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Research Article

Refined Upper Solution Bound of the Continuous Coupled Algebraic Riccati Equation

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The continuous coupled algebraic Riccati equation (CCARE) has wide applications in control theory and linear systems. In this paper, by a constructed positive semidefinite matrix, matrix inequalities, and matrix eigenvalue inequalities, we propose a new two-parameter-type upper solution bound of the CCARE. Next, we present an iterative algorithm for finding the tighter upper solution bound of CCARE, prove its boundedness, and analyse its monotonicity and convergence. Finally, corresponding numerical examples are given to illustrate the superiority and effectiveness of the derived results.

1. Introduction

Consider the following optimal control of jump linear system described by

$$\dot{x}(t) = A(r(t))x(t) + B(r(t))u(t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is the plant state and $u(t) \in R^m$ is the control vector. $A(r(t)) \in R^{n\times n}$, $B(r(t)) \in R^{n\times m}$, and $r(t)$ is a finite state Markov jump process on $S = \{1, 2, \ldots, s\}$, where $s \geq 2$. The quadratic performance index of system (1) is

$$J_u = E\left\{\int_0^\infty \left(x^T(t)Q(r(t))x(t) + u^T(t)R(r(t))u(t)\right)\mathrm{d}t \,\middle|\, x_0, r_0\right\}. \tag{2}$$

We have $A(r(t)) = A_i, B(r(t)) = B_i, Q(r(t)) = Q_i \in R^{n\times n}$, and $R(r(t)) = R_i \in R^{m\times m}$ with $Q_i \geq 0$ and $R_i > 0$ when $r(t) = i \in S$. The optimal state feedback controller to minimize the quadratic performance index (2) is $u(t) = -R_i^{-1}B_i^T P_i x(t)$, where P_i is the symmetric positive semidefinite solution of the continuous coupled algebraic Riccati equation (CCARE):

$$A_i^T P_i + P_i A_i - P_i B_i R_i^{-1} B_i^T P_i + \sum_{j\neq i} d_{ij} P_j + Q_i = 0, \tag{3}$$

where d_{ij} are real constants with the properties $d_{ii} < 0, d_{ij} \geq 0\,(i \neq j)$, and $\sum_{j\in S} d_{ij} = 0$. When $R = I$, the CCARE (3) changes to the following common form:

$$A_i^T P_i + P_i A_i - P_i B_i B_i^T P_i + \sum_{j\neq i} d_{ij} P_j + Q_i = 0. \tag{4}$$

CCARE (4) is usually encountered in robust and optimal control [1–8], filter design [9], time-delay systems controller design [10], stability analysis [11–16], etc. And in these fields, it often suffices to estimate the tighter solution bounds of the algebraic Riccati equation rather than get the exact solution. For example, in [17, 18], the authors have proposed solution bounds of the continuous algebraic Riccati equation, given their applications of the new bounds in redundant control input systems, and obtained several sufficient conditions to decrease the controller gain of the new systems after the control input extension. In recent years, bounds' estimation of solution for the CCARE has become an attractive topic, and many research approaches have been devoted to this topic. In [19–21], upper matrix bounds for the solution of the CCARE have been presented and iterative algorithms have been proposed to derive tighter upper matrix bounds. And there are many other works for studying the solutions of the CCARE, such as matrix bounds and properties [22–27], matrix eigenvalue bounds [28–30], numerical solution [31–33], and the explicit solution [34, 35].

In this paper, we construct a positive semidefinite matrix, propose a new two-parameter-type upper solution bound of the CCARE by matrix inequalities and matrix eigenvalue inequalities. Examples illuminate that the upper bound improves some recent related results. Then, according to the derived solution bound, we give an iterative algorithm, which can guarantee tighter upper solution bound of the CCARE, and prove its boundedness. Subsequently, we analysis its monotonicity and convergence.

Through this paper, let $R^{n\times n}$ ($C^{n\times n}$) denote the set of $n\times n$ real (complex) matrices. For $A=[a_{ij}]\in R^{n\times n}$, the notation $A>0\,(A\geq 0)$ is used to denote that A is a symmetric positive definite (semidefinite) matrix. Inequality $A>(\geq)B$ means matrix $(A-B)$ is positive (semi) definite. For $i=1,2,\ldots,n$, let $\lambda_i(A)$ be the nonincreasing order eigenvalues of A. We assume A^T, A^{-1}, and $\|A\|_2$ are the transpose, the inverse, and the spectral norm of A, respectively. A is called a Z-matrix if all its off-diagonal elements are nonpositive. It is obvious that any Z-matrix A can be written as $sI-B$ with B is nonnegative. A Z-matrix A is called an M-matrix if $s>\rho(B)$. For two vectors $a=(a_1,a_2,\ldots,a_n)^T\in R^n$ and $b=(b_1,b_2,\ldots,b_n)^T\in R^n$, $a\geq b$ is equivalent to $a_i\geq b_i\,(i=1,2,\ldots,n)$. Symbol $\mu(A)$ represents the matrix measure of A and is defined as $\mu(A)=\lambda_1((A+A^T)/2)$.

Lemma 1 (see [36]). *For any given positive semidefinite matrix $A\in R^{n\times n}$ and $B\in R^{n\times n}$,*

$$U=B^TA+AB\leq 0, \tag{5}$$

if and only if the matrix B^T+B is negative semidefinite.

Lemma 2 (see [37]). *Let $A\in R^{n\times n}$ satisfy that $A^T+A<0$, and let $P\in R^{n\times n}$ be symmetric. Then,*

$$A^TP+PA\leq(<)0, \tag{6}$$

if and only if the matrix $P\geq(>)0$.

Lemma 3 (see [38]). *For any symmetric matrix X, the following inequality holds:*

$$\lambda_n(X)I\leq X\leq\lambda_1(X)I. \tag{7}$$

Lemma 4 (see [38]). *Suppose $X,Y\in R^{n\times n}$ are symmetric matrices and $1\leq i,j\leq n$; then,*

$$\begin{aligned}&\lambda_{i+j-1}(X+Y)\leq\lambda_j(X)+\lambda_i(Y),\quad i+j\leq n+1,\\&\lambda_{i+j-n}(X+Y)\geq\lambda_j(X)+\lambda_i(Y),\quad i+j\geq n+1.\end{aligned} \tag{8}$$

Lemma 5 (see [39]). *If $X\in R^{n\times n}$ is an M-matrix, then X^{-1} is nonnegative.*

Lemma 6 (see [40]). *Let $x=(x_1,\ldots,x_n)^T$, $y=(y_1,\ldots,y_n)^T\in R^n$ and $x\geq y$, $A=(a_{ij})\in R^{n\times n}$ be nonnegative; then, $Ax\geq Ay$.*

Lemma 7 (see [41]). *Let $A,B\in R^{n\times n}$ both be symmetric with $A\geq B$. Then,*

$$\lambda_i(A)\geq\lambda_i(B), \tag{9}$$

for all $1\leq i\leq n$.

Lemma 8 (see [20]). *Let $\{W_k\}_{k\in N}\subset R^{n\times n}$ be a given sequence of positive semidefinite matrices. Assume that there exists a positive semidefinite $W_0\in R^{n\times n}$ such that for all k,*

$$\begin{aligned}W_{k+1}&\leq W_k,\\W_k&\geq W_0,\end{aligned} \tag{10}$$

and then $\{W_k\}_{k\in N}$ converges to a unique positive semidefinite $\overline{W}\in R^{n\times n}$.

2. Upper Solution Bounds for the CCARE

In this section, we will propose new upper matrix bounds of the solution for the CCARE (4), which improve the recent results.

Theorem 1. *Let P_i be the positive semidefinite solution of the CCARE (4). If there exist some positive definite matrices K_i, real numbers $n_i\geq(1/2)$, $\gamma_i<0$, and $\gamma_i<\beta_i\leq-\gamma_i$ such that*

$$A_i+A_i^T<\left(B_iB_i^T\right)^{n_i}K_i+K_i\left(B_iB_i^T\right)^{n_i}, \tag{11}$$

and suppose W is an M-matrix; then, for $i=1,\ldots,s$,

$$\begin{aligned}P_i\leq{}&\frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i+\gamma_iI\right)^{-T}\left(\widetilde{A}_i+\beta_iI\right)^TG_i\left(\widetilde{A}_i+\beta_iI\right)\left(\widetilde{A}_i+\gamma_iI\right)^{-1}\\&-\left(\widetilde{A}_i+\gamma_iI\right)^{-T}\left[\left(\beta_i^2-\gamma_i^2\right)\frac{G_i}{-2\mu(A_i)}-(\beta_i-\gamma_i)G_i\right]\\&\cdot\left(\widetilde{A}_i+\gamma_iI\right)^{-1}\equiv P_{ui},\end{aligned} \tag{12}$$

where $\widetilde{A}_i=A_i-(B_iB_i^T)^{n_i}K_i$, $G_i=\sum_{j\neq i}d_{ij}\eta_jI+Q_i+K_i(B_iB_i^T)^{2n_i-1}K_i$,

$$W=\begin{pmatrix}1&-d_{12}r_1&\cdots&-d_{1s}r_1\\-d_{21}r_2&1&\cdots&-d_{2s}r_2\\\vdots&&\ddots&\vdots\\-d_{s1}r_s&-d_{s2}r_s&\cdots&1\end{pmatrix},$$

$$\begin{pmatrix}\eta_1\\\eta_2\\\vdots\\\eta_s\end{pmatrix}\overset{\text{def}}{=}W^{-1}\begin{pmatrix}\xi_1\\\xi_2\\\vdots\\\xi_s\end{pmatrix},$$

$$
\begin{aligned}
r_i &= \lambda_1 \Bigg\{ \frac{1}{-2\mu(\widetilde{A}_i)} (\widetilde{A}_i + \gamma_i I)^{-T} (\widetilde{A}_i + \beta_i I)^T (\widetilde{A}_i + \beta_i I)(\widetilde{A}_i + \gamma_i I)^{-1} \\
&\quad - \left[\frac{\beta_i^2 - \gamma_i^2}{-2\mu(\widetilde{A}_i)} - (\beta_i - \gamma_i) \right] (\widetilde{A}_i + \gamma_i I)^{-T} (\widetilde{A}_i + \gamma_i I)^{-1} \Bigg\}, \\
\xi_i &= \lambda_1 \Bigg\{ \frac{1}{-2\mu(\widetilde{A}_i)} (\widetilde{A}_i + \gamma_i I)^{-T} (\widetilde{A}_i + \beta_i I)^T \\
&\quad \cdot \left[Q_i + K_i (B_i B_i^T)^{2n_i - 1} K_i \right] (\widetilde{A}_i + \beta_i I)(\widetilde{A}_i + \gamma_i I)^{-1} \\
&\quad - \left[\frac{\beta_i^2 - \gamma_i^2}{-2\mu(\widetilde{A}_i)} - (\beta_i - \gamma_i) \right] (\widetilde{A}_i + \gamma_i I)^{-T} \\
&\quad \cdot \left[Q_i + K_i (B_i B_i^T)^{2n_i - 1} K_i \right] (\widetilde{A}_i + \gamma_i I)^{-1} \Bigg\}.
\end{aligned}
\tag{13}
$$

Proof

$$
\begin{aligned}
&P_i B_i B_i^T P_i + \varepsilon P_i^2 - K_i (B_i B_i^T + \varepsilon I)^{n_i} P_i \\
&- P_i (B_i B_i^T + \varepsilon I)^{n_i} K_i + K_i (B_i B_i^T + \varepsilon I)^{2n_i - 1} K_i \\
&= \left[P_i - (B_i B_i^T + \varepsilon I)^{n_i - 1} K_i \right]^T (B_i B_i^T + \varepsilon I) \\
&\cdot \left[P_i - (B_i B_i^T + \varepsilon I)^{n_i - 1} K_i \right] \ge 0,
\end{aligned}
\tag{14}
$$

where ε is any sufficiently small positive constant. When $\varepsilon \longrightarrow 0$, from (14), we obtain

$$
\begin{aligned}
0 &\le P_i B_i B_i^T P_i - K_i (B_i B_i^T)^{n_i} P_i \\
&- P_i (B_i B_i^T)^{n_i} K_i + K_i (B_i B_i^T)^{2n_i - 1} K_i \equiv S_i.
\end{aligned}
\tag{15}
$$

And let $\widetilde{Q}_i = Q_i + \sum_{j \ne i} d_{ij} P_j + K_i (B_i B_i^T)^{2n_i - 1} K_i \ge 0$. According to CARE (4), we obtain

$$
\begin{aligned}
\widetilde{A}_i^T P_i + P_i \widetilde{A}_i &= \left[A_i - (B_i B_i^T)^{n_i} K_i \right]^T P_i + P_i \left[A_i - (B_i B_i^T)^{n_i} K_i \right] \\
&= A_i^T P_i + P_i A_i - K_i (B_i B_i^T)^{n_i} P_i - P_i (B_i B_i^T)^{n_i} K_i \\
&= P_i B_i B_i^T P_i - Q_i - \sum_{j \ne i} d_{ij} P_j - K_i (B_i B_i^T)^{n_i} P_i \\
&\quad - P_i (B_i B_i^T)^{n_i} K_i \\
&= S_i - Q_i - \sum_{j \ne i} d_{ij} P_j - K_i (B_i B_i^T)^{2n_i - 1} K_i = S_i - \widetilde{Q}_i.
\end{aligned}
\tag{16}
$$

Let

$$
\begin{aligned}
D_i &= \frac{1}{-2\mu(\widetilde{A}_i)} (\widetilde{A}_i + \gamma_i I)^{-T} (\widetilde{A}_i + \beta_i I)^T \left[\sum_{j \ne i} d_{ij} P_j + Q_i + K_i (B_i B_i^T)^{2n_i - 1} K_i \right] (\widetilde{A}_i + \beta_i I)(\widetilde{A}_i + \gamma_i I)^{-1} \\
&\quad - (\widetilde{A}_i + \gamma_i I)^{-T} (\beta_i^2 - \gamma_i^2) \frac{\sum_{j \ne i} d_{ij} P_j + Q_i + K_i (B_i B_i^T)^{2n_i - 1} K_i}{-2\mu(\widetilde{A}_i)} (\widetilde{A}_i + \gamma_i I)^{-1} \\
&\quad + (\widetilde{A}_i + \gamma_i I)^{-T} (\beta_i - \gamma_i) \left[\sum_{j \ne i} d_{ij} P_j + Q_i + K_i (B_i B_i^T)^{2n_i - 1} K_i \right] (\widetilde{A}_i + \gamma_i I)^{-1} \\
&= (\widetilde{A}_i + \gamma_i I)^{-T} (\widetilde{A}_i + \beta_i I)^T \frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)} (\widetilde{A}_i + \beta_i I)(\widetilde{A}_i + \gamma_i I)^{-1} \\
&\quad - (\widetilde{A}_i + \gamma_i I)^{-T} (\beta_i^2 - \gamma_i^2) \frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)} (\widetilde{A}_i + \gamma_i I)^{-1} + (\widetilde{A}_i + \gamma_i I)^{-T} (\beta_i - \gamma_i) \widetilde{Q}_i (\widetilde{A}_i + \gamma_i I)^{-1}.
\end{aligned}
\tag{17}
$$

Then, by (16), we have

$$
\begin{aligned}
&\tilde{A}_i^T\left[D_i - P_i\right] + \left[D_i - P_i\right]\tilde{A}_i = -S_i + \tilde{Q}_i + \tilde{A}_i^T D_i + D_i\tilde{A}_i \\
&= -S_i - \frac{1}{\beta_i - \gamma_i}\left[\left(\tilde{A}_i + \gamma_i I\right)^T D_i\left(\tilde{A}_i + \gamma_i I\right) - \left(\tilde{A}_i + \beta_i I\right)^T D_i\left(\tilde{A}_i + \beta_i I\right) + \left(\beta_i^2 - \gamma_i^2\right)D_i - (\beta_i - \gamma_i)\tilde{Q}_i\right] \\
&= -S_i - \frac{1}{\beta_i - \gamma_i}\left\{\left[\left(\tilde{A}_i + \beta_i I\right)^T \frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right) - \left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)} + (\beta_i - \gamma_i)\tilde{Q}_i\right]\right. \\
&\quad - \left\{\left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^{2T}\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right)^2\left(\tilde{A}_i + \gamma_i I\right)^{-1}\right. \\
&\quad - \left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^T\left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right)\left(\tilde{A}_i + \gamma_i I\right)^{-1} \\
&\quad \left. + \left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^T(\beta_i - \gamma_i)\tilde{Q}_i\left(\tilde{A}_i + \beta_i I\right)\left(\tilde{A}_i + \gamma_i I\right)^{-1}\right\} \\
&\quad + \left(\beta_i^2 - \gamma_i^2\right)\left[\left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right)\left(\tilde{A}_i + \gamma_i I\right)^{-1} - \left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \gamma_i I\right)^{-1}\right. \\
&\quad \left.\left. + \left(\tilde{A}_i + \gamma_i I\right)^{-T}(\beta_i - \gamma_i)\tilde{Q}_i\left(\tilde{A}_i + \gamma_i I\right)^{-1}\right] - (\beta_i - \gamma_i)\tilde{Q}_i\right\} \\
&= -S_i - \frac{1}{\beta_i - \gamma_i}\left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^T\left[\left(\tilde{A}_i + \gamma_i I\right)^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \gamma_i I\right)\right. \\
&\quad - \left(\tilde{A}_i + \beta_i I\right)^{-T}\left(\tilde{A}_i + \gamma_i I\right)^T\left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \gamma_i I\right)\left(\tilde{A}_i + \beta_i I\right)^{-1} \\
&\quad - \left(\tilde{A}_i + \beta_i I\right)^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right) + \left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)} - (\beta_i - \gamma_i)\tilde{Q}_i + \left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)} \\
&\quad - \left(\tilde{A}_i + \beta_i I\right)^{-T}\left(\beta_i^2 - \gamma_i^2\right)^2\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right)^{-1} \\
&\quad \left. + \left(\tilde{A}_i + \beta_i I\right)^{-T}\left(\beta_i^2 - \gamma_i^2\right)(\beta_i - \gamma_i)\tilde{Q}_i\left(\tilde{A}_i + \beta_i I\right)^{-1}\right]\left(\tilde{A}_i + \beta_i I\right)\left(\tilde{A}_i + \gamma_i I\right)^{-1} \\
&= -S_i - \frac{1}{\beta_i - \gamma_i}\left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^T\left\{\left[\left(\tilde{A}_i + \gamma_i I\right)^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \gamma_i I\right) - \left(\tilde{A}_i + \beta_i I\right)^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right)\right.\right. \\
&\quad \left. + \left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)} - (\beta_i - \gamma_i)\tilde{Q}_i\right] - \left(\beta_i^2 - \gamma_i^2\right)\left(\tilde{A}_i + \beta_i I\right)^{-T}\left[\left(\tilde{A}_i + \gamma_i I\right)^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \gamma_i I\right)\right. \\
&\quad \left.\left. - \left(\tilde{A}_i + \beta_i I\right)^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\left(\tilde{A}_i + \beta_i I\right) + \left(\beta_i^2 - \gamma_i^2\right)\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)} - (\beta_i - \gamma_i)\tilde{Q}_i\right]\left(\tilde{A}_i + \beta_i I\right)^{-1}\right\}\left(\tilde{A}_i + \beta_i I\right)\left(\tilde{A}_i + \gamma_i I\right)^{-1} \\
&= -S_i + \left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^T\left[\left(\tilde{A}_i^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)} + \frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\tilde{A}_i + \tilde{Q}_i\right)\right. \\
&\quad \left. - \left(\beta_i^2 - \gamma_i^2\right)\left(\tilde{A}_i + \beta_i I\right)^{-T}\left(\tilde{A}_i^T\frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)} + \frac{\tilde{Q}_i}{-2\mu\left(\tilde{A}_i\right)}\tilde{A}_i + \tilde{Q}_i\right)\left(\tilde{A}_i + \beta_i I\right)^{-1}\right]\left(\tilde{A}_i + \beta_i I\right)\left(\tilde{A}_i + \gamma_i I\right)^{-1} \\
&= -S_i + \left(\tilde{A}_i + \gamma_i I\right)^{-T}\left(\tilde{A}_i + \beta_i I\right)^T\left[\overline{Q}_i - \left(\beta_i^2 - \gamma_i^2\right)\left(\tilde{A}_i + \beta_i I\right)^{-T}\overline{Q}_i\left(\tilde{A}_i + \beta_i I\right)^{-1}\right]\left(\tilde{A}_i + \beta_i I\right)\left(\tilde{A}_i + \gamma_i I\right)^{-1},
\end{aligned}
\tag{18}
$$

where $\overline{Q}_i = \widetilde{A}_i^T (\widetilde{Q}_i/(-2\mu(\widetilde{A}_i))) + (\widetilde{Q}_i/(-2\mu(\widetilde{A}_i)))\widetilde{A}_i + \widetilde{Q}_i$. According to condition (11), we get $\widetilde{A}_i + \widetilde{A}_i^T = A_i + A_i^T - (B_iB_i^T)^{n_i}K_i - K_i(B_iB_i^T)^{n_i} < 0$; then,

$$\frac{\widetilde{A}_i^T}{-2\mu(\widetilde{A}_i)} + \frac{I}{2} + \frac{\widetilde{A}_i}{-2\mu(\widetilde{A}_i)} + \frac{I}{2} = \frac{\widetilde{A}_i + \widetilde{A}_i^T}{-2\mu(\widetilde{A}_i)} + I \le \frac{2\mu(\widetilde{A}_i)I}{-2\mu(\widetilde{A}_i)} + I = 0. \tag{19}$$

Considering $\widetilde{Q}_i \ge 0$, using Lemma 1 to (19) yields

$$\begin{aligned}\overline{Q}_i &= \widetilde{A}_i^T \frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)} + \frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)}\widetilde{A}_i + \widetilde{Q}_i \\ &= \left(\frac{\widetilde{A}_i^T}{-2\mu(\widetilde{A}_i)} + \frac{I}{2}\right)\widetilde{Q}_i + \widetilde{Q}_i\left(\frac{\widetilde{A}_i}{-2\mu(\widetilde{A}_i)} + \frac{I}{2}\right) \le 0.\end{aligned} \tag{20}$$

Substituting (20) into (18) with $\beta_i^2 - \gamma_i^2 \le 0$ and $S_i \ge 0$, we get the right-hand side of (18) is negative semidefinite. Moreover, since $\widetilde{A}_i + \widetilde{A}_i^T < 0$, according to Lemma 2, we conclude $D_i - P_i \ge 0$, that is,

$$\begin{aligned}P_i \le D_i = {} & \frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T \\ & \cdot \left[\sum_{j\ne i} d_{ij}P_j + Q_i + K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right]\left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ & - \left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\beta_i^2 - \gamma_i^2\right)\frac{\sum_{j\ne i} d_{ij}P_j + Q_i + K_i\left(B_iB_i^T\right)^{2n_i-1}K_i}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ & + \left(\widetilde{A}_i + \gamma_i I\right)^{-T}(\beta_i - \gamma_i)\left[\sum_{j\ne i} d_{ij}P_j + Q_i + K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right]\left(\widetilde{A}_i + \gamma_i I\right)^{-1}.\end{aligned} \tag{21}$$

Using Lemma 3 to (21), we obtain

$$\begin{aligned}P_i \le {} & \left\{\frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T\left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1}\right. \\ & \left. - \left[\frac{\beta_i^2 - \gamma_i^2}{-2\mu(\widetilde{A}_i)} - (\beta_i - \gamma_i)\right]\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \gamma_i I\right)^{-1}\right\} \sum_{j\ne i} d_{ij}\lambda_1\left(P_j\right) \\ & + \frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T\left[Q_i + K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right]\left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ & - \left[\frac{\beta_i^2 - \gamma_i^2}{-2\mu(\widetilde{A}_i)} - (\beta_i - \gamma_i)\right]\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left[Q_i + K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right]\left(\widetilde{A}_i + \gamma_i I\right)^{-1}.\end{aligned} \tag{22}$$

Applying Lemma 4 to (22) yields

$$\begin{aligned}\lambda_1(P_i) \le \lambda_1 \Bigg\{ & \frac{1}{-2\mu(\widetilde{A}_i)}(\widetilde{A}_i+\gamma_i I)^{-T}(\widetilde{A}_i+\beta_i I)^T(\widetilde{A}_i+\beta_i I)(\widetilde{A}_i+\gamma_i I)^{-1} \\ & -\left[\frac{\beta_i^2-\gamma_i^2}{-2\mu(\widetilde{A}_i)}-(\beta_i-\gamma_i)\right](\widetilde{A}_i+\gamma_i I)^{-T}(\widetilde{A}_i+\gamma_i I)^{-1}\Bigg\}\sum_{j\ne i} d_{ij}\lambda_1(P_j) \\ & +\lambda_1\Bigg\{\frac{1}{-2\mu(\widetilde{A}_i)}(\widetilde{A}_i+\gamma_i I)^{-T}(\widetilde{A}_i+\beta_i I)^T\left[Q_i+K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right](\widetilde{A}_i+\beta_i I)(\widetilde{A}_i+\gamma_i I)^{-1} \\ & -\left[\frac{\beta_i^2-\gamma_i^2}{-2\mu(\widetilde{A}_i)}-(\beta_i-\gamma_i)\right](\widetilde{A}_i+\gamma_i I)^{-T}\left[Q_i+K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right](\widetilde{A}_i+\gamma_i I)^{-1}\Bigg\}.\end{aligned} \tag{23}$$

Therefore, from (23), we obtain $\lambda_1(P_i)-r_i\sum_{j\ne i} d_{ij}\lambda_1(P_j)\le \xi_i, i=1,2,\ldots,s$, i.e.,

$$Wx\le\xi, \tag{24}$$

where $x=(\lambda_1(P_1),\lambda_1(P_2),\ldots,\lambda_1(P_s))$ and $\xi=(\xi_1,\xi_2,\ldots,\xi_s)$. As the coefficient matrix W of (24) is an M-matrix, W^{-1} is nonnegative in terms of Lemma 5. Consequently, (24) is equivalent to

$$x\le W^{-1}\xi=\eta, \tag{25}$$

by Lemma 6, where $\eta=(\eta_1,\eta_2,\ldots,\eta_s)$. Substituting (25) into (22) completes the proof. □

Corollary 1. *We show that*

(1) $P_i\le \dfrac{1}{-2\mu(\widetilde{A}_i)}\left[\sum_{j\ne i} d_{ij}\widetilde{\eta}_j I+Q_i+K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right]\equiv P'_{ui}$,

(2) $P_{ui}\le P'_{ui}$, (26)

where $\widetilde{\gamma}_i=(1/(-2\mu(\widetilde{A}_i)))$, $\widetilde{\xi}_i=(1/(-2\mu(\widetilde{A}_i)))\lambda_1[Q_i+K_i(B_iB_i^T)^{2n_i-1}K_i]$,

$$\begin{pmatrix}\widetilde{\eta}_1\\ \widetilde{\eta}_2\\ \vdots\\ \widetilde{\eta}_s\end{pmatrix} \overset{\text{def}}{=} \widetilde{W}^{-1}\begin{pmatrix}\widetilde{\xi}_1\\ \widetilde{\xi}_2\\ \vdots\\ \widetilde{\xi}_s\end{pmatrix},$$

$$\widetilde{W}=\begin{pmatrix}1 & -d_{12}\widetilde{\gamma}_1 & \cdots & -d_{1s}\widetilde{\gamma}_1\\ -d_{21}\widetilde{\gamma}_2 & 1 & \cdots & -d_{2s}\widetilde{\gamma}_2\\ \vdots & & \ddots & \vdots\\ -d_{s1}\widetilde{\gamma}_s & -d_{s2}\widetilde{\gamma}_s & \cdots & 1\end{pmatrix}, \tag{27}$$

is an M-matrix.

Proof. (1) Since

$$\begin{aligned}&\widetilde{A}_i^T\left(\frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)}-P_i\right)+\left(\frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)}-P_i\right)\widetilde{A}_i\\ &\quad=-S_i+\widetilde{Q}_i+\widetilde{A}_i^T\frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)}+\frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)}\widetilde{A}_i.\end{aligned} \tag{28}$$

Substituting (20) into (28) and combining $S_i\ge 0$, we get the right-hand side of (28) is negative semidefinite. Since $\widetilde{A}_i+\widetilde{A}_i^T<0$, according to Lemma 2, we conclude $(\widetilde{Q}_i/(-2\mu(\widetilde{A}_i)))-P_i\ge 0$, that is,

$$\begin{aligned}P_i&\le\frac{\widetilde{Q}_i}{-2\mu(\widetilde{A}_i)}=\frac{1}{-2\mu(\widetilde{A}_i)}\left[\sum_{j\ne i}d_{ij}P_j+Q_i+K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right]\\ &\le\frac{1}{-2\mu(\widetilde{A}_i)}\left[\sum_{j\ne i}d_{ij}\lambda_1(P_j)I+Q_i+K_i\left(B_iB_i^T\right)^{2n_i-1}K_i\right].\end{aligned} \tag{29}$$

Using Lemma 4 to (29) yields

$$\begin{aligned}\lambda_1(P_i)\le&\frac{1}{-2\mu(\widetilde{A}_i)}\sum_{j\ne i}d_{ij}\lambda_1(P_j)\\ &+\frac{1}{-2\mu(\widetilde{A}_i)}\lambda_1\left[Q_i+K_i\left(B_iB_i^T\right)^{2n-1}K_i\right],\end{aligned} \tag{30}$$

that is, $\lambda_1(P_i)-\widetilde{\gamma}_i\sum_{j\ne i}d_{ij}\lambda_1(P_j)\le\widetilde{\xi}_i$, i.e.,

$$\widetilde{W}x\le\widetilde{\xi}, \tag{31}$$

where $\widetilde{\xi}=(\xi_1,\xi_2,\ldots,\xi_s)$. As the coefficient matrix $\widetilde{W}$ of (31) is an M-matrix, $\widetilde{W}^{-1}$ is nonnegative in terms of Lemma 5. Consequently, (31) is equivalent to

$$x\le\widetilde{W}^{-1}\widetilde{\xi}=\widetilde{\eta}, \tag{32}$$

by Lemma 6, where $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_s)$. Substituting (32) into (29), we get $P_i \le P'_{ui}$.

(2)

(i) First, let us compare P_{ui} with $(G_i/(-2\mu(\tilde{A}_i)))$:

$$\begin{aligned} P_{ui} - \frac{G_i}{-2\mu(\tilde{A}_i)} &= \frac{1}{-2\mu(\tilde{A}_i)}(\tilde{A}_i + \gamma_i I)^{-T}(\tilde{A}_i + \beta_i I)^T G_i(\tilde{A}_i + \beta_i I)(\tilde{A}_i + \gamma_i)^{-1} \\ &\quad - (\tilde{A}_i + \gamma_i I)^{-T}\left[(\beta_i^2 - \gamma_i^2)\frac{G_i}{-2\mu(\tilde{A}_i)} - (\beta_i - \gamma_i)G_i\right](\tilde{A}_i + \gamma_i I)^{-1} - \frac{G_i}{-2\mu(\tilde{A})} \\ &= (\tilde{A}_i + \gamma_i I)^{-T}\left\{(\tilde{A}_i + \beta_i I)^T\frac{G_i}{-2\mu(\tilde{A})}(\tilde{A}_i + \beta_i I) - (\beta_i^2 - \gamma_i^2)\frac{G_i}{-2\mu(\tilde{A}_i)} + (\beta_i - \gamma_i)G_i \right. \\ &\quad \left. - (\tilde{A}_i + \gamma_i I)^T\frac{G_i}{-2\mu(\tilde{A})}(\tilde{A}_i + \gamma_i I)\right\}(\tilde{A}_i + \gamma_i I)^{-1} \\ &= (\beta_i - \gamma_i)(\tilde{A}_i + \gamma_i I)^{-T}\left\{\tilde{A}^T\frac{G_i}{-2\mu(\tilde{A})} + \frac{G_i}{-2\mu(\tilde{A})}\tilde{A} + G_i\right\}(\tilde{A}_i + \gamma_i I)^{-1}. \end{aligned} \tag{33}$$

Considering $G_i \ge 0$, using Lemma 1 to (19) yields

$$\begin{aligned} &\tilde{A}^T\frac{G_i}{-2\mu(\tilde{A})} + \frac{G_i}{-2\mu(\tilde{A})}\tilde{A} + G_i \\ &= \left(\frac{\tilde{A}_i^T}{-2\mu(\tilde{A}_i)} + \frac{I}{2}\right)G_i + G_i\left(\frac{\tilde{A}_i}{-2\mu(\tilde{A}_i)} + \frac{I}{2}\right) \le 0. \end{aligned} \tag{34}$$

Substituting (34) into (33) with $\beta_i - \gamma_i > 0$, we get the right-hand side of (33) is negative semidefinite. Thus,

$$P_{ui} \le \frac{G_i}{-2\mu(\tilde{A}_i)}. \tag{35}$$

(ii) Next, let us compare $G_i/(-2\mu(\tilde{A}_i))$ with P'_{ui}. For (35), assume $G_i = I$; then, (35) changes to

$$\begin{aligned} &\frac{1}{-2\mu(\tilde{A}_i)}(\tilde{A}_i + \gamma_i I)^{-T}(\tilde{A}_i + \beta_i I)^T(\tilde{A}_i + \beta_i I)(\tilde{A}_i + \gamma_i)^{-1} \\ &- \left[\frac{\beta_i^2 - \gamma_i^2}{-2\mu(\tilde{A}_i)} - (\beta_i - \gamma_i)\right](\tilde{A}_i + \gamma_i I)^{-T}(\tilde{A}_i + \gamma_i I)^{-1} \\ &\le \frac{1}{-2\mu(\tilde{A}_i)}I. \end{aligned} \tag{36}$$

By Lemma 7, it is easily to see $r_i \le \tilde{\gamma}_i$, then $W \ge \tilde{W}$. As W and $\tilde{W}$ are M-matrix, we obtain

$$W^{-1} \le \tilde{W}^{-1}. \tag{37}$$

On the contrary, for G_i of (35), assume $\sum_{j\neq i} d_{ij}\eta_j I = 0$; then, (35) changes to

$$\begin{aligned} &\frac{1}{-2\mu(\tilde{A}_i)}(\tilde{A}_i + \gamma_i I)^{-T}(\tilde{A}_i + \beta_i I)^T\left[Q_i + K_i(B_iB_i^T)^{2n_i-1}K_i\right] \\ &\cdot(\tilde{A}_i + \beta_i I)(\tilde{A}_i + \gamma_i I)^{-1} \\ &- \left[\frac{\beta_i^2 - \gamma_i^2}{-2\mu(\tilde{A}_i)} - (\beta_i - \gamma_i)\right](\tilde{A}_i + \gamma_i I)^{-T} \\ &\cdot\left[Q_i + K_i(B_iB_i^T)^{2n_i-1}K_i\right](\tilde{A}_i + \gamma_i I)^{-1} \\ &\le \frac{1}{-2\mu(\tilde{A}_i)}\left[Q_i + K_i(B_iB_i^T)^{2n_i-1}K_i\right]. \end{aligned} \tag{38}$$

Applying Lemma 7 to (38), we conclude $\xi_i \le \tilde{\xi}_i$. Thus, we obtain

$$\eta_i \le \tilde{\eta}_i, \tag{39}$$

with (37). Subsequently, we obtain

$$\frac{G_i}{-2\mu(\tilde{A}_i)} \le \frac{1}{-2\mu(\tilde{A}_i)}\left[\sum_{j\neq i} d_{ij}\tilde{\eta}_j I + Q_i + K_i(B_iB_i^T)^{2n_i-1}K_i\right] = P'_{ui}. \tag{40}$$

Lastly, combining (35) with (40), we derive $P_{ui} \le P'_{ui}$. □

Remark 1. Due to the different proof methods, it is very hard to compare the upper bounds of Theorem 1 with the parallel results theoretically, and we will present examples to illustrate that our upper bounds are tighter than the recent results for some cases.

3. Iterative Algorithm

According to Section 2, we give an iterative algorithm as following. The algorithm is based on upper bounds (12) from Theorem 1. And we will discuss the following conclusions in the same condition as Theorem 1.

Algorithm 1. Let $X_i^{(0)} = \eta_i I, i \in S$, and η_i is from Theorem 1. For $k = 1, 2, \ldots$,

$$X_i^{(k+1)} = \frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T U_i^{(k)} \cdot \left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1} - \left(\beta_i^2 - \gamma_i^2\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\frac{U_i^{(k)}}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-1} + (\beta_i - \gamma_i)\left(\widetilde{A}_i + \gamma_i I\right)^{-T} U_i^{(k)}\left(\widetilde{A}_i + \gamma_i I\right)^{-1}, \quad (41)$$

where $U_i^{(k)} = \sum_{j \neq i} d_{ij} X_j^{(k)} + Q_i + K_i (B_i B_i^T)^{2n_i - 1} K_i$.

Next, we will discuss the boundedness of Algorithm 1 and analyse its monotonicity and convergence.

Theorem 2. *Let $X_i^{(k)}$ be the iteration in Algorithm 1. If the CCARE (4) has positive semidefinite solution P_i, then*

$$P_i \leq X_i^{(k)}, \quad k = 0, 1, 2, \ldots. \quad (42)$$

Proof. We proof the result by induction.

(i) From (25), we get $P_i \leq \lambda_1(P_i) I \leq \eta_i I = X_i^{(0)}$. Furthermore,

$$X_i^{(1)} = \frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T U_i^{(0)} \cdot \left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1} - \left(\beta_i^2 - \gamma_i^2\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\frac{U_i^{(0)}}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-1} + (\beta_i - \gamma_i)\left(\widetilde{A}_i + \gamma_i I\right)^{-T} U_i^{(0)}\left(\widetilde{A}_i + \gamma_i I\right)^{-1}, \quad (43)$$

where $U_i^{(0)} = Q_i + \sum_{j \neq i} d_{ij}\ X_j^{(0)} + K_i (B_i B_i^T)^{2n_i - 1} K_i = Q_i + \sum_{j \neq}\ i d_{ij} \eta_i I + K_i (B_i B_i^T)^{2n_i - 1} K_i$. According to Theorem 1, we get $P_i \leq P_{ui} = X_i^{(1)}$.

(ii) Suppose $P_i \leq X_i^{(k)}$ for all $k \geq 1$, then $\widetilde{Q}_i = Q_i + \sum_{j \neq i} d_{ij} P_j + K_i (B_i B_i^T)^{2n_i - 1} K_i \leq Q_i + \sum_{j \neq i} d_{ij} X_j^{(k)} + K_i (B_i B_i^T)^{2n_i - 1} K_i = U_i^{(k)}$. In a similar way to the proof of (18), we have

$$\begin{aligned} &\widetilde{A}_i^T\left[X_i^{(k+1)} - P_i\right] + \left[X_i^{(k+1)} - P_i\right]\widetilde{A}_i \\ &= -\left(\widetilde{A}_i^T P_i + P_i \widetilde{A}_i\right) + \widetilde{A}_i^T X_i^{(k+1)} + X_i^{(k+1)}\widetilde{A}_i = -S_i + \widetilde{Q}_i + \widetilde{A}_i^T X_i^{(k+1)} + X_i^{(k+1)}\widetilde{A}_i \\ &\leq -S_i + U_i^{(k)} + \widetilde{A}_i^T X_i^{(k+1)} + X_i^{(k+1)}\widetilde{A}_i \\ &= -S_i - \frac{1}{\beta_i - \gamma_i}\left[\left(\widetilde{A}_i + \gamma_i I\right)^T X_i^{(k+1)}\left(\widetilde{A}_i + \gamma_i I\right) - \left(\widetilde{A}_i + \beta_i I\right)^T X_i^{(k+1)}\left(\widetilde{A}_i + \beta_i I\right) + \left(\beta_i^2 - \gamma_i^2\right) X_i^{(k+1)} - (\beta_i - \gamma_i) U_i^{(k)}\right] \\ &= -S_i + \left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T\left[\overline{U_i^{(k)}} - \left(\beta_i^2 - \gamma_i^2\right)\left(\widetilde{A}_i + \beta_i I\right)^{-T}\overline{U_i^{(k)}}\left(\widetilde{A}_i + \beta_i I\right)^{-1}\right]\left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1}, \end{aligned} \quad (44)$$

where $\overline{U_i^{(k)}} = \widetilde{A}_i^T (U_i^{(k)}/(-2\mu(\widetilde{A}_i))) + (U_i^{(k)}/(-2\mu(\widetilde{A}_i)))\widetilde{A}_i + U_i^{(k)}$. Considering $U_i^{(k)} \geq 0$, using Lemma 1 to (19) yield

$$\overline{U_i^{(k)}} = \widetilde{A}^T \frac{U_i^{(k)}}{-2\mu(\widetilde{A})} + \frac{U_i^{(k)}}{-2\mu(\widetilde{A})}\widetilde{A} + U_i^{(k)} = \left(\frac{\widetilde{A}_i^T}{-2\mu(\widetilde{A}_i)} + \frac{I}{2}\right) U_i^{(k)} + U_i^{(k)}\left(\frac{\widetilde{A}_i}{-2\mu(\widetilde{A}_i)} + \frac{I}{2}\right) \leq 0. \quad (45)$$

Substituting (45) into (44) with $\beta_i^2 - \gamma_i^2 \leq 0$ and $S_i \geq 0$, we get the right-hand side of (44) is negative semidefinite. Moreover, since $\widetilde{A}_i^T + \widetilde{A}_i < 0$, according to Lemma 2, we get $P_i \leq X_i^{(k+1)}$. This completes the induction. □

Theorem 3. *Let $X_i^{(k)}$ be the iteration in Algorithm 1. If $P_{ui} \leq \eta_i I, i \in S$, where P_{ui} and η_i are from Theorem 1, then the sequence $X_i^{(k)}$ is monotone decreasing, i.e.,*

$$X_i^{(k+1)} \leq X_i^{(k)}, \quad k = 0, 1, 2, \ldots. \quad (46)$$

Proof. We proof the result by induction.

(i) $X_i^{(1)} = P_{ui} \leq \eta_i I = X_i^{(0)}$.

(ii) Suppose $X_i^{(k)} \leq X_i^{(k-1)}$ for all $k \geq 1$, then

$$\begin{aligned} X_i^{(k+1)} - X_i^{(k)} &= \frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T\left(U_i^{(k)} - U_i^{(k-1)}\right)\left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ &\quad - \left(\beta_i^2 - \gamma_i^2\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\frac{U_i^{(k)} - U_i^{(k-1)}}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ &\quad + (\beta_i - \gamma_i)\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(U_i^{(k)} - U_i^{(k-1)}\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ &= \frac{1}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left(\widetilde{A}_i + \beta_i I\right)^T\left[\sum_{j\neq i} d_{ij}\left(X_j^{(k)} - X_j^{(k-1)}\right)\right]\left(\widetilde{A}_i + \beta_i I\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ &\quad - \left(\beta_i^2 - \gamma_i^2\right)\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\frac{\sum_{j\neq i} d_{ij}\left(X_j^{(k)} - X_j^{(k-1)}\right)}{-2\mu(\widetilde{A}_i)}\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ &\quad + (\beta_i - \gamma_i)\left(\widetilde{A}_i + \gamma_i I\right)^{-T}\left[\sum_{j\neq i} d_{ij}\left(X_j^{(k)} - X_j^{(k-1)}\right)\right]\left(\widetilde{A}_i + \gamma_i I\right)^{-1} \\ &\leq 0. \end{aligned} \tag{47}$$

Therefore, the proof is complete.

According to Lemma 8, combining Theorems 2 and 3, we get the following conclusion. □

Theorem 4. *Let $X_i^{(k)}$ be the iteration in Algorithm 1. If $P_{ui} \leq \eta_i I, i \in S$, where P_{ui} and η_i are from Theorem 1, then the sequence $X_i^{(k)}$ is monotone decreasing, and converges to the unique positive semidefinite matrices $\overline{X}_i$.*

Now, we present examples to illustrate the effectiveness of the main results, and the stopping criterion in Algorithm 1 is set as

$$\left\|X_i^{(k+1)} - X_i^{(k)}\right\|_2 < 10^{-8}. \tag{48}$$

Example 1 (see [42]). Consider the CCARE (4) with

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}, \\ s &= \{1, 2\}, \\ B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \\ Q_1 &= \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}, \\ Q_2 &= \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}, \\ d_{ij} &= \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}. \end{aligned} \tag{49}$$

Choose $n_1 = 1, \gamma_1 = -1, \beta_1 = 1, K_1 = [2.66, 0.9; 0.9, 1.6]$; $n_2 = 1, \gamma_2 = -1, \beta_2 = 1$, and $K_2 = [0.23, 0.1; 0.1, 0.15]$, and by Theorem 1, we obtain

$$\begin{aligned} \eta &= \begin{bmatrix} 7.3194 \\ 0.9499 \end{bmatrix}, \\ W &= \begin{bmatrix} 1.0000 & -0.0652 \\ -0.0111 & 1.0000 \end{bmatrix}, \\ P_{u1} &= \begin{bmatrix} 6.5085 & 2.1689 \\ 2.1689 & 1.5128 \end{bmatrix}, \\ P_{u2} &= \begin{bmatrix} 0.8005 & 0.2798 \\ 0.2798 & 0.4234 \end{bmatrix}. \end{aligned} \tag{50}$$

By Corollary 1, we obtain

$$\begin{aligned} \widetilde{W} &= \begin{bmatrix} 1.0000 & -0.0652 \\ -0.0111 & 1.0000 \end{bmatrix}, \\ P'_{u1} &= \begin{bmatrix} 7.2910 & 3.5194 \\ 3.5194 & 3.2029 \end{bmatrix}, \\ P'_{u2} &= \begin{bmatrix} 0.8363 & 0.2777 \\ 0.2777 & 0.4849 \end{bmatrix}. \end{aligned} \tag{51}$$

By Theorem 3.2 in [21], taking $\tilde{a}_1 = 3, \tilde{a}_2 = 0.1$, $T_i = \tilde{a}_i I, a_i = 3$, and $n = 1$, we obtain

$$\widetilde{P}_1 = \begin{bmatrix} 8.9463 & 4.9736 \\ 4.9736 & 3.5837 \end{bmatrix}, \quad \widetilde{P}_2 = \begin{bmatrix} 1.3337 & 0.5084 \\ 0.5084 & 0.6973 \end{bmatrix}. \tag{52}$$

By Theorem 2.2 in [42], taking $K_1' = [2.600, 0.8950], K_2' = [0.5900, 0.1467]$, $\gamma_1' = \gamma_2' = -1$, and $\beta_2' = \beta_2' = 1$, we obtain

$$\overline{P}_1 = \begin{bmatrix} 7.4634 & 1.8628 \\ 1.8628 & 1.7131 \end{bmatrix}, \quad \overline{P}_2 = \begin{bmatrix} 1.0739 & 0.3657 \\ 0.3657 & 0.9246 \end{bmatrix}. \tag{53}$$

Obviously, we see $P_{ui} \prec P'_{ui} \prec \widetilde{P}_i$ and $P_{ui} \prec P'_{ui} \prec \overline{P}_i$, that is, our bounds in Theorem 1 are tighter than [21, 42].

Since

$$X_1^{(1)} = P_{u1} = \begin{bmatrix} 6.5085 & 2.1689 \\ 2.1689 & 1.5128 \end{bmatrix} < 7.3194I = \eta_1 I = X_1^{(0)},$$
$$X_2^{(1)} = P_{u2} = \begin{bmatrix} 0.8005 & 0.2798 \\ 0.2798 & 0.4234 \end{bmatrix} < 0.9499I = \eta_2 I = X_2^{(0)}, \tag{54}$$

then Algorithm 1 is expected to be strictly monotone decreasing and converges to tighter upper bounds. By computation, the upper solution bounds of CCARE (4) are

$$X_1^{(6)} = \begin{bmatrix} 6.4928 & 2.1746 \\ 2.1746 & 1.4956 \end{bmatrix}, \quad X_2^{(6)} = \begin{bmatrix} 0.7906 & 0.3022 \\ 0.3022 & 0.3674 \end{bmatrix}, \tag{55}$$

after 6 iterations, which are tighter than $X_1^{(0)}$ and $X_2^{(0)}$, respectively.

Example 2. Consider the CCARE (4) with

$$A_1 = \begin{bmatrix} 1.8 & 1.2 \\ 3.2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2.2 & 1.3 \\ 1.4 & -3.1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1.5 & 0.6 \\ 0.25 & -0.4 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 3.6 \\ 0.8 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3.6 \\ 4.3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1.5 \\ 0.2 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 6 & 3 \\ 3 & 4 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 8 & 5 \\ 5 & 7 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix},$$
$$d_{ij} = \begin{bmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix},$$
$$R_1 = R_2 = R_3 = I, \quad s = \{1, 2, 3\}. \tag{56}$$

Choosing $n_1 = 1, \gamma_1 = -2, \beta_1 = 2, K_1 = [0.675, 0.135; 0.135, 0.21]$, $n_2 = 1, \gamma_2 = -4, \beta_2 = 4, K_2 = [0.115, 0.05; 0.05, 0.25]$, and $n_3 = 1, \gamma_3 = -2$, $\beta_3 = 2, K_3 = [0.93, 0.25; 0.25, 0.65]$, by Theorem 1, we obtain

$$\eta = \begin{bmatrix} 1.5421 \\ 1.1661 \\ 7.2392 \end{bmatrix}, \quad W = \begin{bmatrix} 1.0000 & -0.0142 & -0.0142 \\ -0.0145 & 1.0000 & -0.0145 \\ -0.1098 & -0.1098 & 1.0000 \end{bmatrix},$$
$$P_{u1} = \begin{bmatrix} 1.2524 & 0.4801 \\ 0.4801 & 0.6508 \end{bmatrix}, \quad P_{u2} = \begin{bmatrix} 1.0065 & 0.2021 \\ 0.2021 & 0.7353 \end{bmatrix}, \quad P_{u3} = \begin{bmatrix} 6.0565 & 1.7940 \\ 1.7940 & 4.2349 \end{bmatrix}. \tag{57}$$

By Corollary 1, we obtain

$$P'_{u1} = \begin{bmatrix} 1.9408 & 0.6613 \\ 0.6613 & 0.8035 \end{bmatrix}, \quad P'_{u2} = \begin{bmatrix} 1.3952 & 0.8388 \\ 0.8388 & 1.4212 \end{bmatrix}, \quad P'_{u3} = \begin{bmatrix} 8.2740 & 2.9966 \\ 2.9966 & 5.1641 \end{bmatrix}. \tag{58}$$

By Theorem 3.1 in [21], taking $\tilde{a}_1 = 0.5, \tilde{a}_2 = 1.6, \tilde{a}_3 = 1.0$, $T_i = \tilde{a}_i I, a_i = 2$, and $n = 1$, we obtain

$$\widetilde{P}_1 = \begin{bmatrix} 4.7453 & 0.4324 \\ 0.4324 & 3.1767 \end{bmatrix},$$
$$\widetilde{P}_2 = \begin{bmatrix} 2.6529 & 0.5609 \\ 0.5609 & 2.8528 \end{bmatrix}, \quad (59)$$
$$\widetilde{P}_3 = \begin{bmatrix} 8.5305 & 1.2422 \\ 1.2422 & 9.4916 \end{bmatrix}.$$

By Theorem 2.2 in [42], taking $\gamma_1' = -0.2, \beta_1' = 0.2, K_1' = [3.55, 2.9]$, $\gamma_2' = -0.4, \beta_2' = 0.4, K_2' = [0.11, 0.75]$, $\gamma_3' = -0.68$, $\beta_3' = 0.68$, and $K_3' = [3.33, 1.95]$, we obtain

$$\overline{P}_1 = \begin{bmatrix} 2.8058 & 0.1316 \\ 0.1316 & 2.5964 \end{bmatrix},$$
$$\overline{P}_2 = \begin{bmatrix} 3.3744 & 0.0925 \\ 0.0925 & 3.5902 \end{bmatrix}, \quad (60)$$
$$\overline{P}_3 = \begin{bmatrix} 8.3780 & 5.2328 \\ 5.2328 & 9.8737 \end{bmatrix}.$$

Obviously, we see $P_{ui} \prec P_{ui}' \prec \widetilde{P}_i$ and $P_{ui} \prec P_{ui}' \prec \overline{P}_i$, that is, our bounds in Theorem 1 are tighter than [21, 42].

Since

$$X_1^{(1)} = P_{u1} = \begin{bmatrix} 1.2524 & 0.4801 \\ 0.4801 & 0.6508 \end{bmatrix} < 1.5421I = \eta_1 I = X_1^{(0)},$$
$$X_2^{(1)} = P_{u2} = \begin{bmatrix} 1.0065 & 0.2021 \\ 0.2021 & 0.7353 \end{bmatrix} < 1.1661I = \eta_2 I = X_2^{(0)},$$
$$X_3^{(1)} = P_{u3} = \begin{bmatrix} 6.0565 & 1.7940 \\ 1.7940 & 4.2349 \end{bmatrix} < 7.2392I = \eta_3 I = X_3^{(0)}, \quad (61)$$

then Algorithm 1 is expected to be strictly monotone decreasing and converges to tighter upper bounds. By computation, the upper solution bounds of CCARE (4) are

$$X_1^{(7)} = \begin{bmatrix} 1.2391 & 0.4960 \\ 0.4960 & 0.5958 \end{bmatrix},$$
$$X_2^{(7)} = \begin{bmatrix} 0.9809 & 0.2295 \\ 0.2295 & 0.6990 \end{bmatrix}, \quad (62)$$
$$X_3^{(7)} = \begin{bmatrix} 6.0123 & 1.8695 \\ 1.8695 & 4.0732 \end{bmatrix},$$

after 7 iterations, which are tighter than $X_1^{(0)}$, $X_2^{(0)}$, and $X_3^{(0)}$, respectively.

4. Conclusions

In this paper, a new two-parameter-type upper solution bound of the CCARE has been proposed. Next, an iterative algorithm for finding the tighter upper solution bound of CCARE has been presented, and its boundedness, monotonicity, and convergence have been proved. Finally, corresponding numerical examples are given to illustrate the superiority and effectiveness of the derived results.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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