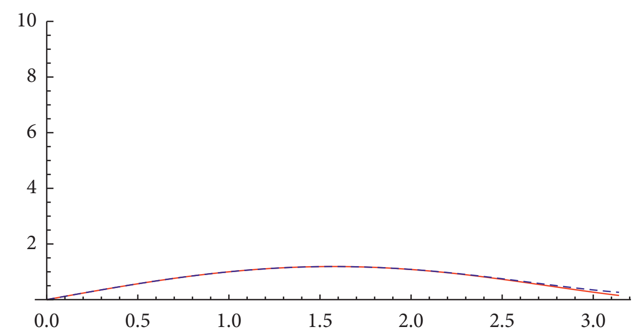

FIGURE 4: Comparison of the classical DTM solution for $\mu = 1$ (red line) and the numerical α-parametrized solution for $\alpha = (1/4)$ (blue line), with $\mu = 1$.

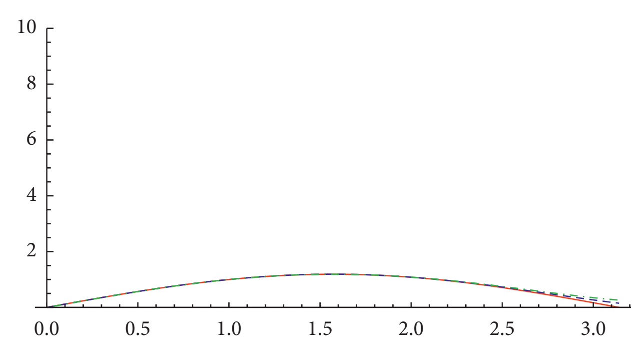

FIGURE 5: Comparison of the exact solution for $\mu = 1$ (red line), the classical DTM solution for $\mu = 1$ (blue line), and the numerical α-parametrized solution for $\alpha = (1/4)$ (green line) with $\mu = 1$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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