Research Article

Research on Extreme Signed Graphs with Minimal Energy in Tricyclic Signed Graphs $S(n, n + 2)$

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A signed graph is acquired by attaching a sign to each edge of a simple graph, and the signed graphs have been widely used as significant computer models in the study of complex systems. The energy of a signed graph S can be described as the sum of the absolute values of its eigenvalues. In this paper, we characterize tricyclic signed graphs with minimal energy.

1. Introduction

There are various complex systems, such as the society, biology, and information technology systems. In these complex systems, oppositional relationships are universal and can be described abstractly as signed networks. The research of signed networks is of great significance for the accurate understanding of complex systems and their application, which has attracted more and more attention. For example, combining positive and negative information can allow a more accurate identification of topics in sensor networks and make more effective recommendations on social networks. Since 1953 [1], Harary put forward signed graphs for a quantitative analysis of the balance of all parties in social relations. Experimental results show that the use of signed networks is effective to achieve the prediction of major social events. Since then, the signed graph has become an important mathematical model to describe the topological structure of signed networks. Meanwhile, the signed graph model also enriches the research content of spectral graph theory. Then, many research studies about unsigned graphs are extended to signed graphs.

In the field of chemical research, chemists have found that the energy of the molecular graphs of most hydrocarbon molecules is linearly related to the total π -electrical energy of the molecule. Due to this significant application in physical

and chemical sciences, the research on energy of graphs has attracted the attention of many chemical and mathematical researchers. The goal of this paper is to investigate the tricyclic signed graphs with minimal energy.

Formally, a signed graph S is a pair (G, σ) , where $G = (V, E)$ is referred to the underlying graph of S and $\sigma: E \rightarrow \{+, -\}$ is the sign function or signature. The signed graphs mentioned in this paper have simple underlying graphs. We draw the positive edge with a plain line and the negative edge with a dotted line. A Signed graph is referred to as a generalization of a graph because a graph can be considered as a signed graph where each edge is positive.

Let C be a cycle of a signed graph S . The sign of C , denoted by $\text{sgn}(C)$, is defined as $\text{sgn}(C) = \prod_{e \in E(C)} \sigma(e)$. If $\text{sgn}(C) = +$ or $\text{sgn}(C) = -$, then the signed cycle is said to be positive or negative, respectively. It is easy to see that $\text{sgn}(C) = +$, if it contains even number of negative edges, and $\text{sgn}(C) = -$ otherwise. A signed graph S is said to be balanced if all cycles in S are positive; otherwise, it is said to be unbalanced. Throughout this paper, a positive cycle of order n is denoted by C_n^+ , a negative cycle of order n is denoted by C_n^- , and we denote by C_n a cycle of order n which is either positive or negative. Let $\mathcal{S}(n, m)$ be the set of connected signed graphs on n vertices and m edges. In particular, for $m = n, n + 1, n + 2$, $\mathcal{S}(n, m)$ is the set of connected unicyclic signed graphs, bicyclic signed graphs,

and tricyclic signed graphs, respectively. The signed tree, the signed path, and the signed star on n vertices are denoted by T_n , P_n , and S_n , respectively. K_{n_1, n_2} denotes the complete bipartite signed graph with parts of size n_1 and n_2 . The degree of a vertex v in a signed graph S is the number of edges that are incident to v in S denoted by $d(v)$. A basic figure is a graph in which each component is either an edge or a cycle.

The adjacency matrix of a signed graph S , denoted by $A(S) = (a_{ij}^\sigma)$, is an $n \times n$ matrix with $a_{ij}^\sigma = \sigma(v_i v_j) a_{ij}$, where a_{ij} is the number of edges between v_i and v_j . In the case that $\sigma = +$, namely, the sign of every edge in S is positive, $A(S) = (a_{ij}^+)$ is exactly the classical adjacency matrix of G . The characteristic polynomial of the adjacency matrix $A(S)$, $\det(xI - A(S))$, is called the characteristic polynomial of the signed graph S and is denoted by $\phi(S, x)$. The eigenvalues of $A(S)$, denoted by $x_1, x_2, \dots, x_n$, are called the eigenvalues of S . Since $A(S)$ is real symmetric, all eigenvalues of S are real. The set of distinct eigenvalues of S together with their multiplicities is called its spectrum.

The energy of a graph G was defined as $E(G) = \sum_{i=1}^n |\lambda_i|$ by Gutman [2] in 1978, where λ_i , $i = 1, 2, \dots, n$, are the eigenvalues of G . This concept was extended to signed graphs in [3] where the energy of a signed graph S was defined as the sum of absolute values of eigenvalues of S , namely, $E(S) = \sum_{j=1}^n |x_j|$. Furthermore, this concept of energy has been extended to digraphs in [4] and signed digraphs in [5].

The spectral criterion for the balance of signed graphs given by Acharya [6] is as follows.

Lemma 1 (see [6]). *A signed graph S is balanced if and only if S and its underlying graph have the same spectrum.*

Suppose $\theta: V(G) \rightarrow \{+, -\}$ is a sign function. Switching S by θ means forming a new signed graph $S^\theta = (G, \sigma^\theta)$ whose underlying graph is the same as G , but whose sign function is defined on an edge uv by $\sigma^\theta(uv) = \theta(u)\sigma(uv)\theta(v)$. In other words, if we define a diagonal signature matrix D^θ with $d_v = \theta(v)$ for each $v \in V(G)$, then $A(S^\theta) = D^\theta A(S) D^\theta$. If there exists a switching function θ such that $S_1 = S_2^\theta$, or equivalently $A(S_2) = D^\theta A(S_1) D^\theta$, we say the signed graphs S_1 and S_2 are switching equivalent. Switching equivalent is an equivalence relation on the sign of a fixed underlying graph. An equivalence class is called a switching class. Switching a signed graph does not change the sign or the balance of cycles in S [7]. That is to say, if two signed graphs are switching equivalent, then they have the same set of positive cycles, and they are either both balanced or both unbalanced. Furthermore, switching equivalent preserves the spectrum. Thus, as long as spectrum are concerned, we use a single signed graph to represent a switching class. Many results on the minimal energy have been obtained for various classes of graphs. The authors of [8] gave the following conjecture.

Conjecture 1 (see [8]). *Among all connected graphs G with $n \geq 6$ vertices and $n - 1 \leq m \leq 2(n - 2)$ edges, the graphs with minimum energy are stars with $m - n + 1$ additional edges all connected to the same vertex for $m \leq n + \lfloor (n - 7)/2 \rfloor$ and*

bipartite graphs with two vertices on one side, one of which is connected to all vertices on the other side otherwise.

The conjecture is true for unicyclic, bicyclic, and tricyclic graphs [9–12]. A signed analogue of this conjecture is true when $m = n$ for $n \geq 6$ in [13] and when $m = n + 1$ for $n \geq 12$ in [14]. In this paper, we consider the above conjecture for signed graphs, in the case $m = n + 2$ for $n \geq 14$.

We also give the following definitions for signed graphs. Let S_n be a signed star on n vertices and $\mathcal{S}_{n,m}$ be a set of signed graphs which are obtained from S_n by adding $m - n + 1$ signed edges joining one pendant vertex of S_n to other $m - n + 1$ pendant vertices. If a signed graph $S_{n,m} \in \mathcal{S}_{n,m}$, then the signed graph $-S_{n,m} \in \mathcal{S}_{n,m}$ is obtained by changing the sign of every edge in $\mathcal{S}_{n,m}$ to the opposite. For $m = n$, there are two switching classes in $\mathcal{S}_{n,n}$. One is positive containing cycle C_3^+ denoted by $S_{n,n}^1$, and the other is negative containing cycle C_3^- denoted by $S_{n,n}^2$. For $m = n + 1$, there are three switching classes in $\mathcal{S}_{n,n+1}$. If the representative is balanced, we denote it by $S_{n,n+1}^1$. If the representative contains C_3^+ , C_3^- , and C_4^- , then we denote it by $S_{n,n+1}^2$, while $S_{n,n+1}^3$ has two C_3^- and one C_4^+ . For $m = n + 2$, there are four switching classes in $\mathcal{S}_{n,n+2}$, respectively, expressed as $S_{n,n+2}^1$, $S_{n,n+2}^2$, $S_{n,n+2}^3$, and $S_{n,n+2}^4$. $S_{n,n+2}^1$ is balanced and $S_{n,n+2}^2$ has two positive cycles of length 3, C_3^+ , C_4^+ , and two negative cycles of length 4. Meanwhile, $S_{n,n+2}^3$ contains two negative cycles of length 3, C_3^- , C_4^- , and two negative cycles of length 4. And, $S_{n,n+2}^4$ is switching equivalent to $-S_{n,n+2}^1$ with three negative cycles of length 3 and three positive cycles of length 4. For illustration, see Figure 1 for these four representatives in $\mathcal{S}_{n,n+2}$.

Let $\mathcal{K}_{2,n-2}$ be the set of complete bipartite signed graphs formed by joining $n - 6$ ($n \geq 6$) signed pendant edges to a vertex of degree 4 of $K_{2,4}$. There are three switching classes in $\mathcal{K}_{2,n-2}$. We use separately $K_{2,n-2}^1$, $K_{2,n-2}^2$, and $K_{2,n-2}^3$ as the representatives of the three switching classes. $K_{2,n-2}^1$ is balanced, $K_{2,n-2}^2$ contains three positive cycles of length 4 and three negative cycles of length 4, and $K_{2,n-2}^3$ contains two positive cycles of length 4 and four negative cycles of length 4. The three representatives are illustrated in Figure 2.

2. Preliminaries

In this section, we shall give some definitions and notations and state some results, which will be needed to prove our main results.

Theorem 1 (see [6]). *If S is a signed graph with characteristic polynomial*

$$\phi(S, x) = x^n + a_1(S)x^{n-1} + \dots + a_{n-1}(S)x + a_n(S), \quad (1)$$

then

$$a_j(S) = \sum_{L \in \Phi_j} (-1)^{p(L)} 2^{|c(L)|} \prod_{Z \in c(L)} s(Z), \quad (2)$$

for all $j = 1, 2, \dots, n$, where Φ_j is the set of all basic figures L of S of order j , $p(L)$ denotes the number of components of L , $c(L)$ denotes the set of all cycles of L , and $s(Z)$ denotes the sign of cycle Z .

FIGURE 1: The four representatives of switching classes in tricyclic signed graphs $\mathcal{S}_{n,n+2}$. (a) $S_{n,n+2}^1$. (b) $S_{n,n+2}^2$. (c) $S_{n,n+2}^3$. (d) $S_{n,n+2}^4$.

FIGURE 2: Bipartite tricyclic signed graphs with two vertices on one side. (a) $K_{2,n-2}^1$. (b) $K_{2,n-2}^2$. (c) $K_{2,n-2}^3$.

From Theorem 1, it is clear that the spectrum of a signed graph remains invariant by changing the signs of noncyclic edges.

Theorem 2 (see [13]). *Let S be a signed graph on n vertices with characteristic polynomial $\phi(S, x) = x^n + a_1(S)x^{n-1} + \dots + a_{n-1}(S)x + a_n(S)$. Then,*

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \log \left[\left(\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a_{2j}(S) x^{2j} \right)^2 + \left(\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a_{2j+1}(S) x^{2j+1} \right)^2 \right] dx. \quad (3)$$

If we put $b_{2j}(S) = |(-1)^j a_{2j}(S)|$ and $b_{2j+1}(S) = |(-1)^j a_{2j+1}(S)|$, then the above integral formula for signed graphs takes the form

$$E(S) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \log \left[\left(\sum_{j=0}^{\lfloor n/2 \rfloor} b_{2j}(S) x^{2j} \right)^2 + \left(\sum_{j=0}^{\lfloor n/2 \rfloor} b_{2j+1}(S) x^{2j+1} \right)^2 \right] dx. \quad (4)$$

It is easy to see that $E(S)$ is a monotonously increasing function of $b_j(S)$. Let S_1 and S_2 be two signed graphs of order n .

- (1) If $b_{2j}(S_1) = b_{2j}(S_2)$ and $b_{2j+1}(S_1) = b_{2j+1}(S_2)$ for all $j \geq 0$, then $E(S_1) = E(S_2)$ and we write $S_1 \sim S_2$.

- (2) If $b_{2j}(S_1) \geq b_{2j}(S_2)$ and $b_{2j+1}(S_1) \geq b_{2j+1}(S_2)$ for all $j \geq 0$, we see that $E(S_1) \geq E(S_2)$ and we write $S_1 \succcurlyeq S_2$.
- (3) If $S_1 \succcurlyeq S_2$ and there exists some j_0 such that $b_{2j_0}(S_1) > b_{2j_0}(S_2)$ or $b_{2j_0+1}(S_1) > b_{2j_0+1}(S_2)$, then we write $S_1 \succ S_2$. Clearly, $\succ$ is a quasiorder relation, that is, reflexive and transitive on coefficient. Therefore,

$$S_1 \succ S_2 \implies E(S_1) > E(S_2). \quad (5)$$

Note that the order of signed graphs does not have to be the same in a quasi-order comparison.

Let $\mathcal{S}(n, n+2)$ be the set of all connected tricyclic signed graphs with $m = n+2$. It is straightforward to check that a signed graph $S \in \mathcal{S}(n, n+2)$ has at least 3 signed cycles and at most 7 signed cycles. We classify $\mathcal{S}(n, n+2)$ into the following two categories.

- (i) *Case 1.* There exists a noncyclic path P connecting one cycle with another denoted by $\mathcal{S}_1(n, n+2)$. For illustration, see Figure 3, where we have not shown noncyclic edges which can be present.
- (ii) *Case 2.* There exist two edges' disjoint paths between any two vertices which are on some cycles, denoted by $\mathcal{S}_2(n, n+2)$. For illustration, see Figure 4.

If V_1 is a subset of the vertex set V of the signed graph S , then $S - V_1$ represents the signed graph obtained from S by removing the vertices in V_1 . Let E_1 be a subset of the edge set E of the signed graph S ; then, by $S - E_1$, we mean the signed graph obtained from S by deleting the edges in E_1 . In addition, we refer to E_1 as a cut set of a signed graph S if $S - E_1$ is disconnected.

FIGURE 3: Seven underlying graphs of tricyclic signed graphs in case 1.

FIGURE 4: Eight underlying graphs of tricyclic signed graphs in case 2.

Let M be an $m \times n$ complex matrix and M^* be the conjugate transpose of M . The singular values $\sigma_1(M), \sigma_2(M), \dots, \sigma_n(M)$ of a matrix M are the positive square roots of the eigenvalues of MM^* . Since $M = A(S)$ is a real symmetric matrix, $A(S)A(S)^* = A(S)^2$ and $E(S) = \sum_i^n \sigma_i(A(S))$. For a partitioned matrix $A = \begin{bmatrix} B & X \\ Y & C \end{bmatrix}$, where both B and C are square matrices, the following inequality about singular value can be find in [15].

$$\sum_i \sigma_i(B) + \sum_i \sigma_i(C) \leq \sum_i \sigma_i(A). \quad (6)$$

It implies that $E(B) + E(C) \leq E(A)$. Proceeding exactly in a similar way as in [15], the following lemma illustrates the change in graph energy when a cut set is deleted.

Lemma 2 (see [15]). *If E_1 is a cut set of a signed graph S , then $E(S - E_1) \leq E(S)$. Moreover, if E_1 consists of a single edge, then $E(S - E_1) < E(S)$.*

The following three lemmas are needed in our paper.

Lemma 3 (see [2]). *$E(S_n) \leq E(T_n) \leq E(P_n)$, with equality on left if and only if $T_n = S_n$ and equality on right if and only if $T_n = P_n$.*

Lemma 4 (see [13]). *Among all unicyclic signed graphs with $n \geq 6$ vertices, the signed graphs $\mathcal{S}_{n,n}$ have the minimal energy.*

Moreover, for $n = 3, 4, 5$, the unicyclic signed graphs with the minimal energy are shown in Figure 5.

Lemma 5 (see [14]). *Among all bicyclic signed graphs with $n \geq 8$ vertices, $S_{n,n+1}^1$ and $S_{n,n+1}^3$ have the minimal energy. For $n = 6, 7$, $B_{n,n+1}^1$ shown in Figure 6 has the minimal energy. For $n = 5$, $B_{5,6}^1$ has the minimal energy and $S_{5,6}^1$ and $S_{5,6}^3$ have the second minimal energy. For $n = 4$, $S_{4,5}^1$ has the minimal energy.*

3. Main Results

In this section, we shall consider the signed tricyclic graphs in $\mathcal{S}(n, n+2)$ ($n \geq 14$) having the minimal energy.

Lemma 6. *For $n \geq 7$, $E(S_{n,n+2}^1) = E(S_{n,n+2}^4) < E(S_{n,n+2}^2) = E(S_{n,n+2}^3)$.*

Proof. It is obvious that $E(S_{n,n+2}^1) = E(S_{n,n+2}^4)$ and $E(S_{n,n+2}^2) = E(S_{n,n+2}^3)$ because $S_{n,n+2}^1$ is switching equivalent to $-S_{n,n+2}^4$ and $S_{n,n+2}^2$ is switching equivalent to $-S_{n,n+2}^3$. Therefore, to compare the energy of $S_{n,n+2}^r$, for $r = 1, 2, 3, 4$, it is enough to compare the energy of $S_{n,n+2}^1$ and $S_{n,n+2}^3$. By Theorem 1, we have

$$\begin{aligned} \phi(S_{n,n+2}^1) &= x^n - (n+2)x^{n-2} - 6x^{n-3} + (3n-15)x^{n-4}, \\ \phi(S_{n,n+2}^3) &= x^n - (n+2)x^{n-2} + 2x^{n-3} + (3n-7)x^{n-4}. \end{aligned} \quad (7)$$

FIGURE 5: Unicyclic signed graphs with minimal energy. (a) C_3^+ . (b) C_3^- . (c) C_4^+ . (d) S_0 . (e) $S_{n,n}^1$. FIGURE 6: Bicyclic signed graphs with minimal energy. (a) $S_{n,n+1}^1$. (b) $S_{n,n+1}^3$. (c) $B_{n,n+1}^1$.

However, we find that the coefficients of signed graph $S_{n,n+2}^1$ and $S_{n,n+2}^3$ are not quasi-order comparable. We will directly solve the integrals to compare the energy using Theorem 2, and we have

$$E(S_{n,n+2}^3) - E(S_{n,n+2}^1) = \frac{1}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log \frac{(1 + (n+2)x^2 + (3n-7)x^4)^2 + 4x^6}{(1 + (n+2)x^2 + (3n-15)x^4)^2 + 36x^6} dx. \quad (8)$$

Put $\alpha(x) = (1 + (n+2)x^2 + (3n-7)x^4)^2 + 4x^6$ and $\beta(x) = (1 + (n+2)x^2 + (3n-15)x^4)^2 + 36x^6$, and we get $\alpha(x) - \beta(x) = 16(3n-11)x^8 + 16nx^6 + 16x^4 > 0$ for $n \geq 7$ and $x > 0$. Thus, $E(S_{n,n+2}^3) > E(S_{n,n+2}^1)$. $\square$

Lemma 7. For $n \geq 7$, $E(K_{2,n-2}^1) < E(K_{2,n-2}^2) < E(K_{2,n-2}^3)$.

Proof. The characteristic polynomials of $\phi(K_{2,n-2}^r)$, $r = 1, 2, 3$, are given by

$$\begin{aligned} \phi(K_{2,n-2}^1) &= x^n - (n+2)x^{n-2} + (4n-24)x^{n-4}, \\ \phi(K_{2,n-2}^2) &= x^n - (n+2)x^{n-2} + (4n-12)x^{n-4}, \\ \phi(K_{2,n-2}^3) &= x^n - (n+2)x^{n-2} + (4n-8)x^{n-4}. \end{aligned} \quad (9)$$

Clearly, $K_{2,n-2}^1 < K_{2,n-2}^2 < K_{2,n-2}^3$. Therefore, $E(K_{2,n-2}^1) < E(K_{2,n-2}^2) < E(K_{2,n-2}^3)$. $\square$

Lemma 8 (see [9])

- (i) For $n \geq 11$, $E(S_{n,n+2}^1) < E(K_{n,n-2}^1)$.
- (ii) For $7 \leq n \leq 10$, $E(K_{n,n-2}^1) < E(S_{n,n+2}^1)$.

Lemma 9 (see [13]). For $n_1, n_2 \geq 6$ and $n = n_1 + n_2$, $E(S_{n_1,n_1}^1 \cup S_{n_2,n_2}^1) \geq E(S_{n-6,n-6}^1 \cup S_{6,6}^1)$. The equality holds if and only if $n_1, n_2 \in \{6, n-6\}$.

Theorem 3

- (1) $E(S_{n,n+1}^{r_1}) > E(S_{n,n}^{r_2})$ for $r_1 = 1, 2, 3$, $r_2 = 1, 2$, $n \geq 6$.
- (2) $E(S_{n-6,n-6}^r \cup S_{6,6}^r) > E(K_{2,n-2}^3)$ for $r = 1, 2$, $n \geq 12$.

Proof

- (1) Note that $S_{n,n}^1 \sim S_{n,n}^2$ and $S_{n,n+1}^1 \sim S_{n,n+1}^3 < S_{n,n+1}^2$. It is enough to prove that

$$E(S_{n,n+1}^1) > E(S_{n,n}^1). \quad (10)$$

Again, by Theorem 1, we have

$$\begin{aligned} \phi(S_{n,n+1}^1) &= x^n - (n+1)x^{n-2} - 4x^{n-3} + 2(n-4)x^{n-4}, \\ \phi(S_{n,n}^1) &= x^n - nx^{n-2} - 2x^{n-3} + (n-3)x^{n-4}. \end{aligned} \quad (11)$$

As $n \geq 6$, $2(n-4) > n-3 > 0$, we see that $S_{n,n+1}^1 > S_{n,n}^1$. So, $E(S_{n,n+1}^1) > E(S_{n,n}^1)$.

(2) The characteristic polynomial of $K_{2,n-2}^3$ is given by

$$\phi(K_{2,n-2}^3) = x^n - (n+2)x^{n-2} + (4n-8)x^{n-4}. \quad (12)$$

By the characteristic polynomial of $S_{n,n}^1$, we get

$$\phi(S_{n-6,n-6}^1) = x^{n-6} - (n-6)x^{n-8} - 2x^{n-9} + (n-9)x^{n-10}. \quad (13)$$

Note that $E(S_{6,6}^1) \approx 6.1722 > 6$. In this case, we define $\alpha(x) = x^4 - (n+2)x^2 + (4n-8)$ and $\beta(x) = x^4 - (n-6)x^2 - 2x + (n-9)$. It is easy to see that $\alpha(2) = 0$, $\alpha(\sqrt{n-2}) = 0$. Also, $\alpha(-x) = \alpha(x)$; therefore, $\alpha(x)$ has two positive and two negative zeros. We observe that $\beta(0) > 0$, $\beta(-\sqrt{n}) > 0$, $\beta(-2) < 0$, $\beta(-1) = 0$, $\beta(-\sqrt{n}) > 0$, $\beta(-2) < 0$, $\beta(-1) = 0$, $\beta(1/2) > 0$, $\beta(\sqrt{n-7}) < 0$, and $\beta(\sqrt{n-6}) > 0$. Therefore, $\beta(x)$ has two positive and two negative zeros too. Let the positive zeros of $\beta(x)$ be b_1 and b_2 . Using the fact that the energy of a signed graph is twice the sum of the positive eigenvalues, we have

$$\begin{aligned} E(S_{n-6,n-6}^r \cup S_{6,6}^r) &= E(S_{n-6,n-6}^1 \cup S_{6,6}^1) = E(S_{n-6,n-6}^1) + E(S_{6,6}^1) \\ &> 2(b_1 + b_2) + 6 > 2\left(\frac{1}{2} + \sqrt{n-7}\right) + 6 \\ &= 7 + 2\sqrt{n-7} > 2(2 + \sqrt{n-2}) \\ &= E(K_{2,n-2}^3), \end{aligned} \quad (14)$$

for all $n \geq 12$. This completes the proof. $\square$

Theorem 4. If $S \in \mathcal{S}_1(n, n+2)$, $n \geq 14$, then $E(S) > E(K_{2,n-2}^3)$.

Proof. Let e_1 be an edge of the noncyclic path P in $S \in \mathcal{S}_1(n, n+2)$. Then, $S - e_1 = S_1 \cup S_2$, where S_1 is a bicyclic signed graph on n_1 vertices and S_2 is a unicyclic signed graph on n_2 vertices. The following cases arise. (1) $n_1 \geq 8, n_2 \geq 6$; (2) $n_1 \geq 9, n_2 = 5$; (3) $n_1 \geq 10, n_2 = 4$; (4) $n_1 \geq 11, n_2 = 3$; (5) $n_1 = 7, n_2 \geq 7$; (6) $n_1 = 6, n_2 \geq 8$; (7) $n_1 = 5, n_2 \geq 9$; and (8) $n_1 = 4, n_2 \geq 10$.

Case (1): $n_1 \geq 8, n_2 \geq 6$. By Lemmas 2 and 9 and Theorem 3, we have

$$\begin{aligned} E(S) &> E(S - e) = E(S_1 \cup S_2) = E(S_1) + E(S_2) \\ &\geq E(S_{n_1, n_1+1}^1) + E(S_{n_2, n_2}^1) \\ &> E(S_{n_1, n_1}^1) + E(S_{n_2, n_2}^1) \\ &\geq E(S_{n-6, n-6}^1) + E(S_{6,6}^1) \\ &> E(K_{2, n-2}^3). \end{aligned} \quad (15)$$

Case (2): $n_1 \geq 9, n_2 = 5$. Proceeding similarly as in Case (1), we can prove that $E(S) > E(S_{n-5, n-5}) + E(S_0)$, where S_0 is the signed graph with the minimal energy for $n = 5$ shown in Figure 5. Thus, we need to prove that $E(S_{n-5, n-5}) + E(S_0) > E(K_{2, n-2}^3)$. Note that $E(S_0) \approx 5.5960 > 5.5$. We have

$$\begin{aligned} E(S_{n-5, n-5}^1) + E(S_0) &> 2\left(\frac{1}{2} + \sqrt{n-6}\right) + 5.5 = 6.5 + 2\sqrt{n-6} \\ &> 2(2 + \sqrt{n-2}) = E(K_{2, n-2}^3). \end{aligned} \quad (16)$$

Case (3): $n_1 \geq 10, n_2 = 4$. Note that $E(C_4^+) = 4$. Then,

$$\begin{aligned} E(S_{n-4, n-4}^1) + E(C_4^+) &> 2\left(\frac{1}{2} + \sqrt{n-5}\right) + 4 = 5 + 2\sqrt{n-5} \\ &> 2(2 + \sqrt{n-2}) = E(K_{2, n-2}^3). \end{aligned} \quad (17)$$

Case (4): $n_1 \geq 11, n_2 = 3$. Note that $E(C_3^+) = 4$. Then,

$$\begin{aligned} E(S_{n-3, n-3}^1) + E(C_3^+) &> 2\left(\frac{1}{2} + \sqrt{n-4}\right) + 4 = 5 + 2\sqrt{n-4} \\ &> 2(2 + \sqrt{n-2}) = E(K_{2, n-2}^3). \end{aligned} \quad (18)$$

Case (5): $n_1 = 7, n_2 \geq 7$. Note that $E(B_{7,8}^1) \approx 7.1831 > 7$. Then,

$$\begin{aligned} E(B_{7,8}^1) + E(S_{n-7, n-7}^1) &> 7 + 2\left(\frac{1}{2} + \sqrt{n-8}\right) = 8 + 2\sqrt{n-8} \\ &> 2(2 + \sqrt{n-2}) = E(K_{2, n-2}^3). \end{aligned} \quad (19)$$

Case (6): $n_1 = 6, n_2 \geq 8$. Note that $E(B_{6,7}^1) \approx 6.4696 > 6$. Then,

$$\begin{aligned} E(B_{6,7}^1) + E(S_{n-6, n-6}^1) &> 6 + 2\left(\frac{1}{2} + \sqrt{n-7}\right) = 7 + 2\sqrt{n-7} \\ &> 2(2 + \sqrt{n-2}) = E(K_{2, n-2}^3). \end{aligned} \quad (20)$$

Case (7): $n_1 = 5, n_2 \geq 9$. Note that $E(B_{5,6}^1) = 2\sqrt{6}$. Then,

$$\begin{aligned} E(B_{5,6}^1) + E(S_{n-5, n-5}^1) &> 4.9 + 2\left(\frac{1}{2} + \sqrt{n-6}\right) = 5.9 + 2\sqrt{n-6} \\ &> 2(2 + \sqrt{n-2}) = E(K_{2, n-2}^3). \end{aligned} \quad (21)$$

Case (8): $n_1 = 4, n_2 \geq 10$. Note that $E(S_{4,5}^1) \approx 5.1231 > 5$. Then,

$$\begin{aligned}
E(S_{4,5}^1) + E(S_{n-4,n-4}^1) &> 5 + 2\left(\frac{1}{2} + \sqrt{n-5}\right) = 6 + 2\sqrt{n-5} \\
&> 2(2 + \sqrt{n-2}) = E(K_{2,n-2}^3).
\end{aligned} \tag{22}$$

The theorem now holds. $\square$

Lemma 10. Let S_{n_1} be a signed star of order n_1 and S_{n_2,n_2+1} be a bicyclic signed graph of order n_2 , where $n_1 \geq 3$, $n_2 \geq 8$ and $n_1 + n_2 = n$. Then,

$$E(S_{n_1} \cup S_{n_2,n_2+1}^r) \geq E(S_3 \cup S_{n_2,n_2+1}^1) \geq E(S_3 \cup S_{n-3,n-2}^1). \tag{23}$$

Proof. By Lemmas 3 and 5, we have $E(S_{n_1} \cup S_{n_2,n_2+1}^r) \geq E(S_3 \cup S_{n_2,n_2+1}^1)$. Then, we consider the following characteristic polynomial:

$$\begin{aligned}
\phi(S_{n_1}, x) \phi(S_{n_2,n_2+1}^1, x) &= [x^{n_1} - (n_1 - 1)x^{n_1-2}] \\
&\cdot [x^{n_2} - (n_2 + 1)x^{n_2-2} - 4x^{n_2-3} + 2(n_2 - 4)x^{n_2-4}] \\
&= x^n - nx^{n-2} - 4x^{n-3} + (n_1n_2 + n - 9)x^{n-4} \\
&\quad + 4(n_1 - 1)x^{n-5} - 2(n_1 - 1)(n_2 - 4)x^{n-6}.
\end{aligned} \tag{24}$$

Applying the integral formula given in Theorem 2, we have

$$\begin{aligned}
E(S_{n_1} \cup S_{n_2,n_2+1}^1) &= \frac{1}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log f(x) dx \\
&\geq \frac{1}{\pi} \int_0^{+\infty} \frac{1}{x^2} \log g(x) dx \\
&= E(S_3 \cup S_{n-3,n-2}^1),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
f(x) &= (1 + nx^2 + (n_1n_2 + n - 9)x^4 + 2(n_1 - 1)(n_2 - 4)x^6)^2 \\
&\quad + (4x^3 + 4(n_1 - 1)x^5)^2,
\end{aligned} \tag{26}$$

$$g(x) = (1 + nx^2 + (4n - 18)x^4 + 4(n - 7)x^6)^2 + (4x^3 + 8x^5)^2. \tag{27}$$

Since $n_1 \geq 3$ and $n_1 + n_2 = n$, we observe that $n_1n_2 \geq 3(n - 3)$, with equality if and only if $n_1, n_2 \in \{3, n - 3\}$. $\square$

Lemma 11

- (i) $E(S_{n-7} \cup B_{7,8}^1) > E(K_{2,n-2}^3)$ for all $n \geq 14$.
- (ii) $E(S_{n-6} \cup B_{6,7}^1) > E(K_{2,n-2}^3)$ for all $n \geq 14$.
- (iii) $E(S_{n-5} \cup B_{5,6}^1) > E(S_{n,n+2}^1)$ for all $n \geq 14$.
- (iv) $E(S_{n-4} \cup S_{4,5}^1) > E(K_{2,n-2}^3)$ for all $n \geq 14$.
- (v) $E(S_3 \cup S_{n-3,n-2}^1) > E(K_{2,n-2}^3)$ for all $n \geq 14$.

Proof

- (i) Note that $\text{spec}(S_n) = \{\pm \sqrt{n-1}, 0^{[n-2]}\}$. Therefore, $E(S_{n-7} \cup B_{7,8}^1) > 2\sqrt{n-8} + 7 > 2(2 + \sqrt{n-2}) = E(K_{2,n-2}^3)$, (28)

for all $n \geq 14$.

- (ii) We have

$$E(S_{n-6} \cup B_{6,7}^1) > 2\sqrt{n-7} + 6 > 2(2 + \sqrt{n-2}) = E(K_{2,n-2}^3), \tag{29}$$

for all $n \geq 14$.

- (iii) By direct calculations, we have verified $E(S_{n,n+2}^1) < 2\sqrt{n-6} + 2\sqrt{6}$ for $n = 14$ to 26 , where $E(B_{5,6}^1) = 2\sqrt{6}$. For $n \geq 27$, we consider the function

$$\alpha(x) = x^4 - (n+2)x^2 - 6x + 3n - 15. \tag{30}$$

We see $\alpha(\sqrt{2}) > 0$, $\alpha(\sqrt{3}) < 0$, $\alpha(\sqrt{n-1}) < 0$, and $\alpha(\sqrt{n+1}) > 0$ for all $n \geq 27$. Therefore, $\alpha(x)$ has two positive zeros in the interval $(\sqrt{2}, \sqrt{3})$ and $(\sqrt{n-1}, \sqrt{n+1})$, respectively. We have

$$E(S_{n-5} \cup B_{5,6}^1) = 2\sqrt{n-6} + 2\sqrt{6} > 2(\sqrt{3} + \sqrt{n+1}) > E(S_{n,n+2}^1), \tag{31}$$

for all $n \geq 27$.

- (iv) Note that $E(S_{4,5}^1) > 5$. We have

$$E(S_{n-5} \cup S_{4,5}^1) > 2\sqrt{n-5} + 5 > 2(2 + \sqrt{n-2}) = E(K_{2,n-2}^3), \tag{32}$$

for all $n \geq 14$.

- (v) We can easily verify $E(S_3) + E(S_{n-3,n-2}^1) > E(K_{2,n-2}^3)$ for $n = 14$. When $n \geq 15$, we consider the function

$$\alpha(x) = x^4 - (n-2)x^2 - 4x + 2(n-7). \tag{33}$$

We see $\alpha(1) \geq 0$, $\alpha(\sqrt{2}) < 0$, $\alpha(\sqrt{n-4}) < 0$, and $\alpha(\sqrt{n-2}) > 0$ for all $n \geq 15$. Note that $E(S_3) = 2\sqrt{2}$. Similar to (iii), we have

$$\begin{aligned}
E(S_3) + E(S_{n-3,n-2}^1) &> 2\sqrt{2} + 2(1 + \sqrt{n-4}) \\
&= 2\sqrt{2} + 2 + 2\sqrt{n-4} > 4 + 2\sqrt{n-2} \\
&= E(K_{2,n-2}^3).
\end{aligned} \tag{34}$$

The lemma holds.

Let $H \in \mathcal{S}_2(n, n+2)$ such that there exists a cut set E_1 of size two whose deletion will result in a signed tree with order at least 3 and a bicyclic signed graph and $\mathcal{H}$ be the set of all signed graphs H . $\square$

Theorem 5. If $H \in \mathcal{H}$ with $n \geq 14$, then $E(S) > E(S_{n,n+2}^1)$.

Proof. Let $H \in \mathcal{H}$. Choosing the cut set $E_1 = \{e_1, e_2\}$, $H - \{e_1, e_2\}$ has two components, say H_1 and H_2 , where H_1 is a

signed tree on $n_1 \geq 3$ vertices and H_2 is a bicyclic signed graph with n_2 vertices such that $n_1 + n_2 = n \geq 14$. By Lemma 3, $E(H_1) \geq E(S_{n_1})$. Then, the result follows by Lemmas 2, 5, 10, and 11.

By Theorem 5, we immediately have the following corollary. $\square$

Corollary 1. *Let S be a tricyclic signed graph in $\mathcal{S}_2(n, n+2)$ satisfying one of the following properties; then, $E(S) > E(S_{n, n+2}^1)$.*

- (i) *There exists a path $v_0, v_1, \dots, v_k$ with $k \geq 4$, $d(v_0) \geq 3$, $d(v_k) \geq 3$, and $d(v_i) = 2$, $i = 1, \dots, k-1$.*
- (ii) *There exists a vertex u other than common vertices with $d(u) \geq 4$.*
- (iii) *There exist two adjacent vertices other than common vertices denoted by w, z with $d(w) \geq 3$ or $d(z) \geq 3$.*
- (iv) *There exists a vertex v not on the cycles with $d(v) \geq 3$.*
- (v) *There exists a pendant path of length at least 3 with one endpoint on the cycles or there exists a pendant path of length at least 2 with one endpoint not on the cycles.*

4. Conclusions

The spectral properties of signed graphs are of great significance in the application of spectral graph theory as well as the topology of complex systems. Generally, graph energy is an important combinatorial spectral form. We characterized the structure of tricyclic signed graphs in $\mathcal{H}$, where $\mathcal{H}$ is a subset of $\mathcal{S}_2(n, n+2)$. Our results show that if $S \in \mathcal{S}_1(n, n+2) \cup \mathcal{H}$, then $E(S) > E(S_{n, n+2}^1)$ which is a key to solving the conjecture (changed graphs to signed graphs) for $m = n+2$ and lays a foundation for future research of tricyclic signed graphs with minimal energy.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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