Research Article

Asymptotic Dynamics of a Stochastic SIR Epidemic System Affected by Mixed Nonlinear Incidence Rates

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This paper considers a stochastic SIR epidemic system affected by mixed nonlinear incidence rates. Using Markov semigroup theory and the Fokker–Planck equation, we explore the asymptotic dynamics of the stochastic system. We first investigate the existence of a positive solution and its uniqueness. Furthermore, we prove that the stochastic system has an asymptotically stable stationary distribution. In addition, the sufficient conditions for disease extinction are also obtained, which imply that the white noise can suppress and control the spread of infectious diseases. Finally, in order to illustrate the analytical results, we give some numerical simulations.

1. Introduction

In 1927, the threshold conditions for disease transmission were established by Kermack and McKendrick [1]. In recent decades, many studies have applied threshold theory to various epidemic systems, and there have been a large amount of results related to the dynamical behaviors for various models. In addition, nonlinear incidence rates are very significant and are used frequently in the dynamics of epidemic models [2–11]. Hethcote [12] introduced an SIR epidemic system affected by bilinear incidence rate λSI :

$$\begin{cases} dS = (u - uS - \lambda SI)dt, \\ dI = (\lambda SI - uI - \eta I)dt, \\ dR = (\eta I - uR)dt, \end{cases} \quad (1)$$

where the parameters u , λ , and η are positive constants. S , I , and R represent the density of susceptible, infectious, and recovered individuals, respectively. Suppose that the recruitment rate is the same as the natural death rate, denoted by u . λ is the transmission coefficient, η represents the recovery rate, and $\lambda_1 SI$ represents the bilinear incidence rate. Furthermore, some researchers [13] proposed SIR epidemic

models with saturated incidence ($\lambda SI/(1 + mI)$) similar to the following form:

$$\begin{cases} dS = \left(u - uS - \frac{\lambda SI}{1 + mI} \right) dt, \\ dI = \left(\frac{\lambda SI}{1 + mI} - uI - \eta I \right) dt, \\ dR = (\eta I - uR) dt, \end{cases} \quad (2)$$

where m is a positive constant. ($\lambda_2 SI/(1 + mI)$) represents the saturated incidence rate. Particularly, Liu et al. [14] studied an SIS epidemic system with general nonlinear incidence rate effect by employing Markov semigroup theory. Inspired by previous works, we consider that the infected person may have immunity to return to recovery class after being cured or return to the susceptible class again. And the epidemic affected by mixed nonlinear incidence rates is more realistic compared to epidemic affected by a single nonlinear incidence rate. Thus, when two different incidence rates (bilinear and saturated incidence rates) are considered at the same time, the following

deterministic SIR epidemic system affected by two different nonlinear incidence rates is studied:

$$\begin{cases} dS = \left[u - uS - p\lambda_1 SI - (1-p)\frac{\lambda_2 SI}{1+mI} \right] dt + [(1-\alpha)\eta I] dt, \\ dI = \left[p\lambda_1 SI + (1-p)\frac{\lambda_2 SI}{1+mI} - uI - \eta I \right] dt, \\ dR = (\alpha\eta I - uR) dt, \end{cases} \quad (3)$$

where the parameters α , λ_1 , λ_2 , and p are positive constants. α denotes the probability that the infected returns to recovery class, $1 - \alpha$ denotes the probability that the infected returns to susceptible class correspondingly, λ_1 and λ_2 are the transmission coefficients, p represents the probability that bilinear incidence rate occurs, and $1 - p$ represents the probability that saturated incidence rate occurs correspondingly. The basic reproduction number $R_0 = ((p\lambda_1 + (1-p)\lambda_2)/(u + \eta))$ is the threshold of model (3).

However, the population systems with stochastic effect present some complex dynamics, which attracts the

attention of widespread researchers [15–18]. There are many kinds of noise in the environment, among which the white noise is a relatively stable noise in the process of propagation. It has been widely used and studied in physics and has a relatively complete system in mathematical analysis and application. In order to consider a more realistic disease model and make it more practical to study the influence of environmental noise on infectious diseases, the white noise is used. There is a long history of using the white noise to depict the influence of environmental randomness on the spread of disease; some researchers [19, 20] put forward that the environmental white noise disturbs the system parameters stochastically, and parameter λ_i ($i = 1, 2$) is an important parameter for the spread of disease. The researchers [21–29] investigated the effect of environment on the dynamic behaviors by introducing stochastic perturbation into deterministic models. Based on the discussion above, we assume that $\lambda_i \rightarrow \lambda_i + \sigma_i B_i(t)$, where $B_i(t)$ represents a standard Brownian motion with intensity $\sigma_i > 0$ ($i = 1, 2$). Now, the stochastic SIR epidemic system corresponding to system (3) is as follows:

$$\begin{cases} dS = \left[u - uS - p\lambda_1 SI - (1-p)\frac{\lambda_2 SI}{1+mI} \right] dt + [(1-\alpha)\eta I] dt - p\sigma_1 SI dB_1(t) - (1-p)\frac{\sigma_2 SI}{1+mI} dB_2(t), \\ dI = \left[p\lambda_1 SI + (1-p)\frac{\lambda_2 SI}{1+mI} - uI - \eta I \right] dt + p\sigma_1 SI dB_1(t) + (1-p)\frac{\sigma_2 SI}{1+mI} dB_2(t), \\ dR = (\alpha\eta I - uR) dt. \end{cases} \quad (4)$$

Since $d(S + I + R) = [u - u(S + I + R)]dt$, for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$ and $S(0) + I(0) + R(0) = 1$, we always have $S(t) + I(t) + R(t) = 1$. Therefore, $(S(t), I(t), R(t) \in \mathbb{R}_+^3 | S(t) + I(t) + R(t) = 1)$ is a positive invariant set

of system (4). For simplicity, we let $R(t) = 1 - S(t) - I(t)$; then, the dynamics of system (4) is equivalent to the following two-dimensional system:

$$\begin{cases} dS = \left[u - uS - p\lambda_1 SI - (1-p)\frac{\lambda_2 SI}{1+mI} \right] dt + [(1-\alpha)\eta I] dt - p\sigma_1 SI dB_1(t) \\ \quad - (1-p)\frac{\sigma_2 SI}{1+mI} dB_2(t), \\ dI = \left[p\lambda_1 SI + (1-p)\frac{\lambda_2 SI}{1+mI} - uI - \eta I \right] dt + p\sigma_1 SI dB_1(t) + (1-p)\frac{\sigma_2 SI}{1+mI} dB_2(t). \end{cases} \quad (5)$$

Next, we study the dynamical behaviors of system (5) which are affected by mixed nonlinear incidence rates. Particularly, as the main purpose, we study the asymptotically stable stationary distribution and extinction of

epidemic by establishing the corresponding sufficient conditions.

In this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ represents a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the

usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). Define $B(t)$ is on the probability space. Denote $\mathbb{R}_+^2 = \{x_i > 0, i = 1, 2\}$.

2. Preliminary Knowledge

2.1. Markov Semigroups and Fokker–Planck Equation. In this section, we provide some definitions about Markov semigroups [14], asymptotic properties [30–34], and Fokker–Planck equation to verify our results.

Denote $\mathbb{X}$, Σ , and m to be a metric space, σ -algebra of Borel sets, and the Lebesgue measure on $(\mathbb{X}, \Sigma)$, respectively. $\mathbb{D} = \{h \in L^1: h \geq 0, \|h\| = 1\}$. $\{P(t)\}_{t \geq 0}$ represents an integral Markov semigroup with a continuous kernel $\mathcal{K}(t; x; y)$ for $t > 0$, which satisfies $\int_{\mathbb{X}} \mathcal{K}(x, y) m(dx) = 1$ for $\forall y \in \mathbb{X}$. For the diffusion process $(S(t), I(t))$, i.e., the transition probability function is $\mathcal{P}(t, x, y, A) = \text{Prob}\{(S(t), I(t)) \in A\}$. Define $\varphi_1(x, y) = x^2 y^2$, $\varphi_2(x, y) = (x^2 y^2 / (1 + my)^2)$, and

$$\begin{cases} f_1(x, y) = u - ux - p\lambda_1 xy - (1-p) \frac{\lambda_2 xy}{1+my} + (1-\alpha)\eta y, \\ f_2(x, y) = p\lambda_1 xy + (1-p) \frac{\lambda_2 xy}{1+my} - uy - \eta y. \end{cases} \quad (6)$$

Remark 1. If for $\forall h \in \mathbb{D}$, we have

$$\int_0^\infty P(t)h dt > 0, \quad \text{a.e.} \quad (7)$$

Then, this semigroup is asymptotically stable or sweeping with respect to compact sets [30, 31]; this property is called the Foguel alternative [34].

Remark 2. If the distribution of $(S(t), I(t))$ is absolutely continuous with respect to the Lebesgue measure with the density $U(t, x, y)$, then U satisfies the Fokker–Planck equation [32]:

$$\begin{aligned} \frac{\partial U}{\partial t} = & \frac{\sigma_1^2 p^2}{2} \left(\frac{\partial^2 (\varphi_1 U)}{\partial x^2} - 2 \frac{\partial^2 (\varphi_1 U)}{\partial x \partial y} + \frac{\partial^2 (\varphi_1 U)}{\partial y^2} \right) \\ & + \frac{\sigma_2^2 (1-p)^2}{2} \left(\frac{\partial^2 (\varphi_2 U)}{\partial x^2} - 2 \frac{\partial^2 (\varphi_2 U)}{\partial x \partial y} + \frac{\partial^2 (\varphi_2 U)}{\partial y^2} \right) \\ & - \frac{\partial(f_1 U)}{\partial x} - \frac{\partial(f_2 U)}{\partial y}. \end{aligned} \quad (8)$$

Remark 3. Denote $P(t)V(x, y) = U(x, y, t)$ for $V \in \mathbb{D}$, and $\mathcal{A}$ represents the infinitesimal generator of semigroup $\{P(t)\}_{t \geq 0}$; then, we have

$$\begin{aligned} \mathcal{A}^* V = & \frac{(1-p)^2 \sigma_2^2 \varphi_2}{2} \left(\frac{\partial^2 (V)}{\partial x^2} - 2 \frac{\partial^2 (V)}{\partial x \partial y} + \frac{\partial^2 (V)}{\partial y^2} \right) \\ & + \frac{p^2 \sigma_1^2 \varphi_1}{2} \left(\frac{\partial^2 (V)}{\partial x^2} - 2 \frac{\partial^2 (V)}{\partial x \partial y} + \frac{\partial^2 (V)}{\partial y^2} \right) \\ & + \frac{\partial(f_1 V)}{\partial x} + \frac{\partial(f_2 V)}{\partial y}. \end{aligned} \quad (9)$$

2.2. Some Lemmas about the Asymptotically Stable Stationary Distribution. In this section, five lemmas are given to study the asymptotically stable stationary distribution of system (5).

Lemma 1. The transition probability function $\mathcal{P}(t, x_0, y_0, A)$ has a continuous density $\mathcal{K}(t, x, y; x_0, y_0)$ with respect to the Lebesgue measure.

Proof. Denote

$$\begin{aligned} \vec{a}(x, y) &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \\ \vec{b}^1(x, y) &= \begin{bmatrix} -p\sigma_1 xy \\ p\sigma_1 xy \end{bmatrix}, \\ \vec{b}^2(x, y) &= \begin{bmatrix} -(1-p) \frac{\lambda_2 xy}{1+my} \\ (1-p) \frac{\lambda_2 xy}{1+my} \end{bmatrix}, \end{aligned} \quad (10)$$

where $a_1 = u - ux - p\lambda_1 xy - (1-p) \frac{\lambda_2 xy}{1+my} + (1-\alpha)\eta y$, $a_2 = p\lambda_1 xy + (1-p) \frac{\lambda_2 xy}{1+my} - uy - \eta y$, and $(x, y) \in \mathbb{R}_+^2$. $[\vec{a}, \vec{b}^i] (i=1, 2)$ is defined as

$$[\vec{a}, \vec{b}^i]_j(x) = \sum_{k=1}^2 \left(a_k \frac{\partial b_j^i}{\partial x_k}(x) - b_k^i \frac{\partial a_j}{\partial x_k}(x) \right), \quad j = 1, 2. \quad (11)$$

Then,

$$[\vec{a}, \vec{b}^i] = \begin{pmatrix} a_1 \frac{\partial b_1^i}{\partial x} - b_1^i \frac{\partial a_1}{\partial x} + a_2 \frac{\partial b_1^i}{\partial y} - b_2^i \frac{\partial a_1}{\partial y} \\ a_1 \frac{\partial b_2^i}{\partial x} - b_1^i \frac{\partial a_2}{\partial x} + a_2 \frac{\partial b_2^i}{\partial y} - b_2^i \frac{\partial a_2}{\partial y} \end{pmatrix}. \quad (12)$$

Denote $b_1^i = -b_2^i$, and by direct calculation, we have

$$\begin{aligned}
\left| \vec{b}^1 \left[\vec{a}, \vec{b}^1 \right] \right| &= \left| \begin{matrix} b_1^1 & a_1 \frac{\partial b_1^1}{\partial x} - b_1^1 \frac{a_1}{\partial x} + a_2 \frac{\partial b_1^1}{\partial y} - b_2^1 \frac{a_1}{\partial y} \\ b_2^1 & a_1 \frac{\partial b_2^1}{\partial x} - b_1^1 \frac{a_2}{\partial x} + a_2 \frac{\partial b_2^1}{\partial y} - b_2^1 \frac{a_2}{\partial y} \end{matrix} \right| \\
&= \alpha \eta p^2 \sigma_1^2 x^2 y^2 > 0, \\
\left| \vec{b}^2 \left[\vec{a}, \vec{b}^2 \right] \right| &= \left| \begin{matrix} b_1^2 & a_1 \frac{\partial b_1^2}{\partial x} - b_1^2 \frac{a_1}{\partial x} + a_2 \frac{\partial b_1^2}{\partial y} - b_2^2 \frac{a_1}{\partial y} \\ b_2^2 & a_1 \frac{\partial b_2^2}{\partial x} - b_1^2 \frac{a_2}{\partial x} + a_2 \frac{\partial b_2^2}{\partial y} - b_2^2 \frac{a_2}{\partial y} \end{matrix} \right| \\
&= \alpha \eta (1-p)^2 \frac{\sigma_2^2 x^2 y^2}{(1+m y)^2} > 0,
\end{aligned} \tag{13}$$

which shows that $\vec{b}^i$ and $[\vec{a}, \vec{b}^i]$ are independent on $\mathbb{R}_+^2$. Thus, according to Hörmander theorem (see [35]), $\mathcal{P}(t, x_0, y_0, A)$ has a continuous density $\mathcal{K}(t, x, y; x_0, y_0)$ and $\mathcal{K} \in C^\infty((0, \infty) \times \mathbb{R}_+^2 \times \mathbb{R}_+^2)$. The proof is completed. In addition, we can obtain that the semigroup $\{P(t)\}_{t \geq 0}$ is an integral Markov semigroup. $\square$

Lemma 2. For each $(x_0, y_0) \in E$ and $(x, y) \in E$, where $\text{supp} U_* = \{(x, y) \in \mathbb{R}_+^2 : (u/(u + \alpha \eta)) < x + y < 1\} := E$, there exists $T > 0$ such that $\mathcal{K}(T, x, y; x_0, y_0) > 0$.

Proof. Fix a point $(x_0, y_0) \in \mathbb{R}_+^2$ and a continuous function $\phi \in L^2([0, T]; \mathbb{R})$; the following system is first considered:

$$\begin{cases} x_\phi(t) = x_0 + \int_0^t (f_1(x_\phi(s), y_\phi(s)) - p\sigma_1 \phi x_\phi(s) y_\phi(s)) ds - \int_0^t (1-p) \frac{\sigma_2 \phi x_\phi(s) y_\phi(s)}{1+m y_\phi(s)} ds, \\ y_\phi(t) = y_0 + \int_0^t (f_2(x_\phi(s), y_\phi(s)) + p\sigma_1 \phi x_\phi(s) y_\phi(s)) ds + \int_0^t (1-p) \frac{\sigma_2 \phi x_\phi(s) y_\phi(s)}{1+m y_\phi(s)} ds. \end{cases} \tag{14}$$

Step 1. Find $D_{x_0, y_0; \phi}^i$ by the perturbation method. Denote $\mathbf{f}'$, $\mathbf{g}^1$, and $\mathbf{g}^2$ are the Jacobians of

$$\begin{aligned}
\mathbf{f} &= \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix}, \\
\mathbf{g}^1 &= \begin{pmatrix} -p\sigma_1 x y \\ p\sigma_1 x y \end{pmatrix}, \\
\mathbf{g}^2 &= \begin{pmatrix} -(1-p) \frac{\sigma_2 x y}{1+m y} \\ (1-p) \frac{\sigma_2 x y}{1+m y} \end{pmatrix},
\end{aligned} \tag{15}$$

respectively; then,

$$F^i = \mathbf{f}'(x_\phi(t), y_\phi(t)) + \mathbf{g}^i(x_\phi(t), y_\phi(t))\phi. \tag{16}$$

For $0 \leq t_0 \leq t \leq T$, let $Q^i(t_0, t_0) = I$ and $(\partial Q^i(t, t_0)/\partial t) = F^i(t)Q^i(t, t_0)$; we get

$$D_{x_0, y_0; \phi}^i h = \int_0^T Q^i(T, s) \mathbf{g}^i(s) h(s) ds. \tag{17}$$

Assume that the rank of $D_{x_0, y_0; \phi}^i$ is 2 for some $\phi \in L^2([0, T]; \mathbb{R})$; then, for $(x, y) = (x_\phi(T), y_\phi(T))$, we have $\mathcal{K}(T, x, y; x_0, y_0) > 0$.

Step 2. Prove that the derivative $D_{x_0, y_0; \phi}^i$ has rank 2. Replace system (14) with

$$\begin{cases} x'_\phi(t) = y'_\phi(t) = \begin{cases} f_1(x_\phi(t), y_\phi(t)) - p\sigma_1 \phi x_\phi(t) y_\phi(t) \\ -(1-p) \frac{\sigma_2 \phi x_\phi(t) y_\phi(t)}{1+m y_\phi(t)}, \\ f_2(x_\phi(t), y_\phi(t)) + p\sigma_1 \phi x_\phi(t) y_\phi(t) \\ +(1-p) \frac{\sigma_2 \phi x_\phi(t) y_\phi(t)}{1+m y_\phi(t)}. \end{cases} \end{cases} \tag{18}$$

Define $\epsilon \in (0, T)$ and $h(t) = (\mathbf{1}_{[T-\epsilon, T]}/x_\phi(t) y_\phi(t) f(x_\phi(t), y_\phi(t)))$ for $t \in [0, T]$; here, $\mathbf{1}_{[T-\epsilon, T]}$ denotes the characteristic function of interval $[T-\epsilon, T]$. Using Taylor expansion, we obtain

$$Q^i(T, s) = I + F^i(T)(T-s) + o(T-s). \tag{19}$$

Then, we get

$$\begin{aligned} D_{x_0, y_0; \phi}^i h &= \int_0^T [I + F^i(T)(T-s) + o(T-s)] g^i(s) h(s) ds \\ &= \epsilon \mathbf{v} + \frac{\epsilon^2}{2} F^i(T) \mathbf{v} + o(\epsilon^2), \end{aligned} \tag{20}$$

and $\mathbf{v}^1 = (-\sigma_1, \sigma_1)^T$, $\mathbf{v}^2 = (-\sigma_2, \sigma_2)^T$.

Then,

$$\begin{aligned} F^1(T)\mathbf{v}^1 &= \sigma_1 \begin{bmatrix} A \\ B \end{bmatrix}, \\ |\mathbf{v}^1 F^1(T)\mathbf{v}^1| &= \begin{vmatrix} -\sigma_1 & \sigma_1 A \\ \sigma_1 & \sigma_1 B \end{vmatrix} = \sigma_1^2 \alpha \eta > 0, \\ F^2(T)\mathbf{v}^2 &= \sigma_2 \begin{bmatrix} C \\ D \end{bmatrix}, \\ |\mathbf{v}^2 F^2(T)\mathbf{v}^2| &= \begin{vmatrix} -\sigma_2 & \sigma_2 C \\ \sigma_2 & \sigma_2 D \end{vmatrix} = \sigma_2^2 \alpha \eta > 0, \end{aligned} \quad (21)$$

where

$$\begin{aligned} A &= p(\lambda_1 + \sigma_1 \phi)(y - x) \\ &\quad + (1 - p)\lambda_2 \frac{-x + y + my^2}{(1 + my)^2} + (1 - \alpha)\eta + u, \\ B &= p(\lambda_1 + \sigma_1 \phi)(x - y) + (1 - p)\lambda_2 \frac{x - y - my^2}{(1 + my)^2} - (u + \eta), \\ C &= u + p\lambda_1 x + (1 - p) \frac{\lambda_2 x}{(1 + my)^2} - (1 - p) \frac{\sigma_2 x}{(1 + my)^2} \end{aligned}$$

$$\begin{aligned} &+ (1 - \alpha)\eta - p\lambda_1 y - (1 - p) \frac{\lambda_2 y}{1 + my} \\ &+ (1 - p) \frac{\sigma_2 y}{1 + my}, \\ D &= -u - p\lambda_1 x - (1 - p) \frac{\lambda_2 x}{(1 + my)^2} + (1 - p) \frac{\sigma_2 x}{(1 + my)^2} \\ &\quad - \eta + p\lambda_1 y + (1 - p) \frac{\lambda_2 y}{1 + my} - (1 - p) \frac{\sigma_2 y}{1 + my}. \end{aligned} \quad (22)$$

Therefore, the rank of $D_{x_0, y_0; \phi}^i$ is 2.

Step 3. Verify that, for any $(x_0, y_0) \in E$ and $(x, y) \in E$, there exist control function ϕ and $T > 0$ such that $(x_\phi(0), y_\phi(0)) = (x_0, y_0)$ and $(x_\phi(T), y_\phi(T)) = (x, y)$. Let $z_\phi = x_\phi + y_\phi$, and

$$\begin{cases} g_1(x, z) = u - ux - p\lambda_1 x(z - x) - (1 - p) \frac{\lambda_2 x(z - x)}{1 + m(z - x)}, \\ g_2(x, z) = u + \alpha \eta x - (u + \alpha \eta)z. \end{cases} \quad (23)$$

System (18) becomes

$$\begin{cases} x'_\phi(t) = g_1(x_\phi(t), z_\phi(t)) - p\sigma_1 \phi x_\phi(t)(z_\phi(t) - x_\phi(t)) - (1 - p) \frac{\sigma_2 \phi x_\phi(t)(z_\phi(t) - x_\phi(t))}{1 + m(z_\phi(t) - x_\phi(t))}, \\ z'_\phi(t) = g_2(x_\phi(t), z_\phi(t)). \end{cases} \quad (24)$$

Denote

$$E_0 = \left\{ (x, z) \in \mathbb{R}_+^2 : 0 < x < 1, \frac{u}{u + \alpha \eta} < z < 1, x < z \right\}. \quad (25)$$

Using the same proof method of Lemma 3.2 in [36], we can obtain that, for any $(x_0, z_0) \in E_0$ and $(x, z) \in E_0$, there exist control function ϕ and $T > 0$ such that $(x_\phi(0), z_\phi(0)) = (x_0, z_0)$ and $(x_\phi(T), z_\phi(T)) = (x, z)$. Now, the positivity of $\mathcal{K}$ is proved by support theorems (see [37–39]).

Lemma 3. If $R_0 - ((p^2 \sigma_1^2 + (1 - p)^2 \sigma_2^2)/2(u + \eta)) > 1$, for every density f and the semigroup $\{P(t)\}_{t \geq 0}$, the following conclusion holds:

$$\lim_{t \rightarrow \infty} \iint_E P(t)f(x, y) dx dy = 1. \quad (26)$$

Proof. Substitute $Z(t) = S(t) + I(t)$; we can obtain that

$$u - (u + \alpha \eta)Z(t) < \frac{dZ(t)}{dt} < u - uZ(t), \quad t \in (0, \infty) \text{ a.s.} \quad (27)$$

Next, in the following three cases, (i) $Z(0) \in (0, (u/(u + \alpha \eta)))$, (ii) $Z(0) \in ((u/(u + \alpha \eta)), 1)$, and (iii) $Z(0) \in (1, \infty)$, we verify that there is $t_0 = t_0(\omega)$ such that

$$\frac{u}{u + \alpha \eta} < Z(t, \omega) < 1, \quad t > t_0, \quad (28)$$

for almost every $\omega \in \Omega$; this proof is completed.

(i) If inequality (28) is not true, then there exists $\Omega' \subset \Omega$ with $\text{Prob}(\Omega') > 0$ which satisfies

$$Z(t, \omega) \in \left(0, \frac{u}{u + \alpha \eta}\right). \quad (29)$$

On the one hand, we can get that $(dZ(t)/dt) > u - (u + \alpha \eta)Z(t) > 0$ from (27). Thus, $Z(t, \omega)$ is strictly increasing on $[0, \infty)$ for any $\omega \in \Omega'$, and then

$\lim_{t \rightarrow \infty} Z(t, \omega) = (u/(u + \alpha\eta))$. In view of the expression of g_2 , we have

$$\begin{aligned} \lim_{t \rightarrow \infty} S(t, \omega) &= 0, \\ \lim_{t \rightarrow \infty} I(t, \omega) &= \frac{u}{u + \alpha\eta}. \end{aligned} \quad (30)$$

Namely,

$$\lim_{t \rightarrow \infty} \frac{\ln I(t) - \ln I(0)}{t} = 0, \quad \omega \in \Omega'. \quad (31)$$

On the other hand, according to Itô's formula, we get

$$\begin{aligned} d \ln I(t) &= \left[p\lambda_1 S + (1-p) \frac{\lambda_2 S}{1+mI} - u - \eta \right] dt \\ &\quad - \left[\frac{1}{2} \left(p^2 \sigma_1^2 S^2 + (1-p)^2 \frac{\sigma_2^2 S^2}{(1+mI)^2} \right) \right] dt \\ &\quad + p\sigma_1 S dB_1(t) + (1-p) \frac{\sigma_2 S}{1+mI} dB_2(t). \end{aligned} \quad (32)$$

Therefore,

$$\begin{aligned} \frac{\ln I(t) - \ln I(0)}{t} &= \frac{1}{t} \int_0^t p\lambda_1 S(s) + (1-p) \frac{\lambda_2 S(s)}{1+mI(s)} ds \\ &\quad - \frac{1}{t} \int_0^t \frac{1}{2} p^2 \sigma_1^2 S^2(s) ds \\ &\quad - \frac{1}{t} \int_0^t \frac{1}{2} (1-p)^2 \frac{\sigma_2^2 S^2(s)}{(1+mI(s))^2} ds \\ &\quad + \frac{1}{t} \int_0^t (-u - \eta) ds + \frac{M_1(t)}{t} + \frac{M_2(t)}{t}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} M_1(t) &= p\sigma_1 \int_0^t S(s) dB_1(s), \\ M_2(t) &= (1-p)\sigma_2 \int_0^t \frac{S(s)}{1+mI(s)} dB_2(s), \\ \langle M_1, M_1 \rangle_t &= p^2 \sigma_1^2 \int_0^t S^2(s) ds, \\ \langle M_2, M_2 \rangle_t &= (1-p)^2 \sigma_2^2 \int_0^t \left(\frac{S(s)}{1+mI(s)} \right)^2 ds. \end{aligned} \quad (34)$$

Clearly,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\langle M_1, M_1 \rangle_t}{t} &= \limsup_{t \rightarrow \infty} \frac{p^2 \sigma_1^2 \int_0^t S^2(s) ds}{t} \\ &< \frac{p^2 \sigma_1^2 \int_0^t 1^2 ds}{t} < p^2 \sigma_1^2 < \infty, \quad \text{a.s.}, \\ \limsup_{t \rightarrow \infty} \frac{\langle M_2, M_2 \rangle_t}{t} &= \limsup_{t \rightarrow \infty} \frac{(1-p)^2 \sigma_2^2 \int_0^t ((S(s)/(1+mI(s))))^2 ds}{t} \\ &< \frac{(1-p)^2 \sigma_2^2 \int_0^t 1^2 ds}{t} < (1-p)^2 \sigma_2^2 \\ &< \infty, \quad \text{a.s.} \end{aligned} \quad (35)$$

By the strong law of large numbers (see [40]), we get

$$\lim_{t \rightarrow \infty} \frac{M_1(t)}{t} = \lim_{t \rightarrow \infty} \frac{M_2(t)}{t} = 0, \quad \text{a.s.} \quad (36)$$

Namely,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{\ln I(t) - \ln I(0)}{t} &= \lim_{t \rightarrow \infty} \left(-u - \eta + \frac{p\lambda_1 \int_0^t S(s) ds}{t} \right) \\
&+ \lim_{t \rightarrow \infty} \frac{(1-p)\lambda_2 \int_0^t (S(s)/(1+mI(s))) ds}{t} \\
&- \lim_{t \rightarrow \infty} \frac{p^2 \sigma_1^2 \int_0^t S^2(s) ds}{2t} \\
&- \lim_{t \rightarrow \infty} \frac{(1-p)^2 \sigma_2^2 \int_0^t (S(s)/(1+mI(s))) ds}{2t} \\
&= -(u + \eta), \quad \text{a.s. on } \Omega',
\end{aligned} \tag{37}$$

this contradiction arises. Hence, the result follows for $Z(0) \in (0, (u/(u + \alpha\eta)))$.

(ii) From (27), we obtain that

$$Z(t) > \frac{u}{u + \alpha} + \left[\left(Z_0 - \frac{u}{u + \alpha} \right) e^{(u+\alpha)t_0} \right] e^{-(u+\alpha)t}, \tag{38}$$

$$Z(t) < 1 + [(Z_0 - 1)e^{ut_0}] e^{-ut} < Z(t),$$

Clearly, conclusion (28) holds in the case of $Z(0) \in ((u/(u + \alpha\eta)), 1)$.

(iii) Suppose that inequality (28) does not hold. On the one hand, using the same way to (ii), we have $\lim_{t \rightarrow \infty} Z(t, \omega) = 1$, $\omega \in \Omega'$, and

$$\begin{aligned}
\lim_{t \rightarrow \infty} S(t, \omega) &= 1, \\
\lim_{t \rightarrow \infty} I(t, \omega) &= 0.
\end{aligned} \tag{39}$$

Clearly,

$$\lim_{t \rightarrow \infty} \frac{\ln I(t) - \ln I(0)}{t} = 0, \quad \omega \in \Omega'. \tag{40}$$

On the other hand, if $R_0 - 1 > ((p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2)/2(u + \eta))$, we obtain

$$\begin{aligned}
\frac{\ln I(t) - \ln I(0)}{t} &= p\lambda_1 + (1-p)\lambda_2 - u - \eta \\
&- \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} \\
&= (u + \eta) \left[R_0 - 1 - \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2(u + \eta)} \right] \\
&> 0, \quad \text{a.s. on } \Omega',
\end{aligned} \tag{41}$$

so there is a contradiction. Thus, the result holds for $Z(0) \in (1, \infty)$. This proof is completed. $\square$

Lemma 4. If $R_0 - ((p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2)/2(u + \eta)) > 1$, the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable or sweeping with respect to compact sets.

Proof. Using the same proof method of Lemma 4.5 in [14], this means that our result holds. We omit it here. $\square$

Lemma 5. The semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable if $R_0 - (p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2/2(u + \eta)) > 1$.

Proof. By system (5), we obtain that

$$\begin{aligned}
L(-\ln I) &= -p\lambda_1 S - (1-p) \frac{\lambda_2 S}{1+mI} + u + \eta \\
&+ \frac{p^2 \sigma_1^2 S^2 + (1-p)^2 \sigma_2^2 ((S/(1+mI)))^2}{2} \\
&\leq -p\lambda_1 S - (1-p) \frac{\lambda_2 S}{1+mI} + u + \eta \\
&+ \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} \\
&= -p\lambda_1 S - (1-p)\lambda_2 S + u + \eta \\
&+ \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} + (1-p)\lambda_2 S \\
&- (1-p) \frac{\lambda_2 S}{1+mI} \\
&= -p\lambda_1 S - (1-p)\lambda_2 S + u + \eta \\
&+ \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} \\
&+ \frac{mI}{1+mI} (1-p)\lambda_2 S.
\end{aligned} \tag{42}$$

Denote

$$\begin{cases} f_1(S, I) = u - uS - p\lambda_1 SI - (1-p)\frac{\lambda_2 SI}{1+mI} + (1-\alpha)\eta I, \\ f_2(S, I) = p\lambda_1 SI + (1-p)\frac{\lambda_2 SI}{1+mI} - uI - \eta I, \end{cases} \quad (43)$$

$$\mathcal{A}^* V = F \left[\frac{\partial^2(V)}{\partial S^2} - 2 \frac{\partial^2(V)}{\partial S \partial I} + \frac{\partial^2(V)}{\partial I^2} \right] + \frac{\partial(f_1 V)}{\partial S} + \frac{\partial(f_2 V)}{\partial I}, \quad (44)$$

where $F = (S^2 I^2 [p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2 ((1/(1+mI)))^2]/2)$, and we call V a Khasminskii function [41].

$\{P(t)\}_{t \geq 0}$ satisfies the Foguel alternative according to Lemma 4. We can exclude sweeping by constructing a nonnegative C^2 -function V and a closed set $U \in \Sigma$ satisfying

$$\sup_{(S,I) \in \Sigma \setminus U} \mathcal{A}^* V(S, I) < -1, \quad (45)$$

□

Step 1. Construct a nonnegative C^2 -function V . For $(S, I) \in E$, define

$$\begin{aligned} g(S, I) = & G \left(-\ln I + \frac{p\lambda_1 + (1-p)\lambda_2}{u} (-S - I) \right) \\ & - \ln S - \ln(1 - S - I) - \ln \left(S + I - \frac{u}{u + \alpha\eta} \right), \end{aligned} \quad (46)$$

where $G > 0$ such that

$$\begin{aligned} & -G(u + \eta) \left(R_0 - 1 - \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2(u + \eta)} \right) + 3u + \alpha\eta \\ & + (p\lambda_1 + (1-p)\lambda_2) + \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} \leq -2. \end{aligned} \quad (47)$$

Let

$$\begin{aligned} V(S, I) &= g(S, I) - g(\bar{S}, \bar{I}) \\ &= G \left(-\ln I + \frac{p\lambda_1 + (1-p)\lambda_2}{u} (-S - I) \right) - \ln S \\ &\quad - \ln(1 - S - I) - \ln \left(S + I - \frac{u}{u + \alpha\eta} \right) - g(\bar{S}, \bar{I}). \end{aligned} \quad (48)$$

Denote

$$\begin{aligned} V_1 &= -\ln I + \frac{p\lambda_1 + (1-p)\lambda_2}{u} (-S - I), \\ V_2 &= -\ln S, \\ V_3 &= -\ln(1 - S - I), \\ V_4 &= -\ln \left(S + I - \frac{u}{u + \alpha\eta} \right) - g(\bar{S}, \bar{I}). \end{aligned} \quad (49)$$

Step 2. Denote a closed set $U \in \Sigma$. For a sufficiently small positive number ε , we denote

$$U = \{(S, I) \in E: \varepsilon \leq S, \varepsilon \leq I, K_1 \leq S + I \leq K_2\}, \quad (50)$$

where $K_1 = (u/(u + \alpha\eta)) + \varepsilon^2$, $K_2 = 1 - \varepsilon^2$. In the set $E \setminus U$, let ε be sufficiently small such that

$$Gm\varepsilon(1-p)\lambda_2 + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u)\varepsilon < 1, \quad (51)$$

$$-\frac{u}{\varepsilon} + Gm(1-p)\lambda_2 + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u) + H < -1, \quad (52)$$

$$-\frac{\alpha\eta}{\varepsilon} + Gm(1-p)\lambda_2 + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u) + H < -1. \quad (53)$$

Step 3. Prove

$$\sup_{(S,I) \in \Sigma \setminus U} \mathcal{A}^* V(S, I) < -1,$$

$$\begin{aligned} \mathcal{A}^* V_1 &\leq -[p\lambda_1 + (1-p)\lambda_2]S + u + \eta + \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} \\ &\quad + \frac{mI}{1+mI} (1-p)\lambda_2 S - [p\lambda_1 + (1-p)\lambda_2] \\ &\quad + [p\lambda_1 + (1-p)\lambda_2]S + \frac{p\lambda_1 + (1-p)\lambda_2}{u} (\alpha\eta + u)I \\ &= -[p\lambda_1 + (1-p)\lambda_2] + u + \eta + \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} \\ &\quad + \frac{mI}{1+mI} (1-p)\lambda_2 S + \frac{p\lambda_1 + (1-p)\lambda_2}{u} (\alpha\eta + u)I \\ &\leq -[p\lambda_1 + (1-p)\lambda_2] + u + \eta + \frac{p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2}{2} \\ &\quad + \frac{mI}{1+mI} (1-p)\lambda_2 + \frac{p\lambda_1 + (1-p)\lambda_2}{u} (\alpha\eta + u)I \\ &:= \frac{mI}{1+mI} (1-p)\lambda_2 + \frac{p\lambda_1 + (1-p)\lambda_2}{u} (\alpha\eta + u)I \\ &\quad - \lambda, \end{aligned} \quad (54)$$

where $\lambda = [p\lambda_1 + (1-p)\lambda_2] - u - \eta - ((p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2)/2) = (u + \eta)[R_0 - 1 - ((p^2 \sigma_1^2 + (1-p)^2 \sigma_2^2)/2(u + \eta))] > 0$.

$$\begin{aligned}
\mathcal{A}^*V_2 &= -\frac{u}{S} + u + p\lambda_1 + (1-p)\frac{\lambda_2 I}{1+mI} - \frac{(1-\alpha)\eta I}{S} \\
&\quad + \frac{p^2\sigma_1^2 I^2 + (1-p)^2(\sigma_2^2 I^2 / (1+mI)^2)}{2} \\
&\leq -\frac{u}{S} + u + p\lambda_1 + (1-p)\lambda_2 + \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2} \\
&= -\frac{u}{S} + [p\lambda_1 + (1-p)\lambda_2] + u + \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2},
\end{aligned} \tag{55}$$

$$\mathcal{A}^*V_3 = \frac{u - u(S+I) - \alpha\eta I}{1-S-I} = u - \frac{\alpha\eta I}{1-S-I}, \tag{56}$$

$$\begin{aligned}
\mathcal{A}^*V_4 &= \frac{u - uS - uI - \alpha\eta I}{S+I - (u/(u+\alpha\eta))} \\
&= (u + \alpha\eta) - \frac{\alpha\eta S}{S+I - (u/(u+\alpha\eta))}.
\end{aligned} \tag{57}$$

Thus, according to (54)–(57), we have

$$\begin{aligned}
\mathcal{A}^*V &\leq -G\lambda + \frac{GmI}{1+mI} (1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u)I - \frac{u}{S} \\
&\quad - \frac{\alpha\eta I}{1-S-I} - \frac{\alpha\eta S}{S+I - (u/(u+\alpha\eta))} + H,
\end{aligned} \tag{58}$$

where $H = 3u + \alpha\eta + [p\lambda_1 + (1-p)\lambda_2] + (p^2\sigma_1^2 + (1-p)^2\sigma_2^2/2)$.

Now, we prove that $\mathcal{A}^*V(S, I) < -1$ on the following four domains which are equivalent to $\mathcal{A}^*V(S, I) < -1$ on the $E \setminus U$.

(i) $U_1 = \{(S, I) \in E: 0 < I < \varepsilon\}$. If $(S, I) \in U_1$, by (47) and (51), we have

$$\begin{aligned}
\mathcal{A}^*V &\leq -G\lambda + \frac{GmI}{1+mI} (1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u)I + H \\
&< -G\lambda + Gm\varepsilon(1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u)\varepsilon + H \\
&< 1 + (-2) \\
&= -1.
\end{aligned} \tag{59}$$

(ii) $U_2 = \{(S, I) \in E: 0 < S < \varepsilon\}$. If $(S, I) \in U_2$, using (52), we get

$$\begin{aligned}
\mathcal{A}^*V &\leq -\frac{u}{S} + \frac{GmI}{1+mI} (1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u)I + H \\
&< -\frac{u}{\varepsilon} + Gm(1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u) + H \\
&< -1.
\end{aligned} \tag{60}$$

(iii) $U_3 = \{(S, I) \in E: \varepsilon \leq I, 1 - \varepsilon^2 < S + I\}$. If $(S, I) \in U_3$, in view of (53), we derive

$$\begin{aligned}
\mathcal{A}^*V &\leq -\frac{\alpha\eta I}{1-S-I} + \frac{GmI}{1+mI} (1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u)I + H \\
&< -\frac{\alpha\eta}{\varepsilon} + Gm(1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u) + H \\
&< -1.
\end{aligned} \tag{61}$$

(iv) $U_4 = \{(S, I) \in E: \varepsilon \leq S, S + I < (u/(u+\alpha\eta)) + \varepsilon^2\}$. If $(S, I) \in U_4$, according to (53), we obtain

$$\begin{aligned}
\mathcal{A}^*V &\leq -\frac{\alpha\eta S}{S+I - (u/(u+\alpha\eta))} + \frac{GmI}{1+mI} (1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u)I + H \\
&< -\frac{\alpha\eta}{\varepsilon} + Gm(1-p)\lambda_2 \\
&\quad + \frac{G[p\lambda_1 + (1-p)\lambda_2]}{u} (\alpha\eta + u) + H \\
&< -1.
\end{aligned} \tag{62}$$

In conclusion, for a sufficiently small positive number ε ,

$$\sup_{(S, I) \in E \setminus U} \mathcal{A}^*V(S, I) < -1. \tag{63}$$

We apply similar arguments to those in [41]; this means that the semigroup is not sweeping from the set U . Thus, the semigroup $\{P(t)\}_{t \geq 0}$ is asymptotically stable by Remark 1.

3. Dynamics of System (5)

3.1. The Property of the Positive Solution. The first result is the existence and uniqueness of the positive solution of system (5).

Theorem 1. System (5) has a unique positive solution $(S(t), I(t))$ on $t \geq 0$ with any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$ almost surely.

Proof. By system (5), we obtain that

$$S(t) + I(t) \leq 1 + e^{-ut} (S(0) + I(0) - 1) \leq K, \quad (64)$$

where $K = \max\{S(0) + I(0), 1\}$.

Define

$$V(S, I) = (S - 1 - \ln S) + (I - 1 - \ln I). \quad (65)$$

Clearly, $V(S, I)$ is nonnegative. Using Itô's formula and the fact $S(t) + I(t) \leq K$, we get

$$\begin{aligned} dV(S, I) &= LV(S, I)dt + [p\sigma_1(I - S)]dB_1(t) \\ &\quad + \left[\frac{(1-p)\sigma_2}{1+mI} (I - S) \right] dB_2(t), \end{aligned} \quad (66)$$

$$\begin{aligned} LV(S, I) &= u - uS + (1-\alpha)\eta I - \frac{u}{S} + u - \frac{(1-\alpha)\eta I}{S} \\ &\quad + p\lambda_1 I + \frac{(1-p)\lambda_2 I}{1+mI} - uI - \eta I + u + \eta + \frac{I^2}{2} \left[p^2\sigma_1^2 + \frac{(1-p)^2\sigma_2^2}{(1+mI)^2} \right] - p\lambda_1 S \\ &\quad - \frac{(1-p)\lambda_2 S}{1+mI} + \frac{S^2}{2} \left[p^2\sigma_1^2 + \frac{(1-p)^2\sigma_2^2}{(1+mI)^2} \right] \\ &\leq 3u + p\lambda_1 I + \frac{(1-p)\lambda_2 I}{1+mI} + \eta + \frac{I^2}{2} \left[p^2\sigma_1^2 + \frac{(1-p)^2\sigma_2^2}{(1+mI)^2} \right] + \frac{S^2}{2} \left[p^2\sigma_1^2 + \frac{(1-p)^2\sigma_2^2}{(1+mI)^2} \right] \\ &\leq 3u + p\lambda_1 K + \frac{(1-p)\lambda_2 K}{1+mI} + \eta + \frac{K^2}{2} \left[p^2\sigma_1^2 + \frac{(1-p)^2\sigma_2^2}{(1+mI)^2} \right] + \frac{K^2}{2} \left[p^2\sigma_1^2 + \frac{(1-p)^2\sigma_2^2}{(1+mI)^2} \right] \\ &\leq 3u + p\lambda_1 K + (1-p)\lambda_2 K + \eta + \frac{K^2}{2} [p^2\sigma_1^2 + (1-p)^2\sigma_2^2] + \frac{K^2}{2} [p^2\sigma_1^2 + (1-p)^2\sigma_2^2] \\ &= 3u + p\lambda_1 K + (1-p)\lambda_2 K + \eta + K^2 p^2\sigma_1^2 + K^2 (1-p)^2\sigma_2^2 \\ &:= K_0. \end{aligned} \quad (67)$$

Here, K_0 is a positive constant. Then, refer to [42], and we complete the proof. $\square$

3.2. Asymptotically Stable Stationary Distribution and Extinction of System (5)

Theorem 2. Denote $(S(t), I(t))$ a solution of system (5) with any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$; the distribution of $(S(t), I(t))$ has a density $U(t, x, y)$ for every $t > 0$. If $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u + \eta)) > 1$, then there is a unique density $U_*(x, y)$ satisfying

$$\lim_{t \rightarrow \infty} \iint_{\mathbb{R}_+^2} |U(t, x, y) - U_*(x, y)| dx dy = 0, \quad (68)$$

$$\text{supp} U_* = \left\{ (x, y) \in \mathbb{R}_+^2 : \frac{u}{u + \alpha\eta} < x + y < 1 \right\} := E.$$

Proof. The proof of Theorem 2 is the following steps:

Step 1. According to Hö rmander theorem [35], we prove that the transition function of the process $(S(t), I(t))$ is absolutely continuous (see Lemma 1)

Step 2. We verify the positivity of the density of the transition function on E by support theorems [37–39] (refer to Lemma 2)

Step 3. We prove that the Markov semigroup satisfies the Foguel alternative (see Lemmas 3 and 4)

Step 4. Excluding sweeping by verifying there is a Khasminskii function (refer to Lemma 5)

Notice that the above strategies can be proved by Lemmas 1–5; thus, this proof is complete. $\square$

Theorem 3. For any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$, $(S(t), I(t))$ is a solution of system (5). If one of the following conditions holds, the epidemic $I(t)$ becomes extinct with probability one.

(i)

$$\begin{aligned}\sigma_1^2 &\leq \frac{\lambda_1}{p}, \\ \sigma_2^2 &\leq \frac{\lambda_2}{1-p},\end{aligned}\quad (69)$$

$$R_0 - \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2(u+\eta)} < 1.$$

(ii)

$$\frac{\lambda_1^2}{2\sigma_1^2} + \frac{\lambda_2^2}{2\sigma_2^2} < u + \eta. \quad (70)$$

Particularly, if (i) holds,

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} &\leq (u + \eta) \left[R_0 - 1 - \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2(u+\eta)} \right] \\ &< 0, \quad \text{a.s.}\end{aligned}\quad (71)$$

If (ii) holds,

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} \leq \frac{\lambda_1^2}{2\sigma_1^2} + \frac{\lambda_2^2}{2\sigma_2^2} - u - \eta < 0, \quad \text{a.s.} \quad (72)$$

Proof. If $\sigma_1^2 \leq (\lambda_1/p)$, $\sigma_2^2 \leq (\lambda_2/(1-p))$, and $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u+\eta)) < 1$, we obtain

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} &\leq -\frac{p^2\sigma_1^2}{2} \left(1 - \frac{\lambda_1}{p\sigma_1^2} \right)^2 + \frac{\lambda_1^2}{2\sigma_1^2} \\ &\quad - \frac{(1-p)^2\sigma_2^2}{2} \left(1 - \frac{\lambda_2}{(1-p)\sigma_2^2} \right)^2 + \frac{\lambda_2^2}{2\sigma_2^2} - u - \eta \\ &= -\frac{p^2\sigma_1^2}{2} + p\lambda_1 - \frac{(1-p)^2\sigma_2^2}{2} + (1-p)\lambda_2 - u - \eta \\ &= (u + \eta) \left(R_0 - 1 - \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2(u+\eta)} \right),\end{aligned}\quad (73)$$

and notice that $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u+\eta)) < 1$; then,

$$(u + \eta) \left(R_0 - 1 - \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2(u+\eta)} \right) < 0. \quad (74)$$

Then,

$$\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} < 0, \quad \text{a.s.} \quad (75)$$

If $(\lambda_1^2/2\sigma_1^2) + (\lambda_2^2/2\sigma_2^2) < u + \eta$, we easily get that

$$\begin{aligned}\limsup_{t \rightarrow \infty} \frac{\ln I(t)}{t} &= \limsup_{t \rightarrow \infty} \left[\frac{p\lambda_1 \int_0^t S(s)ds}{t} - \frac{p^2\sigma_1^2 \int_0^t S^2(s)ds}{2t} \right] \\ &\quad + \limsup_{t \rightarrow \infty} \left[\frac{(1-p)\lambda_2 \int_0^t (S(s)/(1+mI(s)))ds}{t} - u - \eta \right] \\ &\quad - \limsup_{t \rightarrow \infty} \left[\frac{(1-p)^2\sigma_2^2 \int_0^t (S(s)/(1+mI(s)))^2 ds}{2t} \right] \\ &\quad + \limsup_{t \rightarrow \infty} \left[\frac{M_1(t)}{t} + \frac{M_2(t)}{t} \right] \\ &\leq \limsup_{t \rightarrow \infty} \left[\frac{p\lambda_1 \int_0^t S(s)ds}{t} \right] \\ &\quad - \limsup_{t \rightarrow \infty} \left[\frac{p^2\sigma_1^2}{2} \left(\frac{\int_0^t S^2(s)ds}{t} \right)^2 \right]\end{aligned}$$

$$\begin{aligned}
& + \limsup_{t \rightarrow \infty} \left[\frac{(1-p)\lambda_2 \int_0^t (S(s)/(1+mI(s)))ds}{t} - u - \eta \right] \\
& - \limsup_{t \rightarrow \infty} \left[\frac{(1-p)^2\sigma_2^2}{2} \left(\frac{\int_0^t (S(s)/(1+mI(s)))ds}{t} \right)^2 \right] \\
& = -\frac{p^2\sigma_1^2}{2} \liminf_{t \rightarrow \infty} \left(\frac{\int_0^t S(s)ds}{t} - \frac{\lambda_1}{p\sigma_1^2} \right)^2 + \frac{\lambda_1^2}{2\sigma_1^2} \\
& \quad - \frac{(1-p)^2\sigma_2^2}{2} \liminf_{t \rightarrow \infty} J^2 + \frac{\lambda_2^2}{2\sigma_2^2} - u - \eta \\
& \leq \frac{\lambda_1^2}{2\sigma_1^2} + \frac{\lambda_2^2}{2\sigma_2^2} - u - \eta \\
& < 0, \quad \text{a.s.,}
\end{aligned} \tag{76}$$

where $J = (\int_0^t (S(s)/(1+mI(s)))ds/t) - (\lambda_2/(1-p)\sigma_2^2)$. This proof is completed. $\square$

Remark 4. $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u+\eta))$ is the threshold of system (5) when $\sigma_1^2 \leq (\lambda_1/p)$ and $\sigma_2^2 \leq (\lambda_2/(1-p))$.

4. Simulations and Conclusion

Now, in order to illustrate the analytical results, we make some numerical simulations. For systems (3) and (4), the parameter values are $u = 0.2$, $\alpha = 0.4$, $p = 0.6$, $\lambda_1 = 0.5$, $\lambda_2 = 0.6$, $\eta = 0.2$, and $m = 0.2$; then, $R_0 = ((p\lambda_1 + (1-p)\lambda_2)/(u+\eta)) = 1.35 > 1$.

We employ the following discrete equations:

$$\begin{cases}
S_k + 1 = \left[u - uS_k - p\lambda_1 S_k I_k - (1-p) \frac{\lambda_2 S_k I_k}{1+mI_k} \right] \Delta t + [(1-\alpha)\eta I_k] \Delta t - \left[(1-p) \frac{\sigma_2 S_k I_k}{1+mI_k} \right] \sqrt{\Delta t} \xi_k \\
\quad - \left[(1-p) \frac{\sigma_2^2 S_k I_k}{2(1+mI_k)} \right] (\xi_k^2 - 1) \Delta t + S_k - (p\sigma_1 S_k I_k) \sqrt{\Delta t} \xi_k - \left(\frac{p\sigma_1^2 S_k I_k}{2} \right) (\xi_k^2 - 1) \Delta t, \\
I_k + 1 = \left[p\lambda_1 S_k I_k + (1-p) \frac{\lambda_2 S_k I_k}{1+mI_k} - uI_k - \eta I_k \right] \Delta t + (p\sigma_1 S_k I_k) \sqrt{\Delta t} \xi_k + \left(\frac{p\sigma_1^2 S_k I_k}{2} \right) (\xi_k^2 - 1) \Delta t \\
\quad + I_k + \left[(1-p) \frac{\sigma_2 S_k I_k}{1+mI_k} \right] \sqrt{\Delta t} \xi_k + \left[(1-p) \frac{\sigma_2^2 S_k I_k}{2(1+mI_k)} \right] (\xi_k^2 - 1) \Delta t, \\
R_k + 1 = R_k + (\alpha\eta I_k - uR_k) \Delta t,
\end{cases} \tag{77}$$

where $\xi_k (k = 1, 2, \dots, n)$ are $N(0, 1)$ -distributed independent Gaussian random variables.

Example 1. Let $\sigma_1 = 0.2$ and $\sigma_2 = 0.3$; then,

$$R_0 - 1 = 0.35 > 0.0065 = \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2(u+\eta)}, \tag{78}$$

which satisfies the condition in Theorem 2; this means that epidemic $I(t)$ is persistent for long time. Figures 1(a) and 1(b) are the paths of individuals in deterministic model (3) and stochastic model (4), respectively. Figure 1(c) is the phase portrait of Figures 1(a) and 1(b). Figures 2(a)–2(c) are the distribution diagrams of S, I and R , respectively.

FIGURE 1: (a) Deterministic system, (b) corresponding stochastic systems of (a), and (c) the phase portrait of (b). (a) $\sigma_1 = 0, \sigma_2 = 0$; (b) $\sigma_1 = 0.1, \sigma_2 = 0.1$.

FIGURE 2: Continued.

FIGURE 2: (a, b, and c) The distribution diagram of S, I, and R, respectively. $\sigma_1 = 0.1, \sigma_2 = 0.1$.

Example 2. Case 1: let $\sigma_1 = 0.95$ and $\sigma_2 = 0.95$; then,

$$\frac{\lambda_1^2}{2\sigma_1^2} + \frac{\lambda_2^2}{2\sigma_2^2} = 0.33795 < 0.4 = u + \eta, \quad (79)$$

which satisfies condition (ii) in Theorem 3; this means that epidemic $I(t)$ becomes extinct with probability one. Figures 3(a) and 3(b) show the paths of individuals in deterministic model (3) and stochastic model (4), respectively. Figure 3(c) shows the phase portrait of Figures 3(a) and 3(b).

Case 2: let $\sigma_1 = 0.85$ and $\sigma_2 = 0.85$; then,

$$\begin{aligned} \sigma_1^2 &= 0.7225 \leq 0.8333 = \frac{\lambda_1}{p}, \\ \sigma_2^2 &= 0.7225 \leq 1.5 = \frac{\lambda_2}{1-p}, \end{aligned} \quad (80)$$

$$R_0 - 1 = 0.35 < 0.46962 = \frac{p^2\sigma_1^2 + (1-p)^2\sigma_2^2}{2(u+\eta)},$$

which satisfies condition (i) in Theorem 3; this means that epidemic $I(t)$ becomes extinct with probability one. Figures 4(a) and 4(b) show the paths of individuals in deterministic model (3) and stochastic model (4), respectively. Figure 4(c) is the phase portrait of Figures 4(a) and 4(b).

We have studied a stochastic SIR epidemic system affected by mixed nonlinear incidence rates in this paper. Using Markov semigroup theory and Fokker-Planck equation, we proved that the value of $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u+\eta))$ is the threshold of system (5) when $\sigma_1^2 \leq (\lambda_1/p)$ and $\sigma_2^2 \leq (\lambda_2/(1-p))$. If $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u+\eta)) > 1$, the epidemic is persistent, and there is

a unique asymptotically stable stationary distribution. If $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u+\eta)) < 1$, the sufficient conditions for disease extinction are established, that is, if $\sigma_1^2 \leq (\lambda_1/p)$ and $\sigma_2^2 \leq (\lambda_2/p)$, then $\limsup_{t \rightarrow \infty} (\ln I(t)/t) < 0$; this means that the epidemic becomes extinct with probability one. Moreover, if $(\lambda_1^2/2\sigma_1^2) + (\lambda_2^2/2\sigma_2^2) < u + \eta$, the epidemic is also extinct.

By the conclusion of Theorem 1, it is worthy to point out that two different incidence rates which are considered at the same time will not destroy a great property that existence and uniqueness of the positive solution. From Theorems 2 and 3, we know that when two infection rates are considered at the same time, in addition to the values of σ_1 and σ_2 , the value of p also affects the threshold value $R_0 - ((p^2\sigma_1^2 + (1-p)^2\sigma_2^2)/2(u+\eta))$. In addition, from the sufficient conditions for disease extinction, parameter p is an important variable related to the extinction of the population. Especially, when the parameters $p = 1$ and $\alpha = 1$, system (3) becomes (1) in [12], and system (3) is similar to (2) in [13] when the parameters $p = 0$ and $\alpha = 1$. To some extent, our model is more realistic than considering the epidemic affected by a single nonlinear incidence rate.

Since the proposed system is degenerate, we use Markov semigroup theory to study the stationary distribution and ergodicity of the system, and we can also study color noise and other noises in the future. For other noises, this method is feasible as long as it conforms to the relevant properties of Gaussian white noise. In addition, two aspects can be used as a guide for further research. First of all, we can investigate some other systems, for example, one can consider the systems with the impulsive perturbation effects [43, 44]. In addition, it is interesting to study the chemostat as well as population dynamics systems [45–51]. Notice that it is a meaningful question to investigate whether the way used in this article is applied to other epidemic systems. These questions are worthy of further study.

FIGURE 3: (a) Deterministic system, (b) corresponding stochastic systems of (a), and (c) the phase portrait of (b). (a) $\sigma_1 = 0, \sigma_2 = 0$; (b) $\sigma_1 = 0.95, \sigma_2 = 0.95$.

FIGURE 4: Continued.

FIGURE 4: (a) Deterministic system, (b) corresponding stochastic systems of (a), and (c) the phase portrait of (b). (a) $\sigma_1 = 0, \sigma_2 = 0$; (b) $\sigma_1 = 0.85, \sigma_2 = 0.85$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors read and approved the final manuscript.

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