

## Retraction

# Retracted: Robust Adaptive Control for a Class of T-S Fuzzy Nonlinear Systems with Discontinuous Multiple Uncertainties and Abruptly Changing Actuator Faults

### Complexity

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

- (1) Discrepancies in scope
- (2) Discrepancies in the description of the research reported
- (3) Discrepancies between the availability of data and the research described
- (4) Inappropriate citations
- (5) Incoherent, meaningless and/or irrelevant content included in the article
- (6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.


The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

### References

- [1] X. Ning, Y. Zhang, and Z. Wang, "Robust Adaptive Control for a Class of T-S Fuzzy Nonlinear Systems with Discontinuous Multiple Uncertainties and Abruptly Changing Actuator Faults," *Complexity*, vol. 2020, Article ID 8863073, 16 pages, 2020.

## Research Article

# Robust Adaptive Control for a Class of T-S Fuzzy Nonlinear Systems with Discontinuous Multiple Uncertainties and Abruptly Changing Actuator Faults

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In the complex environment, the suddenly changing structural parameters and abrupt actuator failures are often encountered, and the negligence or improper handling method may induce undesired or unacceptable results. In this paper, taking the suddenly changing structural parameters and abrupt actuator failures into consideration, we focus on the robust adaptive control design for a class of heterogeneous Takagi–Sugeno (T-S) fuzzy nonlinear systems subjected to discontinuous multiple uncertainties. The key point is that the switch modes not only vary with the system time but also vary with the system states, and the intrinsic heterogeneous characteristics make it difficult to design stable controllers. Firstly, the concepts of differential inclusion are introduced to describe the heterogeneous fuzzy systems. Meanwhile, a fundamental lemma is provided to demonstrate the criteria of the boundness for a Filippov solution. Then, by using the set-valued Lie derivative of the Lyapunov function and introducing a vector of specific continuous functions, the closed-loop T-S fuzzy differential inclusion systems are proved to be ultimately bounded. The sufficient conditions for system stability are derived in term of linear matrix inequalities (LMIs), which can be solved directly. Finally, a numerical example is provided to illustrate the effectiveness of the proposed control algorithm.

## 1. Introduction

As it is well-known, the T-S fuzzy model is a powerful tool for the analysis and control design of nonlinear systems [1–4]. Therefore, a great wealthy of results has been achieved for T-S fuzzy systems in the past decades. In [5], the parameterized linear matrix inequality technique has been investigated for T-S fuzzy control systems. In [6], by using the LMIs and sum-of-squares-based approach, an output regulator has been constructed for the polynomial fuzzy control systems. In [7, 8], two fuzzy sliding mode control methods have been developed for the T-S fuzzy systems suffering from both matched and unmatched uncertainties. The fault-tolerant control approaches for T-S fuzzy nonlinear systems have been investigated in [9, 10]. As a further

development, a finite-time fault-tolerant control structure of T-S fuzzy nonlinear systems has been synthesized in [11]. The event-triggered control structures for T-S fuzzy nonlinear systems can be found in [12, 13]. In [14, 15], the robust filters have been constructed for continuous and discrete-time T-S fuzzy systems, respectively. For the delayed T-S fuzzy systems with and without stochastic perturbations, the stability analysis and stabilization methods have been provided in [16, 17]. Chang et al. [18] focused on the robust adaptive control design for a class of heterogeneous T-S fuzzy nonlinear systems subjected to discontinuous multiple uncertainties.

It is also well-known that the uncertainties and disturbances are often encountered in many practical systems. In [19, 20], the active disturbance rejection control methods

have been reported for the nonlinear systems with uncertainties. In [21–24], several antidisturbance controllers have been synthesized by constructing the disturbance observers. The composite antidisturbance controllers can be found in [25–29]. Aiming at the uncertainties existing in the quantized control systems, several robust and adaptive controllers have been proposed in [30–32]. For the unknown nonlinearities existing in the control systems, fruitful results have been reported. In [33], a fuzzy adaptive output feedback controller has been proposed for the multi-input and multioutput nonlinear systems with completely unknown nonlinear functions. For a class of switched stochastic nonlinear systems in a pure-feedback form, a fuzzy observer is constructed to approximate unmeasurable states in [34, 35]. In [36], an adaptive fuzzy tracking control problem has been investigated for a class of nonstrict-feedback systems with unmeasured states and unknown nonlinearities. For the state constrained control systems with unknown nonlinear functions, two adaptive control results have been reported in [37, 38]. In [39], the problem of adaptive neural finite-time tracking control for uncertain nonstrict-feedback nonlinear systems with input saturation has been studied.

In spite of the progress, in the aforementioned control results, a vital problem on heterogeneous uncertainties, in the sense that the parameters and the structures of the uncertainties keep switching with both the system time and the system states, was omitted. This kind of uncertainties and system nonlinearities are often encountered in many physical systems [40,41], and it is of significant importance to develop an effective controller for the heterogeneous uncertain nonlinear systems. When system-state-based switching uncertainties and system structures are taken into consideration, the traditional analysis and control methods become invalid. Furthermore, the control approaches of conventional switched systems cannot be applied neither because most of the switched systems are required to be

purely time-based [42–44]. Moreover, when the fuzzy modeling methods and fuzzy control strategies are introduced in the heterogeneous uncertain system, the control design problem becomes more challenging and interesting. As far as the authors know, no results have been reported for the T-S fuzzy systems subjected to heterogeneous uncertainties and system structures. For the purpose of improving the practicability of the proposed algorithm, the abruptly changing actuator faults are also considered. Motivated by the above considerations, this paper is committed to develop an effective control structure for the T-S fuzzy heterogeneous systems with discontinuous multiple uncertainties and abruptly changing actuator faults. Compared to the existing literature, the main contributions of this paper are as follows:

- (i) To the best of the authors' knowledge, it is the first control solution for the T-S fuzzy systems subjected to heterogeneous uncertainties and system structures, which keep switching with both the system time and the system states.
- (ii) A fundamental lemma is provided to demonstrate the criteria of the boundness for a Filippov solution, establishing the mathematical fundamentals for the adaptive control of differential inclusion systems.
- (iii) By proposing a specific vector of continuous functions, the closed-loop multivariable T-S fuzzy differential inclusion systems are proved to be ultimately bounded for the first time.

## 2. Problem Formulation and Preliminaries

**2.1. Problem Statement.** Consider the following T-S fuzzy nonlinear systems.

**Plant Rule:** IF  $\theta_{i,1}(t)$  is  $\mu_{i,1}$ ,  $\theta_{i,2}(t)$  is  $\mu_{i,2}$ ,  $\dots$  and  $\theta_{i,p}(t)$  is  $\mu_{i,p}$ , THEN

$$\dot{x}(t) = \begin{cases} f_i^1(t, x(t)) + A_i^1 x + B_i u, & \text{if } (t, x(t)) \in G_1, \\ f_i^2(t, x(t)) + A_i^2 x + \Delta A_i(t)x + B_i(u + d(t)), & \text{if } (t, x(t)) \in G_2, \\ f_i^3(t, x(t)) + A_i^3 x + \Delta A_i(t)x + B_i(\Lambda(t)u + \varepsilon_u(t) + d(t)), & \text{if } (t, x(t)) \in G_3, \end{cases} \quad (1)$$

where  $u(t) \in \mathbb{R}^m$  is the input signal of the system and  $x(t) \in \mathbb{R}^n$  are the system state vector.  $\theta(t) = [\theta_1(t), \theta_2(t), \dots, \theta_p(t)]$  are the premise variables.  $\mu_{i,1}, \mu_{i,2}, \dots, \mu_{i,p}$ ,  $i \in \mathcal{Y} = \{1, 2, \dots, r\}$  are the fuzzy sets.  $r$  is the number of IF-THEN rules.  $G_k$ ,  $k = 1, 2, 3$  are the open connected sets satisfying that  $\cup_{k=1}^3 G_k = \mathbb{R}_+ \times \mathbb{R}^n$  and  $G_{k_1} \cap G_{k_2} = \emptyset$ ,  $\forall k_1, k_2 \in \{1, 2, 3\}, k_1 \neq k_2$ .  $S = \cup_{k=1}^3 \partial G_k$  is of Lebesgue measure zero, where  $\partial \Omega$  represents the boundary set of a set  $\Omega$ . For  $\forall i \in \mathcal{Y}$ ,  $k = 1, 2, 3$ ,  $A_i^k \in \mathbb{R}^{n \times n}$ ,  $B_i \in \mathbb{R}^{n \times m}$ , and  $D \in \mathbb{R}^{n \times p_3}$  are all known matrices.  $f_i^k(t, x(t))$  is a vector of unknown nonlinear functions.  $\Delta A_i(t) \in \mathbb{R}^{n \times n}$  is an unknown matrix varying with the time.  $\Lambda(t)$  is a time-varying diagonal matrix of remanent actuator effectiveness and  $\varepsilon_u(t)$  is a vector of

actuator deviations when the actuator failure occurs.  $d(t)$  represents the external disturbance. The symbols used in the paper can be found in Table 1.

*Remark 1.* It should be highlighted that system model (1) can reflect the actual situation of many practical engineering systems and possesses important research significance. If  $(t, x(t))$  stays in  $G_1$ , the system is under a normal condition. When  $(t, x(t))$  enters  $G_2$  from  $G_1$ , the system uncertainties  $\Delta A_i(t)$  suddenly appear and external disturbances  $d(t)$  grow rapidly. Finally, if the system states travel into  $G_3$ , the actuator faults abruptly come out. Accompanied by  $(t, x(t))$  entering into different regions, the system matrix and the nonlinear function vector will change.

TABLE 1: The symbols used in the paper.

Variable	Implication
$\theta(t)$	Premise variables
$G_k$	Open connected sets
$\partial G_k$	Lebesgue measure zero
$\partial \Omega$	Boundary set of a set $\Omega$
$f_i^k(t, x(t))$	Unknown nonlinear functions
$A_i^k, B_i, D$	Known matrices
$\Delta A_i(t)$	Unknown matrix varying with time
$\Lambda(t)$	Matrix of remanent actuator effectiveness
$\varepsilon_u(t)$	Vector of actuator deviations
$d(t)$	External disturbance

It is supposed that  $\theta_i(t)$  never depends on  $u(t)$ ,  $\varepsilon(t)$ , and  $d(t)$ . Define

$$h_i(\theta(t)) = \frac{\prod_{j=1}^p \mu_{i,j}(\theta_j(t))}{\sum_{i=1}^r \prod_{j=1}^p \mu_{i,j}(\theta_j(t))}. \quad (2)$$

$\forall i \in \Upsilon, j = 1, 2, \dots, p, \mu_{i,j}(\theta_j(t))$  denote the degree of membership  $\theta_j(t)$  in  $\mu_{i,j}$ .  $h_i(\theta(t))$  is the fuzzy basis function. Clearly, for any  $i \in \Upsilon, h_i(\theta(t)) \geq 0$  and  $\sum_{i=1}^r h_i(\theta(t)) = 1$ . Therefore, the dynamics of system (1) can be rewritten as follows:

$$\dot{x}(t) = g(t, x(t)),$$

$$g(t, x(t)) = \begin{cases} \sum_{i=1}^r h_i(\theta(t)) g_i^1(t, x(t)) & \text{if } (t, x(t)) \in G_1, \\ \sum_{i=1}^r h_i(\theta(t)) g_i^2(t, x(t)) & \text{if } (t, x(t)) \in G_2, \\ \sum_{i=1}^r h_i(\theta(t)) g_i^3(t, x(t)) & \text{if } (t, x(t)) \in G_3, \end{cases} \quad (3)$$

where

$$\begin{aligned} g_i^1(t, x(t)) &= f_i^1(t, x(t)) + A_i^1 x + B_i u, \\ g_i^2(t, x(t)) &= f_i^2(t, x(t)) + A_i^2 x + \Delta A_i(t) x + B_i(u + d(t)), \\ g_i^3(t, x(t)) &= f_i^3(t, x(t)) + A_i^3 x + \Delta A_i(t) x + B_i(\Lambda(t) u \\ &\quad + \varepsilon_u(t) + d(t)). \end{aligned} \quad (4)$$

Our control objective is to design an adaptive controller such that system states  $x(t)$  can converge into a desired compact set in the presence of the discontinuous multiple uncertainties and abruptly changing actuator faults.

To achieve the control objective, the following assumptions are necessary.

*Assumption 1.* For any  $i \in \Upsilon, k = 1, 2, 3, f_i^k(t, x(t))$  satisfy  $f_i^k(0, x(0)) = 0$  and

$$\|f_i^k(t, x_1(t)) - f_i^k(t, x_2(t))\| \leq \|U_i^k(x_1(t) - x_2(t))\|. \quad (5)$$

*Assumption 2.* For any  $i \in \Upsilon,$

$$\Delta A_i(t) = M_i \Xi_i(t) N_i, \quad (6)$$

where  $\Xi_i(t)$  are unknown time-varying matrices satisfying  $\Xi_i^T(t) \Xi_i(t) \leq I$  and  $M_i$  and  $N_i$  are known matrices.

*Assumption 3.* The discontinuous disturbances  $d_i(t)$  are Lebesgue measurable and locally bounded, i.e.,  $\|d_i(t)\| \leq D_i$ , where  $D_i$  is an unknown positive constant.

*Assumption 4.* It is supposed that  $\Lambda(t)$  and  $\varepsilon_u(t)$  are both Lebesgue measurable, and there exist lower and upper bounds for  $\Lambda(t)$  and  $\varepsilon_u(t)$ , respectively. In other words,  $\lambda_{\min}[\Lambda(t)] > \varepsilon_\Lambda > 0, \|\varepsilon_u(t)\| \leq D_u$ .  $\varepsilon_\Lambda$  and  $D_u$  are positive constants.

*2.2. Preliminaries.* Consider a nonlinear system:

$$\dot{x} = g(x(t)), \quad (7)$$

where  $x \in \mathbb{R}, g: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  is Lebesgue measurable, essentially locally bounded and uniformly in  $t$ . Moreover, there exist discontinuities in  $g(x(t))$ .

*Definition 1.* A vector  $x(t)$  is called a Filippov solution of differential equation (7) over  $[0, \infty)$  if  $x(t)$  is absolutely continuous, and for almost everywhere  $t \in [0, \infty)$ ,

$$\dot{x} \in F[g](x(t)), \quad (8)$$

where  $\mathcal{F}[g](x(t))$  is a upper semicontinuous set-valued map defined by

$$F[g](x(t)) \stackrel{\Delta}{=} \bigcap_{\rho > 0} \bigcap_{\mu(N)=0} \bar{co} g\left(\frac{B(x(t), \rho)}{N}\right), \quad (9)$$

where  $\bigcap_{\mu(N)=0}$  denotes the intersection over sets of Lebesgue measure zero,  $\bar{co}$  represents the convex closure, and  $B(x(t), \rho) = \{z \in \mathbb{R}^n \mid \|x(t) - z\| < \rho\}$

*Definition 2* (see [45]). Given a locally Lipschitz function  $V(x(t))$ , the generalized gradient of  $V(x(t))$  is defined by

$$\partial V(x(t)) = \bar{co} \{\lim \nabla V(x^*(t)) \mid x^*(t) \rightarrow x(t), x^*(t) \notin \Omega_V\}, \quad (10)$$

where  $\Omega_V$  is the set of measure zero and  $\nabla V$  is not defined. Moreover, the set-valued Lie derivative of a  $V(x(t))$  is defined as

$$DV(x(t)) \stackrel{\Delta}{=} \bigcap_{\xi \in \partial V(x(t))} \xi^T F[g](x(t)). \quad (11)$$

*Definition 3.* In this paper, the generalized sections for variables, vectors, and matrices are defined. For  $a, b \in \mathbb{R}, a < b$ , define  $[b, a] = [a, b] = [\min\{a, b\}, \max\{a, b\}]$ . For  $a, b \in \mathbb{R}^n$ , define  $[a, b] = [b, a] = ([a_i, b_i])_n$ , where  $a_i$  is the  $i$ th component of  $a$ . For  $A, B \in \mathbb{R}^{m \times n}$ , define  $[A, B] = [B, A] = ([A_{ij}, B_{ij}])_{m \times n}$

**Lemma 1.** Consider nonlinear system (7). Suppose  $g(\cdot)$  is Lebesgue measurable and  $x \rightarrow g(x(t))$  is bounded. Let

$V: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  be locally Lipschitz and regular such that

$$\begin{aligned} \alpha_1(x(t)) \leq V(x(t)) \leq \alpha_2(x(t)), \quad \forall t \geq 0, \\ DV(x(t)) \leq -\gamma_1 V(x(t)) + \gamma_2, \end{aligned} \quad (12)$$

where  $\alpha_1(\cdot)$  and  $\alpha_2(\cdot)$  are  $K_\infty$  functions,  $\gamma_1 > 0$  and  $\gamma_2 > 0$  are constants,  $x(t)$  is a Filippov solution of (7) with initial value  $x(t_0) \in \Omega_2 = \{x \in \mathbb{R}^n | \|x\| < r, \alpha_2(x) \leq \rho\}$ , and  $r > 0$  and  $\rho > 0$  are constants satisfying that  $\gamma_2/\gamma_1 < \rho < \min_{\|x(t)\|=r} \alpha_1(x(t))$ . Then,  $x(t)$  is bounded and converges to a compact set:

$$\Omega_f = \left\{ x \in \mathbb{R}^n | V(x) \leq \frac{\gamma_2}{\gamma_1} \right\}. \quad (13)$$

*Proof.* Define  $\Omega_V = \{x \in \mathbb{R}^n | \|x\| < r, V(x) \leq \rho\}$ . Since  $\alpha_2(x) \leq \rho \implies V(x) \leq \rho$ , it is obvious that  $\Omega_2 \subset \Omega_V$ . On the contrary, with the aid of  $V(x) \leq \rho \implies \alpha_1(x) \leq \rho$ , we can easily get that  $\Omega_V \subset \Omega_1$ . Moreover, considering  $\rho < \min_{\|x(t)\|=r} \alpha_1(x(t))$ , we know that  $\Omega_1 = \{x \in \mathbb{R}^n | \alpha_1(x) \leq \rho\}$  is an interior set of  $B_r = \{x \in \mathbb{R}^n | \|x\| < r\}$ . Hence,

$$\Omega_2 \subset \Omega_V \subset \Omega_1 \subset B_r. \quad (14)$$

If  $V(x(t)) \leq \gamma_2/\gamma_1$ , it is clear that  $x(t)$  stays in  $\Omega_f$ , which is  $x(t) \in \Omega_f \subset \Omega_V \subset B_r$ . On the contrary, for any  $x(t) \notin \Omega_f$ , we can obtain that  $V(x(t)) > \gamma_2/\gamma_1$  and  $\dot{V} < a.e. 0$ . Hence,  $V(x(t))$  is nonincreasing. Furthermore, for any  $x(t_0) \in \Omega_2$ , the solution of (7) stays in  $\Omega_V$ , which is  $x(t) \in \Omega_V \subset B_r$ . Therefore, it can be concluded that the solution of system (7) is bounded for all  $t \geq t_0$ . Furthermore, it follows from (12) that

$$\begin{aligned} \int_{t_0}^t [-\gamma_1 V(x(\tau)) + \gamma_2] d\tau \leq \int_{t_0}^t \dot{V}(x(\tau)) d\tau = V(x(t_0)) \\ - V(x(t)) \leq V(x(t_0)). \end{aligned} \quad (15)$$

Therefore,  $\int_{t_0}^t [-\gamma_1 V(x(\tau)) + \gamma_2] d\tau$  is bounded for all  $t \geq t_0$ . It should be noted that the existence of  $\int_{t_0}^t [-\gamma_1 V(x(\tau)) + \gamma_2] d\tau$  can be guaranteed because  $-\gamma_1 V(x(t)) + \gamma_2$  is locally Lipschitz and regular by definition. According to Barbalat's Lemma [46], it can be obtained that  $-\gamma_1 V(x(t)) + \gamma_2 \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore, we know that  $x(t)$  will converge into  $\Omega_f$  finally. The proof is complete.  $\square$

**Lemma 2.** Given any constant  $\varepsilon > 0$  and any vector  $\xi \in \mathbb{R}^n$ , the following inequality holds:

$$\|\xi\| < \frac{\xi^T \xi}{\sqrt{\xi^T \xi + \varepsilon^2}} + \varepsilon. \quad (16)$$

*Proof.* Since  $\varepsilon > 0$  and  $\sqrt{\xi^T \xi + \varepsilon^2} > 0$ , it can be easily get that

$$\begin{aligned} \left[ \xi^T \xi + \varepsilon \sqrt{\xi^T \xi + \varepsilon^2} \right]^2 - \left[ \|\xi\| \sqrt{\xi^T \xi + \varepsilon^2} \right]^2 \\ = 2\varepsilon \xi^T \xi \sqrt{\xi^T \xi + \varepsilon^2} + \varepsilon^4 > 0. \end{aligned} \quad (17)$$

Hence, we know that

$$\|\xi\| \sqrt{\xi^T \xi + \varepsilon^2} < \xi^T \xi + \varepsilon \sqrt{\xi^T \xi + \varepsilon^2}. \quad (18)$$

By dividing  $\sqrt{\xi^T \xi + \varepsilon^2}$  in both sides of (18), inequality (16) can be obtained. The proof is completed.  $\square$

### 3. Main Results

**3.1. Control Design.** In the following text, the robust adaptive control problem for the concerned T-S fuzzy discontinuous nonlinear systems will be addressed.

In view of (3), according to Definition 1, we can get the following differential inclusion:

$$\begin{aligned} \dot{x} \in \sum_{i=1}^r h_i(\theta(t)) \{ F[f_i](t, x(t)) + F[A_i]x(t) \\ + F[\Delta \bar{A}_i(t)]x(t) + B_i F[\bar{\Lambda}(t)]u(t) + B_i F[\bar{d}(t)] \}, \end{aligned} \quad (19)$$

where

$$\begin{aligned} f_i &= f_i^k(t, x(t)) \quad \text{if } (t, x(t)) \in G_k, \quad k = 1, 2, 3, \\ A_i &= A_i^k \quad \text{if } (t, x(t)) \in G_k, \quad k = 1, 2, 3, \\ \Delta \bar{A}_i(t) &= \begin{cases} 0 & \text{if } (t, x(t)) \in G_1, \\ \Delta A_i(t) & \text{if } (t, x(t)) \in G_2 \cup G_3, \end{cases} \\ \bar{\Lambda}(t) &= \begin{cases} I & \text{if } (t, x(t)) \in G_1 \cup G_2, \\ \Lambda(t) & \text{if } (t, x(t)) \in G_3, \end{cases} \\ \bar{d}(t) &= \begin{cases} 0 & \text{if } (t, x(t)) \in G_1, \\ d(t) & \text{if } (t, x(t)) \in G_2, \\ \varepsilon_u(t) + d(t) & \text{if } (t, x(t)) \in G_3. \end{cases} \end{aligned} \quad (20)$$

Define

$D = \sup_{t \geq 0} \|\varepsilon_u(t) + d(t)\|$ ,  $\vartheta = 1/\inf_{t \geq 0} [\lambda_{\min}(\Lambda(t))]$ . Considering system (19), the control law is designed as follows.

**Controller Rule**  $R_{p,i}$ : IF  $\theta_1(t)$  is  $\mu_{i,1}$ ,  $\theta_2(t)$  is  $\mu_{i,2}$  and ... and  $\theta_p(t)$  is  $\mu_{i,p}$ , THEN

$$\begin{aligned} u(t) &= -\hat{\vartheta} \varphi_{\vartheta}(x(t), v(t)), \\ v(t) &= K_i x(t) - \hat{D} \varphi_D(x(t)), \end{aligned} \quad (21)$$

where  $K_i$  is the control gain matrix to be designed and  $\hat{\vartheta}$  and  $\hat{D}$  are the estimations of  $\vartheta$  and  $D$ , respectively. The continuous functions  $\varphi_{\vartheta}(x(t), v(t))$  and  $\varphi_D(x(t))$  are defined as



$$\begin{aligned}\varphi_{\vartheta}(x(t), v(t)) &= [\varphi_{\vartheta,1}(t), \varphi_{\vartheta,2}(t), \dots, \varphi_{\vartheta,m}(t)]^T, \\ \varphi_{\vartheta,l}(x(t), v(t)) &= \frac{v_l(t)[z_l(t)\hat{\vartheta}v_l(t)]}{\sqrt{[z_l(t)\hat{\vartheta}v_l(t)]^2 + \varepsilon_v^2}}, \quad l = 1, 2, \dots, m, \\ \varphi_D(x(t)) &= \frac{B^T P x(t)}{\sqrt{x^T(t) P B B^T P x(t) + \varepsilon_D^2}},\end{aligned}\quad (22)$$

where

$$B = \sum_{i=1}^r h_i(\theta(t)) B_i, \quad (23)$$

$P \in \mathbb{R}^{n \times n}$  is a symmetric positive matrix.  $\varepsilon_v, \varepsilon_D > 0$  are design constants.  $z_l(t)$  is the  $l$ th component of  $z(t) = [x^T(t) P B]^T$ . The adaptive parameters are updated by

$$\begin{aligned}\dot{\hat{D}} &= \eta_D x^T(t) P B \varphi_D(x(t)) - \eta_D \sigma_D \hat{D}, \\ \dot{\hat{\vartheta}} &= -\eta_{\vartheta} x^T(t) P B v(t) - \eta_{\vartheta} \sigma_{\vartheta} \hat{\vartheta},\end{aligned}\quad (24)$$

where  $\eta_D, \eta_{\vartheta} > 0$  are the gains of adaptive laws and  $\sigma_D, \sigma_{\vartheta} > 0$  are design constants.

*Remark 2.* Note that it is improper to separately design the control laws for each  $G_k$ ,  $k = 1, 2, 3$ , and a universal controller has to be developed for all the three modes. In practical, the condition of multiple uncertainties suddenly changes or actuator failure abruptly occurs which cannot be determined easily. Moreover, the condition is concerned not only with the system time but also with the system states, which makes this problem more complex. Since the boundaries of  $G_k$  are unknown in practical, the separately design methods cannot be applied and a universal control law which is applicable for all the three modes is necessary. In this paper,  $G_k$ ,  $k = 1, 2, 3$ , are only used for analysis, but are not used in control design.

*Remark 3.* The considered system cannot be controlled by using the controllers of conventional switched systems possessing switching signals those only vary with time. The three modes of the concerned system are distinguished by using the conditions concerned with both the system time and the system states. In this paper, the switchings among the three modes are more intrinsic and are difficult to be dealt with. In fact, the proposed controller can degrade into an asynchronous control law if  $G_k$ ,  $k = 1, 2, 3$  is only concerned with time. For the considered  $G_k$ , the proposed controller can be thought of as a deep asynchronous controller.

**3.2. Stability Analysis.** Define  $\tilde{\vartheta}(t) = \hat{\vartheta}(t) - \vartheta(t)$ ,  $\tilde{D}(t) = \hat{D}(t) - D(t)$ . Combining (19)–(21) yields

$$\begin{aligned}\dot{x} \in & \sum_{i=1}^r h_i(\theta(t)) \sum_{j=1}^r h_j(\theta(t)) \{F[f_i](x(t)) + F[A_i + B_i K_j] \\ & \cdot (x(t)) + F[\Delta \bar{A}_i(t)](x(t)) + B_i F[\bar{d}(t)] - B_i \hat{D} \varphi_D \\ & \cdot (x(t)) + B_i F[\bar{\Lambda}(t)](u(t)) - B_i v(t)\}.\end{aligned}\quad (25)$$

**Theorem 1.** Consider the closed-loop fuzzy differential inclusion (25) under Assumptions 1–4. The fuzzy controller is designed as (21) and the adaptive parameters are updated by (24). Given scalars  $\alpha, \lambda > 0$ . For any  $i, j \in Y$  and  $k \in \{1, 2, 3\}$ , if there exist matrices  $P, K$  such that

$$\begin{aligned}\Pi_{i,j}^k &< 0, \quad i = 1, 2, \dots, r, \\ \Pi_{i,j}^k + \Pi_{i,j}^k &< 0, \quad 1 \leq i < j \leq r,\end{aligned}\quad (26)$$

where

$$\Pi_{i,j}^k = \begin{bmatrix} \Pi_{i,j,11}^k & P \\ P & -\frac{1}{\lambda^2} I \end{bmatrix}, \quad (27)$$

$$\begin{aligned}\Pi_{i,j,11}^k &= P(A_i^k + B_i K_j) + (A_i^k + B_i K_j)^T P \\ &+ \frac{1}{\lambda^2} [U_i^k]^T U_i^k + P M_i M_i^T P + N_i^T N_i + \alpha P,\end{aligned}$$

then, for any initial conditions, the Filippov solution of closed-loop fuzzy differential inclusion (25) is bounded and converge to a compact set:

$$\Omega_f = \{x \in \mathbb{R}^n | V(x) \leq \varepsilon_f / \beta\}, \quad (28)$$

where

$$\varepsilon_f = \frac{\sigma_D D^2}{2} + \frac{\sigma_{\vartheta} \vartheta}{2} + D \varepsilon_D + \frac{m \varepsilon_v}{\vartheta}, \quad (29)$$

$$\beta = \min\{\alpha, \eta_D \sigma_D, \eta_{\vartheta}, \sigma_{\vartheta}\}.$$

*Proof.* Select a Lyapunov functional candidate as follows:

$$V(x(t)) = \frac{1}{2} x^T(t) P x(t) + \frac{1}{2 \eta_D} \tilde{D}^2 + \frac{1}{2 \eta_{\vartheta}} \tilde{\vartheta}^2, \quad (30)$$

where  $P$  is a positive definite matrix. According to Definition 2, the set-valued Lie derivative of a  $V(x(t))$  can be taken as

$$DV(x(t)) = \bigcap_{\xi \in \partial V(x(t))} \xi^T [\psi(x(t), \tilde{D}, v(t)), \dot{\tilde{D}}, \dot{\tilde{\vartheta}}]^T, \quad (31)$$

where

$$\begin{aligned}
\psi(x(t), \widehat{D}, v(t)) &= \sum_{i=1}^r h_i(\theta(t)) \sum_{j=1}^r h_j(\theta(t)) \{F[f_i](x(t)) \\
&\quad + F[A_i + B_i K_j]x(t) + F[\Delta \bar{A}_i(t)]x(t) \\
&\quad + F[\bar{d}(t)] - B_i \widehat{D} \varphi_D(x(t)) \\
&\quad + B_i F[\bar{\Lambda}(t)]u(t) - B_i v(t)\}.
\end{aligned} \tag{32}$$

By defining  $\pi_{f_i}, \pi_{A_{ij}}, \pi_{\Delta A_i}, \pi_{\Lambda}$ , and  $\pi_d$  such that

$$\begin{aligned}
\pi_{f_i} &\in \begin{cases} f_i^k & \text{if } (t, x(t)) \in G_k, \\ [f_i^{k_1} f_i^{k_2}] & \text{if } (t, x(t)) \in \partial G_{k_1} \cap \partial G_{k_2}, \end{cases} \\
\pi_{A_{ij}} &\in \begin{cases} A_i^k + B_i K_j & \text{if } (t, x(t)) \in G_k, \\ [A_i^{k_1} + B_i K_j, A_i^{k_2} + B_i K_j] & \text{if } (t, x(t)) \in \partial G_{k_1} \cap \partial G_{k_2}, \end{cases} \\
\pi_{\Delta A_i} &\in \begin{cases} 0 & \text{if } (t, x(t)) \in G_1, \\ \Delta A_i & \text{if } (t, x(t)) \in G_2 \cup G_3, \\ [0, \Delta A_i] & \text{if } (t, x(t)) \in \partial G_1 \cap \partial [G_2 \cup G_3], \end{cases} \\
\pi_{\Lambda} &\in \begin{cases} I & \text{if } (t, x(t)) \in G_1 \cup G_2, \\ \Lambda(t) & \text{if } (t, x(t)) \in G_3, \\ [\Lambda(t), I] & \text{if } (t, x(t)) \in \partial [G_1 \cup G_2] \cap \partial G_3, \end{cases} \\
\pi_d &\in \begin{cases} 0 & \text{if } (t, x(t)) \in G_1, \\ d & \text{if } (t, x(t)) \in G_2, \\ \varepsilon_u + d & \text{if } (t, x(t)) \in G_3, \\ [0, d] & \text{if } (t, x(t)) \in \partial G_1 \cap \partial G_2, \\ [d, \varepsilon_u + d] & \text{if } (t, x(t)) \in \partial G_2 \cap \partial G_3, \\ [0, \varepsilon_u + d] & \text{if } (t, x(t)) \in \partial G_1 \cap \partial G_3, \end{cases}
\end{aligned} \tag{33}$$

we can rewrite (30) as

$$DV(x(t)) = D_1 V(x(t)) + D_2 V(x(t)), \tag{34}$$

where

$$D_1 V(x(t)) = \sum_{i=1}^r h_i(\theta(t)) \sum_{j=1}^r h_j(\theta(t)) \left\{ x^T(t) P \pi_{f_i} + x^T(t) P \pi_{A_{ij}} x(t) + x^T(t) P \pi_{\Delta A_i} x(t) \right\}, \tag{35}$$

$$D_2 V(x(t)) = \sum_{i=1}^r h_i(\theta(t)) \left\{ x^T(t) P B_i \pi_d - x^T(t) P B_i \widehat{D} \varphi_D(x(t)) + x^T(t) P B_i \pi_{\Lambda} u(t) - x^T(t) P B_i v(t) \right\} + \frac{1}{\eta_D} \dot{\widehat{D}} + \frac{1}{\eta_{\theta}} \dot{\widehat{\theta}}. \tag{36}$$

Firstly, the analysis of  $\mathcal{D}_1 V(x(t))$  are given. Since  $P$  is nonsingular, it can be proved that  $P \pi_{\Delta A_i} \in [0, P \Delta A_i]$  holds for all  $(t, x(t))$ . Hence, based on Assumption 2, it can be known that, for any  $(t, x(t))$ ,  $x^T(t) P \pi_{\Delta A_i} x(t) \in [0, x^T(t) P \Delta A_i x(t)]$  and

$$x^T(t) P \pi_{\Delta A_i} x(t) \leq \frac{1}{2} x^T(t) P M_i M_i^T P x(t) + \frac{1}{2} x^T(t) N_i^T N_i x(t). \tag{37}$$

For  $(t, x(t)) \in G_k, k = 1, 2, 3$ , it follows from Assumption 1 that

$$\begin{aligned} \pi_{f_i}^T \pi_{f_i} &\leq x^T(t) [U_i^k]^T U_i^k x(t), \\ P\pi_{A_{ij}} &= \frac{1}{2} \left[ P(A_i^k + B_i K_j) + (A_i^k + B_i K_j)^T P \right]. \end{aligned} \quad (38)$$

By combining (35), (37), and (38), we know that, for  $(t, x(t)) \in G_k$ ,  $k = 1, 2, 3$ , the following inequality holds:

$$\begin{aligned} \Phi_{i,j}^k &= \begin{bmatrix} \Phi_{i,j,11}^k & P \\ P & -\frac{1}{\lambda^2} \end{bmatrix}, \\ \Phi_{i,j,11}^k &= P(A_i^k + B_i K_j) + (A_i^k + B_i K_j)^T P + \frac{1}{\lambda^2} [U_i^k]^T U_i^k \\ &\quad + PM_i M_i^T P + N_i^T N_i. \end{aligned} \quad (40)$$

By using (26), it can be checked that, for  $(t, x(t)) \in G_k$ ,  $k = 1, 2, 3$ ,

$$D_1 V(x(t)) \leq -\frac{\alpha}{2} x^T(t) P x(t). \quad (41)$$

where

On the contrary, for  $(t, x(t)) \in \partial G_{k_1} \cap \partial G_{k_2}$ ,  $k_1, k_2 \in \{1, 2, 3\}$ , it can be proved that

$$\begin{aligned} x^T(t) P \pi_{A_{ij}} x(t) &\in [x^T(t) P (A_i^{k_1} + B_i K_j) x(t), x^T(t) P (A_i^{k_2} + B_i K_j) x(t)], \\ \pi_{f_i}^T \pi_{f_i} &\in [f_i^{k_1}]^T f_i^{k_1}, [f_i^{k_2}]^T f_i^{k_2} \in [x^T(t) [U_i^{k_1}]^T U_i^{k_1} x(t), x^T(t) [U_i^{k_2}]^T U_i^{k_2} x(t)]. \end{aligned} \quad (42)$$

Accordingly, the following inequalities can be obtained:

$$\begin{aligned} x^T(t) P \pi_{A_{ij}} x(t) &\leq \max_{k \in \{1, 2, 3\}} x^T(t) P (A_i^k + B_i K_j) x(t), \\ \pi_{f_i}^T \pi_{f_i} &\leq \max_{k \in \{1, 2, 3\}} x^T(t) [U_i^k]^T U_i^k x(t). \end{aligned} \quad (43)$$

By combining (35), (37), and (43), we know that, for  $(t, x(t)) \in \partial G_{k_1} \cap \partial G_{k_2}$ ,  $k_1, k_2 \in \{1, 2, 3\}$ ,

$$D_1 V(x(t)) \leq \frac{1}{2} \sum_{i=1}^r h_i(\theta(t)) \sum_{j=1}^r h_j(\theta(t)) \begin{bmatrix} x(t) \\ \pi_{f_i} \end{bmatrix}^T \Psi_{i,j}^k \begin{bmatrix} x(t) \\ \pi_{f_i} \end{bmatrix}, \quad (44)$$

where

$$\begin{aligned} \Psi_{i,j}^k &= \begin{bmatrix} \Psi_{i,j,11}^k & P \\ P & -\frac{1}{\lambda^2} \end{bmatrix}, \\ \Psi_{i,j,11}^k &= \max_{k \in \{1, 2, 3\}} \left\{ P(A_i^k + B_i K_j) + (A_i^k + B_i K_j)^T P \right. \\ &\quad \left. + \frac{1}{\lambda^2} [U_i^k]^T U_i^k \right\} + PM_i M_i^T P + N_i^T N_i. \end{aligned} \quad (45)$$

$$D_1 V(x(t)) \leq \frac{1}{2} \sum_{i=1}^r h_i(\theta(t)) \sum_{j=1}^r h_j(\theta(t)) \begin{bmatrix} x(t) \\ \pi_{f_i} \end{bmatrix}^T \Phi_{i,j}^k \begin{bmatrix} x(t) \\ \pi_{f_i} \end{bmatrix}, \quad (39)$$

Since (26) holds for any  $k \in \{1, 2, 3\}$ , we know that, for  $(t, x(t)) \in \partial G_{k_1} \cap \partial G_{k_2}$ ,  $k_1, k_2 \in \{1, 2, 3\}$ ,  $\mathcal{D}_1 V(x(t)) \leq -\alpha/2 x^T(t) P x(t)$  also holds. Then, by considering (41), it can be concluded that, for any  $(t, x(t))$ ,

$$D_1 V(x(t)) \leq -\frac{\alpha}{2} x^T(t) P x(t). \quad (46)$$

Next texts provide the analysis of  $\mathcal{D}_2 V(x(t))$ . From (33), we know that, for  $(t, x(t))$ ,

$$\begin{aligned} \pi_d &\in [0, \varepsilon_u + d], \\ \pi_{\Lambda_i} &\in [\Lambda_i(t), 1], \end{aligned} \quad (47)$$

where  $\pi_{\Lambda_i}$  and  $\Lambda_i(t)$  represent the  $i$ th component of  $\pi_\Lambda$  and  $\Lambda(t)$  on the diagonal line, respectively. Hence, it can be obtained that, for any  $(t, x(t))$ ,

$$\begin{aligned} \sum_{i=1}^r h_i(\theta(t)) x^T(t) P B_i \pi_d &\in [0, x^T(t) P B (\varepsilon_u + d)], \\ \sum_{i=1}^r h_i(\theta(t)) x^T(t) P B_i \pi_{\Lambda_i} u(t) &\in [x^T(t) P B \Lambda(t) u(t), x^T(t) P B u(t)]. \end{aligned} \quad (48)$$

From Lemma 2, it is easy to know that



$$\begin{aligned}
& \sum_{i=1}^r h_i(\theta(t)) \{x^T(t)PB_i\pi_d - x^T(t)PB_i\widehat{D}\varphi_D(x(t))\} \\
& \leq D\|x^T(t)PB\| - x^T(t)PB\widehat{D}\varphi_D(x(t)) \\
& \leq D \frac{x^T(t)PB[x^T(t)PB]^T}{\sqrt{x^T(t)PB[x^T(t)PB]^T + \varepsilon_D^2}} - x^T(t)PB\widehat{D}\varphi_D(x(t)) + D\varepsilon_D \\
& \leq -\widehat{D}x^T(t)PB\varphi_D(x(t)) + D\varepsilon_D.
\end{aligned} \tag{49}$$

Meanwhile, simple computation shows that

$$\begin{aligned}
& \sum_{i=1}^r h_i(\theta(t)) \{x^T(t)PB_i\pi_{\Lambda}u(t) - x^T(t)PB_iv(t)\} \\
& = -\widehat{\vartheta} \sum_{l=1}^m z_l\pi_{\Lambda,l}\varphi_{\vartheta,l}(x(t), v(t)) - x^T(t)PBv(t).
\end{aligned} \tag{50}$$

Since  $0 < 1/\vartheta \leq \pi_{\Lambda,i} \in [\Lambda_i(t), 1]$ , we can get that

$$\begin{aligned}
& -\widehat{\vartheta}z_l\pi_{\Lambda,l}\varphi_{\vartheta,l}(x(t), v(t)) \\
& = \frac{z_l\pi_{\Lambda,l}\widehat{\vartheta}v_l(t)[z_l(t)\widehat{\vartheta}v_l(t)]}{\sqrt{[z_l(t)\widehat{\vartheta}v_l(t)]^2 + \varepsilon_v^2}} \\
& \leq -\frac{1}{\vartheta} \frac{[z_l(t)\widehat{\vartheta}v_l(t)]^2}{\sqrt{[z_l(t)\widehat{\vartheta}v_l(t)]^2 + \varepsilon_v^2}}.
\end{aligned} \tag{51}$$

Hence, it follows from Lemma 2 that

$$\begin{aligned}
& -\widehat{\vartheta} \sum_{l=1}^m z_l\pi_{\Lambda,l}\varphi_{\vartheta,l}(x(t), v(t)) - x^T(t)PBv(t) \\
& \leq \sum_{l=1}^m \left[ \frac{1}{\vartheta} \frac{[z_l(t)\widehat{\vartheta}v_l(t)]^2}{\sqrt{[z_l(t)\widehat{\vartheta}v_l(t)]^2 + \varepsilon_v^2}} - z_l(t)v_l(t) \right] \\
& \leq \sum_{l=1}^m \left[ \frac{1}{\vartheta} (-\|z_l(t)\widehat{\vartheta}v_l(t)\| + \varepsilon_v) - z_l(t)v_l(t) \right] \\
& \leq \sum_{l=1}^m \left[ \frac{\widehat{\vartheta}}{\vartheta} z_l(t)v_l(t) - z_l(t)v_l(t) + \frac{\varepsilon_v}{\vartheta} \right] \\
& \leq \frac{\widehat{\vartheta}}{\vartheta} x^T(t)PBv(t) + \frac{m\varepsilon_v}{\vartheta}.
\end{aligned} \tag{52}$$

By combining (36) and (49)–(52), it can be obtained that

$$\begin{aligned}
D_2V(x(t)) & \leq -\widehat{D}x^T(t)PB\varphi_D(x(t)) + \frac{\widehat{\vartheta}}{\vartheta}x^T(t)PBv(t) \\
& \quad + \frac{1}{\eta_D}\widehat{D}\dot{\widehat{D}} + \frac{1}{\eta_{\vartheta}\vartheta}\widehat{\vartheta}\dot{\widehat{\vartheta}} + D\varepsilon_D + \frac{m\varepsilon_v}{\vartheta},
\end{aligned} \tag{53}$$

where  $m$  is the dimension of the system input signal  $u(t)$ . Substituting (24) into (53) yields

$$D_2V(x(t)) \leq -\sigma_D\widehat{D}\widehat{D} - \frac{\sigma_{\vartheta}\widehat{\vartheta}\widehat{\vartheta}}{\vartheta} + D\varepsilon_D + \frac{m\varepsilon_v}{\vartheta}. \tag{54}$$

By using the following inequalities,

$$\begin{aligned}
2\widehat{D}\widehat{D} & \leq -\widehat{D}^2 + D^2, \\
\widehat{\vartheta}\widehat{\vartheta} & \leq -\widehat{\vartheta}^2 + \vartheta^2.
\end{aligned} \tag{55}$$

We know that

$$D_2V(x(t)) \leq -\frac{\sigma_D}{2}\widehat{D}^2 - \frac{\sigma_{\vartheta}\widehat{\vartheta}^2}{2\vartheta} + \varepsilon_f. \tag{56}$$

By combining (34), (46), and (56), we know that

$$DV(x(t)) \leq -\frac{\alpha}{2}x^T(t)Px(t) - \frac{\sigma_D}{2}\widehat{D}^2 - \frac{\sigma_{\vartheta}\widehat{\vartheta}^2}{2\vartheta} + \varepsilon_f, \tag{57}$$

which means

$$DV(x(t)) \leq -\beta V(x(t)) + \varepsilon_f, \tag{58}$$

where  $\beta$  and  $\varepsilon_f$  are defined in (29). According to Lemma 1, it can be proved that the Filippov solution of closed-loop fuzzy differential inclusion (25) is bounded and converge to  $\Omega_f$ . The proof is complete.  $\square$

*Remark 4.* It should be noted that the final compact set  $\Omega_f$  can be an arbitrarily small neighborhood of the origin by adjusting the control gains and adaptive parameters. Moreover, in most of the adaptive control results, the ultimate boundness of the closed-loop control system is finally ensured. However, for the differential inclusion systems, the criteria of ultimately boundness have never been provided. Hence, it can be concluded that Lemma 1 lays the mathematical fundamentals for the adaptive control of differential inclusion systems.

Next, we will provide the computation method of control gain  $K$  and matrix  $P$  which is necessary in adaptive laws.

**Theorem 2.** Consider the closed-loop fuzzy differential inclusion (25) under Assumptions 1–4. The fuzzy controller is designed as (20), and the adaptive parameters are updated by (24). Given scalars  $\alpha, \lambda > 0$ . For any  $i, j \in Y$  and  $k \in \{1, 2, 3\}$ , if there exist matrices  $Q, R_j$  such that

$$\begin{aligned}
\Gamma_{i,j}^k & < 0, \quad i = 1, 2, \dots, r, \\
\Gamma_{i,j}^k + \Gamma_{i,j}^k & < 0, \quad 1 \leq i < j \leq r,
\end{aligned} \tag{59}$$

where

$$\Gamma_{i,j}^k = \begin{bmatrix} \Gamma_{i,j,11}^k & Q[U_i^k]^T & QN_i^T & I \\ * & -\lambda^2 I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & \frac{1}{\lambda^2} I \end{bmatrix}, \quad (60)$$

$$\Gamma_{i,j,11}^k = A_i^k Q + Q A_i^k + B_i R_j + R_j^T B_i^T + \alpha Q + M_i M_i^T,$$

then, for any initial conditions, the Filippov solution of closed-loop fuzzy differential inclusion (25) is bounded and converges to the compact set  $\Omega_f$ . Moreover, if condition (59) is feasible, it can be obtained that

$$\begin{aligned} K_j &= R_j Q^{-1}, \\ P &= Q^{-1}. \end{aligned} \quad (61)$$

*Proof.* Define  $Q = P^{-1}$  and  $R_j = K_j P^{-1}$ . By performing a congruence transformation to with  $\Pi_{i,j}^k$ , we can get the following matrix:

$$\widehat{\Pi}_{i,j}^k = \begin{bmatrix} \widehat{\Pi}_{i,j,11}^k & I \\ I & -\frac{1}{\lambda^2} I \end{bmatrix}, \quad (62)$$

where

$$\begin{aligned} \widehat{\Pi}_{i,j,11}^k &= A_i^k Q + Q A_i^k + B_i R_j + R_j^T B_i^T + \alpha Q + \frac{1}{\lambda^2} Q [U_i^k]^T U_i^k Q \\ &\quad + M_i M_i^T + Q N_i^T N_i Q. \end{aligned} \quad (63)$$

By performing a Schur complement transformation to (61),  $\Gamma_{i,j}^k$  can be obtained. Since

$$\Gamma_{i,j}^k < 0 \Leftrightarrow \widehat{\Pi}_{i,j}^k < 0 \Leftrightarrow \Pi_{i,j}^k < 0, \quad (64)$$

we know that (26) is satisfied by using (58). According to Theorem 1, the boundedness and convergence of the Filippov solution can be guaranteed. The proof is complete.  $\square$

*Remark 5.* System who involves state-based switching usually has more complex dynamic behaviors which motivates various useful applications, while the construction of rigorous stability for these systems is challenging. Different to classic analysis methods in switching systems such as average dwell time (ADT), differential-inclusion-based approaches provide feasible alternative solutions to the stability analysis of stated-based switching systems. Instead of

analyzing the value of the vector field at individual points, differential-inclusion-based methods focus on the behavior of vector field at the neighborhood of each point. This idea is at the core of constructing continuous Filippov solution which is a general solution to the original differential equations, where the discontinuities is covered by set-valued mapping which is a convex combination of vector field around them. As a result, rigorous stability result can be established for discontinuous vector field as long as the Lebesgue measure of the discontinuities is zero. From the illustration above, it is clear that, for state-based switching systems, the superiority of utilizing differential inclusion is significant.

#### 4. Simulation Study

In this section, we will present a numerical example with two fuzzy subsystems to demonstrate the effectiveness of the proposed control method. The switching regions  $G_i$  of system (1) are defined as

$$\begin{aligned} G_1 &= \{x \in \mathbb{R}^2 \mid \|x\| \geq n1\}, \\ G_2 &= \{x \in \mathbb{R}^2 \mid 0.5t \leq n\|x\|q < h1\}, \\ G_3 &= \{x \in \mathbb{R}^2 \mid \|x\| < n0.5\}. \end{aligned} \quad (65)$$

The system-related matrices are given as follows:

$$\begin{aligned} A_1^1 &= \begin{bmatrix} -2.7 & 1.5 \\ 0.5 & 1.2 \end{bmatrix}, \\ A_2^1 &= \begin{bmatrix} -2.1 & 0.6 \\ 0.2 & 1.8 \end{bmatrix}, \\ A_1^2 &= \begin{bmatrix} -1.9 & 2.3 \\ 1.5 & 2.8 \end{bmatrix}, \\ A_2^2 &= \begin{bmatrix} -3.6 & 2.7 \\ 0.1 & 1.5 \end{bmatrix}, \\ A_1^3 &= \begin{bmatrix} -1.2 & 0.7 \\ 2.1 & 0.5 \end{bmatrix}, \\ A_2^3 &= \begin{bmatrix} -2.8 & 1.7 \\ 3.5 & 1.9 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -1.9 \\ 3.6 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} -1.5 \\ 2.3 \end{bmatrix}. \end{aligned} \quad (66)$$

The unknown nonlinear functions are set as

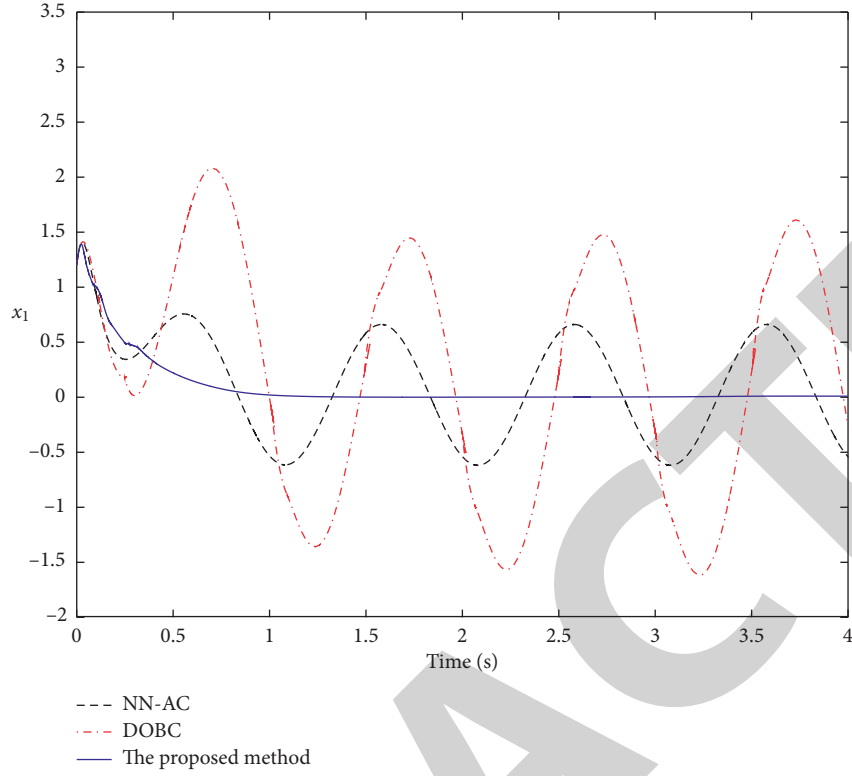


FIGURE 1: The trajectories of the system state  $x_1(t)$  under NN-AC, DOBC, and proposed method.

$$\begin{aligned}
 f_1^1 &= \begin{bmatrix} 0.016 \cos(2x_1x_2) \\ 0.022 \sin(x_1 + x_2) \end{bmatrix}, \\
 f_2^1 &= \begin{bmatrix} 0.018 \sin(x_1^2) \\ 0.02 \cos(x_2^2) \end{bmatrix}, \\
 f_1^2 &= \begin{bmatrix} 0.06 \sin(x_1)\cos(x_2) \\ 0.025 \sin(\|x_1\|) \end{bmatrix}, \\
 f_2^2 &= \begin{bmatrix} 0.028 \cos(x_1)\cos(x_2) \\ 0.025 \cos(\|x_2\|) \end{bmatrix}, \\
 f_1^3 &= \begin{bmatrix} 0.024 \sin(x_1)\cos(x_2) \\ 0.025 \cos(\|x_1\| + \|x_2\|) \end{bmatrix}, \\
 f_2^3 &= \begin{bmatrix} 0.01 \sin(x_1x_2) \\ 0.05 \cos(\|x_1\|x_2) \end{bmatrix}.
 \end{aligned} \tag{67}$$

To compute the control gains, we take

$$\begin{aligned}
 M_1 &= \begin{bmatrix} 0.012 & 0 \\ 0 & 0.08 \end{bmatrix}, \\
 M_2 &= \begin{bmatrix} 0.015 & 0 \\ 0 & 0.06 \end{bmatrix}, \\
 N_1 &= \begin{bmatrix} 0.069 & 0 \\ 0 & 0.087 \end{bmatrix}, \\
 N_2 &= \begin{bmatrix} 0.055 & 0 \\ 0 & 0.072 \end{bmatrix}, \\
 U_1^1 = U_1^2 = U_1^3 &= \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}, \\
 U_2^1 = U_2^2 = U_2^3 &= \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}.
 \end{aligned} \tag{68}$$

By letting  $\alpha = 2$  and  $\lambda = 1$ , we can solve (59) and obtain that

$$\begin{aligned}
 P &= \begin{bmatrix} 4.0590 & 6.9122 \\ 6.9122 & 14.0675 \end{bmatrix}, \\
 K_1 &= \begin{bmatrix} -5.6326 & -10.8441 \end{bmatrix}, \\
 K_2 &= \begin{bmatrix} -9.0548 & -17.6541 \end{bmatrix}.
 \end{aligned} \tag{69}$$

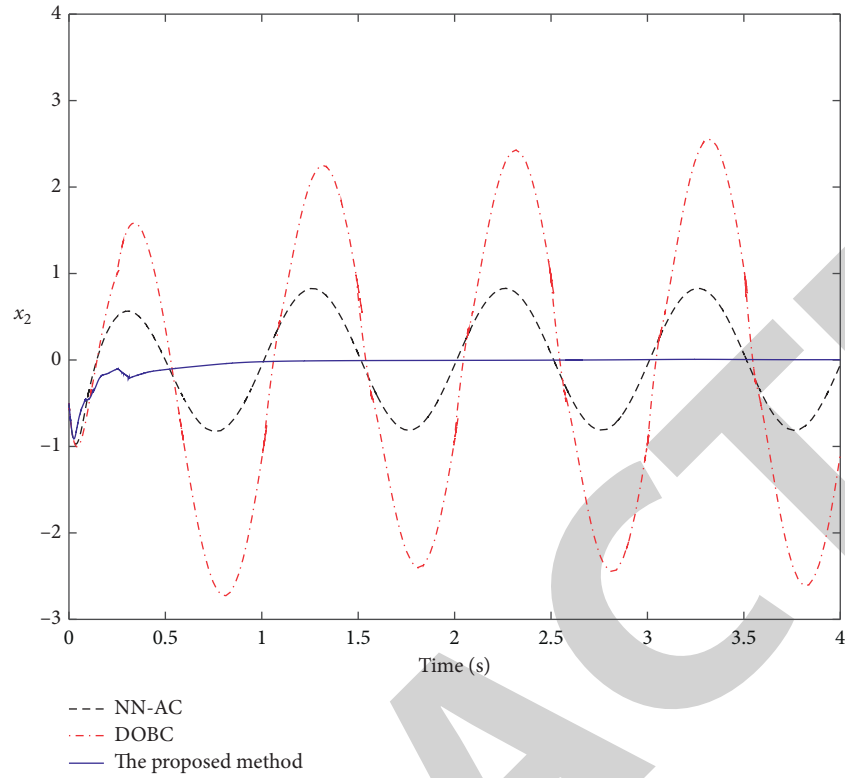


FIGURE 2: The trajectories of the system state  $x_2(t)$  under NN-AC, DOBC, and proposed method.

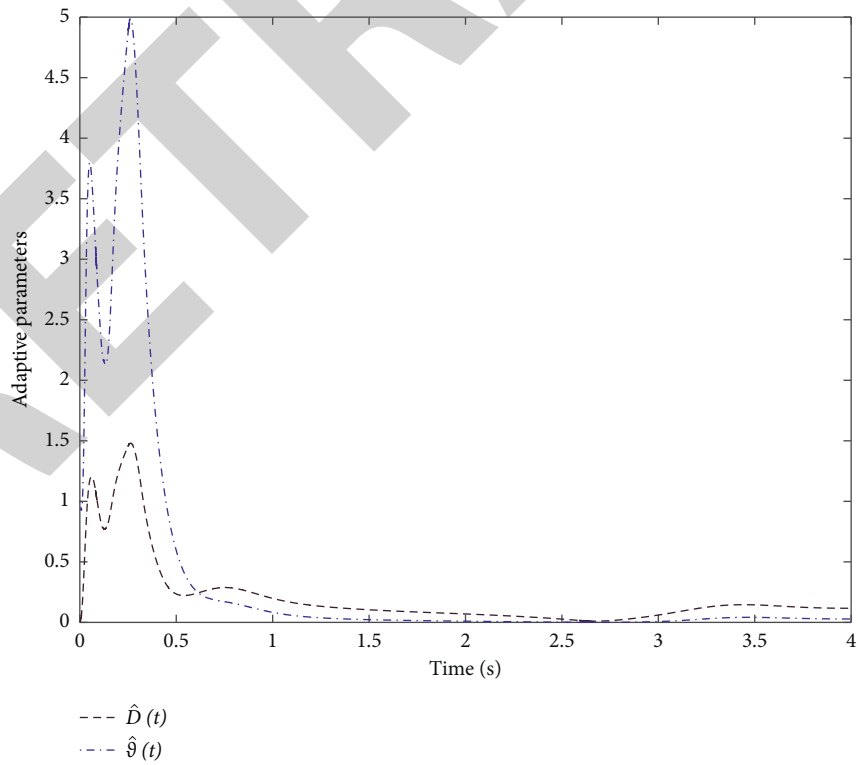
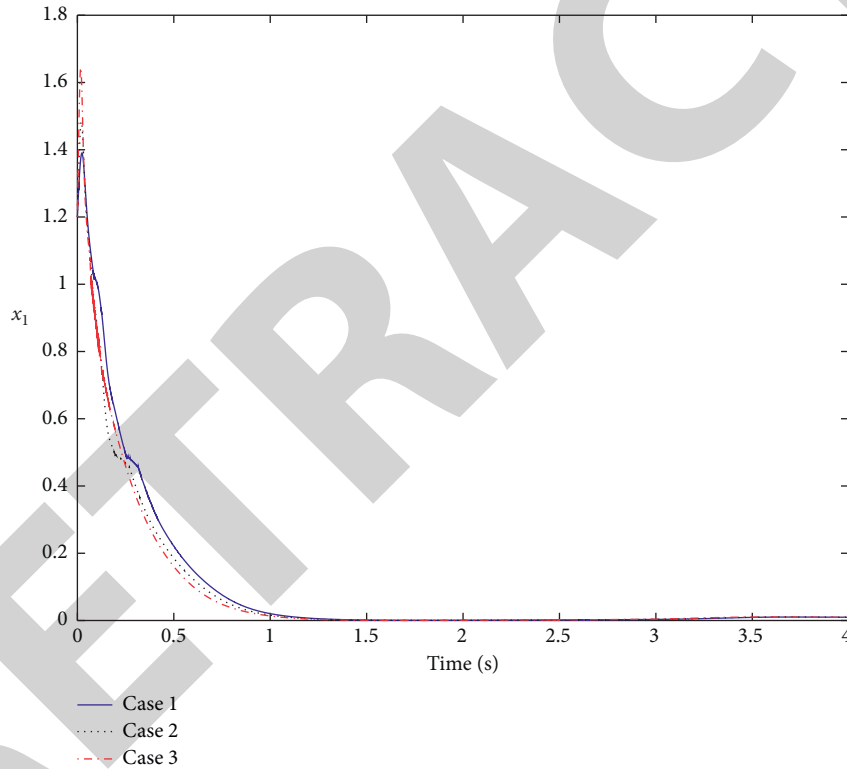


FIGURE 3: The trajectories of the adaptive parameters under the proposed method.

TABLE 2: The parameters of the three simulation cases.

	Switching regions	Fault parameters	Disturbance parameters
Case 1	$G_1 = \{x(t) \in \mathbb{R}^2 \mid \ x\  \geq 1\}$ $G_2 = \{x(t) \in \mathbb{R}^2 \mid 0.5 \leq \ x\  \leq 1\}$ $G_3 = \{x(t) \in \mathbb{R}^2 \mid \ x\  \leq 0.5\}$	$\Lambda(t) = 0.7 + 0.1 \sin(t/2 + t\pi/3)$	$d(t) = 20 \sin(2\pi t - \pi/6) + 0.5$
Case 2	$G_1 = \{x(t) \in \mathbb{R}^2 \mid \ x\  \geq 1.2\}$ $G_2 = \{x(t) \in \mathbb{R}^2 \mid 0.5 \leq \ x\  \leq 1.2\}$ $G_3 = \{x(t) \in \mathbb{R}^2 \mid \ x\  \leq 0.5\}$	$\Lambda(t) = 0.5 + 0.1 \sin(t/2 + t\pi/3)$	$d(t) = 30 \sin(2\pi t - \pi/6) + 0.5$
Case 3	$G_1 = \{x(t) \in \mathbb{R}^2 \mid \ x\  \geq 1.2\}$ $G_2 = \{x(t) \in \mathbb{R}^2 \mid 0.8 \leq \ x\  \leq 1.2\}$ $G_3 = \{x(t) \in \mathbb{R}^2 \mid \ x\  \leq 0.8\}$	$\Lambda(t) = 0.3 + 0.1 \sin(t/2 + t\pi/3)$	$d(t) = 50 \sin(2\pi t - \pi/6) + 0.5$

FIGURE 4: The trajectories of the system state  $x_1(t)$  of the proposed method under three cases.

In the simulation, the initial values of the system states and adaptive parameters are set as  $x_1(0) = 1.2$ ,  $x_2(0) = -0.5$ ,  $\hat{D}(0) = 0$ , and  $\hat{\vartheta}(0) = 1$ . The corresponding adaptive gains and constants are selected as  $\eta_D = \eta_{\vartheta} = 5$ ,  $\sigma_D = \sigma_{\vartheta} = 2$ , and  $\varepsilon_D = \varepsilon_{\vartheta} = 0.001$ . The disturbance is taken as

$$d(t) = 20 \sin\left(2\pi t - \frac{\pi}{6}\right) + 0.5. \quad (70)$$

The actuator-related settings are

$$\Lambda(t) = 0.7 + 0.1 \sin\left(\frac{t}{2} + \frac{\pi}{3}\right); \varepsilon_u = 0.001. \quad (71)$$

Moreover, to reveal the advantages of the proposed method, the advanced neural network based adaptive control method (NN-AC) and the disturbance observer based control method (DOBC) have been employed in the simulation experiments. The parameters of the NN-AC method are set as  $\eta_{\Theta} = 5$  and  $\sigma_{\Theta} = 2$ . The disturbance observation gain of the DOBC method is selected as  $L = [1.2, 0; 0, 1.2]$ . The control gains of the NN-AC and DOBC methods are set as the same as the proposed method.

The simulation results are provided in Figures 1–3. It is obvious that the proposed method can force both  $x_1(t)$  and  $x_2(t)$  to converge towards zero under switching between  $G_i$

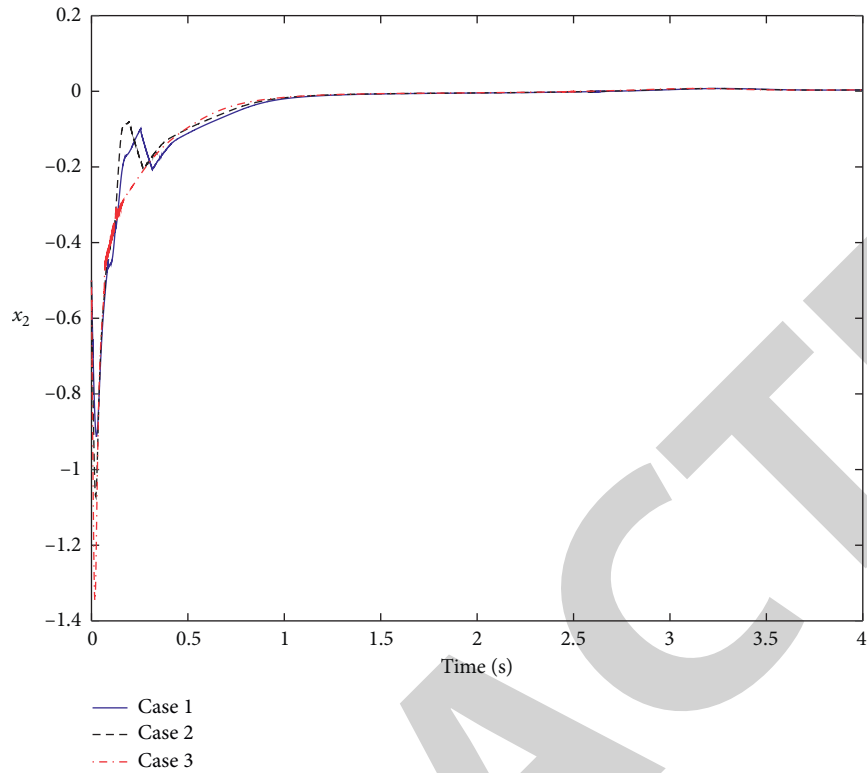


FIGURE 5: The trajectories of the system state  $x_2(t)$  of the proposed method under three cases.

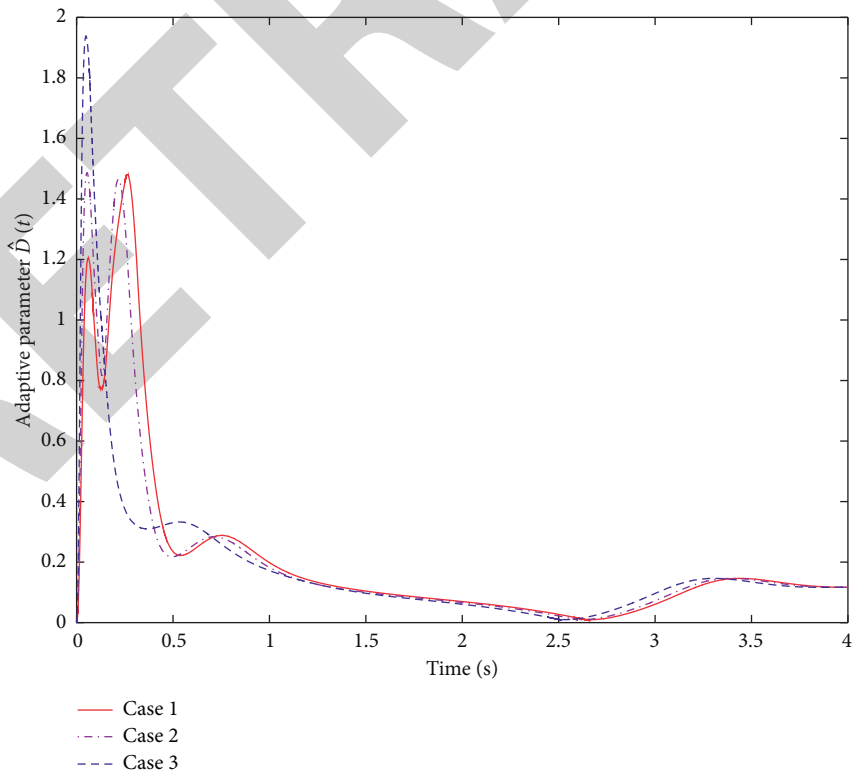


FIGURE 6: The trajectories of the system state  $\hat{D}(t)$  of the proposed method under three cases.



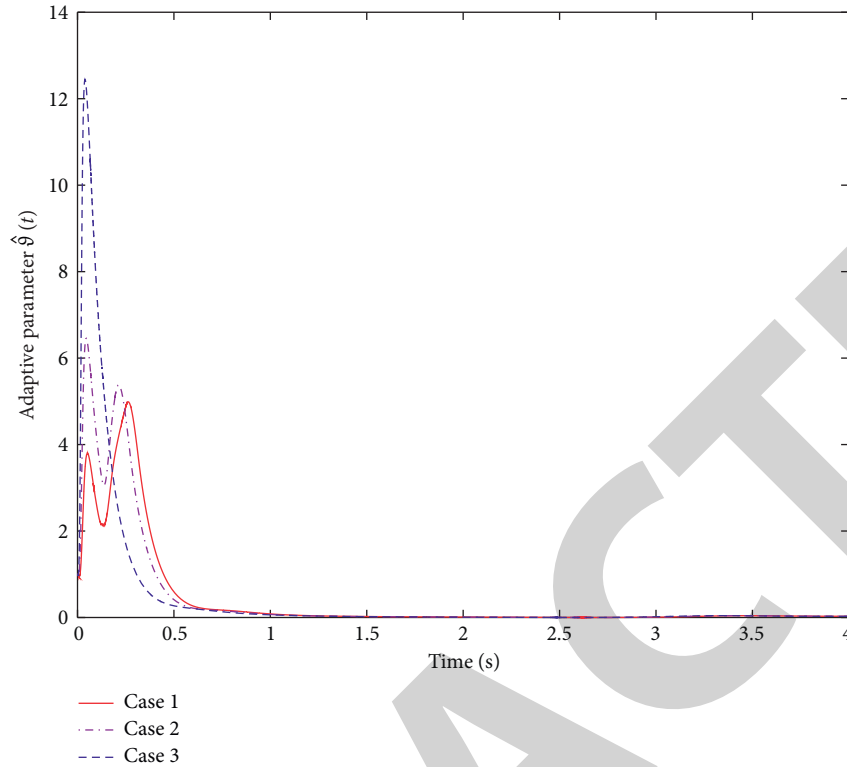


FIGURE 7: The trajectories of the system state  $\hat{\vartheta}(t)$  of the proposed method under three cases.

and finally stay inside a small region near the equilibrium point. Differently, the NN-AC and the DOBC method will cause violent shock, demonstrating that these two methods may possess worse adaptability for the state-dependent switching regions compared with the proposed method. Moreover, it can be found that, by using the proposed method, the adaptive parameters  $\hat{D}(t)$  and  $\hat{\vartheta}(t)$  also converge to a stable value after the initial transient. It can be concluded that, using the proposed method, the closed-loop stability can be guaranteed even under the worst situations (where disturbances and actuator faults both occur). The advantages of the proposed method can be revealed therefore.

Furthermore, to show the robustness of the proposed method, three cases are considered. The parameters of the three cases are given by Table 2.

Under the three cases, the simulation results using the proposed method are given in Figures 4–7. It can be found that although the switching regions, the fault parameters, and the disturbance parameters have changed, the proposed method can still achieve desired control performance.

## 5. Conclusions

A novel robust adaptive controller is given in this paper for solving one of the motivating problems in nonlinear fuzzy systems, that is, to appropriately describe the behavior of the system and to guarantee the stability of the system under discontinuous multiple uncertainties and state-based switching. The proposed differential-inclusion-based

method provides a constructive procedure for the controller design and analysis of a class of heterogeneous T-S fuzzy nonlinear systems with suddenly changing structural parameters and abrupt actuator failures where switching of system dynamics is related with both time and system states. The stability of the resulting closed-loop differential inclusion system is rigorously discussed by virtue of introducing a new fundamental stability lemma for adaptive discontinuous systems, and our results are validated by carefully designed simulations. It should be noted that our control scheme can be easily extended to other T-S fuzzy nonlinear systems with discontinuities and state-based switching, which may provide useful insights for further future research.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## References

- [1] J. Cheng, J. H. Park, L. Zhang, and Y. Zhu, "An asynchronous operation approach to event-triggered control for fuzzy markovian jump systems with general switching policies," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 1, pp. 6–18, 2018.
- [2] J. Lian, S. Li, and J. Liu, "T-S fuzzy control of positive markov jump nonlinear systems," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 4, pp. 2374–2383, 2018.
- [3] H. Shen, F. Li, Z.-G. Wu, J. H. Park, and V. Sreeram, "Fuzzy-model-based nonfragile control for nonlinear singularly perturbed systems with semi-markov jump parameters," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 6, pp. 3428–3439, 2018.
- [4] X.-H. Chang and G.-H. Yang, "Nonfragile  $H_{\infty}$  filtering of continuous-time fuzzy systems," *IEEE Transactions on Signal Processing*, vol. 59, no. 4, pp. 1528–1538, 2011.
- [5] H. D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Transactions on Fuzzy Systems*, vol. 9, no. 2, pp. 324–332, 2001.
- [6] H.-K. Lam, "Output regulation of polynomial fuzzy model-based control systems," *Polynomial Fuzzy Model-Based Control Systems*, vol. 64, no. 2, pp. 137–173, 2016.
- [7] Y. Wang, H. Shen, H. R. Karimi, and D. Duan, "Dissipativity-based fuzzy integral sliding mode control of continuous-time T-S fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 3, pp. 1164–1176, 2018.
- [8] H. Li, J. Yu, C. Hilton, and H. Liu, "Adaptive sliding-mode control for nonlinear active suspension vehicle systems using T-S fuzzy approach," *IEEE Transactions on Industrial Electronics*, vol. 60, no. 8, pp. 3328–3338, 2013.
- [9] B. Jiang, Z. Gao, P. Shi, and Y. Xu, "Adaptive fault-tolerant tracking control of near-space vehicle using takagi-sugeno fuzzy models," *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 5, pp. 1000–1007, 2010.
- [10] J. Tao, R. Lu, H. Su, P. Shi, and Z.-G. Wu, "Asynchronous filtering of nonlinear markov jump systems with randomly occurred quantization via T-S fuzzy models," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 4, pp. 1866–1877, 2017.
- [11] R. Sakthivel, T. Saravanakumar, B. Kaviarasan, and Y. Lim, "Finite-time dissipative based fault-tolerant control of takagi-sugeno fuzzy systems in a network environment," *Journal of the Franklin Institute*, vol. 354, no. 8, pp. 3430–3454, 2017.
- [12] X. Su, Y. Wen, Y.-D. Song, and T. Hayat, "Dissipativity-based fuzzy control of nonlinear systems via an event-triggered mechanism," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 49, no. 6, pp. 1208–1217, 2019.
- [13] J. Cheng, J. H. Park, and H. Wang, "Event-triggered  $H_{\infty}$  control for T-S fuzzy nonlinear systems and its application to truck-trailer system," *ISA Transactions*, vol. 65, pp. 62–71, 2016.
- [14] P. Shi, X. Su, and F. Li, "Dissipativity-based filtering for fuzzy switched systems with stochastic perturbation," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1694–1699, 2016.
- [15] M. Chadli, A. Abdo, and S. X. Ding, "Fault detection filter design for discrete-time Takagi-Sugeno fuzzy system," *Automatica*, vol. 49, no. 7, pp. 1996–2005, 2013.
- [16] Y. Zhao, H. Gao, J. Lam, and B. Du, "Stability and stabilization of delayed T-S fuzzy systems: a delay partitioning approach," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 4, pp. 750–762, 2009.
- [17] X. Yang, L. Wu, H.-K. Lam, and X. Su, "Stability and stabilization of discrete-time T-S fuzzy systems with stochastic perturbation and time-varying delay," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 1, pp. 124–138, 2014.
- [18] X.-H. Chang, Q. Liu, Y.-M. Wang, and J. Xiong, "Fuzzy peak-to-peak filtering for networked nonlinear systems with multipath data packet dropouts," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 3, pp. 436–446, 2019.
- [19] B.-Z. Guo, "Active disturbance rejection control: from ODEs to PDEs\*\*This work was carried out with the support of the national natural science foundation of China and the national research foundation of South Africa," *IFAC-PapersOnLine*, vol. 49, no. 8, pp. 278–283, 2016.
- [20] T. Jiang, C. Huang, and L. Guo, "Control of uncertain nonlinear systems based on observers and estimators," *Automatica*, vol. 59, pp. 35–47, 2015.
- [21] S. Li, J. Yang, W.-H. Chen, and X. Chen, *Disturbance Observer-Based Control: Methods and Applications*, CRC Press, Boca Raton, FL, USA, 2014.
- [22] W.-H. Chen, J. Yang, L. Guo, and S. Li, "Disturbance-observer-based control and related methods-an overview," *IEEE Transactions on Industrial Electronics*, vol. 63, no. 2, pp. 1083–1095, 2016.
- [23] L. Guo and W.-H. Chen, "Disturbance attenuation and rejection for systems with nonlinearity via DOBC approach," *International Journal of Robust and Nonlinear Control*, vol. 15, no. 3, pp. 109–125, 2005.
- [24] W.-H. Chen, "Disturbance observer based control for nonlinear systems," *IEEE/ASME Transactions on Mechatronics*, vol. 9, no. 4, pp. 706–710, 2004.
- [25] L. Guo and S. Cao, "Anti-disturbance control theory for systems with multiple disturbances: a survey," *ISA Transactions*, vol. 53, no. 4, pp. 846–849, 2014.
- [26] X. Wei, N. Chen, and W. Li, "Composite adaptive disturbance observer-based control for a class of nonlinear systems with multisource disturbance," *International Journal of Adaptive Control and Signal Processing*, vol. 27, no. 3, pp. 199–208, 2013.
- [27] H. Sun and L. Guo, "Composite adaptive disturbance observer based control and back-stepping method for nonlinear system with multiple mismatched disturbances," *Journal of the Franklin Institute*, vol. 351, no. 2, pp. 1027–1041, 2014.
- [28] X. Yao and L. Guo, "Composite anti-disturbance control for Markovian jump nonlinear systems via disturbance observer," *Automatica*, vol. 49, no. 8, pp. 2538–2545, 2013.
- [29] X. Wei and N. Chen, "Composite hierarchical anti-disturbance control for nonlinear systems with DOBC and fuzzy control," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 2, pp. 362–373, 2014.
- [30] C. D. Persis, "Robust stabilization of nonlinear systems by quantized and ternary control," *Systems & Control Letters*, vol. 58, no. 8, pp. 602–608, 2008.
- [31] C. De Persis and F. Mazenc, "Stability of quantized time-delay nonlinear systems: a lyapunov-krasowskii-functional approach," *Mathematics of Control, Signals, and Systems*, vol. 21, no. 4, pp. 337–370, 2010.
- [32] T. Liu, Z. P. Jiang, and D. J. Hill, "Small-gain based output-feedback controller design for a class of nonlinear systems with actuator dynamic quantization," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1326–1332, 2012.
- [33] S. Tong, S. Sui, and Y. Li, "Fuzzy adaptive output feedback control of MIMO nonlinear systems with partial tracking errors constrained," *IEEE Transactions on Fuzzy Systems*, vol. 23, no. 4, pp. 729–742, 2015.

- [34] Y. Chang, Y. Wang, F. E. Alsaadi, and G. Zong, "Adaptive fuzzy output-feedback tracking control for switched stochastic pure-feedback nonlinear systems," *International Journal of Adaptive Control and Signal Processing*, vol. 33, no. 10, pp. 1567–1582, 2019.
- [35] L. Ma, N. Xu, X. Huo, and X. Zhao, "Adaptive finite-time output-feedback control design for switched pure-feedback nonlinear systems with average dwell time," *Nonlinear Analysis: Hybrid Systems*, vol. 37, Article ID 100908, 2020.
- [36] L. Wang, M. V. Basin, H. Li, and R. Lu, "Observer-based composite adaptive fuzzy control for nonstrict-feedback systems with actuator failures," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 4, pp. 2336–2347, 2018.
- [37] Y.-J. Liu and S. Tong, "Barrier Lyapunov Functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints," *Automatica*, vol. 64, pp. 70–75, 2016.
- [38] Y.-J. Liu and S. Tong, "Barrier Lyapunov functions for Nussbaum gain adaptive control of full state constrained nonlinear systems," *Automatica*, vol. 76, pp. 143–152, 2017.
- [39] L. Ma, G. Zong, X. Zhao, and X. Huo, "Observed-based adaptive finite-time tracking control for a class of nonstrict-feedback nonlinear systems with input saturation," *Journal of the Franklin Institute*, vol. 357, no. 16, pp. 11518–11544, 2019.
- [40] J. T. Parker, A. Serrani, S. Yurkovich, M. A. Bolender, and D. B. Doman, "Control-oriented modeling of an air-breathing hypersonic vehicle," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 3, pp. 856–869, 2007.
- [41] S. Keshmiri, M. Mirmirani, and R. Colgren, "Six-DOF modeling and simulation of a generic hypersonic vehicle for conceptual design studies," in *Proceedings of the AIAA Modeling and Simulation Technologies Conference and Exhibit*, June 2004.
- [42] S. Yin, H. Yu, R. Shahnazi, and A. Haghani, "Fuzzy adaptive tracking control of constrained nonlinear switched stochastic pure-feedback systems," *IEEE Transactions on Cybernetics*, vol. 47, no. 3, pp. 579–588, 2017.
- [43] L. Long and J. Zhao, "Adaptive output-feedback neural control of switched uncertain nonlinear systems with average dwell time," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, no. 7, pp. 1350–1362, 2015.
- [44] B. Niu, H. R. Karimi, H. Wang, and Y. Liu, "Adaptive output-feedback controller design for switched nonlinear stochastic systems with a modified average dwell-time method," *IEEE Transactions on Systems Man & Cybernetics Systems*, vol. 47, no. 7, pp. 1371–1382, 2017.
- [45] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1910–1914, 1994.
- [46] B. Paden and S. Sastry, "A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Transactions on Circuits and Systems*, vol. 34, no. 1, pp. 73–82, 1987.