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Research Article

The Arcsine Exponentiated-*X* Family: Validation and Insurance Application

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In this paper, we propose a family of heavy tailed distributions, by incorporating a trigonometric function called the arcsine exponentiated-*X* family of distributions. Based on the proposed approach, a three-parameter extension of the Weibull distribution called the arcsine exponentiated-Weibull (ASE-W) distribution is studied in detail. Maximum likelihood is used to estimate the model parameters, and its performance is evaluated by two simulation studies. Actuarial measures including Value at Risk and Tail Value at Risk are derived for the ASE-W distribution. Furthermore, a numerical study of these measures is conducted proving that the proposed ASE-W distribution has a heavier tail than the baseline Weibull distribution. These actuarial measures are also estimated from insurance claims real data for the ASE-W and other competing distributions. The usefulness and flexibility of the proposed model is proved by analyzing a real-life heavy tailed insurance claims data. We construct a modified chi-squared goodness-of-fit test based on the Nikulin–Rao–Robson statistic to verify the validity of the proposed ASE-W model. The modified test shows that the ASE-W model can be used as a good candidate for analyzing heavy tailed insurance claims data.

1. Introduction

Heavy tailed distributions play a significant role in modeling data in applied sciences, particularly in risk management, banking, economics, financial, and actuarial sciences. However, the quality of the procedures primarily depends upon the assumed probability model of the phenomenon under consideration. Among the applied fields, the insurance datasets are usually positive [1], right-skewed [2], unimodal shaped [3], and with heavy tails [4]. Right-skewed data may be adequately modeled by the skewed distributions [5]. Therefore, a number of unimodal positively skewed parametric distributions have been employed to model such datasets [6, 7].

The heavy tailed distributions are those whose right tail probabilities are heavier than the exponential one, that is,

$$x \xrightarrow{\lim} \infty \frac{\exp(-\gamma x) - 1}{1 - F(x)} = 0, \quad \gamma > 0, \tag{1}$$

where $F(x)$ is the cdf of a baseline distribution. More information can be explored in Resnick [8] and Beirlant et al. [9].

Dutta and Perry [10] performed an empirical analysis of loss distributions to estimate the risk via different approaches. They rejected the idea of using the exponential, gamma, and Weibull models because of their poor results and concluded that one would need to use a model that is flexible enough in its structure. These results encouraged the researchers to propose new flexible models providing greater accuracy in data fitting. Therefore, a number of approaches have been proposed to obtain new distributions with heavier tails than the exponential distribution, such as (i) transformation method [11, 12], (ii) composition of two or more distributions [13], (iii) compounding of distributions [14, 15] , and (iv) finite mixture of distributions [16, 17].

The abovementioned approaches are very useful in deriving new flexible distributions; however, they are still subject to some sort of deficiencies, for example, (i) the transformation approach is simple to apply, but its inferences become difficult and many computational work is required to derive the distributional characteristics [18]. (ii) The approach of composition of two or more distributions using a fixed or a priori known mixing weights, and hence they can be very restrictive [19]. To overcome this problem, Scollnik [20] used unrestricted mixing weights. (iii) The density obtained by the compounding approach may not always have a closed form expression which makes the estimation more cumbersome [21]. (iv) Finite mixture models represent a further approach to define very flexible distributions which are also able to capture, for instance, multimodality of the underlying distribution. The price to pay for this greater flexibility is a more complicated and computationally challenging inference [22].

To overcome the problems associated with the above former methods, many authors have proposed new families of distributions, see, for example, Al-Mofleh [23], Jamal and Nasir [24] and Nasir et al. [25], Ahmad et al. [26], Afify et al. [27], Cordeiro et al. [28], Ahmad et al. [29], Afify and Alizadeh [30], and among many others. Therefore, bringing flexibility to the existing distributions by adding additional parameter(s) is a desirable feature and an interesting research topic.

In this regard, Mudholkar and Srivastava [31] introduced the exponentiated family of distributions by adding a shape parameter to obtain more flexible version of the existing distributions. A random variable X is said to follow the exponentiated family, if its cumulative distribution function (cdf) is given by

$$G(x; a, \xi) = F(x; \xi)^a, \quad a > 0, \xi \in \mathbb{R}, x \in \mathbb{R}, \tag{2}$$

where $F(x; \xi)$ is the cdf of the baseline distribution depending on the parameter vector ξ and $a > 0$ is an additional shape parameter. Using equation (2), the exponentiated versions of the existing distributions have been proposed in the literature.

Furthermore, Cordeiro and de Castro [32] proposed another approach known as the Kumaraswamy-generalized (Ku-G) family by adding two additional shape parameters. The cdf of the Ku-G family is

$$G(x; a, b, \xi) = 1 - \{1 - F(x; \xi)^a\}^b, \quad a, b, > 0, \xi \in \mathbb{R}, x \in \mathbb{R}. \tag{3}$$

From equation (3), it is clear that, for $b = 1$, the Ku-G family reduces to the exponentiated family. For a contributed work based on equation (3), we refer to Ahmad et al. [33], Mead and Afify [34], Afify et al. [35], and Mansour et al. [36].

In this paper, we enrich the branch of distribution theory by introducing the heavy tailed arcsine exponentiated-X (ASE-X) family of distributions. A random variable X belongs to the proposed ASE-X family if its cdf is

$$G(x; a, \xi) = \frac{2}{\pi} \text{arcsine}\left(F(x; \xi)^a\right), \quad a > 0, \xi \in \mathbb{R}, x \in \mathbb{R}, \tag{4}$$

where $F(x; \xi)$ is the baseline cdf with a parameter vector ξ and an additional shape parameter a.

The probability density function (pdf) corresponding to equation (4) is given by

$$g(x; a, \xi) = \frac{2}{\pi} \frac{a f(x; \xi) F(x; \xi)^{a-1}}{\sqrt{1 - F(x; \xi)^{2a}}}, \quad a > 0, \xi \in \mathbb{R}, x \in \mathbb{R}. \tag{5}$$

The new pdf is most tractable when $F(x; \xi)$ and $f(x; \xi)$ have simple analytical expressions. Henceforth, a random variable X with pdf equation (5) is denoted by $X \sim \text{ASE} - X(x; a, \xi)$. Moreover, the key motivations for using the ASE-X family in practice are the following:

(i) To improve the characteristics and flexibility of the existing distributions, the special models of this family can provide left-skewed, right-skewed, unimodal, reversed J-shaped and symmetric densities, and decreasing and increasing, bathtub, upside down bathtub, and reversed-J hazard rates (See Figures 1 and 2)

(ii) A very simple and convenient method of adding an additional parameter provide extended heavy tailed distributions which are very useful in modeling data form the insurance field (see Sections 6 and 7)

(iii) To introduce the extended version of a baseline distribution with closed forms for the cdf and hazard rate function (hrf), the special submodels of this family can be used in analyzing censored datasets

(iv) The special cases of the ASE-X approach is capable of modeling heavy tailed datasets in actuarial science as compared with existing competing models (see Sections 6 and 7).

Using the new cdf in equation (4), a number of new flexible distributions can be obtained. Some new contributed models based on the ASE-X approach are presented in Table 1.

The survival function (sf) and hrf of the proposed family are, respectively, given by

$$S(x; a, \xi) = 1 - \frac{2}{\pi} \text{arcsine}\left(F(x; \xi)^a\right), \quad a > 0, \xi \in \mathbb{R}, x \in \mathbb{R},$$

$$h(x; a, \xi) = \frac{a f(x; \xi) F(x; \xi)^{a-1}}{(\pi/2) - \text{arcsine}\left(F(x; \xi)^a\right)\left(\sqrt{1 - F(x; \xi)^{2a}}\right)}, \quad a > 0,\ \xi \in \mathbb{R},\ x \in \mathbb{R}. \tag{6}$$

The paper is outlined as follows. In Section 2, we define the ASE-W distribution and present some plots for its density and hazard functions. We provide some

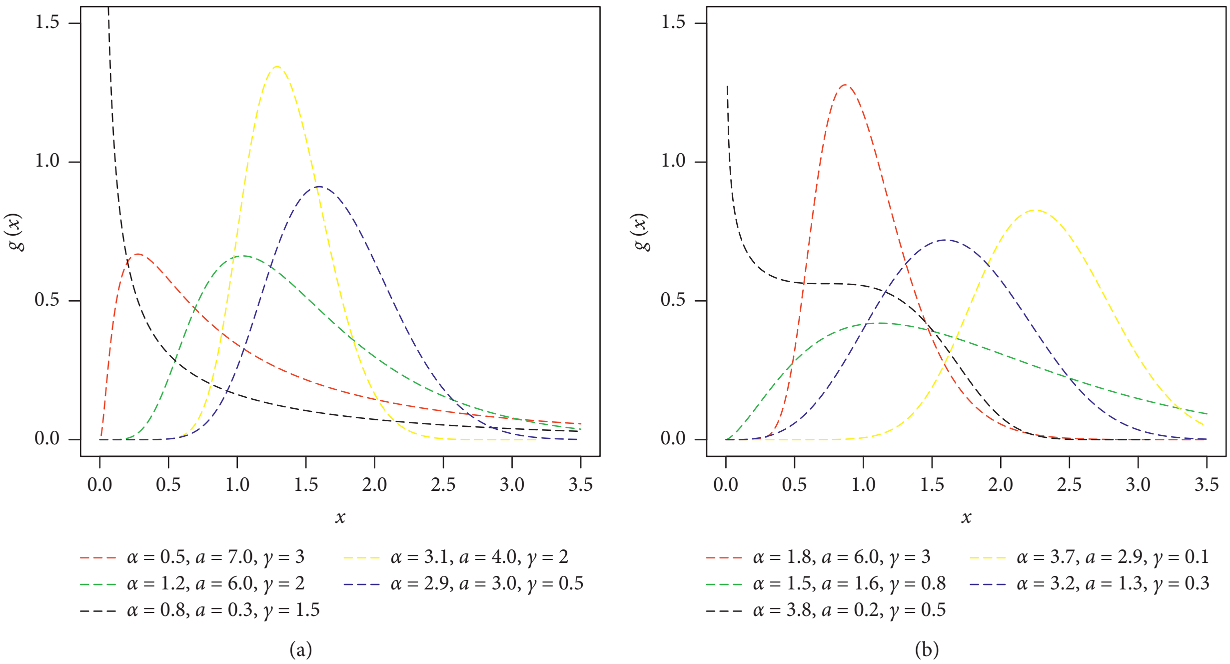


FIGURE 1: Different plots for the pdf of the ASE-W distribution for selected values of its parameters.

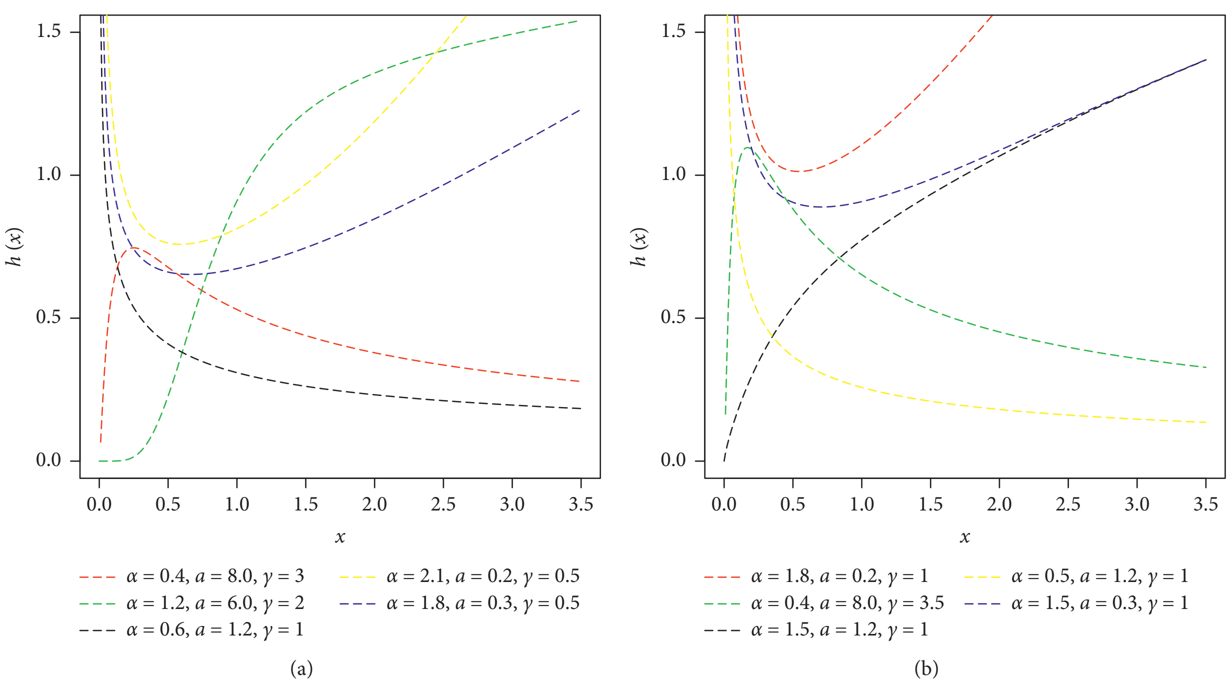


FIGURE 2: Different plots for the hrf of the ASE-W distribution for selected values of its parameters.

mathematical properties of the ASE-X distribution in Section 3. The maximum likelihood estimators (MLEs) of the model parameters are obtained in Section 4. Two Monte Carlo simulation studies to assess the performance of the MLEs are discussed in Section 5. In Section 6, we derive two important risk measures called value at risk and tail value at risk of the ASE-W distribution and perform a simulation study to prove that the ASE-W distribution has a heavier tail than the baseline Weibull distribution. In Section 7, the ASE-W distribution is applied to a real heavy tailed insurance claims data to illustrate its potentiality. Furthermore, the value at risk and tail value at risk measures are estimated for all competing models based on the insurance claims data. A modified goodness-of-fit test using a Nikulin–Rao–Robson statistic test is presented in Section 8. Finally, in Section 9, we provide some concluding remarks.

Table 1: New contributed submodels based on the ASE-X family.

No.	Baseline model	Distribution function	Generated model	Range
1	Weibull	$(2/\pi)\text{arcsine}(\{1-e^{-\gamma x^{\alpha}}\}^{a})$	ASE-Weibull	$x\in\mathbb{R}^{+}, a, \alpha, \gamma>0$
2	Lomax	$(2/\pi)\text{arcsine}(\{1-(1+\gamma x)^{-\alpha}\}^{a})$	ASE-Lomax	$x\in\mathbb{R}^{+}, a, \alpha, \gamma>0$
3	Uniform	$(2/\pi)\text{arcsine}(\{x/\theta\}^{a})$	ASE-Uniform	$0<x<\theta, a, \theta>0$
4	Linear failure rate	$(2/\pi)\text{arcsine}(\{1-e^{-\gamma x^{\alpha}-\theta x}\}^{a})$	ASE-Linear failure rate	$x\in\mathbb{R}^{+}, a, \alpha, \gamma, \theta>0$
5	Exponential	$(2/\pi)\text{arcsine}(\{1-e^{-\gamma x}\}^{a})$	ASE-Exponential	$x\in\mathbb{R}^{+}, a, \gamma>0$
6	Rayleigh	$(2/\pi)\text{arcsine}(\{1-e^{-\gamma x^{2}}\}^{a})$	ASE-Rayleigh	$x\in\mathbb{R}^{+}, a, \gamma>0$
7	Pareto	$(2/\pi)\text{arcsine}(\{1-(x_m/x)^{\alpha}\}^{a})$	ASE-Pareto	$x\in[x_m,\infty), a, \alpha, x_m>0$
8	Burr	$(2/\pi)\text{arcsine}(\{1-(1+x^{k})^{-c}\}^{a})$	ASE-Burr	$x\in\mathbb{R}^{+}, a, c, k>0$
9	Topp Leone	$(2/\pi)\text{arcsine}(\{x^{\alpha}(2-x^{\alpha})\}^{a})$	ASE-Topp Leone	$0<x<1, a, \alpha>0$
10	Log logistics	$(2/\pi)\text{arcsine}(\{1/1+(x/\gamma)^{-\alpha}\}^{a})$	ASE-Log logistics	$x\in\mathbb{R}^{+}, a, \alpha, \gamma>0$
11	Kumaraswamy	$(2/\pi)\text{arcsine}(\{1-(1-x^{a})^{b}\}^{a})$	ASE-Kumaraswamy	$0<x<1, a, b>0$
12	Frechet	$(2/\pi)\text{arcsine}(\{e^{-(x/\gamma)^{-\alpha}}\}^{a})$	ASE-Frechet	$x\in\mathbb{R}^{+}, a, \alpha, \gamma>0$
13	Gamma	$(2/\pi)\text{arcsine}(\{(1/\Gamma\alpha)\gamma(\alpha,\beta x)\}^{a})$	ASE-Gamma	$x\in\mathbb{R}^{+}, a, \alpha, \beta>0$
14	Lindely	$(2/\pi)\text{arcsine}(\{1-((e^{-\theta x}(1+\theta+\theta x))/(1+\theta))\}^{a})$	ASE-Lindely	$x\in\mathbb{R}^{+}, a, \theta>0$
15	Beta	$(2/\pi)\text{arcsine}(\{I_x(a,b)\}^{a})$	ASE-Beta	$a<x<b, a, b>0$
16	Normal	$(2/\pi)\text{arcsine}(\{\Phi(((x-\mu)/\sigma))\}^{a})$	ASE-Normal	$x, \mu\in\mathbb{R}, a, \sigma>0$
17	Gumbel	$(2/\pi)\text{arcsine}(\{e^{-e^{-(x-\mu/\sigma)}}\}^{a})$	ASE-Gumbel	$x, \mu\in\mathbb{R}, a, \sigma>0$
18	Power function	$(2/\pi)\text{arcsine}(\{(x/\gamma)^{\alpha}\}^{a})$	ASE-Power function	$0<x<\gamma, a, \alpha, \gamma>0$
19	Half logistic	$(2/\pi)\text{arcsine}(\{(1-e^{-x})/(1+e^{-x})\}^{a})$	ASE-Half logistic	$x\in\mathbb{R}^{+}, a>0$
20	Erlang	$(2/\pi)\text{arcsine}(\{(1/(k-1)!)\gamma(k,\lambda x)\}^{a})$	ASE-Erlang	$x\in\mathbb{R}^{+}, a, k, \lambda>0$
21	Lévy	$(2/\pi)\text{arcsine}(\{erfc(\sqrt{\sigma/2(x-\mu)})\}^{a})$	ASE-Lévy	$x\in[\mu,\infty), \mu\in\mathbb{R}, \sigma, a>0$
22	Rice	$(2/\pi)\text{arcsine}(\{1-Q_1(\nu/\sigma, x/\sigma)\}^{a})$	ASE-Rice	$x\in\mathbb{R}^{+}, a, \nu, \sigma>0$
23	Shifted Gompertz	$(2/\pi)\text{arcsine}(\{(1-e^{-bx})e^{-\eta e^{-bx}}\}^{a})$	ASE-shifted Gompertz	$x\in\mathbb{R}^{+}, a, b, \eta>0$
24	Dagum	$(2/\pi)\text{arcsine}(\{\{1+(x/\beta)^{-\alpha}\}^{-p}\}^{a})$	ASE-Dagum	$x\in\mathbb{R}^{+}, a, \alpha, \beta, p>0$
25	Beta prime	$(2/\pi)\text{arcsine}(\{I_{(x/1+x)}(\alpha,\beta)\}^{a})$	ASE-Beta prime	$x\in\mathbb{R}^{+}, a, \alpha, \beta>0$
26	Logistic	$(2/\pi)\text{arcsine}(\{1/1+e^{-(x-\mu/\sigma)}\}^{a})$	ASE-Logistic	$x, \mu\in\mathbb{R}, a, \sigma>0$
27	Reciprocal	$(2/\pi)\text{arcsine}(\{(\log_e(x)-\log_e(\alpha))/(\log_e(\beta)-\log_e(\alpha))\}^{a})$	ASE-Reciprocal	$x\in[\alpha,\beta], a, \alpha, \beta>0$
28	Gompertz	$(2/\pi)\text{arcsine}(\{1-e^{-\eta(e^{bx}-1)}\}^{a})$	ASE-Gompertz	$x\in\mathbb{R}^{+}, a, \eta, b>0$
29	Hyperbolic secant	$(2/\pi)\text{arcsine}(\{(2/\pi)\arctan\{e^{(x\pi/2)}\}\}^{a})$	ASE-Hyperbolic secant	$x\in\mathbb{R}, a>0$

2. The ASE-W Distribution

In this section, we introduce the ASE-W distribution and investigate the behavior of its density and hazard functions, for selected values of the parameters. Consider the cdf of the two-parameter Weibull distribution, $F(x;\alpha,\gamma)=1-e^{-\gamma x^{\alpha}}, x\geq 0, \alpha, \gamma>0$. Then, a random variable X is said to follow the ASE-W distribution if its cdf takes the form

$$G(x;\alpha,a,\gamma)=\frac{2}{\pi}\text{arcsine}\left(\left(1-e^{-\gamma x^{\alpha}}\right)^{a}\right),\quad x>0, \alpha, a, \gamma>0. \tag{7}$$

The pdf associated of equation (7) has the form

$$g(x;\alpha,a,\gamma)=\frac{2}{\pi}\frac{a\alpha\gamma x^{\alpha-1}e^{-\gamma x^{\alpha}}\left(1-e^{-\gamma x^{\alpha}}\right)^{a-1}}{\sqrt{1-\left(1-e^{-\gamma x^{\alpha}}\right)^{2a}}}, \quad x>0, \alpha, a, \gamma>0. \tag{8}$$

For $\alpha=1$, the ASE-W distribution reduces to the ASE-exponential distribution with parameter γ, and for $\alpha=2$, it reduces to the ASE-Rayleigh distribution with parameter γ.

For different values of the model parameters, plots of the pdf and hrf of the ASE-W distribution are sketched in

Figures 1 and 2. The two figures reveal that the ASE-W can provide left-skewed, right-skewed, unimodal, reversed J-shaped and symmetric densities, and decreasing and increasing, bathtub, upside down bathtub, and reversed-J hazard rate shapes.

3. Basic Mathematical Properties

In this section, some statistical properties of the ASE-X family are derived.

3.1. Quantile Function. Let X be the ASE-X random variable with pdf equation (5), the quantile function (qf) of X, say $Q(u)$, reduces to

$$x_q = Q(u) = G^{-1}(u) = F^{-1}\left\{\left[\sin\left(\frac{\pi}{2}u\right)\right]^{(1/a)}\right\}, \quad (9)$$

where u has the uniform distribution on the interval (0, 1). From the expression in equation (9), it is clear that the ASE-X family has a closed form solution of its quantile function which makes generating random numbers very simple.

The qf of the ASE-W model follows as

$$Q_{\text{ASE-W}}(u) = \left(\frac{-1}{\gamma}\log\left\{1-\left[\sin\left(\frac{\pi}{2}u\right)\right]^{(1/a)}\right\}\right)^{(1/\alpha)}. \quad (10)$$

3.2. Moments. Moments are very important and play an essential role in statistical analysis. They help to capture important features and characteristics of the distribution (e.g., central tendency, dispersion, skewness, and kurtosis). The r^{th} moment of the ASE-X family is

$$\mu_r' = \int_{-\infty}^{\infty} x^r g(x; a, \xi)\mathrm{d}x. \quad (11)$$

Substituting equation (5) in equation (11), we obtain

$$\mu_r' = \frac{2a}{\pi}\int_{-\infty}^{\infty} x^r \frac{f(x;\xi)(F(x;\xi))^{a-1}}{\sqrt{1-(F(x;\xi)^a)^2}}\mathrm{d}x. \quad (12)$$

Using the binomial expansion, we have

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{n!2^n}x^{2n}. \quad (13)$$

By replacing x with $F(x;\xi)^a$, in equation (13), we obtain

$$\frac{1}{\sqrt{1-(F(x;\xi)^a)^2}} = \sum_{n=0}^{\infty}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{n!2^n}F(x;\xi)^{2an}. \quad (14)$$

By inserting equation (14) in equation (12), we obtain

$$\mu_r' = \frac{a}{\pi}\sum_{n=0}^{\infty}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{n!2^{n-1}}\eta_{r,2an}, \quad (15)$$

where $\eta_{r,2an} = \int_{-\infty}^{\infty} x^r f(x;\xi)F(x;\xi)^{a(2n+1)-1}\mathrm{d}x$.

The moment generating function of the ASE-X class has the form

$$M_X(t) = \frac{a}{\pi}\sum_{r,n=0}^{\infty}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{r!n!2^{n-1}}t^r\eta_{r,2an}. \quad (16)$$

The effects of different values of the parameters α and a on the mean, variance, skewness, and kurtosis of the ASE-W distribution with $\gamma = 1$ are illustrated in Figures 3 and 4.

4. Maximum Likelihood Estimation

In this section, we consider the estimation of the unknown parameters of the ASE-X distribution from complete samples only via the maximum likelihood. Let $X_1, X_2, \ldots, X_n$ be a random sample from the ASE-X family with observed values $x_1, x_2, \ldots, x_n$. The log-likelihood function is

$$\log L(a,\xi) = n\log\left(\frac{2}{\pi}\right) + n\log(a) + \sum_{i=1}^{n}\log f(x_i;\xi) + (a-1)\sum_{i=1}^{n}\log F(x_i;\xi) - \frac{1}{2}\sum_{i=1}^{n}\log\left\{1-F(x;\xi)^{2a}\right\}. \quad (17)$$

The log-likelihood function can be maximized either directly by using the R (AdequecyModel package), SAS (PROC NLMIXED), or the Ox program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations which are obtained by differentiating equation (17) as follows:

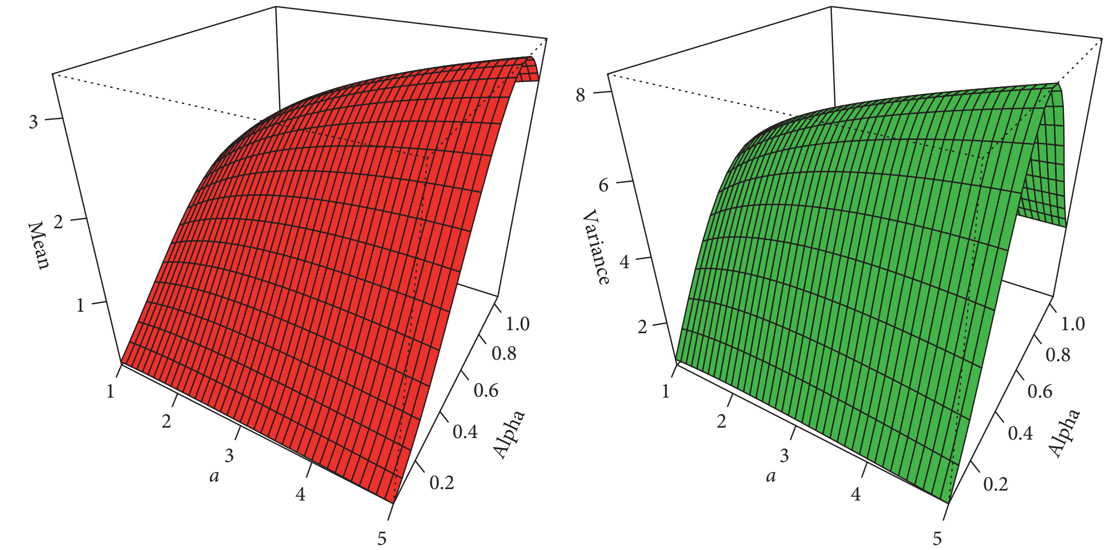
$$\frac{\partial}{\partial a}\log L(a,\xi) = \frac{n}{a} + \sum_{i=1}^{n}\log F(x_i;\xi) + \sum_{i=1}^{n}\left\{\frac{(\log F(x_i;\xi))F(x;\xi)^{2a}}{1-F(x;\xi)^{2a}}\right\},$$

$$\frac{\partial}{\partial \xi}\log L(a,\xi) = \sum_{i=1}^{n}\frac{(\partial F(x_i;\xi)/\partial\xi)}{f(x_i;\xi)} + (a-1)\sum_{i=1}^{n}\frac{(\partial F(x_i;\xi)/\partial\xi)}{F(x_i;\xi)} + a\sum_{i=1}^{n}\left\{\frac{F(x;\xi)^{a-1}(\partial F(x_i;\xi)/\partial\xi)}{1-F(x;\xi)^{2a}}\right\}. \quad (18)$$

The log-likelihood function for the ASE-W model reduces to

$$\ell = n\log\left(\frac{2}{\pi}\right) + n\log(\alpha a\gamma) + (\alpha-1)\sum_{i=0}^{n}\log(x_i) - \gamma\sum_{i=0}^{n}x_i^{\alpha} + (a-1)\sum_{i=0}^{n}\log\left(1-e^{-\gamma x_i^{\alpha}}\right) - \frac{1}{2}\sum_{i=0}^{n}\log\left[1-\left(1-e^{-\gamma x_i^{\alpha}}\right)^{2a}\right]. \quad (19)$$

The nonlinear likelihood equations can be obtained by differentiating the last equation as follows:



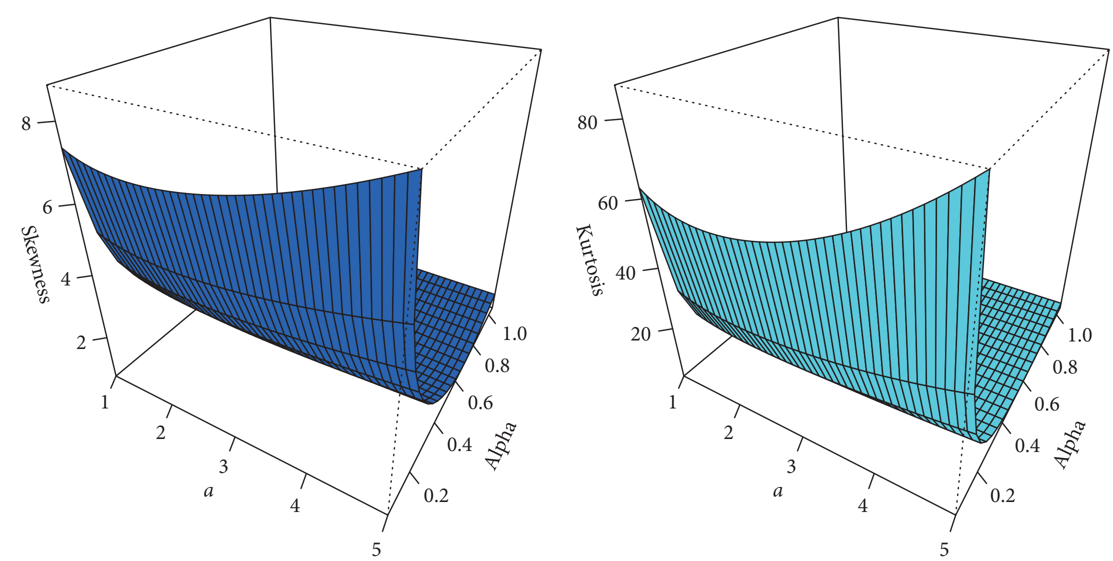
Mean

(a)

Variance

(b)

Figure 3: The mean and variance plots of the ASE-W model.



Skewness

(a)

Kurtosis

(b)

Figure 4: The skewness and kurtosis plots of the ASE-W model.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=0}^{n} \log(x_i) - \gamma \sum_{i=0}^{n} x_i^{\alpha} \log(x_i) + \gamma(a-1) \sum_{i=0}^{n} \frac{x_i^{\alpha} e^{-\gamma x_i^{\alpha}} \log(x_i)}{1 - e^{-\gamma x_i^{\alpha}}}$$

$$+ \gamma a \sum_{i=0}^{n} \frac{x_i^{\alpha} e^{-\gamma x_i^{\alpha}} \left(1 - e^{-\gamma x_i^{\alpha}}\right)^{2a-1} \log(x_i)}{1 - \left(1 - e^{-\gamma x_i^{\alpha}}\right)^{2a}},$$

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} + \sum_{i=0}^{n} \log\left(1 - e^{-\gamma x_i^{\alpha}}\right) + \sum_{i=0}^{n} \frac{\left(1 - e^{-\gamma x_i^{\alpha}}\right)^{2a} \log\left(1 - e^{-\gamma x_i^{\alpha}}\right)}{1 - \left(1 - e^{-\gamma x_i^{\alpha}}\right)^{2a}}, \tag{20}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} - \sum_{i=0}^{n} x_i^{\alpha} + (a-1) \sum_{i=0}^{n} \frac{x_i^{\alpha} e^{-\gamma x_i^{\alpha}}}{1 - e^{-\gamma x_i^{\alpha}}} + a \sum_{i=0}^{n} \frac{x_i^{\alpha} e^{-\gamma x_i^{\alpha}} \left(1 - e^{-\gamma x_i^{\alpha}}\right)^{2a-1}}{1 - \left(1 - e^{-\gamma x_i^{\alpha}}\right)^{2a}}.$$

5. Simulation Results

5.1. Monte Carlo Simulation Study. In this section, we perform a comprehensive simulation study to access the behavior of MLEs of the ASE-W parameters. The random number generation is obtained via the inverse cdf. The inverse process and results of MLEs are obtained using `optim()` R-function with the argument `method L-BFGS-B`. We generate $N = 1000$ samples of size $n = 25$, 100, 300, 600, 900, 1000 from the ASE-W distribution with true parameter values. In this simulation study, we empirically calculate the mean, bias, and mean square error (MSE) of the MLEs for different parameters combinations and each sample.

Coverage probabilities (CPs) are also calculated at the 95% confidence interval (C.I.). The simulation results are provided in Tables 2 and 3. Based on the generated data listed in Tables 2 and 3, the MLEs seem to behave as we expect, that is, the MSE values and the estimated biases decrease as n increases. Furthermore, the mean values of estimates tend to the true values as n increases, showing the consistency property of the MLEs.

5.2. Simulations Using the Barzilai-Borwein Algorithm. In this section, we provide the results of a simulation study for the ASE-W distribution using the Barzilai-Borwein (BB) algorithm [37]. Initial values for the parameters ($\alpha = 1.6$, $\gamma = 0.6$, and $a = 1.9$) are selected and random sample of sizes $n = 50$, 100, 200, and 400 are obtained. Repetitions are made 10,000 times and the averages of the simulated values of the MLEs ($\hat{\alpha}$, $\hat{\gamma}$, $\hat{a}$) along with their MSEs are calculated. The simulation results are provided in Table 4.

From the simulation results provided in Table 4, we can see that the maximum likelihood estimates of the ASE-W parameters are convergent. The graphical sketching of the maximum likelihood estimates of the ASE-W parameters is provided in Figure 5.

From Figure 5, it is clear that all the parameters estimates of the ASE-W distribution converge faster than $n^{-0.5}$. Therefore, we conclude that the MLEs of the ASE-W parameters are $\sqrt{n}$ consistent.

6. Actuarial Measures

One of the most important tasks of financial and actuarial sciences institutions is to evaluate the exposure to market risk in a portfolio of instruments, which arise from changes in underlying variables such as prices of equity, interest rates, or exchange rates. In this section, we derive some important risk measures including value at risk (VaR) and tail value at risk (TVaR) of the ASE-W distribution which play a crucial role in portfolio optimization under uncertainty.

6.1. Value at Risk. The VaR is widely used by practitioners as a standard financial market risk measure. It is also called the quantile premium principle or quantile risk measure. The VaR is always specified with a given degree of confidence say q (typically 90%, 95% or 99%), and it represents the percentage loss in portfolio value that will be equaled or exceeded only X percent of the time. The VaR of a random variable X is the qth quantile of its cdf [38]. Hence, the VaR of the ASE-W distribution is defined as

$$x_q = \left(\frac{-1}{\gamma} \log\left\{1 - \left[\sin\left(\frac{\pi}{2} q\right)\right]^{(1/a)}\right\}\right)^{(1/\alpha)}. \tag{21}$$

6.2. Tail Value at Risk. Another important measure is TVaR which can be used to quantify the expected value of the loss given that an event outside a given probability level has occurred. If X follows the ASE-W distribution, then its TVaR can be defined as

$$\mathrm{TVaR}_q(X) = \frac{1}{1-q} \int_{\mathrm{VaR}_q}^{\infty} x\, g(x; \alpha, a, \gamma) \mathrm{d}x. \tag{22}$$

Substituting equation (8) in equation (22), we obtain

$$\mathrm{TVaR}_q(X) = \frac{1}{1-q} \int_{\mathrm{VaR}_q}^{\infty} \frac{2}{\pi} \frac{a\alpha\gamma x^{\alpha+1-1} e^{-\gamma x^{\alpha}} \left(1 - e^{-\gamma x^{\alpha}}\right)^{a-1}}{\sqrt{1 - \left(1 - e^{-\gamma x^{\alpha}}\right)^{2a}}}. \tag{23}$$

Finally, the TVaR of the ASE-W model takes the form

Table 2: MLE, Bias, MSE, C.I., and CPs of the ASE-W parameters.

		Set 1: $\alpha = 1.4$, $a = 0.9$, $\gamma = 1.2$					
n	Par	MLE	Bias	MSE	C.I.		CPs
25	α	1.7754	0.2788	0.0975	(0.8441	2.1346)	0.8324
	a	1.4123	0.3586	0.6653	(−2.1895	3.3467)	0.8235
	γ	1.8143	0.4786	1.4876	(1.2242	3.6998)	0.8054
100	α	1.6234	0.2167	0.0763	(1.0546	1.8659)	0.8609
	a	1.1899	0.3074	0.5678	(1.7834	3.2089)	0.8542
	γ	1.7758	0.3875	1.2788	(−1.0437	3.4596)	0.8378
300	α	1.5865	0.1976	0.0598	(1.1547	1.7598)	0.8847
	a	1.1079	0.2195	0.3178	(−0.3045	2.2789)	0.8965
	γ	1.6734	0.2765	0.8977	(−0.3546	2.5694)	0.8568
600	α	1.5189	0.1157	0.0376	(1.1654	1.6598)	0.9168
	a	1.0857	0.1834	0.1287	(0.1267	1.6235)	0.9189
	γ	1.5967	0.1865	0.4978	(−0.0956	2.2089)	0.8977
900	α	1.4856	0.1065	0.0254	(1.2908	1.5647)	0.9285
	a	0.9578	0.1004	0.1087	(0.2263	1.3980)	0.9289
	γ	1.3087	0.1267	0.3856	(0.0633	2.1744)	0.9167
1000	α	1.4034	0.0659	0.0187	(1.2734	1.5980)	0.9385
	a	0.9134	0.0288	0.0755	(0.3748	1.3850)	0.9376
	γ	1.2176	0.0775	0.1295	(0.1865	2.0734)	0.9289

Table 3: MLE, Bias, MSE, C.I., and CPs of the ASE-W parameters.

		Set 2: $\alpha = 0.7$, $a = 0.5$, $\gamma = 1.5$					
n	Par	MLE	Bias	MSE	C.I.		CPs
25	α	0.9958	0.1058	0.0478	(0.4998	1.3678)	0.8619
	a	0.8532	0.2008	0.2979	(1.0536	2.6758)	0.8390
	γ	2.741	0.3589	2.976	(1.8797	5.6978)	0.7955
100	α	0.8624	0.0473	0.0342	(0.6053	1.0857)	0.8754
	a	0.7953	0.1853	0.2865	(0.0198	1.7986)	0.8648
	γ	2.1455	0.2653	2.1387	(1.0197	3.5585)	0.8250
300	α	0.8108	0.0304	0.0287	(0.3657	0.9893)	0.8993
	a	0.7275	0.0898	0.2064	(0.0456	1.2238)	0.8829
	γ	1.9753	0.1890	1.4879	(0.1608	2.3839)	0.8434
600	α	0.7958	0.0264	0.0187	(0.7056	0.9492)	0.9202
	a	0.6552	0.0743	0.1876	(0.1489	1.0073)	0.9053
	γ	1.8209	0.1289	0.9562	(0.1535	1.8298)	0.8794
900	α	0.7624	0.0205	0.0137	(0.7056	0.9154)	0.9301
	a	0.5953	0.0465	0.1067	(0.2063	0.8909)	0.9178
	γ	1.7578	0.1056	0.5634	(0.2682	1.8056)	0.8973
1000	α	0.7136	0.0135	0.0098	(0.6198	0.9276)	0.9390
	a	0.5393	0.0289	0.0478	(0.2053	0.9254)	0.9325
	γ	1.5443	0.0189	0.1590	(0.2759	1.7450)	0.9289

Table 4: Maximum likelihood estimates of the parameters (α, γ, a) and their MSEs.

$N = 10000$	$n = 50$	$n = 100$	$n = 200$	$n = 400$
$\widehat{\alpha}$	1.64358	1.63408	1.61008	1.60447
MSE	0.04518	0.03371	0.02456	0.01048
$\widehat{\gamma}$	0.64992	0.64001	0.62115	0.60487
MSE	0.04168	0.03336	0.023007	0.00917
$\widehat{a}$	1.94253	1.93394	1.91108	1.90462
MSE	0.04347	0.03754	0.02647	0.01024

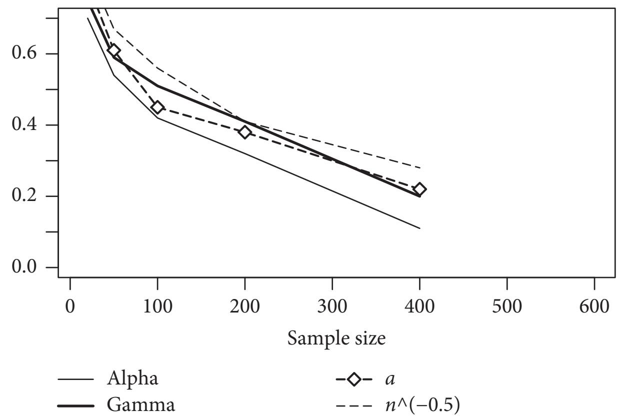


Figure 5: Simulated average absolute errors for MLEs ($\hat{\alpha}$, $\hat{\gamma}$, $\hat{a}$).

Table 5: Simulation results of the actuarial measures at different levels of significance for $\alpha = 1.9$, $\gamma = 0.4$, and $a = 0.9$.

Distribution	Parameters	Level of significance	VaR	TVaR
Weibull	$\alpha = 1.9$, $\gamma = 0.4$	0.700	1.7908	2.4109
		0.750	1.9288	2.5214
		0.800	2.0864	2.6503
		0.850	2.2750	2.8080
		0.900	2.5192	3.0166
		0.950	2.8934	3.3446
		0.975	3.2284	3.6448
		0.999	4.4913	4.8138
ASE-W	$\alpha = 1.9$, $a = 0.9$, $\gamma = 0.4$	0.700	2.3757	5.2320
		0.750	2.6831	5.7737
		0.800	3.0850	6.4986
		0.850	3.6544	7.5479
		0.900	4.5787	9.2856
		0.950	6.5963	13.1515
		0.975	9.3793	18.5495
		0.999	45.9854	90.2646

$$\operatorname{TVaR}_q(X) = \frac{a}{(1-q)\pi} \sum_{n,i=0}^{\infty} (-1)^i \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{n! 2^{n-1} \gamma^{(1/\alpha)} (i+1)^{(1/\alpha)+1}} \cdot \binom{a(2n-1)-1}{i} \times \Gamma\left(\frac{1}{\alpha}+1, \gamma(i+1)\left(\operatorname{VaR}_q\right)^a\right). \tag{24}$$

6.3. Numerical Study of the Actuarial Measures. In this section, we provide some numerical results for the VaR and TVaR for the Weibull and ASE-W distributions for different sets of parameters. The process is described below:

(i) Random sample of size $n = 150$ are generated from the Weibull and ASE-W distributions and parameters have been estimated via the maximum likelihood method.

(ii) 1000 repetitions are made to calculate the VaR and TVaR of the two distributions.

The simulation results of the VaR and TVaR for the Weibull and ASE-W models are provided in Tables 5 and 6. Furthermore, the results in these tables are depicted graphically in Figures 6 and 7, respectively.

The simulation is performed for the Weibull and ASE-W distributions for selected values of their parameters. A model with higher values of VaR and TVaR is said to have a heavier tail. The simulated results in Tables 5 and 6 and the plots in Figures 6 and 7 show that the proposed ASE-W model has higher values of these risk measures than the Weibull model. Hence, the proposed ASE-W model has a heavier tail than the Weibull distribution and can be used effectively to model heavy tailed insurance data.

7. Modeling Heavy Tailed Insurance Claims Data

In this section, we demonstrate the flexibility of the ASE-W distribution by using heavy tailed insurance claims data. Furthermore, we calculate the actuarial measures of the

Table 6: Simulation results of the actuarial measures at different levels of significance for $\alpha = 0.9$, $\gamma = 0.4$, and $a = 1.5$.

Distribution	Parameters	Level of significance	VaR	TVaR
Weibull	$\alpha = 0.9,\ \gamma = 0.4$	0.700	3.3752	6.6801
		0.750	3.9477	7.2857
		0.800	4.6598	8.0348
		0.850	5.5940	9.0120
		0.900	6.9373	10.4090
		0.950	9.2934	12.8434
		0.975	11.7115	15.3275
		0.999	23.5140	27.3507
ASE-W	$\alpha = 0.9,\ a = 1.5,\ \gamma = 0.4$	0.700	7.8442	17.6596
		0.750	8.8805	19.5232
		0.800	10.2387	22.0226
		0.850	12.1687	25.6504
		0.900	15.3140	31.6808
		0.950	22.2204	45.1725
		0.975	31.8140	64.1411
		0.999	160.9151	321.9320

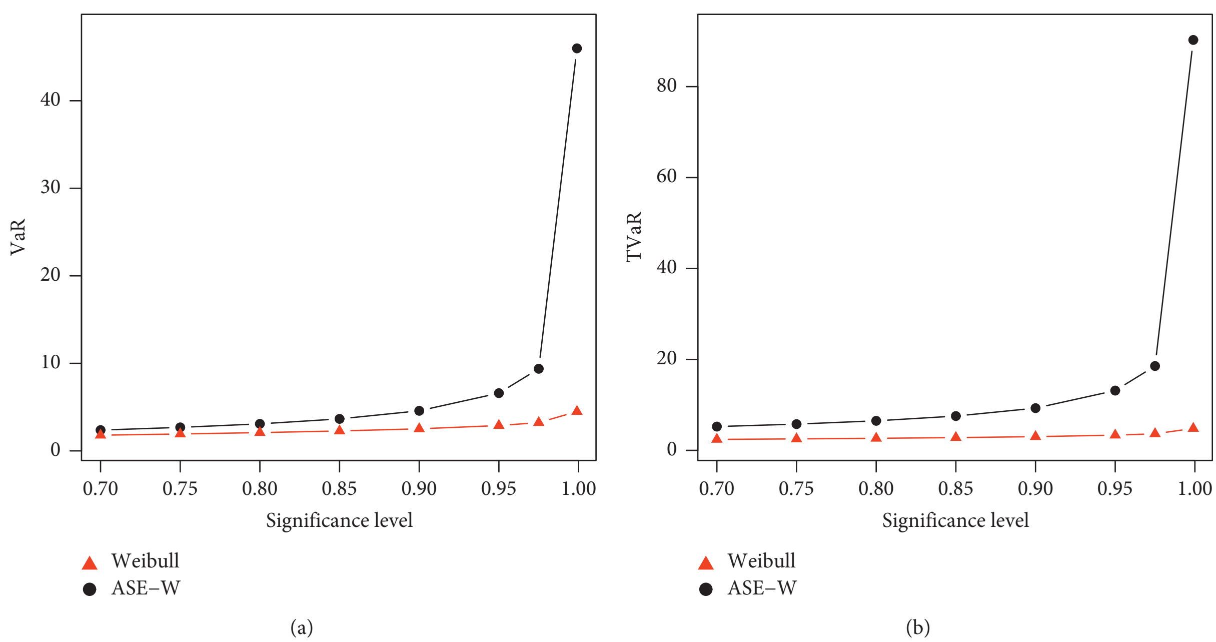


Figure 6: Graphical sketching of the VaR and TVaR using the results in Table 5.

ASE-W and other competing distributions using this real dataset.

7.1. Application of the ASE-W Distribution to Insurance Claims Data. In this section, we consider a dataset from insurance field. This data set represents the unemployment insurance initial claims per month from 1971 to 2018, and it is available at https://data.worlddatany-govns8z-xewg. We compare the goodness-of-fit results of the proposed distribution with some other well-known competing distributions including Weibull, exponentiated exponential (EE), exponentiated Weibull (EW), exponentiated Lomax (EL), Kumaraswamy Weibull (Ku-W), beta Weibull (BW), and new Weibull Burr-XII (NWB-XII) distributions. The distribution functions of these competitive distributions are given by

(1) Weibull distribution:

$$G(x;\alpha,\gamma) = 1 - e^{-\gamma x^{\alpha}}, \quad x \ge 0, \alpha, \gamma > 0. \tag{25}$$

(2) EE distribution:

$$G(x;a,\gamma) = \left(1 - e^{-\gamma x}\right)^{a}, \quad x \ge 0, a, \gamma > 0. \tag{26}$$

(3) EW distribution:

$$G(x,a,\alpha,\gamma) = \left(1 - e^{-\gamma x^{\alpha}}\right)^{a}, \quad x \ge 0, \alpha, \gamma, a > 0. \tag{27}$$

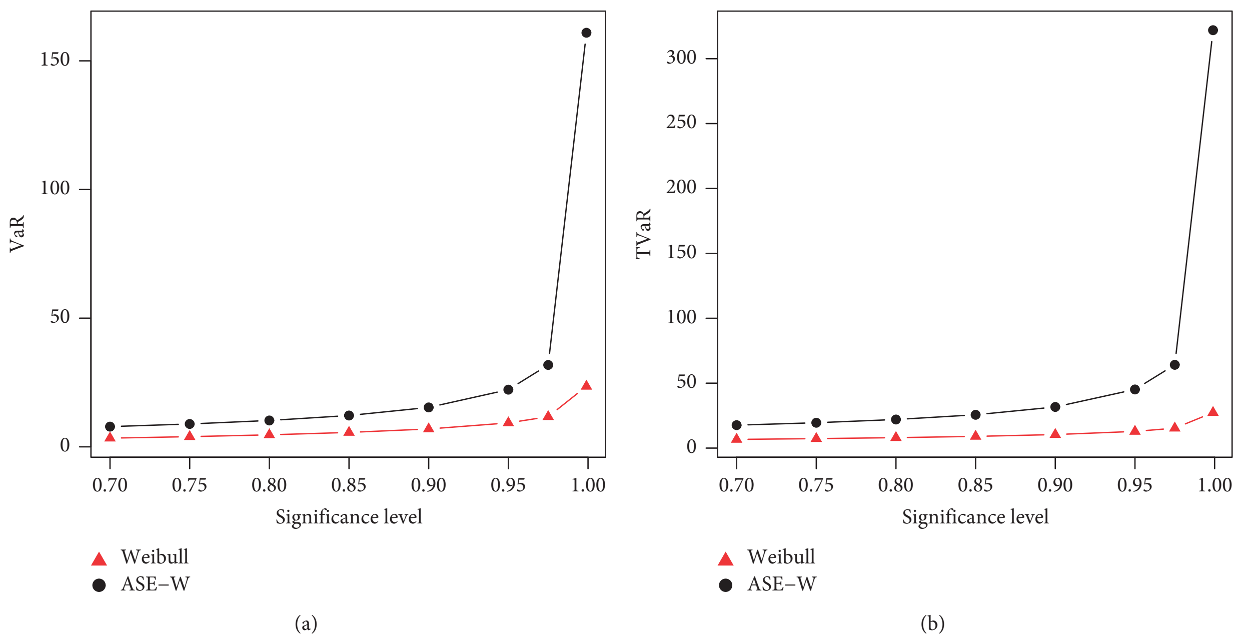


FIGURE 7: Graphical sketching of the VaR and TVaR using the results in Table 6.

(4) EL distribution:

$$G(x;\alpha,a,\gamma)=\left[1-\left(1+\frac{x}{\gamma}\right)^{-\alpha}\right]^{a},\quad x\geq 0,\alpha,a,\gamma>0. \tag{28}$$

(5) Ku-W distribution:

$$G(x;\alpha,\gamma,a,b)=1-\left[1-\left(1-e^{-\gamma x^{\alpha}}\right)^{a}\right]^{b},\quad x\geq 0,\alpha,\gamma,a,b>0. \tag{29}$$

(6) BW distribution:

$$G(x;\alpha,\gamma,a,b)=I_{\left(1-e^{-\gamma x^{\alpha}}\right)}(x;a,b),\quad x\geq 0,\alpha,\gamma,a,b>0. \tag{30}$$

(7) NWB-XII distribution:

$$G(x;\alpha,\gamma,a,b)=1-\exp\{-\gamma[b\log(1+x^{a})]^{\alpha}\},\quad x\geq 0,\alpha,\gamma,a,b>0. \tag{31}$$

The competing models can be compared using some discrimination measures called

(i) The Akaike information criterion (AIC)
(ii) The Bayesian information criterion (BIC)
(iii) The Hannan–Quinn information criterion (HQIC)
(iv) The consistent Akaike information Criterion (CAIC)

In addition to the discrimination measures, we further considered other test statistics called

(i) Anderson Darling (AD)
(ii) Cramér–von Mises (CM)
(iii) The Kolmogorov–Smirnov (KS) with its p value

The formulae for these measures can be explored in Afify et al. Table 7 gives the MLEs and their standard errors. The analytical measures are provided in Tables 8 and 9. The results in these tables indicate that the ASE-W distribution provides better fits than other competing models and could be chosen as an adequate model to analyze the heavy tailed insurance claims data.

Figure 8 displays the fitted pdf and cdf of the proposed distribution which shows that the ASE-W fits the right-skewed heavy tailed distribution very well. The probability-probability (PP) plot and and Kaplan–Meier survival plots are sketched in Figure 9.

7.2. Estimating of VaR and TVaR Measures Using the Insurance Claims Data. In this section, we compute the VaR and TVaR measures of the ASE-W and other competing distributions using the estimated values of the parameters using the insurance claims data. The numerical results for all fitted distributions are reported in Table 10. The results in Table 10 are displayed graphically in Figure 10.

As we have mentioned earlier that a distribution with higher values of the risk measures is said to has a heavier tail. The values in Table 10 and Figure 10 illustrate that the ASE-W distribution has the highest values of VaR and TVaR among all competing models, proving that it has a heavier tail than other competitors for insurance claims data.

Table 7: Estimated parameters along with standard errors (in parenthesis) of the fitted models.

Distribution	α	γ	a	b
ASE-W	1.270 (0.0214)	0.012 (0.0014)	32.780 (4.3526)	
Weibull	1.486 (1.0966)	0.001 (0.0949)		
EE		0.037 (0.0013)	31.713 (3.7912)	
EW	0.941 (0.0500)	0.051 (0.0145)	37.938 (8.8729)	
EL	9.492 (3.9734)	149.038 (105.1275)	94.873 (61.56213)	
Ku-W	0.984 (0.0764)	0.052 (0.0172)	64.933 (0.0633)	0.657 (0.1668)
BW	1.090 (0.0589)	0.031 (0.0070)	40.704 (10.3927)	0.592 (0.1393)
NWB-XII	1.487 (0.1956)	0.238 (0.0276)	65.273 (12.5909)	2.675 (0.1067)

Table 8: Discrimination measures of the ASE-W model and other competing models.

Distribution	AIC	BIC	CAIC	HQIC
ASE-W	5598.818	5611.881	5598.860	5603.913
Weibull	6169.574	6178.290	6169.634	6172.975
EE	5603.732	5612.441	5603.753	5607.129
EW	5604.484	5617.547	5604.526	5609.579
EL	5603.581	5616.644	5603.623	5608.676
Ku-W	5601.734	5619.152	5601.805	5608.527
BW	5602.555	5602.625	5619.972	5609.348
NWB-XII	5743.756	5776.532	5746.452	5761.0897

Table 9: Goodness-of-fit measures of the ASE-W model and other competing models.

Distribution	CM	AD	KS	p value
ASE-W	0.160	0.840	0.030	0.657
Weibull	0.974	2.094	0.896	0.129
EE	0.311	1.576	0.047	0.154
EW	0.297	1.499	0.043	0.217
EL	0.220	1.217	0.046	0.174
Ku-W	0.200	1.037	0.037	0.383
BW	0.221	1.121	0.035	0.471
NWB-XII	1.409	2.498	0.985	0.116

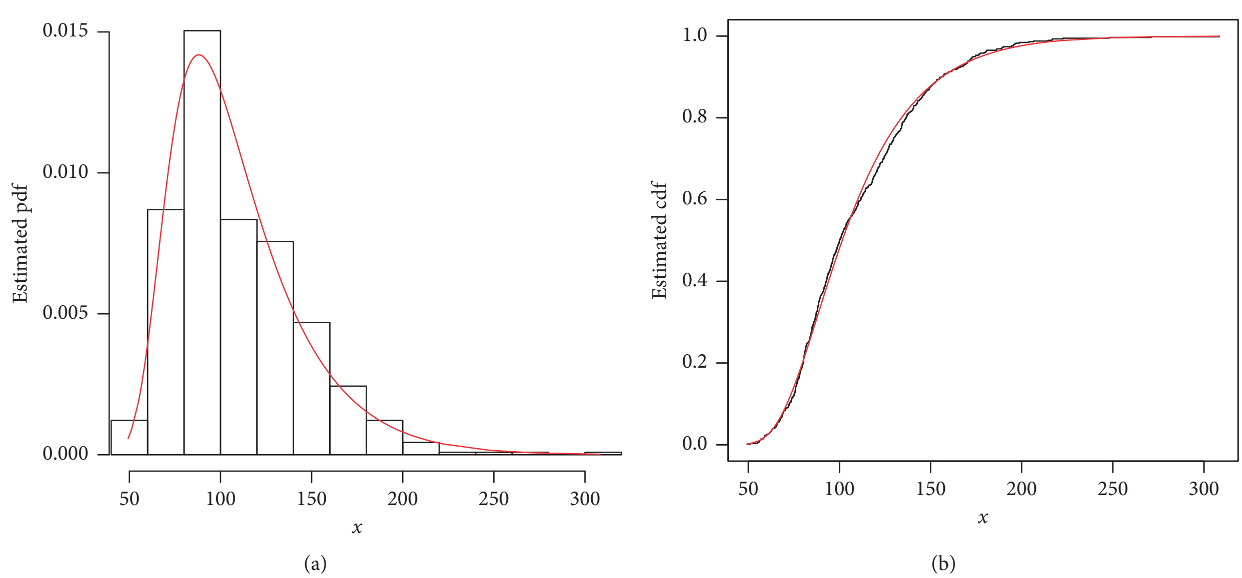


Figure 8: Estimated pdf and cdf of the ASE-W distribution.

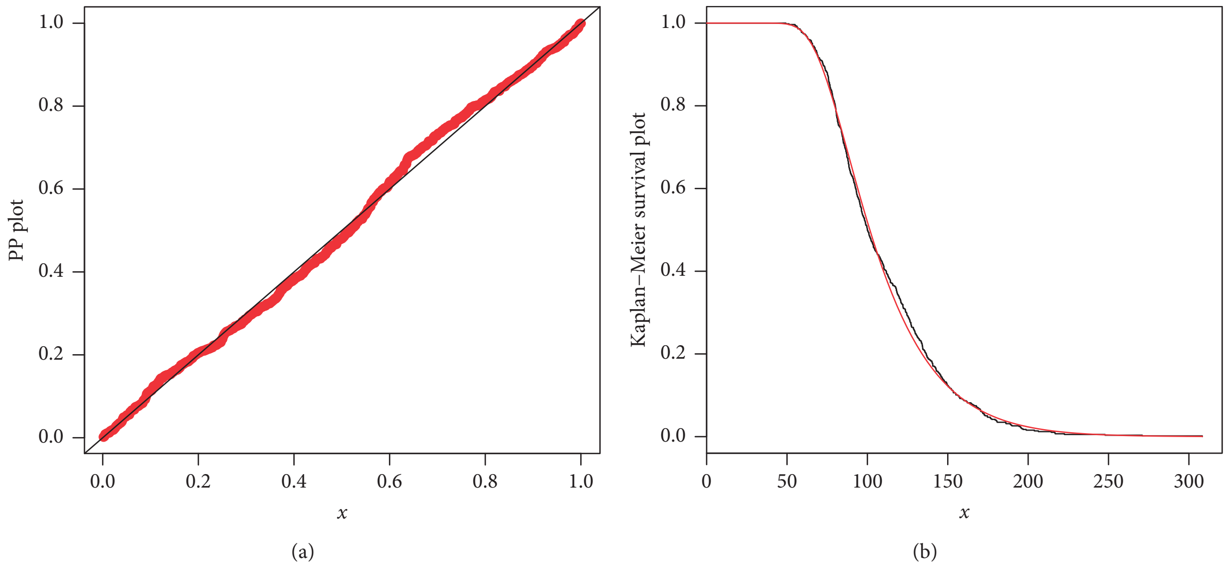


Figure 9: PP plot Kaplan–Meier survival plot of the ASE-W distribution.

Table 10: Numerical results of the VaR and TVaR for the competing models for insurance claims data.

Distribution	Parameters	Level of significance	VaR	TVaR
ASE-W	$\alpha = 1.270$, $a = 32.780$, $\gamma = 0.012$	0.700	230.0434	254.0283
		0.750	238.9876	287.5763
		0.800	251.7543	313.8245
		0.850	287.5642	370.4325
		0.900	327.0358	397.5636
		0.950	376.6542	440.4529
		0.975	478.4263	490.3465
		0.999	532.7983	620.9659
Weibull	$\alpha = 1.9$, $\gamma = 0.4$	0.700	70.3462	89.7568
		0.750	77.7643	101.8539
		0.800	83.8743	115.7943
		0.850	98.3245	135.8764
		0.900	120.5643	156.9875
		0.950	142.6750	179.9876
		0.975	176.7656	201.9875
		0.999	202.6752	230.9786
EE	$\gamma = 0.03$	0.700	86.9876	102.8654
		0.750	92.0896	121.8479
		0.800	102.8569	138.6497
		0.850	115.5865	154.4970
		0.900	131.9724	172.7590
		0.950	152.8598	198.8653
		0.975	198.8569	228.5860
		0.999	252.0789	265.8748
EW	$\alpha = 0.941$, $a = 37.938$, $\gamma = 0.051$	0.700	80.4364	98.7563
		0.750	89.5747	115.0876
		0.800	99.9786	131.5436
		0.850	108.6430	142.0987
		0.900	127.0876	165.4231
		0.950	146.0875	188.4321
		0.975	179.5476	219.3743
		0.999	240.5667	244.9876

Table 10: Continued.

Distribution	Parameters	Level of significance	VaR	TVaR
EL	$\alpha = 9.492,\ a = 94.873,\ \gamma = 149.038$	0.700	91.8392	122.9786
		0.750	99.3874	128.7459
		0.800	109.5623	145.9350
		0.850	123.0348	161.4230
		0.900	139.1245	187.7618
		0.950	161.2498	209.1230
		0.975	210.7923	237.6790
		0.999	269.4320	287.4234
Ku-W	$a = 64.933,\ b = 0.657,\ \alpha = 0.984,\ \gamma = 0.052$	0.700	98.5432	118.6547
		0.750	105.8653	139.8650
		0.800	128.8764	159.8536
		0.850	143.0987	171.9875
		0.900	152.5860	189.8653
		0.950	188.7389	218.0872
		0.975	239.9876	249.2087
		0.999	296.5432	308.3425
BW	$a = 40.704,\ b = 0.592,\ \alpha = 1.090,\ \gamma = 0.031$	0.700	95.8479	112.5987
		0.750	101.7539	132.8763
		0.800	120.8648	148.8958
		0.850	135.8975	163.2634
		0.900	141.5987	178.4658
		0.950	178.6529	209.9509
		0.975	229.8769	238.2038
		0.999	286.7356	289.0236
NWB-XII	$a = 65.273,\ b = 2.675,\ \alpha = 1.487,\ \gamma = 0.238$	0.700	60.4328	68.5490
		0.750	66.0324	79.9782
		0.800	73.0897	95.9823
		0.850	89.4235	109.1032
		0.900	104.8765	124.7074
		0.950	121.0323	138.2398
		0.975	143.9832	149.9876
		0.999	187.4378	189.7654

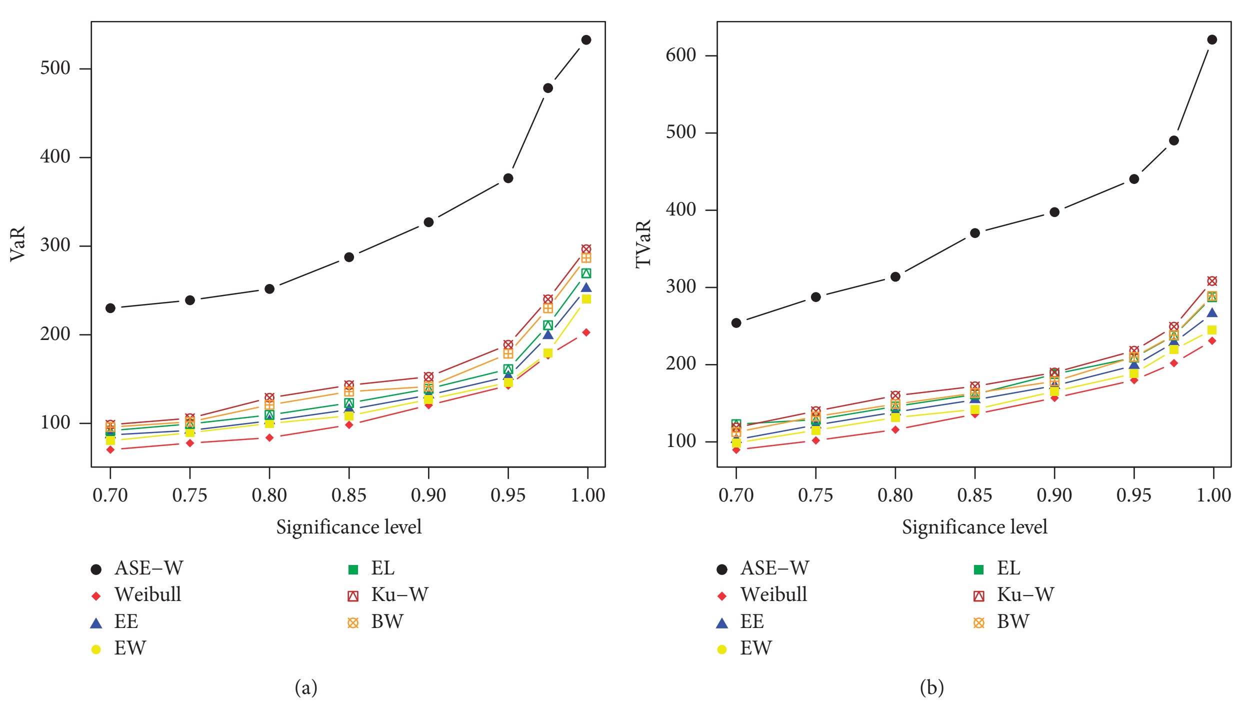


Figure 10: Graphical sketching of the VaR and TVaR using the results in Table 10 for insurance claims data.

8. Validation of the ASE-W Distribution

Goodness-of-fit tests indicate whether or not it is reasonable to assume that a random sample comes from a specific distribution. Statistical techniques often rely on observations obtained from a population that has a distribution of a specific form. Selection of a suitable model in all types of statistical analysis is of a great importance. For this purpose a lot of goodness-of-fit tests are proposed by some researchers. Nikulin [39, 40] proposed a modification in the standard chi-squared Pearson's test for a continuous distribution. Rao and Robson [41] obtained the same result for the exponential family, and later this statistic is well adapted by some researchers with the name as Rao–Robson–Nikulin (RRN) test.

In this section, we use another goodness-of-fit test to show the validity of the ASE-W distribution for heavy tailed insurance data. For this purpose, we use the NRR test statistic to show the utility of the ASE-W distribution in insurance and financial sciences.

8.1. Nikulin–Rao–Robson Test Statistic. So far in the literature, a number of methods have been proposed to verify the adequacy and goodness-of-fit of the statistical models to data. Since the seventies of the last century, researchers have shown a deep interest to propose new modifications of goodness-of-fit test. In this regard, Nikulin [42] and Rao and Robson [41] separately proposed a modification of the Pearson statistic for complete data known as Nikulin–Rao–Robson (NRR) statistic. To test the hypothesis H_0,

$$H_0\colon P\{X_i \le x\} = F(x,\xi)\big|_{\left(x\in\mathbb{R},\xi=\left(\xi_1,\xi_2,\ldots,\xi_s\right)^T\right)}, \tag{32}$$

where ξ represents the vector of unknown parameters, the NRR statistic is denoted by Y^2, and it is defined as follows.

Suppose observations $X_1, X_2, \ldots, X_n$ are grouped in r subintervals $\mathbf{I}_1, \mathbf{I}_2, \ldots, \mathbf{I}_r$, mutually disjoint:

$$\mathbf{I}_j = \left]a_j - 1; a_j\right], \quad j = 1,2,\ldots,r. \tag{33}$$

The limits a_j of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\xi)|_{(j=1,2,\ldots,r)} = \int_{a_{j-1}}^{a_j} g(x,,\xi)\mathrm{d}x, \tag{34}$$

where

$$a_j\big|_{(j=1,\ldots,r-1)} = G^{-1}\left(\frac{j}{r}\right). \tag{35}$$

If

$$\nu_j = \left(\nu_1, \nu_2, \ldots, \nu_r\right)^T, \tag{36}$$

is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals,

$$\nu_j = \sum_{i=1}^{n} 1_{\{x_i\in\mathbf{I}_j\}}|(j=1,\ldots,r). \tag{37}$$

The NRR statistic is given by

$$Y^2\left(\hat{\xi}_n\right) = X_n^2\left(\hat{\xi}_n\right) + n^{-1}\mathbf{L}^T\left(\hat{\xi}_n\right)\left(\mathbf{I}\left(\hat{\xi}_n\right) - \mathbf{J}\left(\hat{\xi}_n\right)\right)^{-1}\mathbf{L}\left(\hat{\xi}_n\right), \tag{38}$$

where

$$X_n^2(\xi) = \left(\frac{\nu_1 - np_1(\xi)}{\sqrt{np_1(\xi)}}, \frac{\nu_2 - np_2(\xi)}{\sqrt{np_2(\xi)}}, \ldots, \frac{\nu_r - np_r(\xi)}{\sqrt{np_r(\xi)}}\right)^T, \tag{39}$$

and $\mathbf{J}(\xi)$ is the information matrix for the grouped data defined by

$$\mathbf{J}(\xi) = B(\xi)^T B(\xi), \tag{40}$$

with

$$B(\xi)|_{(i=1,2,\ldots,r \text{ and } k=1,\ldots,s)} = \left[\frac{1}{\sqrt{p}_i}\frac{\partial p_i(\xi)}{\partial\mu}\right]_{r\times s},$$

$$\mathbf{L}(\xi) = \left(\mathbf{L}_1(\xi), \ldots, \mathbf{L}_s(\xi)\right)^T, \tag{41}$$

$$\mathbf{L}_k(\xi) = \sum_{i=1}^{r}\frac{\nu_i}{p_i}\frac{\partial}{\partial\xi_k}p_i(\xi),$$

where $\mathbf{I}_n(\hat{\xi}_n)$ represents the estimated Fisher information matrix and $\hat{\xi}_n$ is the MLE of the parameter vector. The Y^2 statistic follows a chi square χ^2 distribution with $(r-1)$ degrees of freedom.

8.2. Modified Chi-Squared Test for the ASE-W Distribution. A modified chi-squared goodness-of-fit test is constructed by fitting the Y^2 statistic developed in the previous section to verify if a sample $X = (X_1, X_2, \ldots, X_n)^T$ is distributed according to the ASE-W model, $P\{X_i \le x\} = G_{\text{ASE-W}}(x,\xi)$, with unknown parameters $\xi = (\alpha, \gamma, a)^T$. The MLEs $\hat{\xi}_n$ of the unknown parameters of the ASE-W distribution are computed using the insurance claims data. The statistic Y^2 does not depend on the parameters, we can, therefore, use the estimated Fisher information matrix $\mathbf{I}_n(\hat{\xi}_n)$.

To test the null hypothesis H_0 that the insurance claims data came from the ASE-W distribution, we use the Y^2 statistic. To conduct the analysis, we use the BB algorithm in R software to compute the maximum likelihood estimates given by $\hat{\alpha} = 1.24578, \hat{\gamma} = 0.98534$, and $\hat{a} = 1.24861$. For the insurance claims data, the estimated Fisher information matrix is

$$\mathbf{I}(\hat{\xi}) = \begin{pmatrix} 1.257884 & 0.945112 & 2.01445 \\ 0.94511 & 0.885471 & 3.006794 \\ 2.01445 & 3.006794 & 1.00578 \end{pmatrix}. \tag{42}$$

The value of the NRR statistic is given by $Y^2 = 26.524781$, whereas the critical value is $\chi^2_{0.05}(23-1) = 33.92444$.

We can see that the value of Y^2 statistic is less than the critical value. Therefore, we conclude that the insurance claims data follow the ASE-W model.

Table 11: Empirical levels and corresponding theoretical levels ($\varepsilon = 0.01, 0.02, 0.05, 0.10$).

$N = 10000$	$\varepsilon = 0.01$	$\varepsilon = 0.02$	$\varepsilon = 0.05$	$\varepsilon = 0.10$
$n = 50$	0:9926	0:9841	0:9535	0:9031
$n = 100$	0:9920	0:9830	0:9525	0:9024
$n = 200$	0:9909	0:9811	0:9510	0:9012
$n = 400$	0:9905	0:9805	0:9504	0:9006

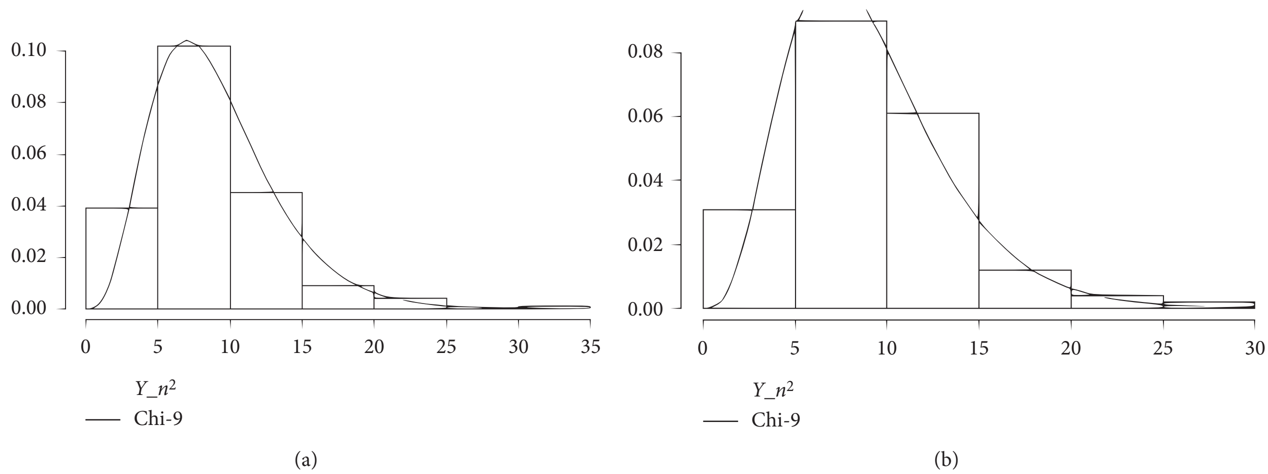


Figure 11: $\widehat{\alpha} = 1.5$, $\widehat{\gamma} = 1.5$, and $\widehat{a} = 0.9$ (a) and $\widehat{\alpha} = 1.5$, $\widehat{\gamma} = 0.5$, and $\widehat{a} = 1.2$ (b).

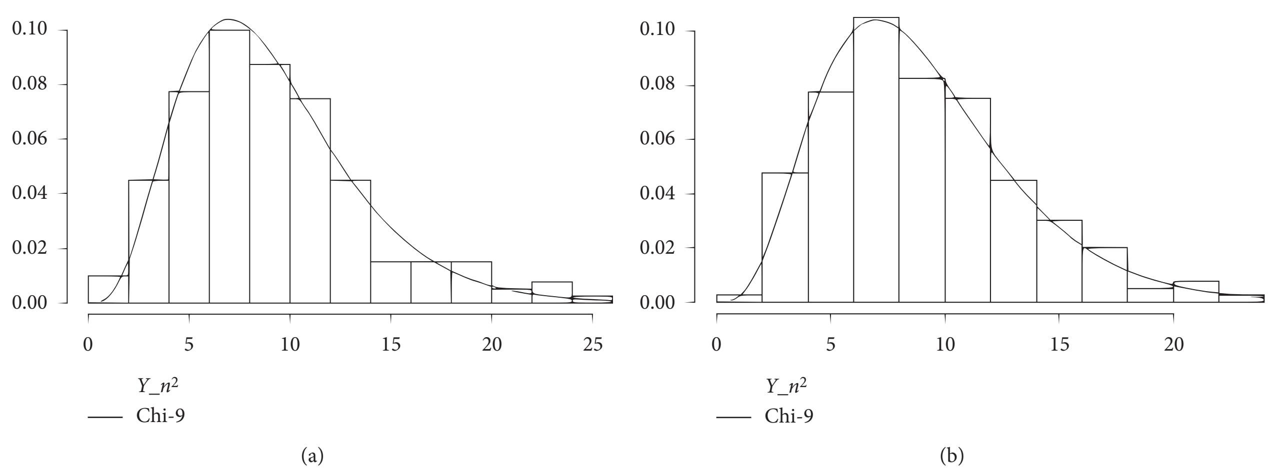


Figure 12: $\widehat{\alpha} = 0.5$, $\widehat{\gamma} = 1.5$, and $\widehat{a} = 0.5$ (a) and $\widehat{\alpha} = 2.5$, $\widehat{\gamma} = 0.9$, and $\widehat{a} = 2$ (b).

8.3. Simulation Study of the ASE-W Distribution Using Y^2 Statistic. To test the null hypothesis H_0 that the sample comes from the ASE-W model, we calculate Y^2 for 10,000 simulated samples with sample sizes $n = 50$, $n = 100$, $n = 200$, and $n = 400$, respectively. For different significance levels ($\varepsilon = 0.01, 0.02, 0.05, 0.1$), we calculate the average of the nonrejections of the null hypothesis, i.e., $Y^2 \leq \chi^2_{\varepsilon}(r-1)$. We present the results of the corresponding empirical and theoretical levels in Table 11. As can be shown, the values of the empirical levels calculated are very close to those of their corresponding theoretical levels. Thus, we conclude that the proposed test provides a good fit to the ASE-W distribution.

8.4. Simulated Distribution of the Y^2 Statistic for the ASE-W Model. The Y^2 statistic follows in the limit chi-squared distribution with $k = r - 1$ degrees of freedom. For demonstrating this fact, we compute $N = 10,000$ times the simulated distribution of $Y^2(\widehat{\xi})$ under the null hypothesis H_0 with different values of parameters and $r = 10$ intervals. We sketch the plots of the chi-squared distribution with $k = r - 1 = 9$ degree of freedom to see the visual representation. The histograms of the Y^2 statistic versus the chi-squared distribution with $k = 9$ degree of freedom are presented in Figures 11 and 12.

From Figures 11 and 12, we observe that the distribution of Y^2 with different values of parameters and different numbers k of grouping cells for different number of equiprobable grouping intervals and different values of parameters in the limit follows a chi-squared distribution with k degrees of freedom within the statistical errors of simulation. Therefore, we can say that the limiting distribution of the generalized chi-squared Y^2 statistic for ASE-W model is distribution free.

9. Concluding Remarks

In this paper, we used the trigonometric function to introduce a new family of heavy tailed distributions called the arcsine exponentiated-X (ASE-X) family of distributions. The ASE-X is very interesting and provides better fits to the heavy tailed insurance data. We define a special submodel called ASE-Weibull (ASE-W) distribution. The maximum likelihood is used to estimate the ASE-W parameters. The simulation results are obtained using the inversion and Barzilai-Borwein algorithms, assessing the performance of the maximum likelihood estimators. We derive two important risk measures called value at risk and tail value at risk of the ASE-W distribution and perform a simulation study to prove that the ASE-W distribution has a heavier tail than the baseline Weibull distribution. A heavy tailed insurance dataset is analyzed showing that the ASE-W distribution provides better fits than some other competing models. Furthermore, the value at risk and tail value at risk measures are estimated for all competing models based on the insurance claims data, proving that the ASE-W distribution performs well than other its competitors. Furthermore, we construct a modified chi-squared goodness-of-fit test statistic for the ASE-W distribution, based on the NRR statistic, to show its validity in modeling financial data. The special cases of Table 1 can be studied in future work. Furthermore, different classical and Bayesian methods can be employed to estimate the unknown parameters of these special submodels.

Data Availability

This work is mainly a methodological development and has been applied on secondary data related to the insurance science data, but if required, data will be provided.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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