Research Article

Additive Eigenvalue Problems of the Laplace Operator with the Prescribed Contact Angle Boundary Condition

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Received 30 November 2019; Revised 12 January 2020; Accepted 21 January 2020; Published 28 April 2020

Guest Editor: Dou Wanchun

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Additive eigenvalue problem appears in ergodic optimal control or the homogenization of Hamilton–Jacobi equations. It has wide applications in several fields including computer science and then attracts the attention. In this paper, we consider the Poisson equations with the prescribed contact angle boundary condition and finally derive the existence and the uniqueness of the solution to the additive problem of the Laplace operator with the prescribed contact angle boundary condition.

1. Introduction

Additive eigenvalue problem appears in ergodic optimal control or the homogenization of Hamilton–Jacobi equations. In ergodic optimal control, the additive eigenvalue corresponds to averaged long-run optimal costs, while it determines the effective Hamiltonian in the homogenization of Hamilton–Jacobi equations. It is usually applied to study the large time behavior of the Cauchy problem of Hamilton–Jacobi equations. As a character of large time behavior, it also appears in the fields of computer science, big data-driven cloud service recommendation, etc. One can refer to the related references such as [1–15]. In pure mathematics, it has been studied by so many mathematicians, such as Lions [16] and Ishii [17]. More introduction can be found in [17] and the references therein.

The Poisson equation is a kind of simple but interesting and important object in the field of partial differential equations, and it appears in lots of mathematical modelling related to the real world. Also, it is always submitted with several kinds of boundary conditions. Among various boundary conditions, Gilbarg and Trudinger boundary and Neumann boundary conditions are already included in the classical theory, and one can refer to the study [18]. For the capillary-type boundary conditions, there are relatively less results. In [19], Xu derived

the gradient estimate for Poisson equations with the prescribed contact angle boundary condition. In this paper, we consider the additive eigenvalue problem, and our main result is related to the additive eigenvalue problem of the Laplace operator with the prescribed contact angle boundary condition.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary, $n \geq 2$, then ν is the inner normal vector of $\partial\Omega$. For any $\theta \in C^\infty(\overline{\Omega})$, and $|\cos \theta| \leq b < 1$, there exist a unique $\tau \in \mathbb{R}$ and a smooth function $w \in C^\infty(\overline{\Omega})$ solving*

$$\begin{cases} \Delta u = \tau, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -\cos \theta \sqrt{1 + |Du|^2}, & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Moreover, the solution to (1) is unique up to a constant.

Our work is motivated by studies [20–22]. In [20], Arisawa got the classical solution to a kind of fully nonlinear uniformly elliptic equation with oblique derivative boundary conditions which comes from the stochastic control fields. In [21], Francesca studied the large time behavior as $t \rightarrow +\infty$ of the viscosity solution χ to a Neumann boundary value problem

$$\begin{cases} \chi_t + F(x, \nabla \chi, \nabla^2 \chi) = \lambda, & \text{in } \Omega \times (0, \infty), \\ \chi(x, 0) = \chi_0(x), & \text{on } \overline{\Omega} \times \{0\}, \\ L(x, \nabla \chi) = \mu, & \text{on } \partial\Omega \times (0, \infty), \end{cases} \quad (2)$$

where F and L are at least continuous functions defined, respectively, on $\overline{\Omega} \times \mathbb{R}^n \times \mathcal{S}^n$ and $\overline{\Omega} \times \mathbb{R}^n$ and got a convergence result $\chi \rightarrow u_\infty$, as $t \rightarrow +\infty$ uniformly in $\overline{\Omega}$, where u_∞ is a solution to

$$\begin{cases} F(x, Du, D^2u) = \lambda, & \text{in } \Omega \times (0, \infty), \\ L(x, Du) = \mu, & \text{on } \partial\Omega \times (0, \infty), \end{cases} \quad (3)$$

under the assumption that F and L satisfy several structure conditions. Especially, we remark that the condition

$$L(x, p + s\nu(x)) - L(x, p) \geq \delta s, \quad (4)$$

is obviously not satisfied by the prescribed contact angle boundary value. Also, it is a bit of regret that the study [21] only discussed the viscosity solution and one may expect for the classical solution even if some more strong conditions are imposed on F . In [22], Gao et al. adopted a blow-up technique to conclude the a priori estimate to

$$\begin{cases} u_t - F(\nabla^2 u) = 0, & \text{in } \Omega \times [0, \infty), \\ u(x, 0) = u_0(x), & \text{on } \overline{\Omega} \times \{0\}, \\ \frac{\partial u}{\partial \nu} = \varphi(x), & \text{on } \partial\Omega \times [0, \infty), \end{cases} \quad (5)$$

and then in [23], Huang and Ye concluded the large time behavior of the solution in the classical senses. They proved that the solution will converge to one of the following additive eigenvalue problems with the Neumann boundary condition as the time goes to infinity:

$$\begin{cases} F(\nabla^2 u) = \lambda, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \varphi(x), & \text{on } \partial\Omega. \end{cases} \quad (6)$$

A similar convergence result was obtained by Ma et al. [24] for the graphic mean curvature flow with the Neumann boundary condition:

$$\begin{cases} u_t = \sum_{i,j=1}^n \left(\delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2} \right) u_{ij}, & \text{in } \Omega \times (0, \infty), \\ u_\nu = \varphi(x), & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{on } \overline{\Omega}, \end{cases} \quad (7)$$

where Ω is a strictly convex bounded domain in $\mathbb{R}^n$ with the smooth boundary for $n \geq 2$ and $u_0(x)$ and $\varphi(x)$ are smooth functions satisfying $u_{0,\nu} = \varphi(x)$ on $\partial\Omega$. They proved that, up to a constant, the solutions converge to a translating solution $\lambda t + w$. In other words, (w, λ) is a solution to

$$\begin{cases} \sum_{i,j=1}^n \left(\delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2} \right) u_{ij} = \lambda, & \text{in } \Omega, \\ u_\nu = \varphi(x), & \text{on } \partial\Omega. \end{cases} \quad (8)$$

In fact, the result in [24] arose from the prescribed contact angle boundary condition for mean curvature-type equations which is the more complicated case, and till now, it is an open problem to get the translating soliton for the graphic mean curvature flow with the prescribed contact angle boundary condition except for the two-dimension case and for the free boundary case. For more details, one can refer to studies [25, 26], etc.

When we try to prove Theorem 1, we actually can follow the procedure in [24] to get the uniform gradient estimate without the C^0 a priori estimate. But for the Laplace case, it seems to be unnatural to require the strict convexity condition of the domain and we have to follow the blow-up technique adopted in [20, 22].

We firstly give some notations.

Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\partial\Omega \in C^3$. Set

$$\begin{aligned} d(x) &= \text{dist}(x, \partial\Omega), \\ \Omega_\sigma &= \{x \in \Omega : d(x) < \sigma\}. \end{aligned} \quad (9)$$

Then, there exists a positive constant $\sigma_1 > 0$ such that $\forall \sigma \leq \sigma_1$, $d(x) \in C^3(\overline{\Omega_\sigma})$. As mentioned by Lieberman [27] or Simon and Spruck [28], we can extend ν as Dd in Ω_σ which is a C^2 vector field. We also have the following formulas:

$$\begin{aligned} |\nabla \nu| + |\nabla^2 \nu| &\leq \tilde{C}(n, \Omega), & \text{in } \Omega_\sigma, \\ |\nu| &= 1, & \text{in } \Omega_\sigma. \end{aligned} \quad (10)$$

Also, in this paper, in order to simplify the proof of the theorems, we write $O(z)$ as an expression that there exists a uniform constant $C > 0$ such that $|O(z)| \leq Cz$.

2. Additive Eigenvalue Problem of the Laplace Operator with the Prescribed Contact Angle Boundary Value

In this section, we study the additive eigenvalue problem of the Poisson equation with the prescribed contact angle boundary value and prove Theorem 1.

To prove Theorem 1, one can find that we cannot get the C^0 estimate of the solutions to (1) since the solutions can differ from each other at least by a constant; thus, we cannot use the classical methods in partial differential equations to conclude the existence of the solutions. And we would like to settle the following perturbed equations and finally let the parameter ϵ go to zero:

$$\begin{cases} \Delta u = \epsilon u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -\cos \theta \sqrt{1 + |Du|^2}, & \text{on } \partial\Omega, \end{cases} \quad (11)$$

where ν is the inner unit normal vector field along $\partial\Omega$. For this equation with fixed ϵ , the gradient estimate is already derived in [19], and the C^0 estimate

$$|\epsilon u^\epsilon| \leq M \quad (12)$$

for some uniform constant $M > 0$ can also be deduced by following the same method in [25]. So for fixed ϵ , we can conclude the existence of the solution to (11) by Schauder theory, and we denote by $u^\epsilon(x)$ the solution to (11).

Let $v^\epsilon(x) = u^\epsilon(x) - u^\epsilon(x_0)$ with some fixed $x_0 \in \Omega$. In the following, we follow [20, 22] to adopt a blow-up technique to give the uniform C^0 estimate of v^ϵ .

Lemma 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary, $n \geq 2$. If $u^\epsilon(x)$ is the smooth solution to (11), $v^\epsilon(x)$ is defined as above, and then there is a constant $A_0 > 0$, independent of ϵ , such that*

$$|v^\epsilon|_{C^0(\Omega)} \leq A_0. \quad (13)$$

Proof. If this is not the case, without loss of generality, we assume that

$$A_\epsilon = |v^\epsilon|_{C^0(\overline{\Omega})} \longrightarrow +\infty \text{ as } \epsilon \longrightarrow 0^+. \quad (14)$$

Let $w^\epsilon = v^\epsilon/A_\epsilon$, then

$$\begin{aligned} w^\epsilon(x_0) &= 0, \\ |w^\epsilon|_{C^0(\overline{\Omega})} &= 1, \end{aligned} \quad (15)$$

and w^ϵ satisfies the following equation:

$$\begin{cases} \Delta w = \epsilon(w + b_\epsilon u^\epsilon(x_0)), \\ w_\nu = -\sqrt{b_\epsilon^2 + |Dw|^2} \cos \theta, \end{cases} \quad (16)$$

where $b_\epsilon = A_\epsilon^{-1}$ for simplicity.

To proceed with the proof, we need the following interior gradient estimate for the Poisson equation. The conclusion is known, and we omit the proof.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 2$. If w satisfying $|w| \leq M$ for some positive constant M is the smooth solution to*

$$\Delta w = f(x, w), \quad \text{in } \Omega, \quad (17)$$

we then have for any $\Omega' \subset\subset \Omega$

$$\sup_{\Omega'} |Dw| \leq C(M, \text{dist}(\Omega', \partial\Omega), \text{diam}(\Omega), |f|_{C^1(\Omega \times [-M, M])}). \quad (18)$$

The following lemma concludes the gradient estimate near the boundary.

Lemma 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary and ν be the inner unit normal vector field along $\partial\Omega$, $n \geq 2$. If w satisfying $|w| \leq M$ for some positive constant M is the smooth solution to*

$$\begin{cases} \Delta w = \epsilon(w + bu^\epsilon(x_0)), \\ w_\nu = -\sqrt{b^2 + |Dw|^2} \cos \theta, \end{cases} \quad (19)$$

where $0 < b \leq 1$ is a constant, we then have for $\sigma \leq \sigma_1$

$$\sup_{\Omega_\sigma} |Dw| \leq C(M, \Omega, |\theta|_{C^2(\overline{\Omega})}). \quad (20)$$

Proof. Denote by $\nu = \sqrt{b^2 + |Dw|^2}$, $\eta = \nu + w_\nu \cos \theta = \nu + \sum_{i=1}^n w_i d_i \cos \theta$. Let $\Phi = \log \eta + h(w) + \alpha d$, where h and α are to be determined later,

We may assume the maximum of Φ on $\overline{\Omega}_{\sigma_0}$ occurs at some point y_0 for some $\sigma_0 \leq \sigma_1$.

Case 1 ($y_0 \in \partial\Omega_{\sigma_0} \cap \Omega$). The estimate follows from the interior gradient estimate stated in Lemma 2.

Case 2 ($y_0 \in \partial\Omega$). Choose a proper coordinate at y_0 ; let n denote the unit inner normal derivative and $1 \leq i \leq n-1$ denote the tangential derivative. Then at y_0 ,

$$d_n = 1, d_i = 0, \quad i = 1, \dots, n-1. \quad (21)$$

Hence,

$$\begin{aligned} 0 &= \Phi_i = \frac{\eta_i}{\eta} + h' w_i + \alpha d_i = \frac{\eta_i}{\eta} + h' w_i, \quad i = 1, 2, \dots, n-1, \\ 0 &\geq \Phi_n = \frac{\eta_n}{\eta} + h' w_n + \alpha d_n = \frac{\eta_n}{\eta} + h' w_n + \alpha. \end{aligned} \quad (22)$$

By a direct computation, we have

$$\begin{aligned} \eta_n &= \nu_n + w_{nn} \cos \theta + w_n (\cos \theta)_n \\ &= \frac{\sum_{i=1}^n w_i w_{in}}{\nu} + w_{nn} \cos \theta + w_n (\cos \theta)_n \\ &= \frac{\sum_{i=1}^{n-1} w_i w_{in}}{\nu} + \frac{w_n w_{nn}}{\nu} + w_{nn} \cos \theta + w_n (\cos \theta)_n \\ &= \frac{\sum_{i=1}^{n-1} w_i w_{ni}}{\nu} + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\nu} + w_n (\cos \theta)_n, \end{aligned} \quad (23)$$

where we denote by k_{ij} the Weingarten matrix.

Differentiating w_n along $\partial\Omega$, we obtain for $i = 1, 2, \dots, n-1$

$$\begin{aligned} w_{ni} &= (-\nu \cos \theta)_i = -\nu_i \cos \theta - \nu (\cos \theta)_i \\ &= -(\eta_i - w_{ni} \cos \theta - w_n (\cos \theta)_i) \cos \theta - \nu (\cos \theta)_i \\ &= -\eta_i \cos \theta + w_{ni} \cos^2 \theta + w_n \cos \theta (\cos \theta)_i - \nu (\cos \theta)_i. \end{aligned} \quad (24)$$

Furthermore, using

$$0 = \Phi_i = \frac{\eta_i}{\eta} + h' w_i, \quad (25)$$

we get

$$w_{ni} = \frac{\eta h' w_i \cos \theta - \nu(1 + \cos^2 \theta)(\cos \theta)_i}{\sin^2 \theta}. \quad (26)$$

Plugging (26) into (23), we have

$$\begin{aligned} \eta_n &= \frac{\sum_{i=1}^{n-1} w_i w_{ni}}{\nu} + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\nu} + w_n (\cos \theta)_n \\ &= \frac{\sum_{i=1}^{n-1} w_i [\eta h' w_i \cos \theta - \nu(1 + \cos^2 \theta)(\cos \theta)_i]}{\nu \sin^2 \theta} \\ &\quad + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\nu} + w_n (\cos \theta)_n. \end{aligned} \quad (27)$$

Then,

$$\begin{aligned} 0 \geq \Phi_n &= \frac{\sum_{i=1}^{n-1} w_i [\eta h' w_i \cos \theta - \nu(1 + \cos^2 \theta)(\cos \theta)_i]}{\eta \nu \sin^2 \theta} \\ &\quad + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\eta \nu} + \frac{1}{\eta} w_n (\cos \theta)_n + h' w_n + \alpha \\ &= \frac{h' \cos \theta \sum_{i=1}^{n-1} w_i^2}{\nu \sin^2 \theta} - \frac{(1 + \cos^2 \theta) \sum_{i=1}^{n-1} w_i (\cos \theta)_i}{\eta \sin^2 \theta} \\ &\quad + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\eta \nu} + \frac{1}{\eta} w_n (\cos \theta)_n + h' w_n + \alpha \\ &= \frac{h' \cos \theta (\nu^2 \sin^2 \theta - b^2)}{\nu \sin^2 \theta} - \frac{(1 + \cos^2 \theta) \sum_{i=1}^{n-1} w_i (\cos \theta)_i}{\eta \sin^2 \theta} \\ &\quad + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\eta \nu} + \frac{1}{\eta} w_n (\cos \theta)_n + h' w_n + \alpha \\ &= h' \nu \cos \theta - \frac{h' b^2 \cos \theta}{\nu \sin^2 \theta} - \frac{(1 + \cos^2 \theta) \sum_{i=1}^{n-1} w_i (\cos \theta)_i}{\eta \sin^2 \theta} \\ &\quad + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\eta \nu} + \frac{1}{\eta} w_n (\cos \theta)_n + h' w_n + \alpha \\ &= \frac{h' b^2 \cos \theta}{\nu \sin^2 \theta} - \frac{(1 + \cos^2 \theta) \sum_{i=1}^{n-1} w_i (\cos \theta)_i}{\eta \sin^2 \theta} \\ &\quad + \frac{\sum_{i,j=1}^{n-1} w_i k_{ij} w_j}{\eta \nu} + \frac{1}{\eta} w_n (\cos \theta)_n + \alpha. \end{aligned} \quad (28)$$

Without loss of generality, we may assume that ν is large such that if α is chosen large enough determined by the geometry of $\partial\Omega$ and θ , the right-hand side of the above inequality will be positive which shows that this case will not occur at all.

Case 3. ($y_0 \in \Omega_{\sigma_0}$). We take a special coordinate around y_0 such that $w_1 = |Dw|$, $w_l = 0$ ($l = 2, 3, \dots, n$), and w_{ij} is diagonal for $i, j \geq 2$. By the assumption, we have

$$0 = \Phi_i = \frac{\eta_i}{\eta} + h' w_i + \alpha d_i, \quad (29)$$

$$0 \geq \Phi_{ij} = \frac{\eta_{ij}}{\eta} - \frac{\eta_i \eta_j}{\eta^2} + h' w_{ij} + h'' w_i w_j + \alpha d_{ij}. \quad (30)$$

Therefore,

$$\begin{aligned} 0 \geq \Delta \Phi &= \frac{\Delta \eta}{\eta} - \sum_{i=1}^n (h' w_i + \alpha d_i)^2 + \alpha \Delta d + h' \Delta w \\ &\quad + h'' |Dw|^2 \\ &= \frac{\Delta \eta}{\eta} + h' \Delta w + [h'' - h'^2] |Dw|^2 - 2h' \alpha d_1 w_1 \\ &\quad + [\alpha \Delta d - \alpha^2 |Dd|^2] \\ &= \text{I} + \text{II} + \text{III} + \text{IV} + \text{V}. \end{aligned} \quad (31)$$

From the first-order condition (29), we get

$$\eta_i = -\eta (h' w_i + \alpha d_i). \quad (32)$$

On the contrary,

$$\begin{aligned} \eta_i &= \nu_i + (w_\nu \cos \theta)_i = \nu_i + \left(\sum_{l=1}^n w_l d_l \cos \theta \right)_i \\ &= \frac{\sum_{l=1}^n w_l w_{li}}{\nu} + \sum_{l=1}^n w_{li} d_l \cos \theta + \sum_{l=1}^n w_l (d_l \cos \theta)_i \\ &= \sum_{l=1}^n \left(\frac{w_l}{\nu} + d_l \cos \theta \right) w_{li} + \sum_{l=1}^n w_l (d_l \cos \theta)_i. \end{aligned} \quad (33)$$

Therefore, for $i > 1$, we get

$$(d_i \cos \theta) w_{ii} + \left(\frac{w_1}{\nu} + d_1 \cos \theta \right) w_{1i} + w_1 (d_1 \cos \theta)_i = -\alpha \eta d_i. \quad (34)$$

Denote

$$T_1 = \frac{w_1}{\nu} + d_1 \cos \theta, \quad (35)$$

without loss of generality, then we may assume that $|Dw|$ is large such that T_1 is larger than 0 due to the assumption that $|\cos \theta| \leq b < 1$. Therefore,

$$\begin{aligned} w_{1i} &= -\frac{(d_i \cos \theta) w_{ii}}{T_1} - \frac{\eta \alpha d_i}{T_1} - \frac{(d_1 \cos \theta)_i w_1}{T_1} \\ &= -\frac{(d_i \cos \theta) w_{ii}}{T_1} + O(\nu). \end{aligned} \quad (36)$$

And for $i = 1$, we get

$$\begin{aligned}
w_{11} &= -\frac{\sum_{l=2}^n (d_l \cos \theta) w_{1l}}{T_1} - \frac{\eta(h'w_1 + \alpha d_1)}{T_1} \\
&\quad - \frac{(d_1 \cos \theta)_1 w_1}{T_1} \\
&= \frac{\sum_{l=2}^n (d_l \cos \theta) [(d_l \cos \theta) w_{1l} / T_1 + O(v)]}{T_1} \\
&\quad - \frac{\eta(h'w_1 + \alpha d_1)}{T_1} - \frac{(d_1 \cos \theta)_1 w_1}{T_1} \\
&= \sum_{l=2}^n \frac{(d_l \cos \theta)^2}{T_1^2} w_{1l} + O(v^2).
\end{aligned} \tag{37}$$

A direct calculation shows that

$$\begin{aligned}
\eta_{ij} &= \sum_{l=1}^n \left(\frac{w_l}{v} + d_l \cos \theta \right) w_{lij} + \sum_{l=1}^n \left(\frac{w_l}{v} + d_l \cos \theta \right)_j w_{li} \\
&\quad + \sum_{l=1}^n (d_l \cos \theta)_i w_{lj} + \sum_{l=1}^n (d_l \cos \theta)_{ij} w_l \\
&= \sum_{l=1}^n \left(\frac{w_l}{v} + d_l \cos \theta \right) w_{lij} + \sum_{l=1}^n \left[\frac{w_{lj}}{v} - \frac{w_l v_j}{v^2} + (d_l \cos \theta)_j \right] w_{li} \\
&\quad + \sum_{l=1}^n (d_l \cos \theta)_i w_{lj} + \sum_{l=1}^n (d_l \cos \theta)_{ij} w_l,
\end{aligned} \tag{38}$$

and thus by equation (19),

$$\begin{aligned}
\Delta \eta &= \sum_{l,i=1}^n \left(\frac{w_l}{v} + d_l \cos \theta \right) (\Delta w)_l + \sum_{l,i=1}^n \frac{w_{li}^2}{v} - \sum_{i=1}^n \frac{w_1^2 w_{1i}^2}{v^3} \\
&\quad + 2 \sum_{l,i=1}^n (d_l \cos \theta)_i w_{li} + w_1 \Delta (d_1 \cos \theta) \\
&\geq \sum_{l,i=1}^n \frac{w_{li}^2}{v} - \sum_{i=1}^n \frac{w_1^2 w_{1i}^2}{v^3} + 2 \sum_{l,i=1}^n (d_l \cos \theta)_i w_{li} \\
&\quad + [w_1 \Delta (d_1 \cos \theta) + w_1 d_1 \cos \theta] \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{39}$$

For the term I_1 ,

$$\begin{aligned}
I_1 &= \sum_{l,i=1}^n \frac{w_{li}^2}{v} - \sum_{i=1}^n \frac{w_1^2 w_{1i}^2}{v^3} = \frac{b^2 w_{11}^2}{v^3} \\
&\quad + \left(\frac{1}{v} + \frac{b^2}{v^3} \right) \sum_{i=2}^n w_{1i}^2 + \frac{\sum_{i=2}^n w_{ii}^2}{v} \geq \frac{\sum_{i=2}^n w_{ii}^2}{v}.
\end{aligned} \tag{40}$$

For the term I_2 ,

$$\begin{aligned}
I_2 &= 2(d_1 \cos \theta)_1 w_{11} + 2 \sum_{l=2}^n [(d_1 \cos \theta)_l + (d_l \cos \theta)_1] w_{1l} \\
&\quad + 2 \sum_{l=2}^n (d_l \cos \theta)_l w_{ll} \\
&= 2(d_1 \cos \theta)_1 \left\{ \sum_{l=2}^n \frac{(d_l \cos \theta)^2}{T_1^2} w_{ll} + O(v^2) \right\} \\
&\quad + 2 \sum_{l=2}^n (d_l \cos \theta)_l w_{ll} - 2 \sum_{l=2}^n [(d_1 \cos \theta)_l + (d_l \cos \theta)_1] \\
&\quad \cdot \left[\frac{(d_l \cos \theta) w_{ll}}{T_1} + O(v) \right] \\
&= \sum_{l=2}^n O(1) w_{ll} + O(v^2),
\end{aligned} \tag{41}$$

and for the last term I_3 ,

$$I_3 = O(v). \tag{42}$$

Combining the results of I_1 , I_2 , and I_3 , we have

$$I = I_1 + I_2 + I_3 \geq \sum_{l=2}^n \left(\frac{w_{ll}^2}{v} + O(1) w_{ll} \right) + O(v^2) \geq O(v^2). \tag{43}$$

Then, we have

$$\bar{I} = \frac{\Delta \eta}{\eta} \geq O(v). \tag{44}$$

Let $h(w) = -\ln \cos(w/M)$, then

$$|II| = |h' \Delta w| = O(1),$$

$$|III| = (h'' - h'^2) |Dw|^2 = \frac{1}{M^2} |Dw|^2,$$

$$|IV| = |-2h' \alpha d_1 w_1| = |2h' \alpha d_1 Dw| \leq \frac{1}{2M^2} |Dw|^2 + O(1),$$

$$|V| = |\alpha \Delta d - \alpha^2| |Dd|^2 = O(1). \tag{45}$$

Therefore, we have at y_0

$$0 \geq \Delta \Phi \geq O(v) + \frac{1}{2M^2} |Dw|^2 + O(1), \tag{46}$$

which shows that

$$|Dw|^2(y_0) \leq C. \tag{47}$$

Then by a standard discussion, we have

$$|Dw|^2(x) \leq C, \quad \forall x \in \overline{\Omega}_{\sigma_0}. \tag{48}$$

And this finishes the proof of Lemma 3.

Now we can proceed with the proof of Lemma 1.

Proof of Lemma 1. Based on the estimates in Lemma 2 and Lemma 3, we can use Schauder theory to consider the limit behavior of problem (16) and finally get that

$$\begin{cases} \Delta w = 0, & \text{in } \Omega, \\ w_\nu = -|Dw|\cos\theta, & \text{on } \partial\Omega. \end{cases} \quad (49)$$

It shows that the maximum of w only occurs on the boundary, denoted by $\bar{x}$. Then at this point, the tangential part of Dw is zero and $w_\nu \neq 0$. It follows that

$$|\cos\theta| = 1. \quad (50)$$

But this contradicts the condition $|\cos\theta| \leq b < 1$, and thus, the proof of Lemma 1 is finished. $\square$

Proof of Theorem 1. Since $v^\epsilon(x) = u^\epsilon(x) - u^\epsilon(x_0)$, the equation v^ϵ satisfies

$$\begin{cases} \Delta v^\epsilon = \epsilon(v^\epsilon + u^\epsilon(x_0)), \\ v^\epsilon_\nu = -\sqrt{1 + |Dv^\epsilon|^2} \cos\theta. \end{cases} \quad (51)$$

To consider this equation, because of (12), we can apply Lemma 2 and Lemma 3 to conclude the uniform gradient estimate of v^ϵ . Combined with Lemma 1, we immediately derive the uniform C^1 bound of v^ϵ independent of ϵ . And then follows the uniform $C^{k,\alpha}$ bound of v^ϵ , for $k \in \mathbb{Z}^+$, $\alpha \in (0, 1)$. Now we come to think about the process as ϵ goes to zero. Obviously, we can assume that v^ϵ will converge to a smooth function v in the $C^{k,\alpha}$ sense for $k \in \mathbb{Z}^+$, $\alpha \in (0, 1)$.

For the function $\epsilon(v^\epsilon + u^\epsilon(x_0)) = \epsilon u^\epsilon(x)$ with the parameter ϵ , we know by (12) that they are uniformly bounded with their gradient; thus by the Arzela-Ascoli theorem, there is a subsequence, denoted also by $\epsilon u^\epsilon(x)$, converging to a C^1 function $v_0(x)$ satisfying $Dv_0(x) = \lim_{\epsilon \rightarrow 0} \epsilon D(v^\epsilon + u^\epsilon(x_0)) = 0$, where we used the fact that $|Dv^\epsilon|$ is uniformly bounded. Thus, $v_0(x)$ is a constant function τ . That is to say, we have solved the following additive eigenvalue problem:

$$\begin{cases} \Delta v = \tau, & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = -\cos\theta \sqrt{1 + |Dv|^2}, & \text{on } \partial\Omega. \end{cases} \quad (52)$$

For the uniqueness, we can assume that $(\bar{v}, \bar{\tau})$ is also a solution to (1), we may suppose that $\tau \leq \bar{\tau}$, and then we have

$$\begin{cases} \Delta(v - \bar{v}) = \tau - \bar{\tau} \leq 0, & \text{in } \Omega, \\ \frac{\partial(v - \bar{v})}{\partial \nu} = -\langle \vec{b}, D(v - \bar{v}) \rangle \cos\theta, & \text{on } \partial\Omega, \end{cases} \quad (53)$$

where $\vec{b} = \int_0^1 (D(tv + (1-t)\bar{v}) / \sqrt{1 + |D(tv + (1-t)\bar{v})|^2}) dt$ is a vector with $|\vec{b}| < 1$. We then can follow the discussion of (49) to discuss the minimum point of $v - \bar{v}$ and finally deduce that $v - \bar{v}$ must be a constant and $\tau = \bar{\tau}$. This finishes the proof of Theorem 1. $\square$

Remark. From Introduction, we can know that the additive eigenvalue problem is usually related to the large time behavior of the corresponding parabolic equation with the same kind of boundary value. Actually, we will come to consider this problem in the forthcoming paper.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported financially by the National Natural Science Foundation of China (11471188) and STPF of Shandong Province (No. J17KA161).

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