Research Article

Fractal Dimension for the Nonautonomous Stochastic Fifth-Order Swift–Hohenberg Equation

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Some dynamics behaviors for the nonautonomous stochastic fifth-order Swift–Hohenberg equation with additive white noise are considered. The existence of pullback random attractors for the nonautonomous stochastic fifth-order Swift–Hohenberg equation with some properties is mainly investigated on the bounded domain and unbounded domain, through the Ornstein–Uhlenbeck transformation and tail-term estimates. Furthermore, on the basis of some conditions, the finiteness of fractal dimension of random attractor is proved.

1. Introduction

Swift and Hohenberg proposed the Swift–Hohenberg (S-H) equation as a model for the convective instability in the Rayleigh–Bénard convection in 1997 [1]. There have been some results for the classical S-H equation [2–6]. Peletier and his collaborators have studied the S-H equation from different aspects, such as the stability of stationary solutions and pattern selections of solutions [7–9]. Recently, some results about pullback attractor [10, 11] and uniform attractor [12] of S-H equation are investigated. As we know, more and more authors investigated the random attractors and have obtained many important results. Meanwhile, there has been tremendous interest in developing the fractal dimension estimate of random attractors in recent years (see [13–19] and the references there in).

We consider the stochastic fifth-order S-H equation driven by additive white noise:

$$\begin{aligned} du + (\Delta^2 u + 2\Delta u + au + u^5 - g(x, t))dt \\ = \phi(x)dW(t), \quad ((x, t) \in D \times (\kappa, +\infty), \kappa \in \mathbb{R}), \end{aligned} \quad (1)$$

with boundary condition

$$u(t, x) = 0, \quad x \in \partial D, \quad (2)$$

and the initial condition

$$u(x, \kappa) = u_\kappa(x), \quad x \in D, \quad (3)$$

where $a > 0$ and $D \subset \mathbb{R}^2$ is a bounded smooth domain.

There are some results about dynamics behaviors for the classical autonomous stochastic S-H equation [20, 21]. Except for the study of the existence of the random attractor [22–24], Zhou et al. have established an efficient theory about the finite fractal dimensions of random attractor [25, 26]. To our knowledge, the fractal dimension estimate has been barely studied for the random attractors of the stochastic fifth-order S-H equation yet. According to the ideas in [21, 25–28], stochastic dynamics behaviors of the random attractor are considered for stochastic equations (1)–(3) in two cases.

Firstly, we mainly give the existence of random attractor for the fifth-order S-H equation corresponding to (1)–(3). A few results about the dynamics behaviors of the fifth-order S-H equation with additive noise have been given when the nonlinear term is five ordered. Due to the increasing order, more difficult terms can be produced to derive uniform estimates for the solution of equations

(1)–(3). In order to overcome these difficulties aroused by the fifth-order term, we mainly use the method of integration by parts after Ornstein–Uhlenbeck transformation. Notice that the uniform estimates are independent of bounded region D , and we can obtain the existence of random attractor by proving that the random dynamical system is asymptotically compact through tail-term estimates on unbounded domain $\mathbb{R}^2$.

Secondly, we are devoted to the finiteness of fractal dimension for the random attractor of (1)–(3). Because of the complexity of proving the boundedness of fractal dimension on unbounded domain, we pay attention to studying the fifth-order S-H equation on bounded domain D . Furthermore, we will discuss the case of unbounded domain. Here, in order to obtain the boundedness of the fractal dimension, some sufficient conditions are proposed [16–18].

2. Preliminaries

We give the theorem related to random attractors. Since these conclusions are classic, we will not discuss them in detail, and readers can check the relevant literature [22–24, 29, 30].

In this paper, we will use $\|\cdot\|$ to denote the norm and $(\cdot, \cdot)$ to denote the inner product in $L^2(\mathbb{R}^2)$ or $L^2(D)$, where D is a bounded smooth domain. In the case of bounded domain, for simplicity, we use the notation H to present the space $L^2(D)$. We will write the norm of $L^p(\mathbb{R}^2)$ ($L^p(D)$) as $\|\cdot\|_{L^p}$ and use $\|\cdot\|_X$ to denote the norm of Banach space X and $\|\cdot\|_s$ to denote the norm in $H^s(D)$ or $H^s(\mathbb{R}^2)$.

The continuous random dynamical system will be showed for the stochastic fifth-order S-H equation on $L^2(\mathbb{R}^2)$ (or $L^2(D)$):

$$du + (\Delta^2 u + 2\Delta u + au + u^5 - g(x, t))dt = \sum_{i=1}^m \phi_i(x) dW_i(t), \quad (4)$$

with boundary condition

$$u(t, x) = 0, \quad x \in \partial D, \quad (5)$$

and with the initial condition

$$u(x, \kappa) = u_\kappa(x), \quad x \in D, \quad (6)$$

where $\phi_i(x) \in H^4(\mathbb{R}^2)$ ($i = 1, \dots, m$) is the smooth enough function. W is a two-sided real-value Wiener process on a probability space and $g(x, \cdot) \in C_b(\mathbb{R}, H)$ is given.

Let $\bar{z}(\omega) = -a \int_{-\infty}^0 e^{a\kappa} \omega(\kappa) d\kappa$, $\omega \in \Omega$, which is a unique stationary solution of the equation [29]:

$$d\bar{z} + a\bar{z}dt = dW. \quad (7)$$

In addition, for each fixed $\omega \in \Omega$, $\bar{z}(\theta_t \omega)$ is pathwise continuous. And there is a tempered function $r(\omega) > 0$:

$$|\bar{z}(\theta_t \omega)|^2 + |\bar{z}(\theta_t \omega)|^p \leq e^{a-6/2|t|} r(\omega), \quad t \in \mathbb{R}, \text{P-a.e. } \omega \in \Omega. \quad (8)$$

Given a translation $v(t, \kappa, \omega, v_\kappa) = u(t, \kappa, \omega, u_\kappa) - z(\theta_t \omega)$, one yields $v_\kappa = u_\kappa - z(\omega)$, where $z(\theta_t \omega) = \phi(x)\bar{z}(\theta_t \omega)$. Then, we obtain

$$\begin{aligned} \frac{dv}{dt} + \Delta^2 v + 2\Delta v + 2\Delta z(\theta_t \omega) + av + \Delta^2 z(\theta_t \omega) + v^5 \\ + z^5(\theta_t \omega) + 5v^4 z(\theta_t \omega) + 10v^3 z^2(\theta_t \omega) + 10v^2 z^3(\theta_t \omega) \\ + 5v z^4(\theta_t \omega) = g(x, t), \end{aligned} \quad (9)$$

$$v(t, x) = 0, \quad x \in \partial D, \quad (10)$$

$$v(\kappa, \kappa, \omega, x) = v_\kappa(\omega) = u_\kappa(x) - z(\omega). \quad (11)$$

By the Galerkin method, for all $v_\kappa \in H$, as proved in [31], one can show that systems (9)–(11) are well-posed for every $\omega \in \Omega$ in H . A continuous cocycle $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \rightarrow H$ is defined by

$$\Phi(t, \kappa, \omega, u_\kappa) = \Phi(t, \kappa, \omega)u_\kappa = u(t + \kappa, \kappa, \theta_{-\kappa}\omega, u_\kappa), \quad (12)$$

and a cocycle $\Psi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \rightarrow H$ by

$$\Psi(t, \kappa, \omega, v_\kappa) = \Psi(t, \kappa, \omega)v_\kappa = v(t + \kappa, \kappa, \theta_{-\kappa}\omega, v_\kappa), \quad (13)$$

where $v_\kappa = u_\kappa - z(\omega)$. Notice that the continuous dynamical systems Φ and Ψ are equivalent. Similarly, $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ is a continuous cocycle.

Let $B = \{B(\kappa, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\}$, which are a family of bound nonempty subsets of H (or $L^2(\mathbb{R}^2)$) and

$$\lim_{t \rightarrow +\infty} e^{-ct} \|B(\kappa + t, \theta_t \omega)\|^2 = 0, \quad \forall c > 0, \kappa \in \mathbb{R}, \omega \in \Omega, \quad (14)$$

where $\|B\| = \sup_{u \in B} \|u\|$. Here, we always assume that $\mathcal{D}$ is the collection of all tempered families of nonempty subsets of H . Furthermore, we prove that there exist $\mathcal{D}$ -pullback attractors for cocycle Φ . In the whole paper, we assume $g(x, \cdot) \in C_b(\mathbb{R}, H)$ with

$$\|g\|^2 = \sup_{t \in \mathbb{R}} \|g(\cdot, t)\|^2 < +\infty. \quad (15)$$

3. Some Uniform Estimates

The uniform estimates are derived for solutions of systems (9)–(11) on bounded domain D and $\mathbb{R}^2$, respectively.

Lemma 1. Suppose $a > 6$ and (8) holds. Then, for every $\kappa \in \mathbb{R}$, $\omega \in \Omega$, and $B = \{B(\kappa, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there is a $T_0(\kappa, \omega, B) \geq 1$ satisfying

$$\begin{aligned} \|v(\kappa, \kappa - t, \theta_{-\kappa}\omega, v_{\kappa-t})\|^2 &\leq r_0(\kappa, \omega), \\ \|u(\kappa, \kappa - t, \theta_{-\kappa}\omega, u_{\kappa-t})\|^2 &\leq r_1(\kappa, \omega), \end{aligned} \quad (16)$$

for $\forall t \geq T_0(\kappa, \omega, B)$, where v is the solution of systems (9)–(11), $v_{\kappa-t} + z(\theta_{-\kappa}\omega) \in B(\kappa - t, \theta_{-\kappa}\omega)$, and $r_0(\kappa, \omega)$ and $r_1(\kappa, \omega)$ are two tempered random variables.

Proof. From (9), we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|^2 + \|\Delta v\|^2 &= -a \|v\|^2 - \|v\|_{L^6}^6 \\ &\quad - (2\Delta v + 2\Delta z(\theta_t \omega) + \Delta^2 z(\theta_t \omega), v) \\ &\quad - (z^5(\theta_t \omega) + 5v^4 z(\theta_t \omega) + 10v^3 z^2(\theta_t \omega), v) \\ &\quad - 5(2v^2 z^3(\theta_t \omega) + v z^4(\theta_t \omega), v) + (g(t), v). \end{aligned} \quad (17)$$

Since

$$\begin{aligned} (10v^3 z^2(\theta_t \omega), v) &= 10 \int_D v^4 z^2(\theta_t \omega) dx \geq 0, \quad (5v z^4(\theta_t \omega), v) \\ &= 5 \int_D v^2 z^4(\theta_t \omega) dx \geq 0, \end{aligned} \quad (18)$$

applying the Hölder inequality and the ε -Young inequality, one gets

$$\begin{aligned} \frac{d}{dt} \|v\|^2 + \|v\|_{L^6}^6 + (a-6) \|v\|^2 + \|\Delta v\|^2 &\leq C(|\tilde{z}(\theta_t \omega)|^{10} \\ &\quad + |\tilde{z}(\theta_t \omega)|^6 + |\tilde{z}(\theta_t \omega)|^2) \\ &\quad + \frac{1}{a} \|g(t)\|^2 \doteq F(\theta_t \omega) \\ &\quad + \frac{1}{a} \|g(t)\|^2, \end{aligned} \quad (19)$$

when $r \geq \kappa - t$, by the Gronwall inequality on $[\kappa - t, r]$ and substituting $\theta_{-\kappa} \omega$ for ω , one has

$$\begin{aligned} \|v(r, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 &\leq e^{-(a-6)(r+t-\kappa)} \|v_{\kappa-t}\|^2 \\ &\quad + \int_{\kappa-t}^r \left(e^{(6-a)(r-s)} F(\theta_{s-\kappa} \omega) + \frac{e^{(6-a)(r-s)}}{a} \|g(s)\|^2 \right) ds \\ &\leq e^{-(a-6)t} \|v_{\kappa-t}\|^2 + \int_{-\infty}^0 \left(e^{-(6-a)s} F(\theta_s \omega) \right. \\ &\quad \left. + \frac{e^{-(6-a)s}}{a} \|g(s+\kappa)\|^2 \right) ds. \end{aligned} \quad (20)$$

From (20), since $v_{\kappa-t} + z(\theta_{-t} \omega) \in B(\kappa - t, \theta_{-t} \omega)$, and $B = \{B(\kappa, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\}$ is tempered, and there is $T_0 = T_0(\kappa, \omega, B) > 1$ satisfying

$$e^{-(a-6)t} \|v_{\kappa-t}\|^2 \leq 1, \quad \forall t \geq T_0. \quad (21)$$

From (20), we can deduce

$$\begin{aligned} \|v(\kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 &\leq 1 + C \int_{-\infty}^0 e^{\frac{(a-6)s}{2}} r(\omega) ds + \frac{1}{a} \int_{-\infty}^0 \|g(s+\kappa)\|^2 e^{(a-6)s} ds \\ &\leq e^{-(a-6)t} \|v_{\kappa-t}\|^2 + \frac{2C}{a-6} r(\omega) + \frac{1}{a} \int_{-\infty}^0 e^{(a-6)s} \|g(s+\kappa)\|^2 ds \\ &\doteq r_0(\kappa, \omega), \quad \forall t \\ &\geq T_0(\kappa, \omega, B). \end{aligned} \quad (22)$$

For all $v_{\kappa-t} + z(\theta_{-t} \omega) \in B(\kappa - t, \theta_{-t} \omega)$, we have

$$\begin{aligned} \|u(\kappa, \kappa - t, \theta_{-\kappa} \omega, u_{\kappa-t})\|^2 &\leq 2\|z(\omega)\|^2 + 2 \\ &\quad + C \left(\int_{-\infty}^0 e^{(a-6)s/2} r(\omega) ds \right. \\ &\quad \left. + \int_{-\infty}^0 \|g(s+\kappa)\|^2 e^{(a-6)s} ds \right) \\ &\doteq r_1(\kappa, \omega), \quad \forall \kappa \in \mathbb{R}, \omega \in \Omega. \end{aligned} \quad (23)$$

Denote

$$K(\kappa, \omega) = \{u \in H: \|u\|^2 \leq r_1(\kappa, \omega), \omega \in \Omega, \kappa \in \mathbb{R}\} \in \mathcal{D}. \quad (24)$$

Then, $K(\kappa, \omega)$ is an absorbing set for Φ in $\mathcal{D}$ $\square$

Remark 1. Using the Gronwall lemma, we can obtain the following results. For every $\kappa \in \mathbb{R}, \omega \in \Omega$, and $B = \{B(\kappa, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there is a $T_1(\kappa, \omega, B) \geq T_0 \geq 1$ satisfying

$$\begin{aligned} \int_{\kappa-1}^{\kappa} \|v(s, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 ds &\leq r_2(\kappa, \omega), \quad \forall t \geq T_1(\kappa, \omega, B), \\ \int_{-t}^0 e^{(a-6)s} \|\Delta v(s + \kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 ds \\ &\quad + \int_{-t}^0 e^{(a-6)s} \|v(s + \kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|_{L^6}^6 ds \\ &\leq r_3(\kappa, \omega), \quad \forall t \geq T_1(\kappa, \omega, B), \end{aligned} \quad (25)$$

where v is the solution of systems (9)–(11), $v_{\kappa-t} + z(\theta_{-t} \omega) \in B(\kappa - t, \theta_{-t} \omega)$, and $r_2(\kappa, \omega)$ and $r_3(\kappa, \omega)$ are tempered random variables.

Lemma 2. Suppose $a > 6$ and (15) holds. Then, for every $\kappa \in \mathbb{R}, \omega \in \Omega$, and $B = \{B(\kappa, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there is a $T_1(\kappa, \omega, B) \geq T_0 \geq 1$ satisfying

$$\|\Delta v(\kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 \leq r_4(\kappa, \omega), \quad \forall t \geq T_1(\kappa, \omega, B), \quad (26)$$

where v is the solution of systems (9)–(11), $v_{\kappa-t} + z(\theta_{-t} \omega) \in B(\kappa - t, \theta_{-t} \omega)$, and $r_4(\kappa, \omega)$ and $r_5(\kappa, \omega)$ are tempered random variables.

Proof. By (9), one obtains

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta v\|^2 &+ (2\Delta v, \Delta^2 v) + (2\Delta z(\theta_t \omega) + \Delta^2 z(\theta_t \omega), \Delta^2 v) \\ &\quad + a \|\Delta v\|^2 + \|\Delta v\|^2 + (v^5, \Delta^2 v) + (z^5(\theta_t \omega), \Delta^2 v) \\ &\quad + ((v^4 z(\theta_t \omega), \Delta^2 v) + (10v^3 z^2(\theta_t \omega), \Delta^2 v) \\ &\quad + (10v^2 z^3(\theta_t \omega), \Delta^2 v) + (5v z^4(\theta_t \omega), \Delta^2 v)) = (g(x, t), \Delta^2 v). \end{aligned} \quad (27)$$

It is easy to get

$$|2(\Delta v, \Delta^2 v)| \leq \frac{1}{4} \|\Delta^2 v\|^2 + 4 \|\Delta v\|^2. \quad (28)$$

It can easily be shown that

$$\begin{aligned} |(2\Delta z(\theta_t \omega), \Delta^2 v)| &\leq \frac{1}{12} \|\Delta^2 v\|^2 + C \|\Delta z(\theta_t \omega)\|^2, \\ |(g(x, t), \Delta^2 v)| &\leq \frac{1}{12} \|\Delta^2 v\|^2 + 3 \|g(t)\|^2. \end{aligned} \quad (29)$$

It is evident that

$$|(\Delta^2 z(\theta_t \omega), \Delta^2 v)| \leq \frac{1}{12} \|\Delta^2 v\|^2 + C \|z(\theta_t \omega)\|^2. \quad (30)$$

Since

$$\begin{aligned} \|v\|_{L^8} &\leq C \|\Delta^2 v\|^{3/16} \|v\|^{13/16}, \\ \|v\|_{L^4} &\leq C \|\Delta^2 v\|^{1/8} \|v\|^{7/8}, \end{aligned} \quad (31)$$

we deduce that

$$\begin{aligned} |(5v^4 z(\theta_t \omega), \Delta^2 v)| &\leq 5 \|v\|_{L^8}^4 \|\Delta^2 v\| \|z(\theta_t \omega)\|_{L^\infty} C \\ &\quad \|\Delta^2 v\|^{7/4} \|v\|^{13/4} \|z(\theta_t \omega)\|_{L^\infty} \leq \frac{1}{12} \|\Delta^2 v\|^2 \\ &\quad + C \|z(\theta_t \omega)\|_{L^\infty}^8, \\ |(5v z^4(\theta_t \omega), \Delta^2 v)| &\leq 5 \|v\|_{L^4} \|\Delta^2 v\| \|z(\theta_t \omega)\|_{L^{16}}^4 \\ &\leq C \|\Delta^2 v\|^{9/8} \|z(\theta_t \omega)\|_{L^{16}}^4 \leq \frac{1}{12} \|\Delta^2 v\|^2 \\ &\quad + C \|z(\theta_t \omega)\|_{L^{16}}^{64/7}, \end{aligned} \quad (32)$$

where we have used the boundedness of $\|v\|$. It shows that

$$|(z^5(\theta_t \omega), \Delta^2 v)| \leq \frac{1}{12} \|\Delta^2 v\|^2 + C \|z(\theta_t \omega)\|_{L^{10}}^{10}. \quad (33)$$

Similarly, we can get

$$\begin{aligned} |(10v^3 z^2(\theta_t \omega), \Delta^2 v)| &\leq 10 \|v\|_{L^{12}}^3 \|\Delta^2 v\| \|z(\theta_t \omega)\|_{L^8}^2 \\ &\leq C \|\Delta^2 v\|^{13/8} \|z(\theta_t \omega)\|_{L^8}^2 \|v\|^{19/8} \leq C \|\Delta^2 v\|^{13/8} \|z(\theta_t \omega)\|_{L^8}^2 \\ &\leq \frac{1}{12} \|\Delta^2 v\|^2 + C \|z(\theta_t \omega)\|_{L^8}^{32/3}, \\ |(10v^2 z^3(\theta_t \omega), \Delta^2 v)| &\leq 10 \|v\|_{L^8}^2 \|\Delta^2 v\| \|z(\theta_t \omega)\|_{L^{12}}^3 \\ &\leq C \|\Delta^2 v\|^{3/8} \|v\|^{13/8} \|\Delta^2 v\| \|z(\theta_t \omega)\|_{L^{12}}^3 \\ &\leq C \|\Delta^2 v\|^{11/8} \|z(\theta_t \omega)\|_{L^{12}}^3 \leq \frac{1}{12} \|\Delta^2 v\|^2 + C \|z(\theta_t \omega)\|_{L^{12}}^{48/5}. \end{aligned} \quad (34)$$

Similarly, we have the following estimates:

$$\begin{aligned} &\int_D (\nabla v)^2 v^3 (\nabla^2 v) dx \\ &\leq \|\nabla v\|_{L^\infty}^2 \|\nabla^2 v\| \|v\|_{L^6}^3 \\ &\leq C \|v\| \|\Delta^2 v\| \|\Delta v\| \|v\|_{L^6}^3 \leq C \|\Delta^2 v\| \|\Delta v\| \|v\|_{L^6}^3 \\ &\leq \frac{1}{240} \|\Delta^2 v\|^2 + C \|\Delta v\|^2 \|v\|_{L^6}^6, \\ (v^5, \Delta^2 v) &= -5 \int_D v^4 \nabla v (\nabla^3 v) dx = 5 \int_D \nabla (v \nabla v^4) (\nabla^2 v) dx \\ &= 5 \int_D (\nabla^2 v v^4 + 4 (\nabla v)^2 v^3) (\nabla^2 v) dx \\ &= 5 \int_D (\nabla^2 v)^2 v^4 dx + 20 \int_D (\nabla v)^2 v^3 (\nabla^2 v) dx, \end{aligned} \quad (35)$$

where $\int_D (\nabla^2 v)^2 v^4 dx \geq 0$.

Finally, we obtain

$$\frac{d}{dt} \|\Delta v\|^2 + (2a - 8) \|\Delta v\|^2 \leq C \|v\|_{L^6}^6 \|\Delta v\|^2 + H(\theta_t \omega) + 3 \|g(t)\|^2, \quad (36)$$

where

$$\begin{aligned} H(\theta_t \omega) &= C \left(|\tilde{z}(\theta_t \omega)|^2 + |\tilde{z}(\theta_t \omega)|^{10} + |\tilde{z}(\theta_t \omega)|^{32/3} \right) \\ &\quad + |\tilde{z}(\theta_t \omega)|^8 + |\tilde{z}(\theta_t \omega)|^{64/7} + |\tilde{z}(\theta_t \omega)|^{48/5}. \end{aligned} \quad (37)$$

Therefore,

$$\frac{d}{dt} \|\Delta v\|^2 \leq C \|v\|_{L^6}^6 \|\Delta v\|^2 + H(\theta_t \omega) + 3 \|g(t)\|^2. \quad (38)$$

For $s \in [\kappa - 1, \kappa]$, choosing $t \geq T_1(\kappa, \omega, B) \geq 1$, by the Gronwall inequality on the interval $[s, \kappa]$, we obtain

$$\begin{aligned} &\|\Delta v(\kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 \\ &\leq e \int_{\kappa-1}^{\kappa} C \|v\|_{L^6}^6 d\sigma \left(\|\Delta v(s, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 \right. \\ &\quad \left. + \int_{\kappa-1}^{\kappa} H(\theta_{r-\kappa} \omega) dr \right. \\ &\quad \left. + 3 \int_{\kappa-1}^{\kappa} \|g(r)\|^2 dr \right). \end{aligned} \quad (39)$$

By Remark 1, one can obtain

$$\int_{\kappa-1}^{\kappa} \|v(s, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|_{L^6}^6 ds \leq C r_3(\kappa, \omega). \quad (40)$$

Now, one can integrate (39) with respect to s . Then,

$$\begin{aligned}
\|\Delta v(\kappa, \kappa - t, \theta_{-\kappa}\omega, v_{\kappa-t})\|^2 &\leq e \int_{\kappa-1}^{\kappa} C \|v\|_{L^6}^6 d\sigma \\
&\left(\int_{\kappa-1}^{\kappa} \|\Delta v(s, \kappa - t, \theta_{-\kappa}\omega, v_{\kappa-t})\|^2 ds + \int_{-1}^0 (H(\theta_r\omega) \right. \\
&\quad \left. + 3\|g(r + \kappa)\|^2) dr \right) \leq e^{Cr_3(\kappa, \omega)} \left(r_2(\kappa, \omega) + \frac{2C}{a-6} r(\omega) \right. \\
&\quad \left. + 3 \int_{-1}^0 \|g(r + \kappa)\|^2 dr \right). \tag{41}
\end{aligned}$$

By Remark 1, (39)–(41), we can obtain

$$\|\Delta v(\kappa, \kappa - t, \theta_{-\kappa}\omega, v_{\kappa-t})\|^2 \leq r_4(\kappa, \omega). \tag{42}$$

Furthermore,

$$\|\Delta u(\kappa, \kappa - t, \theta_{-\kappa}\omega, u_{\kappa-t})\|^2 \leq r_5(\kappa, \omega). \tag{43}$$

Theorem 1. Suppose $a > 6$ and (15) holds. There exists a unique $\mathcal{D}$ -pullback random attractor $\mathcal{A} = \{\mathcal{A}(\kappa, \omega) : \kappa \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $H = L^2(D)$ for continuous cocycle Φ of systems (1) and (2).

In the proof of the above lemmas, we find that all the estimates of solution do not depend on bounded domains D , so these estimates are also valid for unbounded domain.

Lemma 3. Suppose $a > 6$ and (15) holds. Then, for every $\kappa \in \mathbb{R}, \omega \in \Omega$, and $B = \{B(\kappa, \omega) : \kappa \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there are $T_2(\kappa, \omega, B, \epsilon) \geq T_1 \geq 1$ and $k_3(\kappa, \omega, \epsilon) \geq 1$ satisfying

$$\int_{|x| \geq k} |v(\kappa, \kappa - t, \theta_{-\kappa}\omega, v_{\kappa-t})|^2 dx \leq \epsilon, \forall t \geq T_2(\kappa, \omega, B, \epsilon), k \geq k_3, \tag{44}$$

where v is the solution of systems (9)–(11) and $v_{\kappa-t} + z(\theta_{-t}\omega) \in B(\kappa - t, \theta_{-t}\omega)$.

Proof. Let $0 \leq \theta(s) \leq 1$ ($s \in \mathbb{R}^+$) be a smooth function. When $0 \leq s \leq 1$, $\theta(s) = 0$. When $s \geq 2$, $\theta(s) = 1$. Then, $|\theta'(s)| + |\theta''(s)| < C$. Multiplying (21) with $\theta(|x|^2/k^2)v$, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v(\Delta^2 v + 2\Delta v) dx \\
&\quad + \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v(2\Delta z(\theta_t \omega) + av + v^5 + \Delta^2 z(\theta_t \omega)) dx \\
&\quad + 5 \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) vz \left(v^4 + 2v^3 z + 2v^2 z^2 + vz^3 + \frac{1}{5} z^4 \right) dx \\
&\quad = \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v g(x, t) dx. \tag{45}
\end{aligned}$$

For the estimate of $\int_{\mathbb{R}^2} \theta(|x|^2/k^2) v \Delta^2 v dx$, similar to [21], it follows that

$$\begin{aligned}
\int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v \Delta^2 v dx &\geq -\frac{C}{k} (\|v\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2) \\
&\quad + \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |\Delta v|^2 dx. \tag{46}
\end{aligned}$$

In addition, there are

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v \Delta v dx \right| &\leq \frac{1}{2} \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |\Delta v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |v|^2 dx, \\
\left| \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v \Delta^2 z(\theta_t \omega) dx \right| &\leq \frac{a}{2} \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + C \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |\Delta^2 z(\theta_t \omega)|^2 dx, \\
\int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v g(x, t) dx &\leq \frac{1}{2} \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + C \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |g(t)|^2 dx. \tag{47}
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\left| \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) v \Delta z(\theta_t \omega) dx \right| &\leq \frac{1}{2} \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |v|^2 dx \\
&\quad + C \int_{\mathbb{R}^2} \theta\left(\frac{|x|^2}{k^2}\right) |\Delta z(\theta_t \omega)|^2 dx, \tag{48}
\end{aligned}$$

where we have used the boundedness of function θ .

It is easy to obtain the following estimates:

$$\left| \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) v^5 z (\theta_t \omega) dx \right| \leq \frac{1}{10} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |v|^6 dx + C \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |z (\theta_t \omega)|^6 dx, \quad (49)$$

$$\left| \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) v z^5 (\theta_t \omega) dx \right| \leq \frac{1}{2} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |v|^2 dx + C \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |z (\theta_t \omega)|^{10} dx,$$

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) v^3 z^3 (\theta_t \omega) dx \right| &\leq \frac{1}{20} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |v|^6 dx \\ &+ C \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |z (\theta_t \omega)|^6 dx. \end{aligned} \quad (50)$$

Because $\int_{\mathbb{R}^2} \theta (|x|^2/k^2) v^4 z^2 (\theta_t \omega) dx \geq 0$ and $\int_{\mathbb{R}^2} \theta (|x|^2/k^2) v^2 z^4 (\theta_t \omega) dx \geq 0$, one yields

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |v|^2 dx + (a-6) \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |v|^2 dx &\leq \frac{C}{k} (\|v\|^2 + \|\nabla v\|^2 + \|\Delta v\|^2) \\ &+ C \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) (|\Delta z|^2 + |z|^6 + |\Delta^2 z|^2 + |z|^{10}) dx + C \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |g(t)|^2 dx. \end{aligned} \quad (51)$$

Applying the Gronwall lemma on $[\kappa - t, \kappa]$, one gets

$$\begin{aligned} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |v(\kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})|^2 dx &\leq e^{-(a-6)t} \|v_{\kappa-t}\|^2 + \frac{C}{k} \int_{-\kappa}^0 e^{(a-6)s} \left(\|v(s + \kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 \right. \\ &\quad \left. + \|\nabla v(s + \kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 + \|\Delta v(s + \kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})\|^2 \right) ds \\ &+ C \int_{-\infty}^0 e^{(a-6)s} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) (|\Delta z(\theta_s \omega)|^2 + |\Delta^2 z(\theta_s \omega)|^2 + |z(\theta_s \omega)|^6 + |z(\theta_s \omega)|^{10}) dx ds \\ &+ C \int_{-\infty}^0 e^{(a-6)s} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |g(s + \kappa)|^2 dx ds. \end{aligned} \quad (52)$$

Noticing $v_{\kappa-t} + z(\theta_{-t} \omega) \in B(\kappa - t, \theta_{-t} \omega)$ and $B \in \mathcal{D}$, there is a $T_2 \geq T_1(t, \omega, B, \epsilon)$. When $t \geq T_2$, we have

$$e^{-(a-6)t} \|v_{\kappa-t}\|^2 \leq \frac{\epsilon}{4}. \quad (53)$$

From (51), combining with Remark 1, if $\epsilon > 0$, there is a $k_1(\kappa, \epsilon) \geq 1$ satisfying

$$\frac{C}{k} \int_{-\kappa}^0 e^{(a-6)s} (\|v\|^2 + \|\Delta v\|^2) ds \leq \frac{\epsilon}{4}, \forall k \geq k_1(\kappa, \epsilon). \quad (54)$$

And there is a $k_2 = k_2(\epsilon) \geq k_1(\kappa, \epsilon)$ such that

$$\begin{aligned} C \int_{-\infty}^0 e^{(a-6)s} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) (|\Delta z(\theta_s \omega)|^2 + |\Delta^2 z(\theta_s \omega)|^2 \\ + |z(\theta_s \omega)|^6 + |z(\theta_s \omega)|^{10}) dx ds &\leq C \epsilon \int_{-\infty}^0 \\ e^{(a-6)s} (|\tilde{z}(\theta_s \omega)|^2 + |\tilde{z}(\theta_s \omega)|^6 + |\tilde{z}(\theta_s \omega)|^{10}) ds &\leq \frac{\epsilon}{4}, k \geq k_2. \end{aligned} \quad (55)$$

Meanwhile, there is a $k_3(\kappa, \epsilon) \geq k_2$ satisfying

$$C \int_{-\infty}^0 e^{(a-6)s} \int_{\mathbb{R}^2} \theta \left(\frac{|x|^2}{k^2} \right) |g(s+\kappa)|^2 dx ds \leq \frac{\epsilon}{4}, \quad k \geq k_3(\kappa, \epsilon). \quad (56)$$

By (51)–(56), we deduce

$$\int_{|x| \geq k} |v(\kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})|^2 dx \leq \epsilon, \quad \forall k \geq k_3(\kappa, \epsilon). \quad (57)$$

□

Lemma 4. Suppose $a > 6$ and (15) holds. Then, given $\epsilon > 0$ and $B = \{B(\cdot, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, $\exists T_2 = T_2(\kappa, \omega, B, \epsilon) \geq T_1 \geq 1$, and $k_4 = k_4(\omega, \epsilon) > 0$, the solution $u(\kappa, \kappa - t, \theta_{\kappa} \omega, u_{\kappa-t}) = v(\kappa, \kappa - t, \theta_{\kappa} \omega, v_{\kappa-t}) + z(\theta_t \omega)$ satisfies the following inequality:

$$\int_{|x| \geq k'} |u(\kappa, \kappa - t, \theta_{-\kappa} \omega, u_{\kappa-t})|^2 dx \leq \epsilon, \quad \forall t \geq T_2, k' \geq k_4, \quad (58)$$

where $v_{\kappa-t} + z(\theta_{-t} \omega) \in B(\kappa - t, \theta_{-t} \omega)$.

Proof. Since $z(\theta_t \omega) \in H^4(\mathbb{R}^2)$, then there exists a $k_1' > 0$ such that

$$\int_{|x| \geq k_1'} |z(\theta_t \omega)|^2 dx \leq \epsilon |\tilde{z}(\theta_t \omega)|^2. \quad (59)$$

By Lemma 3, $\exists k_2' > k_3 > 0$, then

$$\int_{|x| \geq k_2'} |v(\kappa, \kappa - t, \theta_{-\kappa} \omega, v_{\kappa-t})|^2 dx \leq \frac{\epsilon}{4}, \quad \forall t \geq T_2(\kappa, \omega, B, \epsilon). \quad (60)$$

By (55) and (56), $\forall t \geq T_2$ and $k' \geq k_4 = \max\{k_1', k_2'\}$, one gets

$$\int_{|x| \geq k'} |u(\kappa, \kappa - t, \theta_{-\kappa} \omega, u_{\kappa-t})|^2 dx \leq \epsilon. \quad (61)$$

□

4. Finiteness of Fractal Dimension

Now, we are devoted to the existence of random attractor on $\mathbb{R}^2$ for the random dynamical system Φ .

Lemma 5. Suppose $a > 6$ and (15) holds. Then, $\mathcal{D}$ -pullback asymptotically compact holds in $L^2(\mathbb{R}^2)$ for continuous cocycle Φ of (1) and (2).

Similar to the method in [30], we only give the sketch of the proof for Lemma 5.

Firstly, the weak convergence

$$\Phi(t_n, \kappa - t_n, \theta_{-t_n} \omega, u_{0,n}) \rightharpoonup \xi \quad (62)$$

can be given in $L^2(\mathbb{R}^2)$.

Secondly, by Lemma 4, there exist enough large t and M satisfying

$$\int_{|x| \geq M} |\Phi(t, \kappa - t, \theta_{-t} \omega, u_0)|^2 dx \leq \epsilon. \quad (63)$$

Denote the set $Q_M = \{x \in \mathbb{R}^2: |x| \leq M\}$. By the estimates of Lemma 2, the embedding $H^2(Q_M) \hookrightarrow L^2(Q_M)$ is compact. It follows that the strong convergence

$$\Phi(t_n, \kappa - t_n, \theta_{-t_n} \omega, u_{0,n}) \longrightarrow \xi \quad (64)$$

holds in $L^2(\mathbb{R}^2)$.

According to [30], the following theorem is easily obtained. The proof is omitted.

Theorem 2. Suppose $a > 6$ and (15) holds. Then, the continuous cocycle Φ corresponding to problems (1) and (2) has a unique $\mathcal{D}$ -pullback attractor $\mathcal{A} = \{\mathcal{A}(\kappa, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\mathbb{R}^2)$.

By Theorem 1, similar to the continuous cocycle Φ defined in (12), the random dynamical system Ψ defined in (13) has a unique $\mathcal{D}$ -pullback attractor, denoted by $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(\kappa, \omega): \kappa \in \mathbb{R}, \omega \in \Omega\}$ in $H = L^2(D) \in \mathcal{D}$. The boundedness of fractal dimension is proved for the random dynamical system Ψ . Because of complexity of proof on unbounded domain, we pay attention to studying the case of bounded domain D . Especially, the space $L^2(D)$ is denoted by H .

Assume that $\{\chi(\kappa, \omega)\}_{\kappa \in \mathbb{R}, \omega \in \Omega}$ of X are a family of bounded closed random subsets, which satisfy the below conditions. For $\forall \omega \in \Omega, \kappa \in \mathbb{R}$, the following holds:

H1: there is tempered random variable R_ω , which is not dependent on κ and satisfies the diameter $\|\chi(\kappa, \omega)\|_X$ of $\chi(\kappa, \omega)$ controlled by R_ω , and $R_{\theta_t \omega} (\forall t \in \mathbb{R})$ is continuous.

H2: for $\forall t > 0$, $\chi(t + \kappa, \theta_t \omega) = \Psi(t, \kappa, \omega) \chi(\kappa, \omega)$.

H3: there are random variables $C_0(\omega) \geq 0$, positive numbers λ, δ, t_0 , and projector $P_m: X \longrightarrow P_m X$ satisfying

$$\|P_m \Psi(t_0, \kappa, \omega) v - P_m \Psi(t_0, \kappa, \omega) u\|_X \leq e^{\int_0^{t_0} C_0(\theta_s \omega) ds} \|v - u\|_X, \quad (65)$$

$$\begin{aligned} & \| (I - P_m) \Psi(t_0, \kappa, \omega) v - (I - P_m) \Psi(t_0, \kappa, \omega) u \|_X \\ & \leq \left(e^{-\lambda t_0} + \delta e^{\int_0^{t_0} C_0(\theta_s \omega) ds} \right) \|v - u\|_X, \end{aligned} \quad (66)$$

for any $u, v \in \chi(\kappa, \omega)$, $\kappa \in \mathbb{R}$, and $\omega \in \Omega$, where λ, δ, t_0, m do not depend on κ, ω .

H4: $\lambda, \delta, t_0, C_0(\omega)$ satisfy the following conditions:

$$\begin{cases} 0 \leq E[C_0(\omega)] < \infty, \\ \lambda \geq 32E[C_0(\omega)], \\ 0 < \delta \leq \frac{1}{32}, \\ 4 \ln 2 \leq \lambda t_0 \leq 8 \ln 2, \end{cases} \quad (67)$$

where “ E ” denotes the expectation of random variables.

Once these conditions (H1)–(H4) are satisfied, the finiteness of fractal dimension can be obtained by further calculation [18, 25]. We know that conditions (H1) and (H2) hold.

Lemma 6. Let $\omega \in \Omega, \kappa \in \mathbb{R}, t \geq 0$. We have

$$\begin{aligned} & \|\varphi_1(t + \kappa, \kappa, \theta_{-\kappa}\omega, \varphi_{1\kappa}(\theta_{-\kappa}\omega)) - \varphi_2(t + \kappa, \kappa, \theta_{-\kappa}\omega, \varphi_{2\kappa}(\theta_{-\kappa}\omega))\| \\ & \leq e^{\int_0^t (1-a)ds} \|\varphi_{1\kappa} - \varphi_{2\kappa}\|, \varphi_{1\kappa}, \varphi_{2\kappa} \in \tilde{\mathcal{A}}. \end{aligned} \quad (68)$$

Proof. Let $\varphi_j(r) = \varphi_j(r, \kappa, \omega, \varphi_{j\kappa}(\omega))$, $y(r) = \varphi_1(r) - \varphi_2(r)$, $j = 1, 2$, when $r \geq \kappa$. By equation (9), we have

$$\begin{cases} \frac{dy}{dt} + \Delta^2 y + 2\Delta y + ay + f(\varphi_1) - f(\varphi_2) = 0, & r \geq \kappa, \\ y(r, \omega, x) = 0, & x \in \partial D, \\ y(\kappa, \omega, x) = y_\kappa(\omega, x), & \kappa \in \mathbb{R}, x \in D, \end{cases} \quad (69)$$

where $f(\zeta) = (\zeta + z)^5$. Multiplying equation (69) with $y(r)$, noticing that

$$\int_D y(f(\varphi_1) - f(\varphi_2))dx \geq 0, \quad (70)$$

we have

$$\frac{d}{dr} \|y(r)\|^2 \leq 2(1-a) \|y(r)\|^2, \quad \forall r \geq \kappa. \quad (71)$$

By the Gronwall inequality, we obtain the estimate (68).

Let $\{e_j\}_{j \in \mathbb{N}}$ be the eigenvectors of operator $-\Delta$ corresponding to the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ with $-\Delta e_j = \lambda_j e_j$ for $j \in \mathbb{N}$. Let H_n be the space spanned by $\{e_j\}_{j=1}^n$ and $P_n: H \rightarrow H_n$ be the orthonormal projector. Let $Q_n: H \rightarrow H_n^\perp$ be the defined by $P_n v = v_n$ for $v \in H$, with $v = v_n + v_n^\perp$, where $v_n \in H_n, v_n^\perp \in H_n^\perp$. Then,

$$\lambda_{n+1} \|Q_n v\|^2 \leq \|v\|_1^2. \quad (72)$$

□

Lemma 7. For $\varphi_{1\kappa}, \varphi_{2\kappa} \in \tilde{\mathcal{A}}(\kappa, \omega)$, there are a random variable $C_0(\omega) \geq 0$ and a projector $P_n: H \rightarrow H_n$ satisfying

$$\begin{aligned} & \|(I - P_n)\Psi(t, \kappa, \theta_{-\kappa}\omega)\varphi_{1\kappa}(\theta_{-\kappa}\omega) - (I - P_n) \\ & \Psi(t, \kappa, \theta_{-\kappa}\omega)\varphi_{2\kappa}(\theta_{-\kappa}\omega)\|, \\ & \leq \left(e^{(4-a-1/4\lambda_{n+1}^2)t} + \delta_n e^{\int_0^t C_0(\theta_s\omega)ds} \right) \|\varphi_{1\kappa} - \varphi_{2\kappa}\|, \end{aligned} \quad (73)$$

and

$$\begin{aligned} & \|P_n \Psi(t, \kappa, \theta_{-\kappa}\omega)\varphi_{1\kappa}(\theta_{-\kappa}\omega) - P_n \Psi(t, \kappa, \theta_{-\kappa}\omega)\varphi_{2\kappa}(\theta_{-\kappa}\omega)\| \\ & \leq e^{\int_0^t C_0(\theta_s\omega)ds} \|\varphi_{1\kappa} - \varphi_{2\kappa}\|, \quad \forall \kappa \in \mathbb{R}, \omega \in \Omega, t \geq 0, \end{aligned} \quad (74)$$

where

$$\delta_n = \frac{1}{\sqrt{\lambda_{n+1}}}. \quad (75)$$

Proof. Multiplying equation (69) with $y_n = Q_n y$ in H , it follows that

$$|(2\Delta y, y_n)| \leq \frac{1}{4} \|\Delta y\|^2 + 4 \|y_n\|^2, \quad (76)$$

$$\|\nabla v\|^2 \leq C \|\Delta v\| \|v\| \leq r_1(\omega) + Cr_8(\omega) = M_0^2(\theta_r \omega).$$

Then, there exists a ϑ between $\varphi_1(r)$ and $\varphi_2(r)$ such that

$$\begin{aligned} (f(\varphi_1) - f(\varphi_2), y_n) & \leq 5 \left(\int_D (\vartheta + z(\theta_r \omega))^{12} dx \right)^{1/3} \\ & \cdot \left(\int_D y^2 dx \right)^{1/2} \left(\int_D y_n^6 dx \right)^{1/6} \\ & \leq C \left(\|\vartheta\|_{H_0^4}^4 + \|z(\theta_r \omega)\|_{L^{12}}^4 \right) \|y\| \|y_n\|_{L^6} \\ & \leq C \left(M_0^4(\theta_r \omega) + \|z(\theta_r \omega)\|_{L^{12}}^4 \right)^2 \|y\|^2 \\ & \quad + \frac{1}{2} \|\Delta y_n\|^2 \end{aligned} \quad (77)$$

and

$$\frac{d}{dt} \|y_n\|^2 + \left(\frac{1}{2} \lambda_{n+1}^2 + 2a - 8 \right) \|y_n\|^2 \leq C \left(M_0^4(\theta_r \omega) + \|z(\theta_r \omega)\|_{L^{12}}^4 \right)^2 \|y\|^2. \quad (78)$$

Applying Gronwall lemma to (78) and combining (68), one gets

$$\begin{aligned}
\|y_n(\kappa+t, \kappa, \theta_{-\kappa}\omega, y_\kappa(\theta_{-\kappa}\omega))\|^2 &\leq \|y_n(\kappa)\|^2 e^{\int_\kappa^{\kappa+t} (8-2a-1/2\lambda_{n+1}^2)ds} + \int_\kappa^{\kappa+t} C \left(M_0^4(\theta_{r-\kappa}\omega) + \|z(\theta_{r-\kappa}\omega)\|_{L^{12}}^4 \right)^2 \|\varphi_1(r) - \varphi_2(r)\|^2 \\
&\quad e^{\int_r^{\kappa+t} (8-2a-1/2\lambda_{n+1}^2)ds} dr \leq \|y_n(\kappa)\|^2 e^{\int_0^t (8-2a-1/2\lambda_{n+1}^2)ds} \\
&\quad + \|\varphi_{1\kappa} - \varphi_{2\kappa}\|^2 \int_\kappa^{\kappa+t} C \left(M_0^4(\theta_{r-\kappa}\omega) + \|z(\theta_{r-\kappa}\omega)\|_{L^{12}}^4 \right)^2 e^{\int_\kappa^r 2(1-a)ds} e^{\int_r^{\kappa+t} (8-2a-1/2\lambda_{n+1}^2)ds} dr \leq \|y_n(\kappa)\|^2 \\
&\quad e^{\int_0^t (8-2a-1/2\lambda_{n+1}^2)ds} + \|\varphi_{1\kappa} - \varphi_{2\kappa}\|^2 e^{\int_0^t (8-2a)ds} \int_0^t C \left(M_0^4(\theta_{r-\kappa}\omega) + \|z(\theta_{r-\kappa}\omega)\|_{L^{12}}^4 \right)^2 e^{-1/2\lambda_{n+1}^2(t-r)} dr.
\end{aligned} \tag{79}$$

Noticing $\sqrt{x} < e^x$ ($x \geq 0$), we have

$$\begin{aligned}
\int_0^t C \left(M_0^4(\theta_r\omega) + \|z(\theta_r\omega)\|_{L^{12}}^4 \right)^2 \cdot e^{-1/2\lambda_{n+1}^2(t-r)} dr &\leq \left(\int_0^t C^2 \cdot \left(M_0^4(\theta_r\omega) + \|z(\theta_r\omega)\|_{L^{12}}^4 \right)^4 dr \right)^{\frac{1}{2}} \\
\left(\int_0^t e^{-\lambda_{n+1}^2(t-r)} dr \right)^{1/2} &\leq \frac{1}{\lambda_{n+1}} \int_0^t C^2 \left(M_0^4(\theta_r\omega) + \|z(\theta_r\omega)\|_{L^{12}}^4 \right)^4 dr.
\end{aligned} \tag{80}$$

By (79),

$$\|y_n(\kappa+t, \kappa, \theta_{-\kappa}\omega, y_\kappa(\theta_{-\kappa}\omega))\|^2 \leq \left(e^{\int_0^t (8-2a-1/2\lambda_{n+1}^2)ds} + \frac{1}{\lambda_{n+1}} e^{\int_0^t (8-2a+C^2(M_0^4(\theta_s\omega) + \|z(\theta_s\omega)\|_{L^{12}}^4)ds)} \right) \|\varphi_{1\kappa} - \varphi_{2\kappa}\|^2 / \tag{81}$$

Thus,

$$\|y_n(\kappa+t, \kappa, \theta_{-\kappa}\omega, y_\kappa(\theta_{-\kappa}\omega))\| \leq \left(e^{\int_0^t (4-a-1/4\lambda_{n+1}^2)ds} + \frac{1}{\sqrt{\lambda_{n+1}}} e^{\int_0^t (4-a+C^2/2(M_0^4(\theta_s\omega) + \|z(\theta_s\omega)\|_{L^{12}}^4)ds)} \right) \|\varphi_{1\kappa} - \varphi_{2\kappa}\|, \tag{82}$$

where $C_0(\omega) = 4 - a + C^2/2(M_0^4(\omega) + \|z(\omega)\|_{L^{12}}^4)$. Then, (73) holds. From (68), it follows that (74) holds. $\square$

Lemma 8. *The expectations of $C_0(\omega)$, $C_0^2(\omega)$ are finite, that is,*

$$0 \leq E[C_0(\omega)], E[C_0^2(\omega)] < \infty. \tag{83}$$

Proof. Notice that $\tilde{z}(\theta_t\omega)$ satisfies $E[|\tilde{z}(\theta_t\omega)|^r] = \Gamma(1+r/2)/\sqrt{\pi}$, for all $r > 0, t \in \mathbb{R}$, and Γ satisfies the equality [25]:

$$\begin{aligned}
\Gamma\left(\frac{1+2k}{2}\right) &= \frac{(2k-1)(2k-3)\cdots 3 \cdot 1}{2^k} \\
\sqrt{\pi} \Gamma\left(\frac{1+(2k+1)}{2}\right) &= k!, \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{84}$$

By $M_0^2(\omega) = r_1(\omega) + C \cdot r_8(\omega)$,

$$\begin{aligned}
C_0(\omega) &= \overline{C} \left(1 + r_1^8(\omega) + r_8^8(\omega) + \|z(\theta_t\omega)\|_{L^{12}}^{16} \right), \\
C_0^2(\omega) &= C_1 + C_2 r_1^{16}(\omega) + C_3 r_8^{16}(\omega) + C_4 \|z(\theta_t\omega)\|_{L^{12}}^{32},
\end{aligned} \tag{85}$$

where $\overline{C}, C_1, C_2, C_3, C_4$ are positive constants.

Since $r_1(\omega), r_8(\omega)$ are tempered, we obtain $E[r_1^8(\omega)] < \infty$, $E[r_8^8(\omega)] < \infty$ and $E[r_1^{16}(\omega)] < \infty$, $E[r_8^{16}(\omega)] < \infty$. By the properties of $\tilde{z}(\theta_t\omega)$, we get

$$\begin{aligned}
E\left[\|z(\theta_t\omega)\|_{L^{12}}^{16}\right] &= \|\phi(x)\|_{L^{12}}^{16} \cdot E\left[|\tilde{z}(\theta_t\omega)|^{16}\right] \\
&= \|\phi(x)\|_{L^{12}}^{16} \frac{15 \cdot 13 \cdot 11 \cdots 3 \cdot 1}{2^8} < \infty, \\
E\left[\|z(\theta_t\omega)\|_{L^{12}}^{32}\right] &= \|\phi(x)\|_{L^{12}}^{32} \cdot E\left[|\tilde{z}(\theta_t\omega)|^{32}\right] \\
&= \|\phi(x)\|_{L^{12}}^{32} \frac{31 \cdot 29 \cdot 27 \cdots 3 \cdot 1}{2^{16}} < \infty.
\end{aligned} \tag{86}$$

Then,

$$\begin{aligned} E[C_0(\omega)] &= \overline{C}(1 + E[r_1^8(\omega)] + E[r_8^8(\omega)] \\ &\quad + E[\|z(\theta_t\omega)\|_{L^{12}}^{16}]) < \infty, \\ E[C_0^2(\omega)] &= C_1 + C_2 E[r_1^{16}(\omega)] + C_3 E[r_8^{16}(\omega)] \\ &\quad + C_4 E[\|z(\theta_t\omega)\|_{L^{12}}^{32}] < \infty. \end{aligned} \quad (87)$$

Now, based on the conclusions of Lemmas 7 and 8, the main result is presented. $\square$

Theorem 3. *The finiteness of fractal dimension for $\tilde{\mathcal{A}}(\kappa, \omega)$ holds, that is,*

$$\dim_f \tilde{\mathcal{A}}(\kappa, \omega) \leq \frac{8n_0 \ln(\sqrt{n_0} \lambda_{n_0+1}^{1/2} + 1)}{\ln 2} < \infty, \kappa \in \mathbb{R}, \omega \in \Omega, \quad (88)$$

where $n_0 = \min\{n: \lambda_{n+1} \geq \max\{(128E[C_0(\omega)] + 16 - 4a)^{1/2}, 1024\}\} < \infty$.

Proof. Comparing (66) and (73), we see that

$$\lambda = \frac{1}{4} \lambda_{n+1}^2 + a - 4 \longrightarrow \infty, \delta = \frac{1}{\sqrt{\lambda_{n+1}}} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (89)$$

By (67) and Lemma 8, $\exists n_0 \in \mathbb{N}$, we deduce

$$\frac{1}{4} \lambda_{n+1}^2 + a - 4 \geq 32E[C_0(\omega)], 0 < \frac{1}{\sqrt{\lambda_{n+1}}} < \frac{1}{32}. \quad (90)$$

Take $t = t_0 > 0$ in (73) and (74) such that the following inequality is satisfied:

$$0 < \frac{16 \ln 2}{\lambda_{n+1}^2 + 4a - 16} \leq t_0 \leq \frac{32 \ln 2}{\lambda_{n+1}^2 + 4a - 16} < \infty. \quad (91)$$

If (H1)–(H4) holds, the following estimate is similar to [25]; here, we omit the details. Then,

$$\begin{aligned} \dim_f \tilde{\mathcal{A}}(\kappa, \omega) &\leq \frac{8n_0 \ln(\sqrt{n_0}/\delta + 1)}{\ln 2} = \frac{8n_0 \ln(\sqrt{n_0} \lambda_{n_0+1}^{1/2} + 1)}{\ln 2} \\ &< \infty, \forall \kappa \in \mathbb{R}, \omega \in \Omega. \end{aligned} \quad (92) \quad \square$$

Data Availability

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation, to any qualified researcher.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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