Research Article

Nine Limit Cycles in a 5-Degree Polynomials Liénard System

Junning Cai , Minzhi Wei , and Hongying Zhu

Department of Information and Statistics, Guangxi University of Finance and Economics, Nanning, Guangxi 530003, China

Correspondence should be addressed to Minzhi Wei; xiaoyanxiong123@163.com

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In this article, we study the limit cycles in a generalized 5-degree Liénard system. The undamped system has a polycycle composed of a homoclinic loop and a heteroclinic loop. It is proved that the system can have 9 limit cycles near the boundaries of the period annulus of the undamped system. The main methods are based on homoclinic bifurcation and heteroclinic bifurcation by asymptotic expansions of Melnikov function near the singular loops. The result gives a relative larger lower bound on the number of limit cycles by Poincaré bifurcation for the generalized Liénard systems of degree five.

1. Introduction

Consider the following perturbed Hamiltonian system:

$$\begin{aligned}\dot{x} &= H_y(x, y) + \varepsilon p(x, y, \delta), \\ \dot{y} &= -H_x(x, y) + \varepsilon q(x, y, \delta),\end{aligned}\quad (1)$$

where $H(x, y)$ is a polynomial of degree $n + 1$, $p(x, y, \delta)$ and $q(x, y, \delta)$ are polynomials of degree n , ε is the sufficiently small perturbation parameter, and $\delta \in D \subset \mathbb{R}^N$ with D compact. Taking $\varepsilon = 0$, system (1) becomes a Hamiltonian system:

$$\begin{aligned}\dot{x} &= H_y(x, y), \\ \dot{y} &= -H_x(x, y),\end{aligned}\quad (2)$$

and system (1) is usually called a near-Hamiltonian system. We assume that the unperturbed system (2) has a family of periodic orbits $\{\Gamma_h\}$ defined by $\{(x, y) \mid H(x, y) = h\}$, and the family forms a periodic annulus. The inner boundary of the periodic annulus is a center which may be an elementary one or a nilpotent one, and the outer boundary of the periodic annulus is usually a homoclinic loop or a heteroclinic loop or a polycycle. Most periodic orbits are broken when system (2) is perturbed and only a finite number of periodic orbits persist as limit cycles of system (1), which is the so-called Poincaré bifurcation. The most efficient tool to study

Poincaré bifurcation of system (1) is the following first-order approximation of the Poincaré map:

$$M(h, \delta) = \oint_{\Gamma_h} q dx - p dy, \quad (3)$$

which is a continuous integral over the continuous ovals $\{\Gamma_h\}$ of $H(x, y)$ and is usually called the first-order Melnikov function or Abelian integral. The zeros of $M(h, \delta)$ correspond to the limit cycles by Poincaré bifurcation for system (1) (see [1–3]). It should be noted that studying the maximal number of zeros of $M(h, \delta)$ is the topic of weak Hilbert's 16th problem. We note that the original version of Hilbert's 16th problem asks the maximal number of limit cycles of a general polynomial system:

$$\begin{aligned}\dot{x} &= P_n(x, y), \\ \dot{y} &= Q_n(x, y).\end{aligned}\quad (4)$$

Let $H(n)$ denote the maximal number of limit cycles of system (4) and $H^*(n)$ denote the maximal number of limit cycles of system (1). Most results on the lower bounds of $H(n)$ are obtained in the literature as we know, by studying $H^*(n)$. However, it should be noted that it is also very difficult to determine $H^*(n)$ for a given degree n . We refer the works [4] for the relatively new results on the lower bounds of $H^*(n)$.

In order to reduce the difficulty, researchers usually study some special forms of system (1), such as

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= g(x) + \varepsilon f(x)y,\end{aligned}\quad (5)$$

which is called Liénard system of type (m, n) having the degree $n^* = \max\{n+1, m\}$, where $g(x)$ and $f(x)$ are polynomials of degrees m and n , respectively, and ε is positive and very small. Let $H(n, m)$ denote the maximal number of limit cycles of system (5) and $I(n^*)$ denote the maximal number of limit cycles of system (5) of degree n^* . Dumortier and Li studied four Liénard systems with different portraits of type $(3, 2)$ in a series of papers [5–8] and gave the corresponding sharp bound of number of limit cycles by Poincaré bifurcation. The exact bounds of $H(4, 3)$ and $H(5, 4)$ for some special Liénard systems were reported in [9–14] and references therein. For results on $H(7, 6)$ associated with symmetric system (5), the relatively new works are referred [15–17]. It is usually very difficult to give the exact bound; however, lots of lower bounds of $H(m, n)$ have been obtained by studying the number of limit cycles near the center, homoclinic loops, and heteroclinic loops (see [18]). In particular, Xu and Li [19] investigated a Liénard system of type $(5, n)$ and proved $H(5, 2) \geq 3$, $H(5, 4) \geq 5$, $H(5, 6) \geq 10$, $H(5, 8) \geq 10$. In this paper, we study a type $(5, 4)$ Liénard system which has a polycycle consisting of a homoclinic loop and a heteroclinic loop:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -\frac{1}{7}x(7x+3)(x-1)(x+1)^2 + \varepsilon f(x)y,\end{aligned}\quad (6)$$

with special elliptic Hamiltonian function:

$$\tilde{H}(x, y) = \frac{1}{2}y^2 + \frac{1}{6}x^6 + \frac{2}{7}x^5 - \frac{1}{7}x^4 - \frac{10}{21}x^3 - \frac{3}{14}x^2, \quad (7)$$

where $0 < \varepsilon \ll 1$, $f(x) = \sum_{i=1}^n a_i x^i$, a_i ($i = 0, 1, \dots, n$) are real bounded parameters. System (6) has the degree $n^* = \max\{5, n+1\}$. Let $H^*(5, n)$ denote the maximum number of limit cycles of (6) and $I^*(n^*)$ denote the maximum number of limit cycles of (6) of degree n^* . Our main interest is focused on $H^*(5, 4)$.

The level sets (i.e., $\tilde{H}(x, y) = h$) of Hamiltonian function (7) are sketched in Figure 1. $\tilde{H}(x, y) = h$ defines the periodic orbits of system (6). There are a heteroclinic loop and a homoclinic loop for system (6) $\varepsilon = 0$ defined by $\tilde{H}(x, y) = 0$, denoted by L_1 and L_2 , respectively; let us in the following say $L^* = L_1 \cup L_2$ is a hetero-homoclinic loop. L_1 connects the nilpotent cusp $S_1(-1, 0)$ and the hyperbolic saddle $S_2(0, 0)$ of system (6) $\varepsilon = 0$, L_2 connects a the hyperbolic saddle $S_2(0, 0)$ of system (6) $\varepsilon = 0$. There are three families of clockwise periodic orbits, one family is $L_1^h = \{(x, y) | -25/2688 < \tilde{H}(x, y) < 0\}$ inside L_1 surrounding a center $C_1(-3/7, 0)$ with $H(-3/7, 0) = -25/2688$, one family is $L_2^h = \{(x, y) | -8/21 < \tilde{H}(x, y) < 0\}$ inside L_2 surrounding another center $C_2(1, 0)$

FIGURE 1: The portrait of system (6) when $\varepsilon = 0$.

with $H(1, 0) = -8/21$, and the third family is $L_3^h = \{(x, y) | 0 < \tilde{H}(x, y) < +\infty\}$ surrounding L_1 and L_2 . Three families of periodic orbits form three periodic annuli, the boundaries of which are L_1 and C_1 , L_2 and C_2 , and L^* .

For system (6), we get the following main result.

Theorem 1. *There exist some a_0, a_1, a_2, a_3 such that system (6) has 9 limit cycles for the type $(5, 4)$ system (6). Therefore, $H(5, 4) \geq H^*(5, 4) \geq 9$, $I(5) \geq I^*(5) \geq 9$.*

The rest of this paper is organized as follows. In Section 2, we present some preliminaries which will be used in the next section. In Section 3, we study the asymptotic expansions of the related Melnikov functions for system (6). The proof of the main result is given in Section 4. Conclusion and discussions are drawn in Section 5.

2. Preliminary Lemmas

Consider the Melnikov function $M(h, \delta)$ for the near-Hamiltonian system (1); we suppose the Hamiltonian system (2) has a bounded periodic annulus $\{\Gamma_h\}$ denoted by $\{H(x, y) = h\}$, and the period orbits in the periodic annulus are clockwise. The boundary of $\{\Gamma_h\}$ can be a center, a homoclinic loop, and a heteroclinic loop. We suppose the inner boundary of $\{\Gamma_h\}$ is defined by $H(x, y) = \alpha$ and the outer boundary is defined by $H(x, y) = \beta$; correspondingly, we have

$$M(h, \delta) = \oint_{\Gamma_h} q dx - p dy, \quad \alpha < h < \beta, \quad (8)$$

and $M(h, \delta)$ can be expanded near its boundary (see [20–23]).

When the inner boundary of $\{\Gamma_h\}$ is elementary center, we suppose it is located at $C(x_c, y_c)$ with $H(x_c, y_c) = \alpha$, and for (x, y) near $C(x_c, y_c)$,

$$\begin{aligned}
H(x, y) &= \alpha + \frac{1}{2}(x - x_c)^2 + \frac{1}{2}(y - y_c)^2 \\
&\quad + \sum_{i+j \geq 3} \tilde{h}_{ij}(x - x_c)^i (y - y_c)^j, \\
p(x, y, \delta) &= \sum_{i+j \geq 0} \tilde{a}_{ij}(x - x_c)^i (y - y_c)^j, \\
q(x, y, \delta) &= \sum_{i+j \geq 0} \tilde{b}_{ij}(x - x_c)^i (y - y_c)^j.
\end{aligned} \tag{9}$$

For the expansion of $M(h, \delta)$ near $C(x_c, y_c)$, we have the following.

Lemma 1 (see [20]). *Under the condition we suppose above, for the expansion of $M(h, \delta)$ near the elementary center $C(x_c, y_c)$ ($h = \alpha$), we have*

$$M(h, \delta) = \sum_{j \geq 0} b_j(\delta)(h - \alpha)^{j+1}, \quad \alpha < h \ll 1. \tag{10}$$

Under (9), the formulas of b_j can be obtained by using the programs in [20].

When the outer boundary of $\{\Gamma_h\}$ is homoclinic loop denoted by Γ_β passing through a hyperbolic saddle $S(x_s, y_s)$ satisfying $H(x_s, y_s) = \beta$, for the expansion of $M(h, \delta)$ near the homoclinic loop, we have the following.

Lemma 2.

(i) (see [21]). *Under the condition we suppose above, the function $M(h, \delta)$ has the following expansion:*

$$\begin{aligned}
M(h, \delta) &= c_0(\delta) + c_1(\delta)(h - \beta) \ln|h - \beta| + c_2(\delta)(h - \beta) \\
&\quad + c_3(\delta)(h - \beta)^2 \ln|h - \beta| + O(|h - \beta|^2),
\end{aligned} \tag{11}$$

for $0 < -(h - \beta) \ll 1$, where $c_1(\delta)$ and $c_3(\delta)$ depend on the coefficients of $H(x, y)$, $p(x, y)$, and $q(x, y)$.

$$\begin{aligned}
c_0(\delta) &= M(\beta, \delta) = \oint_{\Gamma_\beta} (qdx - pdy) |_{\varepsilon=0}, \\
c_2(\delta) &= \oint_{\Gamma_\beta} (p_x + q_y - \sigma)dt + bc_1(\delta),
\end{aligned} \tag{12}$$

for $\sigma = (p_x + q_y)|_{x=x_s, y=y_s}$ is divergence at $S(x_s, y_s)$, b is a constant.

(ii) (see [21]). *In particular, for (x, y) near the hyperbolic saddle $S(x_s, y_s)$,*

$$\begin{aligned}
H(x, y) &= \beta + \frac{\lambda}{2}(y - y_s)^2 - \frac{1}{2}(x - x_s)^2 \\
&\quad + \sum_{i+j \geq 2} \bar{h}_{ij}(x - x_s)^i (y - y_s)^j, \\
p(x, y, \delta) &= \sum_{i+j \geq 0} \bar{a}_{ij}(x - x_s)^i (y - y_s)^j, \\
q(x, y, \delta) &= \sum_{i+j \geq 0} \bar{b}_{ij}(x - x_s)^i (y - y_s)^j.
\end{aligned} \tag{13}$$

Then,

$$\begin{aligned}
c_1(\delta) &= -\frac{1}{|\lambda|}(\bar{a}_{10} + \bar{b}_{01}), \\
c_3(\delta) &= \frac{-1}{2|\lambda|\lambda} \{ (-3\bar{a}_{30} - \bar{b}_{21} + \bar{a}_{12} + 3\bar{b}_{03}) \\
&\quad - \frac{1}{\lambda} [(2\bar{b}_{02} + \bar{a}_{11})(3\bar{h}_{03} - \bar{h}_{21}) \\
&\quad + (2\bar{a}_{20} + \bar{b}_{11})(\bar{h}_{30} - \bar{h}_{12})] \}.
\end{aligned} \tag{14}$$

Definition 1. The values $c_1(\delta)$ and $c_3(\delta)$ are, respectively, called the first and second local Melnikov coefficients at the saddle $S(x_s, y_s)$, denoted by $c_1(S_{\text{saddle}}, \delta)$, $c_3(S_{\text{saddle}}, \delta)$, respectively.

Note.

- (i) The local coefficients are presented in the expansion of a Melnikov function, which are local quantities at a singular point depending on the coefficients of the expansions of $H(x, y)$, $p(x, y)$, and $q(x, y)$ at the singular point.
- (ii) When there is a nilpotent cusp $S(x_n, y_n)$ connecting a homoclinic loop, there are similarly some local coefficients at the cusp $S(x_n, y_n)$ presented in the expansion of the corresponding Melnikov function near the homoclinic loop; we denote these first three local coefficients by $c_1(S_{\text{cusp}}, \delta)$, $c_3(S_{\text{cusp}}, \delta)$, and $c_4(S_{\text{cusp}}, \delta)$ (see [22]).

Lemma 3 (see [22]). *Suppose there is a nilpotent cusp $S(x_n, y_n)$ which is a limit point of a homoclinic loop or a heteroclinic loop, and for (x, y) near the nilpotent cusp $S(x_n, y_n)$:*

$$\begin{aligned}
H(x, y) &= \beta + \frac{1}{2}(y - y_n)^2 \\
&\quad + \sum_{i+j \geq 3} h_{ij}(x - x_n)^i (y - y_n)^j, \quad h_{30} < 0,
\end{aligned} \tag{15}$$

$$p(x, y, \delta) = \sum_{i+j \geq 0} a_{ij}(x - x_n)^i (y - y_n)^j,$$

$$q(x, y, \delta) = \sum_{i+j \geq 0} b_{ij}(x - x_n)^i (y - y_n)^j.$$

Then, the first three local coefficients of the corresponding Melnikov functions at the nilpotent cusp $S(x_n, y_n)$ are as follows:

$$\begin{aligned}
c_1(S_{\text{cusp}}, \delta) &= 2\sqrt{2}h_{30}^{-1/3} [a_{10} + b_{01}], \\
c_3(S_{\text{cusp}}, \delta) &= 2\sqrt{2}h_{30}^{-5/3} [h_{30}(2a_{20} + b_{11} - h_{12}(a_{10} + b_{01})) + 1/3(h_{21}^2 - 2h_{40})(a_{10} + b_{01})], \\
c_4(S_{\text{cusp}}, \delta) &= 9\mu_1^{-1}\alpha_{01} - 2\mu_1^{-7}[(20\mu_2^3 - 20\mu_1\mu_2\mu_3 + 4\mu_1^2\mu_4)\alpha_{00} + (4\mu_1^2\mu_3 - 10\mu_1\mu_2^2)\alpha_{10} + 4\mu_1^2\mu_2\alpha_{20} - \mu_1^3\alpha_{30}], \\
\mu_1 &= \sqrt[3]{h_{30}}, \\
\mu_2 &= -\frac{1}{6}h_{30}^{-2/3}(-2h_{40} + h_{21}^2), \\
\mu_3 &= \frac{1}{36}h_{30}^{-5/3}[12h_{30}(h_{50} - h_{31}h_{21} + h_{12}h_{21}^2) - 4h_{40}^2 + 4h_{40}h_{21}^2 - h_{21}^4], \\
\mu_4 &= -\frac{1}{648}h_{30}^{-8/3} \left[\begin{aligned} &5h_{21}^6 - 40h_{40}^3 + 30h_{40}h_{21}^2(2h_{40} - h_{21}^2) + 144h_{30}h_{40}(h_{50} - h_{31}h_{21} + h_{12}h_{21}^2) \\ &+ 72h_{30}h_{21}^2(-h_{50} + h_{31}h_{21} - h_{12}h_{21}^2) + 216h_{30}^2(-h_{60} - h_{22}h_{21}^2 + h_{41}h_{21} + h_{03}h_{21}^3) \\ &+ 108h_{30}^2h_{31}^2 + 432h_{30}^2h_{12}h_{21}(h_{12}h_{21} - h_{31}) \end{aligned} \right], \\
\alpha_{00} &= 2\sqrt{2}(a_{10} + b_{01}), \\
\alpha_{10} &= 2\sqrt{2}[-h_{12}(a_{10} + b_{01}) + 2a_{20} + b_{11}], \\
\alpha_{20} &= 2\sqrt{2}\left[(a_{10} + b_{01})\left(3h_{03}h_{21} - h_{22} + \frac{3}{2}h_{12}^2\right) - 2h_{12}a_{20} - h_{12}b_{11} + 3a_{30} + b_{21} - a_{11}h_{21} - 2b_{02}h_{21}\right], \\
\alpha_{30} &= 2\sqrt{2} \left[\begin{aligned} &(a_{10} + b_{01})\left(3h_{13}h_{21} + 3h_{03}h_{31} + 3h_{12}h_{22} - 15h_{12}h_{03}h_{21} - \frac{5}{2}h_{12}^3 - h_{32}\right) + (3a_{11} + 6b_{02}) \\ &h_{12}h_{21} - 2(b_{12} + a_{21})h_{21} - (a_{11} + 2b_{02})h_{31} + 4a_{40} + b_{31} + (3b_{11} + 6a_{20}) \\ &h_{03}h_{21} - (2a_{20} + b_{11})h_{22} + \left(3a_{20} + \frac{3}{2}b_{11}\right)h_{12}^2 - (3a_{30} + b_{21})h_{12} \end{aligned} \right], \\
\alpha_{01} &= 2\sqrt{2}\left[\frac{2}{3}a_{12} + 2b_{03} - 2h_{03}a_{11} - 4h_{03}b_{02} + (a_{10} + b_{01})(5h_{03}^2 - 2h_{04})\right].
\end{aligned} \tag{16}$$

When the outer boundary of Γ_h is heteroclinic loop denoted by Γ_β passing through a hyperbolic saddle $S(x_s, y_s)$ and a nilpotent cusp $N(x_n, y_n)$ satisfying $H(x_s, y_s) = H(x_n, y_n) = \beta$, for the expansion of $M(h, \delta)$ near the heteroclinic loop Γ_β , we have the following.

Lemma 4 (see [23]). *The expansion of $M(h, \delta)$ near the heteroclinic loop L_β has the form*

$$\begin{aligned}
M(h, \delta) &= \tilde{c}_0(\delta) + B_{00}\tilde{c}_1(\delta)|h - \beta|^{5/6} + \tilde{c}_2(\delta)(h - \beta)\ln|h - \beta| \\
&+ [\tilde{c}_3(\delta) + b_1\tilde{c}_1(\delta) + b_2\tilde{c}_2(\delta)] \\
&\cdot (h - \beta) + B_{10}\tilde{c}_4(\delta)|h - \beta|^{7/6} \\
&+ \tilde{c}_5(\delta)(h - \beta)^2\ln|h - \beta| \\
&- \frac{1}{11}B_{00}\tilde{c}_6(\delta)|h - \beta|^{11/6} + O((h - \beta)^2).
\end{aligned} \tag{17}$$

for $0 < -(h - \beta) \ll 1$, where $B_{00} = 3/5 \int_0^1 dv / \sqrt{v(1 - v^3)} = 3/5 \times 2.4286 \dots$,

$B_{10} = -3/7 \int_0^1 (v^{(3/2)dv} / \sqrt{1 - v^3} (1 + \sqrt{1 - v^3}) - 2) > 0$, b_1 and b_2 are constants, and

$$\begin{aligned}
\tilde{c}_0(\delta) &= M(\beta, \delta) \\
\tilde{c}_1(\delta) &= c_1(N_{\text{cusp}}, \delta), \\
\tilde{c}_2(\delta) &= c_1(S_{\text{saddle}}, \delta), \\
\tilde{c}_4(\delta) &= c_3(N_{\text{cusp}}, \delta), \\
\tilde{c}_5(\delta) &= c_3(S_{\text{saddle}}, \delta), \\
\tilde{c}_6(\delta) &= c_4(N_{\text{cusp}}, \delta),
\end{aligned} \tag{18}$$

and $\tilde{c}_3(\delta) \in \mathbb{R}$. In particular, if $\tilde{c}_1(\delta) = \tilde{c}_2(\delta)$, we have

$$\tilde{c}_3(\delta) = \oint_{\Gamma_\beta} (p_x + q_y) dt. \tag{19}$$

In many cases, the Hamiltonian function is not of the form supposed in the above lemmas. Then, to apply the lemmas, we need to first introduce suitable linear change of variables which will cause a change of the first-order Melnikov function. The following lemma gives the relationship between the old and new Melnikov functions.

Lemma 5 (see [2]). *Under the linear change of variables of the form*

$$\begin{aligned} u &= a(x - x_0) + b(y - y_0), \\ v &= c(x - x_0) + d(y - y_0), \end{aligned} \quad (20)$$

and time rescaling $\tau = kt$, where $D = ad - bc \neq 0$, system (31) becomes

$$\begin{aligned} \frac{du}{d\tau} &= \tilde{H}_v + \varepsilon \tilde{p}, \\ \frac{dv}{d\tau} &= -\tilde{H}_u + \varepsilon \tilde{q}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \tilde{H}(u, v) &= \frac{D}{k} H(x, y), \\ \tilde{p}(u, v, \delta) &= \frac{1}{k} [ap(x, y, \delta) + bq(x, y, \delta)], \\ \tilde{q}(u, v, \delta) &= \frac{1}{k} [cp(x, y, \delta) + dq(x, y, \delta)]. \end{aligned} \quad (22)$$

Let

$$\tilde{M}(h, \delta) = \oint_{\tilde{\Gamma}_h} (\tilde{q} du - \tilde{p} dv) \big|_{\varepsilon=0}, \quad (23)$$

which is the Melnikov function of system (21). Then,

$$M(h, \delta) = \frac{|k|}{D} \tilde{M}\left(\frac{D}{k} h, \delta\right). \quad (24)$$

Remark 1. Usually when we apply Lemma 5, we always take a linear transformation such that $D/k = 1$ because under this condition, the local coefficients between the old and the new ones, respectively, are the same (if $|k|/D = 1$) or only differ from a symbol “−” (if $|k|/D = -1$).

Let us now suppose system (2) has a hetero-homoclinic loop $\Gamma^* = \Gamma_1 \cup \Gamma_2$, where Γ_1 is heteroclinic loop connecting a cusp $S_1(x_1, y_1)$ and a hyperbolic saddle $S_2(x_2, y_2)$ satisfying $H(x_1, y_1) = H(x_2, y_2) = \beta$ and Γ_2 is homoclinic loop passing through the previous hyperbolic saddle $S(x_2, y_2)$. There exist three families of orbits, the first family of periodic orbits Γ_h^1 surrounding a center $C_1(x_3, y_3)$ with $H(x_3, y_3) = \alpha_1$ in Γ_1 , the second family of periodic orbits Γ_h^2 surrounding another center $C_2(x_4, y_4)$ with $H(x_4, y_4) = \alpha_2$ in Γ_2 , and the third family of periodic orbits Γ_h^* surrounding Γ^* . The portrait of Γ^* is just the same as Figure 1.

Correspondingly, we have three Melnikov functions:

$$\begin{aligned} M_i(h, \delta) &= \oint_{\Gamma_h^i} q dx - p dy, \quad \text{for } \alpha_i < h < \beta, \quad i = 1, 2, \\ M_3(h, \delta) &= \oint_{\Gamma_h^*} q dx - p dy, \quad \text{for } \beta < h < +\infty. \end{aligned} \quad (25)$$

From Lemmas 2–4, we have the following.

Lemma 6. *Under the condition supposed above, the three Melnikov functions have the following expansions:*

$$\begin{aligned} M_1(h, \delta) &= e_0(\delta) + B_{00}e_1(\delta)|h - \beta|^{5/6} + e_2(\delta)(h - \beta)\ln|h - \beta| \\ &\quad + [e_3(\delta) + b_1e_1(\delta) + b_2e_2(\delta)](h - \beta) \\ &\quad + B_{10}e_4(\delta)|h - \beta|^{7/6} + e_5(\delta)(h - \beta)^2 \\ &\quad \cdot \ln|h - \beta| - \frac{1}{11}B_{00}e_6(\delta)|h - \beta|^{11/6} + O((h - \beta)^2), \end{aligned} \quad (26)$$

for $0 < -(h - \beta) \ll 1$.

$$\begin{aligned} M_2(h, \delta) &= c_0(\delta) + c_1(\delta)(h - \beta)\ln|h - \beta| + c_2(\delta)(h - \beta) \\ &\quad + c_3(\delta)(h - \beta)^2\ln|h - \beta| + O(|h - \beta|^2), \end{aligned} \quad (27)$$

for $0 < -(h - \beta) \ll 1$.

$$\begin{aligned} M_3(h, \delta) &= \tilde{c}_0(\delta) + B_{00}^*\tilde{c}_1(\delta)|h - \beta|^{5/6} + \tilde{c}_2(\delta)(h - \beta) \\ &\quad \cdot \ln|h - \beta| + [\tilde{c}_3(\delta) + b_3\tilde{c}_1(\delta) + b_4\tilde{c}_2(\delta)](h - \beta) \\ &\quad + B_{10}^*\tilde{c}_4(\delta)|h - \beta|^{7/6} + \tilde{c}_5(\delta)(h - \beta)^2\ln|h - \beta| \\ &\quad - \frac{1}{11}B_{00}^*\tilde{c}_6(\delta)|h - \beta|^{11/6} + O((h - \beta)^2), \end{aligned} \quad (28)$$

for $0 < (h - \beta) \ll 1$, where B_{00} and B_{10} are the same as before, $B_{00}^* < 0$ and $B_{10}^* < 0$, b_1, b_2, b_3 and b_4 are some constants. and $e_1(\delta) = c_1(S_{1_{\text{cusp}}}, \delta)$, $e_2(\delta) = c_1(S_{2_{\text{saddle}}}, \delta)$, $e_4(\delta) = c_3(S_{1_{\text{cusp}}}, \delta)$, $e_5(\delta) = c_3(S_{2_{\text{saddle}}}, \delta)$, $e_6(\delta) = c_4(S_{1_{\text{cusp}}}, \delta)$, $c_1(\delta) = c_1(S_{2_{\text{saddle}}}, \delta)$, $c_3(\delta) = c_3(S_{2_{\text{saddle}}}, \delta)$, $\tilde{c}_1(\delta) = c_1(S_{1_{\text{cusp}}}, \delta)$, $\tilde{c}_2(\delta) = 2c_1(S_{2_{\text{saddle}}}, \delta)$, $\tilde{c}_4(\delta) = c_3(S_{1_{\text{cusp}}}, \delta)$, $\tilde{c}_5(\delta) = 2c_3(S_{2_{\text{saddle}}}, \delta)$, $\tilde{c}_6(\delta) = c_4(S_{1_{\text{cusp}}}, \delta)$, where $c_1(S_{1_{\text{cusp}}}, \delta)$, $c_3(S_{1_{\text{cusp}}}, \delta)$, and $c_4(S_{1_{\text{cusp}}}, \delta)$ are local coefficients at the nilpotent cusp $S_1(x_1, y_1)$ which can be obtained by Lemma 3 and $c_1(S_{2_{\text{saddle}}}, \delta)$ and $c_3(S_{2_{\text{saddle}}}, \delta)$ are local coefficients at the hyperbolic saddle $S_2(x_2, y_2)$ which can be obtained by Lemma 2, while

$$\begin{aligned} e_0(\delta) &= M_1(\beta, \delta) = \oint_{\Gamma_1} q dx - p dy, \quad c_0(\delta) = M_2(\beta, \delta) \\ &= \oint_{\Gamma_2} q dx - p dy, \quad \tilde{c}_0(\delta) = e_0 + c_0, \end{aligned} \quad (29)$$

and $e_3(\delta) \in \mathbb{R}$; in particular, if $c_1(S_{2_{\text{saddle}}}, \delta) = c_1(S_{1_{\text{cusp}}}, \delta) = 0$, we have

$$e_3(\delta) = \oint_{\Gamma_1} (p_x + q_y) dt, \quad (30)$$

and $c_2(\delta) = \oint_{\Gamma_2} (p_x + q_y - a_{10} - b_{01}) \big|_{\varepsilon=0} dt + bc_1(\delta) \in \mathbb{R}$; in particular, if $c_1(S_{2_{\text{saddle}}}, \delta) = 0$, we have

$$c_2(\delta) = \oint_{\Gamma_2} (p_x + q_y) dt, \quad (31)$$

and $\tilde{c}_3(\delta) = \oint_{\Gamma_1} (p_x + q_y) dt + \oint_{\Gamma_2} (p_x + q_y) dt(\delta)$ if $c_1(S_{2_{saddle}}, \delta) = c_1(S_{1_{cusp}}, \delta) = 0$.

3. The Expansions of Melnikov Functions of System (6)

Liénard system (6) is a special form of near-Hamiltonian system (1) with $H(x, y) = \tilde{H}(x, y)$, $p(x, y) = 0$, $q(x, y) = f(x)y = \sum_{i=0}^4 a_i x^i y$. In this section, we take $n = 4$; then, system (6) is a Liénard system of type (5, 4). There are three Melnikov functions corresponding to three periodic annuli of system (6) $\varepsilon = 0$:

$$\begin{aligned} M_1(h, \delta) &= \oint_{L_1^h} f(x)y dx, \quad \text{for } -\frac{25}{2688} < h < 0, \\ M_2(h, \delta) &= \oint_{L_2^h} f(x)y dx, \quad \text{for } -\frac{8}{21} < h < 0, \\ M_3(h, \delta) &= \oint_{L_3^h} f(x)y dx, \quad \text{for } 0 < h < +\infty. \end{aligned} \quad (32)$$

By Theorem 1, we obtain

$$\begin{aligned} M_1(h, \delta) &= e_0(\delta) + B_{00}e_1(\delta)|h|^{5/6} + e_2(\delta)h\ln|h| \\ &\quad + [e_3(\delta) + b_1e_1(\delta) + b_2e_2(\delta)](h) \\ &\quad + B_{10}e_4(\delta)|h|^{7/6} + e_5(\delta)h^2\ln|h| \\ &\quad - \frac{1}{11}B_{00}e_6(\delta)|h|^{11/6} + O(h^2), \end{aligned} \quad (33)$$

for $0 < -h \ll 1$.

$$\begin{aligned} M_2(h, \delta) &= c_0(\delta) + c_1(\delta)h\ln|h| + c_2(\delta)h \\ &\quad + c_3(\delta)h^2\ln|h| + O(h^2), \end{aligned} \quad (34)$$

for $0 < -h \ll 1$.

$$\begin{aligned} M_3(h, \delta) &= \tilde{c}_0(\delta) + B_{00}^*\tilde{c}_1(\delta)h^{5/6} + \tilde{c}_2(\delta)h\ln h \\ &\quad + [\tilde{c}_3(\delta) + b_3\tilde{c}_1(\delta) + b_4\tilde{c}_2(\delta)]h \\ &\quad + B_{10}^*\tilde{c}_4(\delta)h^{7/6} + \tilde{c}_5(\delta)h^2\ln h \\ &\quad + \frac{1}{11}B_{00}^*\tilde{c}_6(\delta)h^{11/6} + O(h^2), \end{aligned} \quad (35)$$

for $0 < h \ll 1$.

By Lemma 1, for $0 < h + 25/2688 \ll 1$,

$$M_1(h, \delta) = \sum_{i=0} b_i(\delta) \left(h + \frac{25}{2688} \right)^{i+1}, \quad (36)$$

and for $0 < h + 8/21 \ll 1$,

$$M_2(h, \delta) = \sum_{i=0} d_i(\delta) \left(h + \frac{8}{21} \right)^{i+1}. \quad (37)$$

In the following, we use the preliminary lemmas to compute the coefficients of the above expansions for $M_1(h, \delta)$, $M_2(h, \delta)$, and $M_3(h, \delta)$. Firstly, we have

$$\begin{aligned} e_0(\delta) &= M_1(0, \delta) = \oint_{L_1} f(x)y dx \\ &= 2 \int_{-1}^0 f(x) \sqrt{-2A(x)} dx = \sum_{i=0}^9 k_i a_i, \end{aligned} \quad (38)$$

where

$$\begin{aligned} k_0 &= \frac{741\sqrt{21}}{4802} - \frac{256\sqrt{3}}{2401}\pi + \frac{512\sqrt{3}}{2401}\arcsin\left(\frac{1}{8}\right), \\ k_1 &= \frac{1191\sqrt{21}}{12005} - \frac{256\sqrt{3}}{2401}\pi + \frac{512\sqrt{3}}{2401}\arcsin\left(\frac{1}{8}\right), \\ k_2 &= \frac{17467\sqrt{21}}{168070} - \frac{4352\sqrt{3}}{50421}\pi + \frac{8704\sqrt{3}}{50421}\arcsin\left(\frac{1}{8}\right), \\ k_3 &= \frac{382302\sqrt{21}}{4117715} - \frac{74496\sqrt{3}}{823543}\pi + \frac{148992\sqrt{3}}{823543}\arcsin\left(\frac{1}{8}\right), \\ k_4 &= \frac{11525967\sqrt{21}}{115296020} - \frac{506112\sqrt{3}}{5764801}\pi + \frac{1012224\sqrt{3}}{5764801}\arcsin\left(\frac{1}{8}\right). \end{aligned} \quad (39)$$

Secondly,

$$\begin{aligned} c_0(\delta) &= M_2(0, \delta) = \oint_{L_2} f(x)y dx \\ &= 2 \int_0^{9/7} f(x) \sqrt{-2A(x)} dx = \sum_{i=0}^9 I_i a_i, \end{aligned} \quad (40)$$

where

$$\begin{aligned} I_0 &= \frac{741\sqrt{21}}{4802} + \frac{256\sqrt{3}}{2401}\pi + \frac{512\sqrt{3}}{2401}\arcsin\left(\frac{1}{8}\right), \\ I_1 &= \frac{1191\sqrt{21}}{12005} + \frac{256\sqrt{3}}{2401}\pi + \frac{512\sqrt{3}}{2401}\arcsin\left(\frac{1}{8}\right), \\ I_2 &= \frac{17467\sqrt{21}}{168070} + \frac{4352\sqrt{3}}{50421}\pi + \frac{8704\sqrt{3}}{50421}\arcsin\left(\frac{1}{8}\right), \\ I_3 &= \frac{382302\sqrt{21}}{4117715} + \frac{74496\sqrt{3}}{823543}\pi + \frac{148992\sqrt{3}}{823543}\arcsin\left(\frac{1}{8}\right), \\ I_4 &= \frac{11525967\sqrt{21}}{115296020} + \frac{506112\sqrt{3}}{5764801}\pi + \frac{1012224\sqrt{3}}{5764801}\arcsin\left(\frac{1}{8}\right). \end{aligned} \quad (41)$$

Therefore,

$$\tilde{c}_0(\delta) = e_0(\delta) + c_0(\delta) = \sum_{i=1}^9 \tilde{I}_i a_i, \quad (42)$$

where $\tilde{I}_i = k_i + I_i$.

In order to get the local coefficients $c_1(S_{1_{\text{cusp}}}, \delta)$ at the cusp $S_1(-1, 0)$, we introduce the transformation $u := x + 1, v := y$; then, system (6) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u^5 + \frac{25}{7}u^4 - \frac{26}{7}u^3 + \frac{8}{7}u^2 + \varepsilon \bar{q}(u, v), \end{aligned} \quad (43)$$

with Hamiltonian function $\bar{H}(u, v) = 1/6u^6 - 5/7u^5 + 13/14u^4 - 8/21u^3 + 1/2v^2$, $h_{30} = -8/21$, $\bar{q}(u, v) = f(x)y|_{x=u-1, y=v}$, $\bar{q}_v|_{u=0, v=0} = a_0 - a_1 + a_2 - a_3 + a_4$.

Applying the formula for $c_1(S_{1_{\text{cusp}}}, \delta)$, $c_3(S_{1_{\text{cusp}}}, \delta)$, $c_4(S_{1_{\text{cusp}}}, \delta)$ in Lemma 3, in accordance with Lemma 6, we have

$$\begin{aligned} e_1(\delta) &= \tilde{c}_1(\delta) = c_1(S_{1_{\text{cusp}}}, \delta) \\ &= -21^{1/3} \sqrt{2} (a_0 - a_1 + a_2 - a_3 + a_4), \\ e_4(\delta) &= \tilde{c}_4(\delta) = c_3(S_{1_{\text{cusp}}}, \delta) \\ &= -\frac{21^{2/3} \sqrt{2}}{16} (13a_0 - 5a_1 - 3a_2 + 11a_3 - 19a_4), \\ e_6(\delta) &= \tilde{c}_6(\delta) = c_4(S_{1_{\text{cusp}}}, \delta) \\ &= \frac{21 \sqrt[3]{21} \sqrt{2}}{4096} \\ &\quad \cdot (11669a_0 - 4765a_1 + 1189a_2 + 83a_3 - 75a_4). \end{aligned} \quad (44)$$

In order to get the local coefficient $c_1(S_{2_{\text{saddle}}}, \delta)$ near the hyperbolic saddle $S_2(0, 0)$, we introduce the transformation $u := \sqrt{3/7}x, v := y, \tau := \sqrt{3/7}t$; then, system (6) becomes

$$\begin{aligned} \frac{du}{d\tau} &= v, \\ \frac{dv}{d\tau} &= -\frac{343}{27}u^5 - \frac{70}{27}\sqrt{21}u^4 + \frac{28}{9}u^3 + \frac{10}{9}\sqrt{21}u^2 + u, \end{aligned} \quad (45)$$

with Hamiltonian system $\hat{H}(u, v) = 343/162u^6 + 14/27\sqrt{21}u^5 - 7/9u^4 - 10/27\sqrt{21}u^3/2u^2 + 1/2v^2$, and $\hat{q}(u, v) = \sqrt{3/7}(\sum_{i=0}^4 a_i x^i y)|_{x=\sqrt{7/3}, y=v}$. Applying the formula for $c_1(S_{2_{\text{saddle}}}, \delta)$ and $c_3(S_{2_{\text{saddle}}}, \delta)$ in Lemma 2, in accordance with Lemma 5, we have

$$\begin{aligned} e_2(\delta) &= c_1(\delta) = \tilde{c}_2(\delta) = c_1(S_{2_{\text{saddle}}}, \delta) = -\frac{1}{3}\sqrt{21}a_0, \\ \tilde{c}_2(\delta) &= 2c_1(S_{2_{\text{saddle}}}, \delta) = -\frac{2}{3}\sqrt{21}a_0, \\ e_5(\delta) &= c_3(\delta) = c_3(S_{2_{\text{saddle}}}, \delta) = -\frac{7\sqrt{21}}{54}(10a_1 - 3a_2), \\ \tilde{c}_5(\delta) &= 2c_3(S_{2_{\text{saddle}}}, \delta) = -\frac{7\sqrt{21}}{27}(10a_1 - 3a_2). \end{aligned} \quad (46)$$

Let $e_1(\delta) = e_2(\delta) = 0$; we have

$$\begin{aligned} a_0 &= 0, \\ a_1 &= a_2 - a_3 + a_4. \end{aligned} \quad (47)$$

Under this case,

$$\begin{aligned} e_3(\delta) &= \oint_{L_1} f(x) dt = \oint_{L_1} \frac{f(x)}{y} dx = \oint_{L_1} \frac{f(x)}{\sqrt{-2A(x)}} dx \\ &= 2 \int_{-1}^0 \frac{f(x)}{\sqrt{-2A(x)}} dx. \end{aligned} \quad (48)$$

Taking (47) into (48), we have with the help of Maple 13,

$$e_3(\delta) = \sum_{i=2}^9 l_i a_i, \quad (49)$$

where

$$\begin{aligned} l_2 &= 2\sqrt{3} \arcsin\left(\frac{1}{8}\right) - \sqrt{3}\pi, \\ l_3 &= -\frac{12}{7}\sqrt{3} \arcsin\left(\frac{1}{8}\right) + \frac{6}{7}\sqrt{3}\pi + \frac{6\sqrt{21}}{7}, \\ l_4 &= \frac{150\sqrt{3}}{49} \arcsin\left(\frac{1}{8}\right) - \frac{75\sqrt{3}}{49}\pi - \frac{33\sqrt{21}}{49}. \end{aligned} \quad (50)$$

Let $c_1(\delta) = 0$, i.e., $a_0 = 0$; we have

$$\begin{aligned} c_2(\delta) &= \oint_{L_1} \frac{f(x)}{y} dx = \oint_{L_2} \frac{f(x)}{\sqrt{-2A(x)}} dx \\ &= 2 \int_0^{9/7} \frac{f(x) - a_0}{\sqrt{-2A(x)}} dx. \end{aligned} \quad (51)$$

Direct computation gives

$$c_2(\delta) = \sum_{i=1}^9 J_i a_i, \quad (52)$$

where

$$\begin{aligned} J_1 &= 3/4\sqrt{21}, \\ J_2 &= 2\sqrt{3} \arcsin\left(\frac{1}{2}\right) - \frac{3}{4}\sqrt{21} + \sqrt{3}\pi, \\ J_3 &= -\frac{12\sqrt{3}}{7} \arcsin\left(\frac{1}{8}\right) + \frac{45\sqrt{21}}{28} - \frac{6\sqrt{3}}{7}\pi, \\ J_4 &= \frac{150\sqrt{3}}{49} \arcsin\left(\frac{1}{8}\right) - \frac{279\sqrt{21}}{196} + \frac{75\sqrt{3}}{49}\pi. \end{aligned} \quad (53)$$

Then, by (49) and (51) and Lemma 6, we have

$$\tilde{c}_3(\delta) = \sum_{i=1}^9 \tilde{J}_i a_i, \quad (54)$$

where $\tilde{J}_1 = J_1, \tilde{J}_i = l_i + J_i, i = 2, 3, \dots, 9$.

Last we compute b_0 , b_1 , d_0 , and d_1 in (36) and (37). Taking a linear transformation $u = 4\sqrt{30}/49(x + 3/7)$, $v = y$

with time scaling $d\tau = 4\sqrt{30}/49 dt$ and applying Lemma 5 and program in [20], we obtain

$$\begin{aligned} b_0 &= \sqrt{30}\pi\left(\frac{49}{60}a_0 - \frac{7}{20}a_1 + \frac{3}{20}a_2 - \frac{9}{140}a_3 + \frac{27}{980}a_4\right), \\ b_1 &= \sqrt{30}\pi\left(\frac{939662563}{99532800}a_0 - \frac{829307801}{165888000}a_1 + \frac{254004191}{55296000}a_2 - \frac{71782697}{18432000}a_3 + \frac{17630543}{6144000}a_4\right). \end{aligned} \quad (55)$$

Taking a linear transformation $u = \sqrt{40/7}(x - 1)$, $v = y$ with time scaling $d\tau = \sqrt{40/7}dt$ and applying Lemma 5 and program in [20], we obtain

$$\begin{aligned} d_0 &= \frac{1}{10}\pi\sqrt{70}(a_0 + a_1 + a_2 + a_3 + a_4), \\ d_1 &= \frac{7\sqrt{70}}{64000}\pi(327a_0 + 111a_1 - 25a_2 - 81a_3 - 57a_4). \end{aligned} \quad (56)$$

4. Proof of Theorem 1

In this section, we prove Theorem 1. Solving the equations $c_0(\delta) = e_0(\delta) = c_1(S_{1_{\text{cusp}}}, \delta) = c_1(S_{2_{\text{saddle}}}, \delta) = 0$ gives

$$\begin{aligned} a_0 &= 0, \\ a_1 &= -\frac{27}{56}, \\ a_2 &= -\frac{27}{28}, \\ a_3 &= \frac{29}{56}. \end{aligned} \quad (57)$$

We take $\delta = (a_0, a_1, a_2, a_3, 1)$ and $\delta_0 = (0, -27/56, -27/28, 29/56, 1)$; then,

$$\begin{aligned} e_3(\delta_0) &= -\frac{3\sqrt{3}}{196}\left(8\pi - 16\arcsin\left(\frac{1}{8}\right) + 15\sqrt{7}\right) \\ &\approx -1.665254593 < 0, \\ c_2(\delta_0) &= -\frac{3\sqrt{3}}{196}\left(-8\pi - 16\arcsin\left(\frac{1}{8}\right) + 15\sqrt{7}\right) \\ &\approx -0.3326673051 < 0, \\ \tilde{c}_3(\delta_0) &= -\frac{3\sqrt{3}}{98}\left(-16\arcsin\left(\frac{1}{8}\right) + 15\sqrt{7}\right) \\ &\approx -1.997921898 < 0. \end{aligned} \quad (58)$$

Therefore, there exist h_{10} , h_{20} , and h_{30} , satisfying $0 < -h_{10} \ll 1$, $0 < -h_{20} \ll 1$, and $0 < h_{30} \ll 1$, respectively, such that

$$\begin{aligned} M_1(h_{10}, \delta_0) &= e_3(\delta_0)h_{10} + O(|h_{10}|^{7/6}) > 0, \\ M_2(h_{20}, \delta_0) &= c_2(\delta_0)h_{20} + O(h_{20}^2 \ln|h_{20}|) > 0, \\ M_3(h_{30}, \delta_0) &= \tilde{c}_3(\delta_0)h_{30} + O(h_{30}^{7/6}) < 0, \end{aligned} \quad (59)$$

while taking $h = 100$,

$$\begin{aligned} M_3(100, \delta_0) &= \oint_{L_{100}^3} f(x, \delta_0) y dx \\ &= \oint_{L_{100}^3} f(x, \delta_0) \sqrt{200 - A(x)} dx \\ &\approx 1678.486757 > 0, \end{aligned} \quad (60)$$

and thus there exists a z_{30} between $h = h_{30}$ and $h = 100$, such that

$$M_3(z_{30}, \delta_0) = 0. \quad (61)$$

It is easy to find

$$\text{Rank} \frac{\partial(c_0(\delta), e_0(\delta), c_1(S_{1_{\text{cusp}}}, \delta), c_1(S_{2_{\text{saddle}}}, \delta))}{\partial(a_0, a_1, a_2, a_3)} = 4. \quad (62)$$

Therefore, $e_0(\delta)$, $c_0(\delta)$, $c_1(S_{1_{\text{cusp}}}, \delta)$, and $c_1(S_{2_{\text{saddle}}}, \delta)$ can be taken as free parameters; we denote them by e_0 , c_0 , $c_1(S_{1_{\text{cusp}}})$, and $c_1(S_{2_{\text{saddle}}})$, respectively. Next, we take these free parameters to perturb 8 zeros.

We take $c_1(S_{2_{\text{saddle}}})$, $c_1(S_{1_{\text{cusp}}})$, c_0 , e_0 , in turn satisfying

$$\begin{aligned} c_1(S_{2_{\text{saddle}}}) &< 0, \\ c_1(S_{1_{\text{cusp}}}) &> 0, \\ c_0 &> 0, \\ e_0 &< 0, \quad 0 < |e_0| \ll |c_0| \\ &\ll |c_1(S_{1_{\text{saddle}}})| \ll |c_1(S_{2_{\text{cusp}}})| \ll 1. \end{aligned} \quad (63)$$

We take δ_1 in $U_1(\delta_0, \epsilon) \triangleq \{\delta | e_0, c_0, c_1(S_{1_{\text{cusp}}}), c_1(S_{2_{\text{saddle}}}) \text{ satisfying (62)}\}$ which is a very small neighborhood of δ_0 .

Therefore, $e_0 < 0$, $e_1 > 0$, $e_2 < 0$, $|e_0| \ll |e_1| \ll |e_2| \ll |e_3(\delta_0)|$ for $M_1(h, \delta_1)$ by Lemma 6; then, there exist 3 zeros of h_{11} , h_{12} and h_{13} of $M_1(h, \delta_1)$ satisfying $0 < -h_{13} < -h_{12} < -h_{11} < -h_{10} \ll 1$.

Under (63), $c_0 > 0$, $c_1 < 0$, $|c_0| \ll |c_1| \ll |c_2(\delta_0)|$ for $M_2(h, \delta_1)$ by Lemma 6; then, there exist 2 zeros h_{21} , h_{22} of $M_2(h, \delta_1)$ satisfying $0 < -h_{22} < -h_{21} < -h_{20} \ll 1$.

Under (63), $\tilde{c}_0 > 0$, $\tilde{c}_1 > 0$, $\tilde{c}_2 < 0$, $|\tilde{c}_0| \ll |\tilde{c}_1| \ll |\tilde{c}_2| \ll |\tilde{c}_3|$ ($|\delta_0|$) for $M_3(h, \delta_1)$ by Lemma 6; then, there exist 3 zeros h_{31} , h_{32} , and h_{33} of $M_3(h, \delta_1)$ satisfying $0 < h_{33} < h_{32} < h_{31} < h_{30} \ll 1$.

Note that

$$M_3(h, \delta_1) = M_3(h, \delta_0) + O(\epsilon), \quad (64)$$

and in accordance with (59), we obtain a z_{30}^* near z_{30} , such that $M_3(z_{30}^*, \delta_1) = 0$; therefore, there is another zero of $M_3(h, \delta_1)$ near z_{30} .

The above 9 zeros are isolated real zeros; therefore, they correspond to 9 limit cycles of system (6). The main theorem is proved.

5. Conclusion and Discussion

In the paper, we have applied the theories of Hopf bifurcation, homoclinic loop bifurcation, and heteroclinic loop bifurcation to detect the limit cycles near the center and polycycle locally. The main routine is to prove the independence of the coefficients of the asymptotic expansions of the Melnikov functions and then treat them as free perturbation parameters. The computational analysis shows that there exist at least 9 limit cycles in the suitably damped system. The result gives a relative larger lower bound on the number of limit cycles for the Liénard system of degree 5. It is interesting to show that the asymptotic expansions of the Melnikov functions not only are the efficient tools to detect limit cycles such as a complicated investigation in [24] but also have been successfully applied to study the existence of periodic traveling waves and the coexistence of periodic solitary traveling waves (see [25, 26]). It may be more interesting to investigate the periodic traveling waves in external perturbation considered in the model given in [27]. It should also be pointed that there exist more questions left to solve for system (5), for example, what is the maximal number of limit cycles, or limit cycles by Poincaré bifurcation, or limit cycles by Hopf, or limit cycles by Hopf bifurcation, how about the global dynamics, and what is the period function of the undamped system. However, it is not easy at all to solve these questions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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