Research Article

Finite-Time Tracking Control for Nonstrict-Feedback State-Delayed Nonlinear Systems with Full-State Constraints and Unmodeled Dynamics

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Abstract

The problem of finite-time tracking control is discussed for a class of uncertain nonstrict-feedback time-varying state delay nonlinear systems with full-state constraints and unmodeled dynamics. Different from traditional finite-control methods, a C1 smooth finite-time adaptive control framework is introduced by employing a smooth switch between the fractional and cubic form state feedback, so that the desired fast finite-time control performance can be guaranteed. By constructing appropriate Lyapunov-Krasovskii functionals, the uncertain terms produced by time-varying state delays are compensated for and unmodeled dynamics is coped with by introducing a dynamical signal. In order to avoid the inherent problem of “complexity of explosion” in the backstepping-design process, the DSC technology with a novel nonlinear filter is introduced to simplify the structure of the controller. Furthermore, the results show that all the internal error signals are driven to converge into small regions in a finite time, and the full-state constraints are not violated. Simulation results verify the effectiveness of the proposed method.

1. Introduction

During the past few decades, great achievements have been proposed for uncertain nonlinear systems based on adaptive control technique, especially for pure-feedback systems (e.g., see [1–5]) and strict-feedback systems (e.g., see [6–9]) with the lower-triangular structure. Lately, the authors in [10] introduced a more general nonlinear system named nonstrict-feedback nonlinear systems. By employing the variable separation method, the tracking control problem has been well solved. Since then, many control techniques for nonstrict-feedback systems and extensions to other fields were achieved (e.g., see [11–17]).

It is known to all that many practical systems encounter the effect of the constraints, such as the temperature of chemical reactor and physical stoppages. Thus, the research about the systems with state constraints is very meaningful and necessary on account of the existence of state constraints which may undermine the stability of the system. In order to tackle the problem of state constraints, some effective control techniques (e.g., model predictive control (MPC) [18, 19], reference governors (RGs) [20], one-to-one nonlinear mapping (NM) [21–23], and barrier Lyapunov functions (BLFs) [24–28]) have been presented. Due to the fact that MPC and RGs require strong online computing capability to guarantee constraints, this requirement restricts their applications in engineering design. Therefore, one-to-one NM and the BLFs-based methods become the main methods to deal with the constrained nonlinear systems. There exist many significant results which focus on lower-triangular structure nonlinear systems with different constraints (e.g., input constraints [3], output constraints [24], partial-state constraints [25], and full-state constraints [21–23, 26, 27]). In addition, the rate of convergence is also an essential consideration for most practical systems. The works mentioned above only obtain asymptotic or exponential stability with infinite time, which cannot meet the requirement of finite-time control in most practical control
systems. As a consequence, a considerable number of meaningful researches (e.g., see [28–33]) have been proposed on finite-time control for nonlinear systems. However, most of the works are to present $C^0$ finite-time controller by using a backstepping technique together with a nonsmooth fractional feedback design method. In order to achieve a faster convergence rate, the authors in [34] originally proposed a $C^1$ smooth finite-time adaptive NN controller by using a smooth switch between the fractional and cubic form state feedback. Moreover, there are other significant results presented in [35–41], such that two globally stable adaptive controllers were proposed in [35, 36]. To obtain the tracking accuracy, a practical adaptive fuzzy tracking controller for a class of perturbed nonlinear systems with backslash nonlinearity has been designed in [37]. An adaptive fuzzy output-feedback tracking control technique for switched stochastic pure-feedback nonlinear systems has been presented in [38]. The authors in [39] proposed an observed-based adaptive finite-time tracking control technique for a class of nonstrict-feedback nonlinear systems with input saturation. An adaptive finite-time output-feedback controller for switched pure-feedback nonlinear systems with average dwell time has been given in [40]. A decentralized event-triggered controller for interconnected systems with unknown disturbances has been proposed in [41].

Furthermore, due to the fact that unmodeled dynamics can severely degrade the closed-loop system performance, dealing with the effects of unmodeled dynamics is essential for practical nonlinear control systems. Therefore, several results were proposed by employing backstepping or DSC in [4, 21–23, 42–47]. Generally, unmodeled dynamics was disposed by introducing a dynamic signal in [4, 21–23, 42–46] or a Lyapunov function description in [47].

In addition, time delays frequently occur in some practical engineering systems. As stated in [48], their existence can deteriorate the transient performance and even can destroy the stability of the control systems. Thus, the research on nonlinear time-delay systems has become one of the hot topics and some meaningful results have been achieved during the past decades [49–53]. For uncertain nonlinear time-delay systems, the effective controller was developed originally in [50] by combining the backstepping technique with Lyapunov-Krasovskii functionals. Soon afterward, this method was extended to nonlinear strict-feedback time-delay system with unknown control gain functions [51] and uncertain multi-input/multi-output nonlinear systems with time delays [52]. Later, some improved control schemes based on [50] were proposed (e.g., see [35, 53, 54]).

Although many significant research results on adaptive neural network control for uncertain nonstrict-feedback systems have been obtained in [11–17], their considered systems did not include unmodeled dynamics or full-state constraints. In [21–28], the effective controllers have been designed for the lower-triangular structure nonlinear systems with state constraints and unmodeled dynamics, but their considered systems did not include state delay and their control methods may be invalid to nonstrict-feedback systems on account of subsystem function which contains the whole state variables. Furthermore, the above-mentioned control methods only obtain asymptotic or exponential stability with infinite time. To the best knowledge of the authors, finite-time tracking control for a class of uncertain nonstrict-feedback time-varying state-delayed nonlinear systems with full-state constraints and unmodeled dynamics has not been fully discussed in the literature, which is still open and remains unsolved. In this paper, we are committed to solving the problem mentioned above. The main contributions of the paper are summarized as follows:

(i) In contrast to the existing results reported in [21–28, 47] where the control methods have been proposed for nonlinear strict-feedback or pure-feedback systems with state or output constraints and unmodeled dynamics, a generalization of the results is proposed for a class of nonstrict-feedback state delay systems with state constraints and unmodeled dynamics of which the subsystem function contains the whole state variables. To the best of authors’ knowledge, it is the first time to develop an adaptive DSC method for uncertain nonstrict-feedback state delay systems with state constraints and unmodeled dynamics.

(ii) Different from the finite-control methods in [31–33], a $C^1$ smooth finite-time adaptive control framework is introduced by employing a smooth switch between the fractional and cubic form state feedback reported in [34], so that the desired fast finite-time control performance can be guaranteed. Moreover, unmodeled dynamics is coped with by introducing a dynamical signal and the uncertain terms produced by time-varying state delays are compensated for by constructing appropriate Lyapunov-Krasovskii functionals. The results show that all the error signals are driven to converge into small regions in a finite time, and the full-state constraints are never violated.

The remainder of this paper is organized as follows. In Section 2, the problem formulation and preliminaries are presented. Adaptive DSC design and stability analysis are given in Section 3. Simulation results verify the effectiveness of the proposed control approach in Section 4, followed by Section 5, which concludes this paper.

Notation. In this paper, $R$ denotes a set of real numbers, $R^+$ denotes a set of nonnegative real numbers, $R^{m \times n}$ denotes a set of $m \times n$ real matrices, $R^n$ denotes a set of $n$-dimensional real vectors, $\sup(\cdot)$ denotes the least upper bound, $\|\cdot\|$ denotes 2-norm of a vector or matrix, $|\cdot|$ denotes an absolute value of a real number, $\exp(\cdot)$ denotes an exponential function of $\cdot$, and $\log(\cdot)$ denotes the natural logarithm of $\cdot$.

2. Problem Formulation and Preliminaries

2.1. Problem Statement. Consider a class of uncertain nonstrict-feedback state-delayed nonlinear systems with unmodeled dynamics for $i = 1, 2, \ldots, n – 1$ in the following form:
where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$ is the state vector, $\xi \in \mathbb{R}^n$ is the unmodeled dynamics, and $u, y, T_i(t)$ denote the system input, the system output, and the unknown time-varying delays, respectively. $f_i(x), g_i(x)$, and $d_i(x, t)$ are the unknown smooth functions. Let $\bar{x}_i = [x_1, x_2, \ldots, x_i]^T$ and $\delta_i(\xi, x, t)$ be the unknown uncertain disturbances. All the states $x_i$ are required to remain in the sets $\Omega_{x_i} = \{x_i : |x_i| < k_{x_i}\}$, where $k_{x_i}$ are positive constants.

Remark 1. System (1) is called a nonstrict-feedback form in which the system function $f_i(\cdot)$ and its bounding function contain all the state variables [10]. Apparently, strict-feedback and pure-feedback structures are the special cases of system (1). The methods proposed in [21–28, 31–33, 47] cannot be directly applied to system (1) on account of its nonstrict-feedback structure.

The control objective of this paper is to construct an adaptive NN controller $u(t)$ to make sure that the output $y$ follows the desired trajectory $y_d$ in a finite time, while every state $x_i \in \Omega_{x_i}$ is never violated.

2.2. RBFNN Approximation. In this paper, for $i = 1, \ldots, n$, the unknown smooth nonlinear functions $\bar{F}_i(Z_i) : \mathbb{R}^m \rightarrow \mathbb{R}$ will be approximated on a compact set $\Omega_i \subset \mathbb{R}^m$ by the following RBFNN:

$$\bar{F}_i(Z_i) = W_i^T S_i(Z_i) + \epsilon_i(Z_i),$$

where $Z_i, W_i, I$ denote input vectors, weight vectors, and NN node number, respectively. $\epsilon_i(Z_i)$ are the NN inherent approximation errors which are bounded over the compact sets; that is, $\epsilon_i(Z_i) \leq \epsilon_i$, where $\epsilon_i$ are unknown constants and $S_i(Z_i) = [s_1(Z_i), \ldots, s_l(Z_i)]^T : \Omega_i \rightarrow \mathbb{R}^l$ are known smooth vector functions with $s_q(Z_i)$ being chosen as the commonly used Gaussian functions, which have the form

$$s_q(Z_i) = \exp\left[-\frac{(Z_i - \mu_q)^T (Z_i - \mu_q)}{\eta_q^2}\right], \quad q = 1, \ldots, l,$$

where $\mu_q = [\mu_{q1}, \ldots, \mu_{qn}]^T$ is the center vector and $\eta_q$ is the spreads of the Gaussian function. The optimal weight vector $W_i$ is defined as

$$W_i = \arg \min_{\bar{W}_i \in \mathbb{R}^l} \left\{ \sup_{Z_i \in \Omega_i} \left| F(Z_i) - \bar{W}_i^T S(Z_i) \right| \right\},$$

where $\bar{W}_i$ is the estimate of $W_i$.

2.3. Key Definition and Lemmas

Definition 1 (see [21]). The unmodeled dynamics $\xi$ is said to be exponentially input-state-practically stable (exp-IspS), that is, for system $\dot{x} = f(x, t)$, if there exist functions $\bar{a}_1, \bar{a}_2$ of class $K_{\infty}$ and a Lyapunov function $V(\xi)$, such that

$$\bar{a}_1(\|\xi\|) \leq V(\xi) \leq \bar{a}_2(\|\xi\|),$$

and there exist two constants $c > 0, d \geq 0$ and a class $K_{\infty}$ function $\gamma(\cdot)$, such that

$$\frac{\partial V(\xi)}{\partial \xi} q(\xi, x, t) \leq -cV(\xi) + \gamma(|x_1|) + d, \quad \forall t \geq 0,$$

where $c$ and $d$ are known positive constants and $\gamma(\cdot)$ is a known function of class $K_{\infty}$.

Lemma 1 (see [21]). If $V$ is an exp-IspS Lyapunov function for a system $\dot{x} = q(\xi, x, t)$, that is, (5) and (6) hold, then, for any constant $\bar{c} \in (0, c)$, any initial instant $t_0 > 0$, any initial condition $x_0 = \xi(t_0)$, and any continuous function $\gamma$, such that $\gamma(|x_1|) \geq 0$, there exist a finite $T_0 = \max\{0, \log(\sqrt{\bar{a}_1(\|\xi_0\|)/r_0/(c-\bar{c})})\} \geq 0$, a nonnegative function $\bar{D}(t_0, t)$ defined for all $t \geq t_0$, and a signal described by

$$\dot{r} = -\bar{c}r + \gamma(\|x_1\|) + d, \quad r(t_0) = r_0,$$

such that $D(t_0, t) = 0$ for $t \geq t_0 + T_0$ and $V(\xi) \leq r(t) + D(t_0, t)$ with $D(t_0, t) = \max\{0, e^{-\bar{c}(t-t_0)}V(z_0) - e^{-\bar{c}(t-t_0)}r_0\}$.

Lemma 2 (see [11]). Let $S(Z)$ be the basis function vector of an RBFNN and $Z$ be the input vector, where $S(Z) = [s_1(Z), \ldots, s_l(Z)]^T$ and $Z = [z_1, z_2]^T$. For any positive integer $m \leq n$, let $Z_m = [z_1, \ldots, z_m]^T$, and the following inequality holds:

$$\|S(Z_m)\|^2 \leq \|S(Z_m)\|^2.$$

Lemma 3 (see [55]). For any real numbers $\zeta_1 > 0, \zeta_2 > 0$ and $0 < h < 1$, an extended Lyapunov condition of finite-time stability can be given in the form of fast terminal sliding mode as $V(x) + \zeta_1 V'(x) + \zeta_2 V''(x) \leq 0$; then, $V(x)$ is in fast time convergence with a finite settling time $T^* \leq (1/\zeta_1 (1-h)) \log((1/\zeta_1 h) + \zeta_2/\zeta_1)$.

Lemma 4 (see [56]). For $x, y \in \mathbb{R}$, if $0 < h < h_1 = h_2 < 1$, where $h_1, h_2 > 0$ are odd integers, then $x^{h} \leq \zeta^{-1} x^{h_i} + \zeta_{2} (x + y)^{1+h}$, where $\zeta_1 = 1/(1+h)$ and $\zeta_2 = (1/(1+h))(1+(2h/(1+h))(2^{-h})(h))$. Where $\zeta_2 = \zeta^{-1} x^{h_i} + \zeta_{2} (x + y)^{1+h}$. Where $\zeta_2 = \zeta^{-1} x^{h_i} + \zeta_{2} (x + y)^{1+h}$.
Lemma 5 (see [34]). Consider the dynamic system
\[ \dot{\phi}(t) = -l_1\phi(t) - l_2\phi_h(t) + \varphi(t), \] (9)
where \( \phi(t) \in R, 0 < h = (h_1 / b_1) < 1 \) (\( h_1 \) and \( h_2 \) are positive odd integers), \( l_1 \) and \( l_2 \) are positive constants, and \( \varphi(t) \) is a positive function. Then, for any given bounded initial condition \( \phi(0) \geq 0 \), one has that \( \phi(t) \geq 0, \forall t \geq 0 \).

Lemma 6 (see [57]). For \( x_i \in R, i = 1, 2, \ldots, n, \) and \( 0 < h \leq 1 \),
\[ \sum_{i=1}^{n} |x_i|^h \leq \sum_{i=1}^{n} |x_i|^h \leq n^{-1} \sum_{i=1}^{n} |x_i|^h. \]

To obtain the control objective, the following assumptions are needed.

Assumption 1. The unmodelled dynamics \( \xi \) is exp-ISpS.

Assumption 2. There exist unknown nonnegative continuous functions \( \varphi_{l_1} \) and nondecreasing continuous functions \( \varphi_{l_2} \) such that
\[ |\delta_i(\xi, x, t)| \leq \varphi_{l_1}(\|\xi\|) + \varphi_{l_2}(\|\xi\|), \quad \forall (\xi, x, t) \in R^{n_0} \times R^n \times R^r, \tag{10} \]
where \( \varphi_{l_2}(0) = 0, \quad i = 1, \ldots, n. \)

Remark 2. From Definition 1 and Assumption 1, we have \( \|\xi\| \leq \bar{a}_1^{-1}(V(\xi)) \). According to Lemma 1, there exists a positive constant \( D_0 \) such that \( \|\xi\| \leq \bar{a}_1^{-1}(r + D_0), \forall t \geq 0 \). This inequality will be used to cope with the uncertain terms in the following controller design.

Assumption 3. The sign of \( g_i(\hat{x}_i) \) is known, and there exist some unknown positive constants \( a_i \) and \( b_i \) such that \( 0 < b_i \leq |g_i(\hat{x}_i)| \leq a_i \). Without loss of generality, this paper assumes that \( g_i(\hat{x}_i) > 0 \).

Assumption 4. The reference trajectory \( y_r(t) \) and its derivatives about time \( y_r \) and \( y_r \) are in a bounded region \( \Omega_{y_r} \) and there exists a known constant \( A_0 \), such that \( |y_r| \leq A_0 < k_{c_1} \).

Assumption 5. The unknown continuous functions \( d_i(\hat{x}_i(t - T_i(t))) \) satisfy the following inequality:
\[ d_i(\hat{x}_i(t - T_i(t))) \leq \sum_{j=1}^{i} \rho_{ij}(x_j(t - T_j(t))), \tag{11} \]
and the time-varying state delays \( T_i(t) \) satisfy the inequalities \( 0 \leq T_i(t) \leq T_{\text{max}} \) and \( T_i(t) \leq T_{\text{max}} < 1 \), where \( \rho_{ij}(x_j(t - T_j(t))) \) are unknown positive smooth functions and \( T_{\text{max}} \) and \( T_{\text{max}} \) are unknown constants.

3. Adaptive DSC Design and Stability Analysis

3.1. Adaptive DSC Design. Similar to traditional backstepping, the backstepping-design procedure with \( n \) steps is developed to construct the adaptive neural controller in this part.

By using the backstepping technique, the proposed adaptive DSC scheme contains \( n \) steps as follows.

Step 1. Define the first surface error \( z_1 = x_1 - y_r \), the time derivative of \( z_1 \) is defined as
\[ \dot{z}_1 = d_1(x_1) + g_1(\hat{x}_1)x_2 + \delta_1(\xi, x, t) + d_1(\hat{x}_1(t - T_1(t))) - y_r. \tag{12} \]

The virtual control law \( a_1 \) and the update law for \( \hat{a}_1 \) are designed as
\[ a_1 = -c_1 z_1 - \frac{z_1}{k_{b_1} - z_1} \xi \hat{a}_1 S_1(\hat{x}_1) S_1(\hat{x}_1) - k_1 \beta_1(z_1), \tag{13} \]
\[ \hat{a}_1 = \rho_1 \left( -\sigma_{11} \hat{a}_1 - \sigma_{12} \hat{a}_1^h + \frac{z_1^2 S_1^T(\hat{x}_1) S_1(\hat{x}_1)}{2l_1(k_{b_1} - z_1^2)} \right), \tag{14} \]
where \( c_1, \sigma_{11}, k_{b_1}, k_1, \rho_1, \sigma_{12}, l_1 \) are positive design parameters, \( \hat{a}_1 \) is an estimate of \( a_1 \), \( \hat{a}_1 = \hat{a}_1 - b_1 \hat{a}_1 \), \( \hat{a}_1 = \|W_1\|^2, b_1 \) is defined in Assumption 3, \( \beta_1(z_1) \) is defined as
\[ \beta_1(z_1) = \begin{cases} \frac{z_1^h (k_{b_1} - z_1^2)^{(1 - h)/2}}{t_{11} z_1 + t_{12} z_1^3}, & \text{if } |z_1| \geq r_1, \\ 1, & \text{if } |z_1| < r_1, \end{cases} \tag{15} \]
where \( 0 < h = (h_1 / b_1) < 1, h_1 \) and \( h_2 \) are the positive odd integers, \( t_{11} = r_1^{1-h} (k_{b_1} - r_1^2)^{(1-(h/2))} - t_{12} r_1^2 + t_{12} = (1/2) r_1^3 \)
\( (h - 1) r_1^2 [(k_{b_1} - r_1^2)^{(1-(h/2))} + r_1^2 (k_{b_1} - r_1^2)^{(1+(h/2))}], \) and \( r_1 < k_{b_1} \) is a small positive constant.

Consider the BLF candidate \( V_{z_1} \) as
\[ V_{z_1} = \frac{1}{2} \log \frac{k_{b_1}^2}{k_{b_1}^2 - z_1^2} + \frac{1}{2b_1 \rho_1} \hat{a}_1. \tag{16} \]

Obviously, \( V_{z_1} \) is positive definite and continuously differentiable. Based on Assumptions 2 and 5 and Young’s inequality, we obtain the time derivative of \( V_{z_1} \) as follows:
\[ V_{\zeta} = \frac{z_1}{k_{b_{1}} - z_1^2} \left[ f_1(x) + g_1(\tilde{x}_1)x_2 + \delta_1(\tilde{x}_1, x, t) + d_1(\tilde{x}_1(t - T_1(t))) - \frac{1}{\rho_1} \tilde{\phi} \tilde{\omega}_1, \right] \]

\[ \leq \frac{z_1}{k_{b_{1}} - z_1^2} \left[ f_1(x) + g_1(\tilde{x}_1)x_2 + \frac{|z_1|}{k_{b_{1}} - z_1^2} \left[ \beta_{11}(\|\tilde{x}_1\|) + \beta_{12}(\tilde{\alpha}_1^{-1}(r + D_{b})) \right] + \rho_{11}(x_1(t - T_1(t))) - \frac{1}{\rho_1} \tilde{\phi} \tilde{\omega}_1, \right] \]

\[ \leq \frac{z_1}{k_{b_{1}} - z_1^2} \left[ f_1(x) + g_1(\tilde{x}_1)x_2 + \frac{z_1^2}{z_1^2} \left[ \beta_{11}(\|\tilde{x}_1\|) + \beta_{12}(\tilde{\alpha}_1^{-1}(r + D_{b})) \right] \right] + \frac{\rho_{11}(x_1(t - T_1(t)))}{\rho_1} + \frac{1}{4} \]

\[ + \frac{z_1^2}{2(k_{b_{1}} - z_1^2)} \left[ \tilde{\phi} \tilde{\omega}_1, \right] \]

\[ \leq \frac{z_1}{k_{b_{1}} - z_1^2} \left[ F_1(Z_1) + g_1(Z_1) + \frac{1}{4} + \frac{\rho_{11}(x_1(t - T_1(t)))}{\rho_1} - \frac{1}{\rho_1} \tilde{\phi} \tilde{\omega}_1, \right] \]

\[ = \frac{z_1}{k_{b_{1}} - z_1^2} \left[ F_1(Z_1) + g_1(Z_1)(z_2 + y_2 + a_1) \right] + \frac{1}{4} + \frac{\rho_{11}(x_1(t - T_1(t)))}{\rho_1} - \frac{1}{\rho_1} \tilde{\phi} \tilde{\omega}_1, \]

(17)

where

\[ F_1(Z_1) = f_1(x) + \frac{z_1}{k_{b_{1}} - z_1^2} \left[ \beta_{11}(\|\tilde{x}_1\|) + \beta_{12}(\tilde{\alpha}_1^{-1}(r + D_{b})) \right] \]

\[ + \frac{z_1^2}{2(k_{b_{1}} - z_1^2)} - \dot{y}_r \]

(18)

Note that \( \tilde{F}_1(Z_1) \) is an unknown continuous function and RBFNN can be used to approximate it. Hence, from (2), the following equation holds:

\[ \tilde{F}_1(Z_1) = W_1^T S_1(Z_1) + \epsilon_1(Z_1), \]

(19)

where \( W_1^T S_1(Z_1) \) is an NN, \( |\epsilon_1(Z_1)| \leq \epsilon_1, Z_1 = [\tilde{x}_n, z_1, r, \dot{y}_r]^T \), and \( \epsilon_1 > 0 \) is any given.

By using Young's inequality and Lemma 2, one has:

\[ \frac{z_1}{k_{b_{1}} - z_1^2} \tilde{F}_1(Z_1) = \frac{z_1}{k_{b_{1}} - z_1^2} \left[ W_1^T S_1(Z_1) + \epsilon_1(Z_1) \right], \]

\[ \leq \frac{|z_1|}{k_{b_{1}} - z_1^2} \|W_1\| S_1(Z_1) + \frac{\epsilon_1(Z_1) z_1}{k_{b_{1}} - z_1^2}, \]

\[ \leq \frac{|z_1|}{k_{b_{1}} - z_1^2} \|W_1\| S_1(Z_1) + \frac{\epsilon_1(Z_1) z_1}{k_{b_{1}} - z_1^2}, \]

\[ \leq \frac{1}{2} \frac{z_1^2}{(k_{b_{1}} - z_1^2)} \|W_1\|^2 \|S_1(Z_1)\|^2 + \frac{l_1}{2}, \]

(20)
where $\Xi_1 = [x_1, z_1, r, y_1]^T$.

Substituting (13), (14), and (20) into (17), we can obtain

\[ V_{z_1} \leq \frac{g_1(\bar{x}_1)z_1z_2}{k_{b_1} - z_1^2} + \frac{g_1(\bar{x}_1)z_1y_2}{k_{b_1} - z_1^2} - \frac{c_1g_1(\bar{x}_1)z_1^2}{k_{b_1} - z_1^2} - \frac{\mu_1g_1(\bar{x}_1)z_1^2}{(k_{b_1} - z_1^2)^2} - \frac{g_1(\bar{x}_1)z_1^2||S_1(\Xi_1)||^2}{2l_1(k_{b_1}^2 - z_1^2)^2} + \frac{l_1}{2} \]

\[ - \frac{k_1g_1(\bar{x}_1)z_1\beta_1(z_1)}{k_{b_1} - z_1^2} + \frac{z_1^2||S_1(\Xi_1)||^2}{2l_1(k_{b_1}^2 - z_1^2)^2} + \frac{1}{4} + \frac{\epsilon_1z_1}{k_{b_1} - z_1^2} + \frac{1}{2}p_{11}^2(x_1(t - T_1(t))) \]

\[ - \frac{1}{\rho_1} \left[ \rho_1 \left( -\sigma_{11}\bar{\omega}_1 - \sigma_{12}\bar{\omega}_1^h + \frac{z_1^2S_1^T(\Xi_1)S_1(\Xi_1)}{2l_1(k_{b_1}^2 - z_1^2)^2} \right) + \sigma_{11}\bar{\omega}_1 + \sigma_{12}\bar{\omega}_1^h \right] \leq \left( \begin{array}{c}
\leq \frac{g_1(\bar{x}_1)z_1z_2}{k_{b_1} - z_1^2} + \frac{g_1(\bar{x}_1)z_1y_2}{k_{b_1} - z_1^2} - \frac{b_1c_1z_1^2}{k_{b_1} - z_1^2} - \frac{\mu_1b_1z_1^2}{(k_{b_1} - z_1^2)^2} - \frac{v_1^2||S_1(\Xi_1)||^2 - z_1^2||S_1(\Xi_1)||^2}{2l_1(k_{b_1}^2 - z_1^2)} + \frac{l_1}{2} \right) \]

\[ + \frac{l_1}{2} \frac{k_1g_1(\bar{x}_1)z_1\beta_1(z_1)}{k_{b_1} - z_1^2} + \frac{v_1^2||S_1(\Xi_1)||^2}{2l_1(k_{b_1}^2 - z_1^2)^2} + \frac{1}{4} + \frac{\epsilon_1z_1}{k_{b_1} - z_1^2} + \frac{1}{2}p_{11}^2(x_1(t - T_1(t))) + \sigma_{11}\bar{\omega}_1 + \sigma_{12}\bar{\omega}_1^h. \]

By utilizing Young’s inequality, the following inequalities can be obtained:

\[ \frac{g_1(\bar{x}_1)z_1y_2}{k_{b_1} - z_1^2} \leq \frac{\mu_1b_1z_1^2}{2(k_{b_1} - z_1^2)} \leq \frac{g_1^2(\bar{x}_1)y_2^2}{2\mu_1b_1} \leq \frac{g_1^2(\bar{x}_1)y_2^2}{2\mu_1b_1} \]

\[ \frac{\epsilon_1z_1}{k_{b_1} - z_1^2} - \frac{\mu_1b_1z_1^2}{2(k_{b_1} - z_1^2)} \leq \frac{\epsilon_1^2}{2\mu_1b_1} \]

Therefore, we have

\[ V_{z_1} \leq \frac{g_1(\bar{x}_1)z_1z_2}{k_{b_1} - z_1^2} + \frac{a_1^2y_2^2}{2\mu_1b_1} + \frac{\epsilon_1^2}{2\mu_1b_1} - \frac{b_1c_1z_1^2}{k_{b_1} - z_1^2} + \frac{l_1}{2} \]

\[ - \frac{k_1g_1(\bar{x}_1)z_1\beta_1(z_1)}{k_{b_1} - z_1^2} + \frac{1}{4} + \frac{1}{2}p_{11}^2(x_1(t - T_1(t))) + \sigma_{11}\bar{\omega}_1 + \sigma_{12}\bar{\omega}_1^h, \]

According to the inequality $2b_1\bar{\omega}_1^h \leq \bar{\omega}_1^2 - \bar{\omega}_1^4$ and Lemma 4, one as

\[ V_{z_1} \leq \frac{g_1(\bar{x}_1)z_1z_2}{k_{b_1} - z_1^2} + \frac{a_1^2y_2^2}{2\mu_1b_1} + \frac{\epsilon_1^2}{2\mu_1b_1} - \frac{b_1c_1z_1^2}{k_{b_1} - z_1^2} + \frac{l_1}{2} \]

\[ - \frac{k_1g_1(\bar{x}_1)z_1\beta_1(z_1)}{k_{b_1} - z_1^2} + \frac{1}{4} + \frac{1}{2}p_{11}^2(x_1(t - T_1(t))) + \sigma_{11}\bar{\omega}_1 + \sigma_{12}\bar{\omega}_1^h, \]

\[ - \frac{\sigma_{11}\zeta_1\bar{\omega}_1^h}{b_1^2} + \frac{\sigma_{12}\zeta_2}{b_1^2}\bar{\omega}_1^h, \]

where $\zeta_1$ and $\zeta_2$ are defined in Lemma 4.

To deal with the time delay in equation (24), define the Lyapunov-Krasovskii functional as follows:

\[ V_{U_1} = \frac{e^{-\gamma(t - T_{max})}}{2(1 - T_{max})} \int_{t - T_1(t)}^t e^{\gamma s}P_1^2(x_1(s))ds, \]

where $\gamma > 0$ is a positive constant. Using Assumption 5, we obtain that the derivative of $V_{U_1}$ is
\[ \dot{V}_{U_1} = \frac{\varepsilon^{-\gamma(t-T_{\text{max}})}}{2(1-T_{\text{max}})} \left[ e^{\frac{\gamma}{2}} \rho_{11}^2 (x_1(t)) - e^{\frac{\gamma(t-T_{\text{max}})}}{\rho_{11}^2 (x_1(t))} \right] \]

\[ + (x_1(t) - T_{\text{max}})) (1 - T_{\text{max}})) - y^2 V_{U_1}, \]

\[ \leq \frac{e^{\gamma(t-T_{\text{max}})}}{2(1-T_{\text{max}})^2} \rho_{11}^2 (x_1(t)) \left( 1 - \frac{1}{\rho_{11}^2 (x_1(t))} \right) \right) - y^2 V_{U_1}. \]

Remark 4. From (29), it can be seen that the proposed filter involves both the linear and fractional terms. In particular, when \( t_i = 0 \) or \( t_{i+1} = 0 \), filter (29) degrades into the fractional filter used in [58] and the linear filter as widely used in the literature [21–23, respectively. \( y_i \) is the output of the following first-order filter:

\[ \ddot{y}_i = -\tau_1y_i - \tau_2y_i^h, \]

where \( \tau_1 \) and \( \tau_2 \) are positive design parameters and \( h \) is defined in (15).

Remark 3. From (29), it can be seen that the proposed filter involves both the linear and fractional terms. In particular, when \( t_i = 0 \) or \( t_{i+1} = 0 \), filter (29) degrades into the fractional filter used in [58] and the linear filter as widely used in the literature [21–23, respectively. It is the key to ensure the fast finite-time stability of the closed-loop system, which will be detailed in the following analysis.

Step 2. (i = 2, 3, …, n - 1) Define the \( i \)th surface error \( z_i = x_i - u_i \); the time derivative of \( z_i \) is defined as

\[ \dot{z}_i = f_i(x) + \sum_{j=1}^{i} \delta_j (\xi, x, t) + d_i(x_i(t) - T_{\text{max}})) - \tilde{u}_i, \]

\[ = f_i(x) + d_i(x_i(t) - T_{\text{max}})) - \tilde{u}_i. \]

The virtual control law \( \alpha_i \) and the update law \( \tilde{\alpha}_i \) are designed as

\[ \alpha_i = -c_i z_i - \mu_i \tilde{z}_i \left( \frac{\tilde{z}_i}{k_{bi} - z_i^2} = \frac{\tilde{z}_i}{k_{bi} - z_i^2} \right) - k_i \beta_i (z_i), \]

\[ \tilde{\alpha}_i = \rho_i \left( -\sigma_i \tilde{\alpha}_i - \sigma_i \tilde{\alpha}_i^2 \left( \frac{\tilde{z}_i^2}{k_{bi} - z_i^2} \right) \right), \]

where \( c_i, \mu_i, k_{bi}, k_{hi}, \sigma_i, \sigma_i, \mu_i, \) and \( z_i \) are positive design parameters, \( \tilde{\alpha}_i \) is an estimate of \( \tilde{\alpha}_i \), \( \tilde{\alpha}_i = \tilde{\alpha}_i - u_i \), \( \alpha_i = \|\tilde{W}_i\| \), \( \beta_i (z_i) \) is defined as

\[ \beta_i (z_i) = \frac{\hat{z}_i}{k_{hi} - z_i^2} (1-h^2), \]

\[ \text{satisfies} \]

\[ \text{where} \]

\[ z_i \leq \frac{z_i}{k_{hi} - z_i^2} \left( \tilde{F}_i (z_i) + g_i (\tilde{x}_i) (z_i+i+y_i+i+\alpha_i) \right) \]

\[ + \frac{1}{4} \beta_i \tilde{\alpha}_i \]

\[ + \frac{1}{2} \sum_{j=1}^{i} \rho_i (x_j(t - T_{\text{max}})) \left( - \frac{g_{i-1}(z_{i-1})}{k_{b(i-1)} - z_{i-1}^2} - \frac{\dot{z}_{i-1}}{k_{b(i-1)} - z_{i-1}^2} \right), \]

where

\[ \tilde{F}_i (z_i) = f_i(x) + \frac{z_i}{k_{ hi - z_i^2}} \left( \tilde{\alpha}_i (\tilde{z}_i) + \tilde{\alpha}_i \right), \]

\[ - \frac{g_{i-1}(z_{i-1})}{k_{b(i-1)} - z_{i-1}^2} - \frac{\dot{z}_{i-1}}{2k_{b(i-1)} - z_{i-1}^2}. \]
Note that $F_i(Z_i)$ is an unknown continuous function and RBFNN can be used to approximate it. Hence, from (2), the following equation holds:

$$F_i(Z_i) = W_i^T S_i(Z_i) + \varepsilon_i(Z_i),$$

(38)

$$Fi = \frac{z_i}{k_{bi}^2 - z_i^2} F_i(Z_i) = \frac{z_i}{k_{bi}^2 - z_i^2} \left[ W_i^T S_i(Z_i) + \varepsilon_i(Z_i) \right],$$

$$\leq \frac{|z_i|}{k_{bi}^2 - z_i^2} \| W_i \| \| S_i(Z_i) \| + \frac{\varepsilon_i(Z_i) z_i}{k_{bi}^2 - z_i^2},$$

$$\leq \frac{|z_i|}{k_{bi}^2 - z_i^2} \| W_i \| \| S_i(\Xi) \| + \frac{\varepsilon_i(Z_i) z_i}{k_{bi}^2 - z_i^2},$$

$$\leq \frac{1}{2 l_i} \left( \frac{z_i^2}{(k_{bi}^2 - z_i^2)^2} \right) \| W_i \|^2 \| S_i(\Xi) \|^2 + \frac{l_i}{4} + \frac{\varepsilon_i(Z_i) z_i}{k_{bi}^2 - z_i^2},$$

(39)

Substituting (32), (33), and (39) into (36), we can obtain

$$\dot{V}_{zi} \leq g_i(\bar{x}_i) z_i z_{i+1} + g_i(\bar{x}_i) z_i y_{i+1} - \beta_i(\bar{x}_i) - \beta_i(\bar{x}_i) z_i + \frac{\mu_i b_i z_i^2}{2 l_i (k_{bi}^2 - z_i^2)^2} - \frac{z_i^2 \tilde{\omega}_i S_i(\Xi)}{2 l_i (k_{bi}^2 - z_i^2)^2} + \frac{l_i}{2} + \frac{1}{4}$$

$$- \kappa_i g_i(\bar{x}_i) z_i y_{i+1} + \frac{z_i^2 \tilde{\omega}_i S_i(\Xi)}{2 l_i (k_{bi}^2 - z_i^2)^2} + \frac{l_i}{4} - \kappa_i g_i(\bar{x}_i) z_i y_{i+1} + \frac{z_i^2 \tilde{\omega}_i S_i(\Xi)}{2 l_i (k_{bi}^2 - z_i^2)^2} + \frac{l_i}{2} + \frac{1}{4}$$

$$+ \sum_{j=1}^{i} \beta_j(x_j(t - T_j(t))),$$

(40)
By utilizing Young’s inequality, the following inequalities can be obtained:

\[
\frac{g_i(\bar{x}_i)z_iy_{i+1}}{k_{bi}^2 - z_i^2} - \frac{\mu_i b_i z_i^2}{2(k_{bi}^2 - z_i^2)} \leq \frac{g_i^2(\bar{x}_i)y_{i+1}^2}{2}\leq \frac{2}{2\mu_i b_i}.
\]

(41)

Therefore, we have

\[
\mathcal{V}_{z_i} \leq \frac{g_i(\bar{x}_i)z_i^2e_{i+1}}{k_{bi}^2 - z_i^2} + \frac{a_i^2y_{i+1}^2}{2\mu_i b_i} + \frac{\epsilon_i^2}{2\mu_i b_i} - \frac{b_i c_i^2}{k_{bi}^2 - z_i^2} - \frac{\kappa_i g_i(\bar{x}_i)z_i^2\beta_i(z_i)}{k_{bi}^2 - z_i^2} + \frac{l_i}{2} + \frac{1}{4} + \sigma_{i1}\bar{\omega}_i \bar{\omega}_i + \sigma_{i2}\bar{\omega}_i \bar{\omega}_i^h
\]

\[
\cdot \sum_{j=1}^{i} \frac{\rho_{ij}(\bar{x}_i(t - T_j(t)))}{2b_j} + \frac{1}{\epsilon_i} \frac{1}{2} + \frac{\epsilon_i^2}{2\mu_i b_i} - \frac{b_i c_i z_i^2}{k_{bi}^2 - z_i^2} - \frac{g_i(\bar{x}_i)z_i^2e_{i-1}}{k_{bi}^2 - z_i^2}
\]

(42)

According to the inequality \(2b_i \bar{\omega}_i \bar{\omega}_i \leq \bar{\omega}_i^2 - \bar{\omega}_i^2\) and Lemma 4, one has

\[
\mathcal{V}_{z_i} \leq \frac{g_i(\bar{x}_i)z_i^2e_{i+1}}{k_{bi}^2 - z_i^2} + \frac{a_i^2y_{i+1}^2}{2\mu_i b_i} + \frac{\epsilon_i^2}{2\mu_i b_i} - \frac{b_i c_i^2}{k_{bi}^2 - z_i^2} - \frac{\kappa_i g_i(\bar{x}_i)z_i^2\beta_i(z_i)}{k_{bi}^2 - z_i^2} + \frac{l_i}{2} + \frac{1}{4} + \sigma_{i1}\bar{\omega}_i \bar{\omega}_i + \sigma_{i2}\bar{\omega}_i \bar{\omega}_i^h
\]

\[
\cdot \sum_{j=1}^{i} \frac{\rho_{ij}(\bar{x}_i(t - T_j(t)))}{2b_j} + \frac{1}{\epsilon_i} \frac{1}{2} + \frac{\epsilon_i^2}{2\mu_i b_i} - \frac{b_i c_i z_i^2}{k_{bi}^2 - z_i^2} - \frac{g_i(\bar{x}_i)z_i^2e_{i-1}}{k_{bi}^2 - z_i^2}
\]

(43)

where \(\zeta_i\) and \(\zeta_2\) are defined in Lemma 4.

To handle the time delay, define the Lyapunov-Krasovskii functional as follows:

\[
\mathcal{V}_{U_i} = \frac{e^{-\gamma(1 - T_{\max})}}{2} \sum_{j=1}^{i} \int_{t-T_j(t)}^{t} e^{\gamma s} \mathcal{P}_{ij}(x_j(s)) ds
\]

(44)

where \(\gamma > 0\) is a positive constant. By using Assumption 5, we obtain that the derivative of \(\mathcal{V}_{U_i}\) is

\[
\dot{\mathcal{V}}_{U_i} = e^{-\gamma(1 - T_{\max})} \sum_{j=1}^{i} \int_{t-T_j(t)}^{t} e^{\gamma s} \mathcal{P}_{ij}(x_j(s)) ds
\]

(45)

From equations (43) and (45), we have

\[
\dot{\mathcal{V}}_{z_i} + \dot{\mathcal{V}}_{U_i} \leq \frac{g_i(\bar{x}_i)z_i^2e_{i+1}}{k_{bi}^2 - z_i^2} + \frac{a_i^2y_{i+1}^2}{2\mu_i b_i} + \frac{\epsilon_i^2}{2\mu_i b_i} - \frac{b_i c_i^2}{k_{bi}^2 - z_i^2} - \frac{\kappa_i g_i(\bar{x}_i)z_i^2\beta_i(z_i)}{k_{bi}^2 - z_i^2} + \frac{l_i}{2} + \frac{1}{4} + \sigma_{i1}\bar{\omega}_i \bar{\omega}_i + \sigma_{i2}\bar{\omega}_i \bar{\omega}_i^h
\]

\[
- \frac{\sigma_{i1}^2\xi_{i-1}}{2b_i} - \frac{\sigma_{i2}^2\xi_{i-1}^h}{b_i} - \frac{g_i(\bar{x}_i)z_i^2e_{i-1}}{k_{bi}^2 - z_i^2} - \frac{\kappa_i g_i(\bar{x}_i)z_i^2\beta_i(z_i)}{k_{bi}^2 - z_i^2} - \frac{\kappa_i g_i(\bar{x}_i)z_i^2\beta_i(z_i)}{k_{bi}^2 - z_i^2} - \frac{\kappa_i g_i(\bar{x}_i)z_i^2\beta_i(z_i)}{k_{bi}^2 - z_i^2} - \frac{\kappa_i g_i(\bar{x}_i)z_i^2\beta_i(z_i)}{k_{bi}^2 - z_i^2}
\]

(46)

where \(\Psi = \sum_{j=1}^{i} (\varepsilon^2(1 - \bar{T}_{\max}^{2})) t_j x_j(t)\).

Similar to the analysis in Remark 4, there exists a continuous function \(\lambda_{i+1}(\bar{x}_{i+1}, \bar{\omega}_{i}, y_{i+1}, y_j, y_j, \cdots, y_{i+1}, y_j, y_j)\) which satisfies

\[
\dot{y}_{i+1} - \lambda_{i+1}(\bar{x}_{i+1}, \bar{\omega}_{i}, y_{i+1}, y_j, \cdots, y_{i+1}, y_j, y_j) = \bar{\omega}_{i+1} \dot{y}_{i+1} + \lambda_{i+1}(\bar{x}_{i+1}, \bar{\omega}_{i}, y_{i+1}, y_j, \cdots, y_{i+1}, y_j, y_j).
\]

(47)

Step 3. Define the nth surface error \(z_n = x_n - w_n\); the time derivative of \(z_n\) is defined as

\[
z_n = f_n(x) + g_n(\bar{x}_n)u + \delta_n(\bar{\omega}_n, y_{i+1}, y_j, \cdots, y_{i+1}, y_j, y_j) + d_n(\bar{x}_n(t - T_n(t))) - \dot{\bar{\omega}}_n.
\]

(48)

The actual control law \(u\) and the update law \(\bar{\omega}_n\) are designed as

\[
u = -c_n z_n - \mu_n \frac{z_n}{2k_{bn}^2 - z_n^2} - \frac{z_n \bar{\omega}_n S_n^T(\bar{\omega}_n) S_n(\bar{\omega}_n)}{2k_{bn}^2 - z_n^2} - \kappa_n \beta_n(z_n).
\]

(49)

\[
\bar{\omega}_n = \rho_n \left( -c_n \bar{\omega}_n - c_n z_n^2 + z_n \bar{\omega}_n S_n^T(\bar{\omega}_n) S_n(\bar{\omega}_n) \right) + \frac{2k_{bn}^2 - z_n^2}{2k_{bn}^2 - z_n^2}.
\]

(50)

where \(c_n, k_{bn}, \rho_n, c_n, \beta_n, l_n\) are positive design parameters, \(\bar{\omega}_n\) is an estimate of \(\omega_n\), \(\bar{\omega}_n = \omega_n - b_n \bar{\omega}_n\), \(\omega_n = ||W||^2, \beta_n(z_n)\) is defined as

\[
\beta_n(z_n) = \left[ \begin{array}{c} \frac{z_n^{2} (2k_{bn}^2 - z_n^2)^{(1-n)/2}}{2n} \frac{\bar{\omega}_n}{2} \end{array} \right] \text{, if } |z_n| \geq \tau_n,
\]

\[
\beta_n(z_n) = \left[ \begin{array}{c} \frac{z_n^{2} (2k_{bn}^2 - z_n^2)^{(1-n)/2}}{2n} \frac{\bar{\omega}_n}{2} \end{array} \right] \text{, if } |z_n| < \tau_n.
\]

(51)
where $h$ is defined in (15), $t_{n} = R_{h}^{-1} (k_{n}^{2} - r_{n}^{2})^{1-h/2} - t_{n2}^{2} H_{h}^{2} h_{n} = (1/2 t_{n}^{2}) (h-1) t_{n}^{2} (k_{n}^{2} - r_{n}^{2})^{1-h/2} + t_{n}^{2} (k_{n}^{2} - r_{n}^{2})^{-(1+h/2)}$, and $t_{n} < k_{h}^{(n)}$ is a small positive constant.

Consider the BLF candidate $V_{n}$ as

$$V_{n} = 1/2 \log k_{b_{n}}^{2} - z_{n}^{2} + 1/2 b_{n}^{2} \rho_{n}^{2},$$

(52)

Similar to (17) and (36), we can obtain the time derivative of $V_{n}$ as follows:

$$\dot{V}_{n} \leq z_{n}^{2} \left[ \tilde{F}_{n}(Z_{n}) + g_{n}(\tilde{x}_{n}) u_{n} \right] + \frac{1}{2} \sum_{j=1}^{n} \rho_{n}^{2}$$

$$\cdot \left( x_{j}(t - T_{j}(t)) - \frac{1}{\rho_{n}} \rho_{n} \right) n_{j} x_{j} t$$

$$\cdot \left( \frac{g_{n-1}(\tilde{x}_{n})}{k_{b_{n}}^{(n-1)} - z_{n}^{2}} - \frac{n}{2} \right)$$

(53)

where

$$\tilde{F}_{n}(Z_{n}) = f_{n}(x) + z_{n} \left[ \frac{1}{k_{b_{n}}^{2} - z_{n}^{2}} \left( v_{n} \left( \| x_{n} \| \right) + v_{n} \left( a_{n}^{1} (r + D_{b}) \right) \right) \right]$$

$$+ \frac{g_{n-1}(\tilde{x}_{n})}{k_{b_{n}}^{(n-1)} - z_{n}^{2}} + \frac{n}{2},$$

(54)

Note that $\tilde{F}_{n}(Z_{n})$ is an unknown continuous function and RBFN can be used to approximate it. Hence, from (2), the following equation holds:

$$\tilde{F}_{n}(Z_{n}) = W_{n}^{T} S_{n}(Z_{n}) + e_{n}(Z_{n}),$$

(55)

where $W_{n}^{T} S_{n}(Z_{n})$ is an NN, $\| e_{n}(Z_{n}) \| \leq e_{n}, Z_{n} = [\tilde{x}_{n}, x_{n-1}, x_{n}, r, w_{n-1}, w_{n}]^{T}$, and $e_{n} > 0$ is any given.

By using Young’s inequality and Lemma 2, one has

$$\frac{z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} F_{n}(Z_{n}) \leq \frac{z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} \left[ W_{n}^{T} S_{n}(Z_{n}) + e_{n}(Z_{n}) \right],$$

$$\leq \frac{z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} \left\| W_{n} \right\| \left\| S_{n}(Z_{n}) \right\| + e_{n}(Z_{n}) z_{n},$$

$$\leq \frac{z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} \left\| W_{n} \right\| \left\| S_{n}(Z_{n}) \right\| + e_{n}(Z_{n}) z_{n},$$

$$\leq \frac{1}{2} \left( \frac{z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} \left\| W_{n} \right\| \left\| S_{n}(Z_{n}) \right\| \right)^{2} + \frac{l_{n}}{2},$$

$$+ e_{n}(Z_{n}) z_{n},$$

(56)

where $Z_{n} = [\tilde{x}_{n}, x_{n-1}, x_{n}, r, w_{n-1}, w_{n}]^{T}$.

Substituting (49), (50), and (56) into (53), we can obtain

$$\dot{V}_{n} \leq -\frac{b_{n} c_{n} z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} - \frac{\mu_{b_{n}} z_{n}^{2}}{2(k_{b_{n}}^{2} - z_{n}^{2})^{2}} - g_{n-1}(\tilde{x}_{n}) z_{n}^{2} + \frac{l_{n}}{2} + e_{n}(Z_{n}) z_{n},$$

$$+ \sum_{j=1}^{n} \rho_{n}^{2} \left( x_{j}(t - T_{j}(t)) \right),$$

$$\leq \frac{g_{n-1}(\tilde{x}_{n}) z_{n}^{2} - g_{n-1}(\tilde{x}_{n}) z_{n}^{2}}{k_{b_{n}}^{(n-1)} - z_{n}^{2}} - \frac{2}{k_{b_{n}}^{2} - z_{n}^{2}} + \frac{l_{n}}{2},$$

(57)

By utilizing Young’s inequality, the following inequality can be obtained:

$$e_{n}(Z_{n}) z_{n}^{2} - \frac{\mu_{b_{n}} z_{n}^{2}}{2(k_{b_{n}}^{2} - z_{n}^{2})^{2}} \leq \frac{e_{n}^{2}(Z_{n})}{2\mu_{b_{n}} z_{n}^{2}} \leq \frac{e_{n}^{2}(Z_{n})}{2\mu_{b_{n}} z_{n}^{2}}.$$  

Therefore, we have

$$\dot{V}_{n} \leq -\frac{b_{n} c_{n} z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} + \frac{e_{n}^{2}(Z_{n})}{2\mu_{b_{n}} z_{n}^{2}} - \frac{\mu_{b_{n}} z_{n}^{2}}{2(k_{b_{n}}^{2} - z_{n}^{2})^{2}} + \frac{l_{n}}{2} + \sum_{j=1}^{n} \rho_{n}^{2} \left( x_{j}(t - T_{j}(t)) \right),$$

(59)

According to the inequality $2b_{n} \tilde{\omega}_{n} \tilde{\omega}_{n} \leq \tilde{\omega}_{n}^{2} - \tilde{\omega}_{n}^{2}$ and Lemma 4, one has

$$\dot{V}_{n} \leq -\frac{b_{n} c_{n} z_{n}^{2}}{k_{b_{n}}^{2} - z_{n}^{2}} + \frac{e_{n}^{2}(Z_{n})}{2\mu_{b_{n}} z_{n}^{2}} - \frac{\mu_{b_{n}} z_{n}^{2}}{2(k_{b_{n}}^{2} - z_{n}^{2})^{2}} + \frac{l_{n}}{2} + \sum_{j=1}^{n} \rho_{n}^{2} \left( x_{j}(t - T_{j}(t)) \right),$$

$$+ \frac{g_{n-1}(\tilde{x}_{n}) z_{n}^{2}}{k_{b_{n}}^{(n-1)} - z_{n}^{2}} - \frac{2}{k_{b_{n}}^{2} - z_{n}^{2}} + \frac{l_{n}}{2},$$

(60)

where $\zeta_{1}$ and $\zeta_{2}$ are defined in Lemma 4.

To handle the time delay, define the Lyapunov-Krasovskii functional as follows:

$$V_{U_{n}} = \frac{1}{2} \left( 1 - T_{\max} \right) \sum_{j=1}^{n} t_{j-T_{j}(t)} e_{n}^{T} \rho_{n} \left( x_{j}(s) \right) ds,$$

(61)
where \( y > 0 \) is a positive constant. By using Assumption 5, we obtain that the derivative of \( V_U \) is

\[
\dot{V}_{U_i} = e^{-y(T_{\text{max}} - t)} \sum_{j=1}^{n} \left[ e^{-\gamma(T_{\text{max}} - t)} \rho_{nj}^2 (x_i(t)) - e^{-\gamma(T_{\text{max}} - t)} \rho_{nj}^2 \right] \\
\cdot \left( x_i(t - T_j(t)) \right) \left( 1 - T_j(t) \right) - yV_{U_i} \\
\leq \sum_{j=1}^{n} \left[ e^{-\gamma(T_{\text{max}} - t)} \rho_{nj}^2 (x_i(t)) - \frac{1}{2} \sum_{j=1}^{n} \rho_{nj}^2 \right] \left( x_i(t - T_j(t)) \right) \left( 1 - T_j(t) \right) - yV_{U_i}.
\]

(62)

From equations (60) and (62), we have

\[
\dot{V}_{z_i} + \dot{V}_{U_i} \leq - \frac{b_{01} \hat{z}_i^2}{\eta_{b0}} + \frac{\hat{e}_i^2}{\eta_{b0}} - \frac{\kappa_{n}g_{n}(\hat{x}_n)}{2\mu_{b0}} z_n \beta_n(z_n) \\
+ \frac{l_2}{2} + \frac{1}{4} \frac{\sigma_{n1} \hat{z}_i^2}{b_{01}^2} + \frac{\sigma_{n1} \hat{z}_i^2}{b_{01}^2} - \frac{g_{n-1}(\hat{x}_n-1)}{k_{b0}^2} z_n - \frac{1}{2} \sum_{j=1}^{n} \rho_{nj}^2 \left( x_i(t - T_j(t)) \right) \\
+ \Phi_n - yV_{U_i},
\]

(63)

where \( \Phi_n = \sum_{j=1}^{n} (e^{-\gamma(T_{\text{max}} - t)} / (2(1 - T_{\text{max}}))) \rho_{nj}^2 (x_i(t)). \)

3.2. Stability Analysis. In this subsection, we present the stability analysis of the resulting closed-loop system. The main results are presented by the following theorem.

**Theorem 1** Consider the nonlinear system (1) with Assumptions 1–5. Let the actual control input and the NN adaptive law be designed as (49) and (50), respectively. If the initial conditions satisfy \( V(0) \leq \Delta, ||z_i(0)|| \leq \kappa_{bi} \) in which \( \Delta > \kappa_{bi} \) is any positive constant for \( i = 1, 2, \ldots, n \) and \( \kappa_{bi} \) are properly chosen, such that \( \kappa_{cl} > \kappa_{b1} + A_0 \) and \( \kappa_{cl} > \bar{w}_i + \kappa_{b1} \) with \( \bar{w}_i = \sup |w_i| \) for \( i = 2, 3, \ldots, n \), one has that all internal signals \( z_i, \hat{z}_i \), and \( y_i \) are globally uniformly ultimately bounded and tracking error will converge into the arbitrarily small regions in a finite time. Meanwhile, each state \( x_i \) will remain in the set \( \Omega_{x_i} \); that is, the full-state constraints are never violated.

**Proof.** Construct the overall Lyapunov function candidate

\[
V = \sum_{i=1}^{n} V_{z_i} + \sum_{i=1}^{n} V_{U_i} + \sum_{j=1}^{n-1} V_{y_j},
\]

(64)

where \( V_{y_j} = (a_j^2 / 2b_j) y_{y_j}^2 \) and \( V_{z_i}, V_{U_i} \) are defined in (35) and (44), respectively.

From (28), (29), and (47), the derivative of \( \sum_{j=1}^{n-1} V_{y_j} \) is

\[
\dot{V}_{y_j} = - \frac{a_j^2}{b_j} \left( \tau_{(i+1)} y_{y_j} + \tau_{(i+1)} y_{y_j}^{1+h} \right) + \sum_{i=1}^{n-1} \frac{a_j^2}{b_j} y_{y_j} \lambda_{0i} \left( x_{z_j}, \hat{\theta}_0, y_2, y_3, \ldots, y_{j-1}, y_j, \hat{y}_j \right).
\]

(65)

Define a compact set as \( \Omega_n = \{ (\hat{x}_n, \hat{\theta}_n, y_2, y_3, \ldots, y_n) : V \leq \Delta \} \) with \( \Delta \) being a positive constant. If \( V \leq \Delta \), together with Assumption 4 and (65), it can be obtained that there exists a positive constant \( \lambda_{0i} \) such that \( \lambda_{0i} \leq \lambda_{0i+1} \) on the compact set \( \Omega_n \times \Omega_d \). Then, applying Young’s inequality to (65) yields

\[
\sum_{j=1}^{n-1} \frac{a_j^2}{b_j} \left( \tau_{(i+1)} - \frac{1}{2\lambda_{0i}} \right) y_{y_j} + \sum_{i=1}^{n-1} \frac{a_j^2}{b_j} y_{y_j} \lambda_{0i} \left( x_{z_j}, \hat{\theta}_0, y_2, y_3, \ldots, y_{j-1}, y_j, \hat{y}_j \right).
\]

(66)

where \( \lambda_{0i} \) are positive constants.

According to the above analysis, we can obtain the derivative of the overall Lyapunov function candidate \( V \) as

\[
\dot{V} \leq - \sum_{j=1}^{n} \frac{b_{01} \hat{z}_i^2}{\eta_{b0}} - \sum_{i=1}^{n} \frac{\kappa_{n} \hat{x}_n \beta_n(z_n)}{2\mu_{b0}} z_n - \sum_{j=1}^{n} \frac{\sigma_{n1} \hat{z}_i^2}{b_{01}^2} - \sum_{i=1}^{n} \frac{g_{n-1}(\hat{x}_n-1)}{k_{b0}^2} z_n - \frac{1}{2} \sum_{j=1}^{n} \rho_{nj}^2 \left( x_i(t - T_j(t)) \right) \\
+ \Phi_n - \gamma V_{U_i} + d_0,
\]

(67)

where

\[
\tau_{(i+1)} = \frac{a_j^2}{b_j} \left( \tau_{(i+1)} - \frac{1}{2\mu_{b0}} - \frac{1}{2\lambda_{0i+1}} \right),
\]

(68)

Here, we choose \( \tau_{(i+1)} > (1/2\mu_{b0}) + (1/2\lambda_{0i+1}) \), such that \( \tau_{(i+1)} > 0 \).

From the definition of \( \beta_i(z_i) \) for \( i = 1, 2, \ldots, n \) in (15), (34), and (51), the following two cases should be considered.

Case 1: When \( ||z_i|| < \tau_i, i = 1, 2, \ldots, n \), substituting \( \beta_i(z_i) = t_1 z_i + t_2 z_i^3 \) into (67) gives
\[
\dot{V} \leq \sum_{i=1}^{n} \left( c_i + \kappa_i \xi_i \right) \frac{b_i z_i^2}{k_{bi} - z_i^2} - \sum_{i=1}^{n} \frac{\sigma_i \zeta_i^2}{k_{bi} - \zeta_i^2} - \sum_{i=1}^{n} \frac{\sigma_i \zeta_i^2}{2b_i} - \sum_{i=1}^{n-1} \frac{\sigma_i \zeta_i^2}{b_i} + \sum_{i=1}^{n-1} \frac{\sigma_i \zeta_i^2}{b_i} + \sum_{i=1}^{n-1} \frac{\sigma_i \zeta_i^2}{b_i} \quad \text{(69)}
\]

Noting (69), we can have
\[
\dot{V} \leq -\nu V + d_0, \quad \text{(70)}
\]

with \( \nu = \min \{ 2b_1 (c_1 + \kappa_1 \xi_1), \ldots, 2b_n (c_n + \kappa_n \xi_n) \}, \rho_1 \sigma_1, \ldots, \rho_n \sigma_n, (2b_1 \tau_{(2,1)} / a_1^2), \ldots, (2b_n \tau_{(n,1)} / a_n^2 - 1 \}, \) which further implies that all the internal signals are uniformly ultimately bounded.

Case 2: When \( \|z_i\| \geq \tau_i, i = 1, 2, \ldots, n, \) substituting \( \beta_i(z_i) = z_i^h (k_{bi}^2 - z_i^2)^{(1-h/2)} \) into (67) gives
\[
\dot{V} \leq -v_1 V - v_2 V^{(1+h/2)} + d_1, \quad \text{(71)}
\]

where
\[
v_1 = \min \left\{ 2b_1 c_1, \ldots, 2b_n c_n, \sigma_1 \rho_1, \ldots, \sigma_n \rho_n, \frac{\gamma}{2} \frac{2b_1 \tau_{(2,1)}}{a_1^2}, \ldots, \frac{2b_n \tau_{(n,1)}}{a_n^2} \right\},
\]

\[
v_2 = \min \left\{ \frac{b_1 c_1}{2} \frac{(1+h/2)}{a_1^2}, \ldots, \frac{b_n c_n}{2} \frac{(1+h/2)}{a_n^2}, \frac{\sigma_1 \kappa_1}{2b_1}, \frac{2b_1 \rho_1}{a_1^2}, \ldots, \frac{2b_n \rho_n}{a_n^2} \right\},
\]

\[
d_1 = d_0 + \frac{\gamma}{2} \frac{1 - h}{2} \frac{(1 + h)}{2} \frac{(1+h/2)}{a_1^2} \]

}\]
By virtue of [59, Th.5.2], there always exists a finite-time $t^*$, such that $V \geq (2d_1/v_{12})^{1/(1+h)}$, for all $t \in [0,t^*]$. Thus, for all $t \in [0,t^*]$, one has $V \leq v_1^{-1} V - (v_1/2)V^{1/(1+h)}$, and it then comes from Lemma 3 that the fast finite-time stability of the closed-loop system can be ensured by a finite settling time $T^* \leq (2/(v_1(1-h)))\log((2v_1V^{1/(1+h)}(0) + v_{12})/v_{12})$. Furthermore, it is readily seen that $t^* \leq T^*$. Therefore, $\forall t > T^*, \ V \leq (2d_1/v_{12})^{2/(1+h)}$. Then, the internal error signals $z_i, \tilde{w}_i$, and $y_{i+1}$ will converge into the following compact sets:

$$
|z_i| \leq k_{b1} \left(1 - e^{-2(2d_1/v_{12})^{1/(1+h)}}\right)^{(1/2)}, \quad i = 1, \ldots, n,
$$

$$
|\tilde{w}_i| \leq 2\rho_b \left(\frac{2d_1}{v_{12}}\right)^{(1/2)} \left(\frac{2d_1}{v_{12}}\right)^{(1/1+h)}), \quad i = 1, \ldots, n,
$$

$$
|y_{i+1}| \leq \left(2\rho_b \left(\frac{2d_1}{v_{12}}\right)^{(1/2)} \left(\frac{2d_1}{v_{12}}\right)^{(1/1+h)}\right)^{(1/2)}, \quad i = 1, \ldots, n-1,
$$

in a finite-time $T^*$ with $T^* \leq (2/(v_1(1-h)))\log((2v_1V^{1/(1+h)}(0) + v_{12})/v_{12})$. It is readily seen that the regions (73) can be made as small as possible by adjusting $(2d_1/v_{12})$ with proper control parameters.

Then, we will prove that the full-state constraints are never violated. According to [60, Lemma 1], we can conclude from (70) and (71) that $|z_i| \leq k_{b1}, i = 1, \ldots, n$, for all $t \geq 0$. Noting that $|y_i| \leq A_{b0}$ from Assumption 4 and $z_i = x_i - y_i$, we have that $|z_i| \leq k_{b1} + A_{b0}$. To get $x_2 \leq k_{b2}$, we need to show the boundedness of $\tilde{w}_2$. From (73), one has that $y_2$ is bounded and $b_2\tilde{w}_2 = \tilde{w}_1 - \tilde{w}_2$ is also bounded. With the proper choices of $t_1$ and $t_2$, $\tilde{w}_2$ is a continuous function of $\tilde{w}_1, x_1$, and $y_2$. Then, there exists an upper bound $\tilde{w}_2$, such that $w_2 = |y_2 + \tilde{a}_1| \leq \tilde{w}_2$. From $z_2 = x_2 - w_2$ and $z_2 < k_{b2}$, we get that $|z_2| \leq |z_1| + |w_2| \leq k_{b2}$. Similarly and iteratively, we have that $a_{i-1}$ and $y_i$ for $i = 3, \ldots, n$ are bounded, which together with $z_i < k_{bi}$ ensures that $|x_i| \leq k_{ci}, i = 3, \ldots, n$. Therefore, each state $x_i, i = 3, \ldots, n$ will remain in the set $\Omega_{x_i}$. The proof is completed.

4. Simulation Results

Example 1. Consider the following nonlinear system:

$$
\begin{align*}
\dot{x} &= -x + 0.5x^2\sin(x_1t), \\
\dot{x}_1 &= x_1e^{-0.5x_1} + (1 + x_1^2)x_2 + \delta_1(\xi, x_1, x_2, t) + 2x_1^2(t - T_1(t)), \\
\dot{x}_2 &= x_1x_2 + 2.5u(t) + \delta_2(\xi, x_1, x_2, t) + 0.2x_1^2(t - T_2(t)), \\
y &= x_1,
\end{align*}
$$

(74)

where $\delta_1(\xi, x_1, x_2, t) = 0.2x_1\sin(x_1t), \delta_2(\xi, x_1, x_2, t) = 0.1\xi^2(0.5x_1^2, T_1(t) = 0.2(4 + \sin t), T_2 = 4 + 0.5\sin t$, and the dynamic signal $r = -r - 2.5x_1^2 + 0.625$. The desired tracking trajectory $y_t = \sin(0.5t)$, $u$ is the control input. The design parameters of the controller are taken as $k_{b1} = 0.4, k_{b2} = 0.4, 0.01, \sigma_{11} = \sigma_{12} = 0.01, \rho_1 = \rho_2 = 0.5, \mu_1 = \mu_2 = 0.5, h = 0.6$. There are 68 nodes with the center placed on $[-2,2] \times [-2,2] \times [-2,2] \times [-2,2]$ and the width of Gaussian functions is $\eta_1 = 1$ in the first RBF vector. There are 85 nodes with the center placed on $[-2,2] \times [-2,2] \times [-2,2] \times [-2,2] \times [-2,2]$ and the width of Gaussian functions is $\eta_2 = 1$ in the second RBF vector. With the initial conditions, $x_1(0) = 0.2, x_2(0) = 0.1, u_1(0) = 0.1, \delta_1 = 0.2, \delta_2 = 0.5, r(0) = 0.1$. Simulation results are shown in Figures 1–6. The profiles of the system output $y$ and the desired signal $y_t$ are shown in Figure 1, which indicates that the output $y$ follows the specified desire trajectory $y_t$. From Figure 2, we know that all state constraints are not violated.

Example 2. A Spring-Mass-Damper system is provided in this part. The system model is as follows:

$$
\begin{align*}
\dot{p} &= V, \\
MV &= -KP - CV + F,
\end{align*}
$$

(75)

where $P, V$ and $F$ are the position, the velocity, and the force applied to the object, respectively. Let $x_1 = P, x_2 = V, u = F$. Assuming that the controller system (75) gives unmodeled dynamics and time delay, let $\delta_1(\xi, x_1, x_2, t) = 0.2\xi x_1 \sin(x_2t), \delta_2(\xi, x_1, x_2, t) = 0.1\xi \cos(0.5x_1^2, T_1(t) = 0.2(4 + \sin t), T_2 = 4 + 0.5\sin t$, and the dynamic signal $r = -r - 2.5x_1^2 + 0.625$. Then, system (75) can be rewritten as

$$
\begin{align*}
\dot{x} &= -x + 0.5x^2\sin(x_1t), \\
\dot{x}_1 &= x_1 + \delta_1(\xi, x_1, x_2, t) + 2x_1^2(t - T_1(t)), \\
\dot{x}_2 &= -Kx_1 - Cx_2 + 1\mu(t) + \delta_2(\xi, x_1, x_2, t) + 0.2x_1^2(t - T_2(t)), \\
y &= x_1,
\end{align*}
$$

(76)

The desired tracking trajectory $y_t = \sin(0.5t) + 0.5\sin(t)$. The design parameters of the controller are taken as $k_{b1} = 0.4, k_{b2} = 0.4, \sigma_{11} = \sigma_{12} = 0.01, \sigma_{11} = \sigma_{12} = 0.01, \rho_1 = \rho_2 = 0.5, \mu_1 = \mu_2 = 0.5, \eta_1 = 0.5, \eta_2 = 0.6$. 

![Figure 1: Output y and desired trajectory y_t.](image-url)
Figure 2: Phase portrait of states $x_1$ and $x_2$.

Figure 3: Profiles of control inputs $u$ and $\alpha_1$.

Figure 4: Profiles of the tracking errors $z_1$ and $z_2$.

Figure 5: Estimated parameters $\widehat{\omega}_1$ and $\widehat{\omega}_2$. 

Complexity
Figure 6: Profiles of the boundary layer error $y_2$.

Figure 7: Output $y$ and desired trajectory $y_r$.

Figure 8: Phase portrait of states $x_1$ and $x_2$.

Figure 9: Profiles of control inputs $u$ and $\alpha_1$. 
There are 68 nodes with the center placed on $[-2,2] \times [-2,2] \times [-2,2] \times [-2,2] \times [-2,2]$ and the width of Gaussian functions is $\eta_1 = 1$ in the first RBF vector. There are 85 nodes with the center placed on $[-2,2] \times [-2,2] \times [-2,2] \times [-2,2] \times [-2,2]$ and the width of Gaussian functions is $\eta_2 = 1$ in the second RBF vector.

With the initial conditions, $x_1(0) = 0.2, x_2(0) = 0.1, \omega_1(0) = 0.1, \hat{\omega}_1 = 2, \hat{\omega}_3 = 0.5, r(0) = 0.1$. Simulation results are shown in Figures 7–12.

5. Conclusions

The problem of finite-time tracking control for a class of uncertain nonstrict-feedback state-delayed nonlinear systems with full-state constraints and unmodeled dynamics has been proposed in this paper. Unmodeled dynamics is dealt with by introducing a dynamical signal and the uncertain terms produced by time-varying state delays are compensated for by constructing appropriate Lyapunov-Krasovskii functionals. By utilizing a smooth switch between the fractional and cubic form state feedback, novel $C^1$ smooth finite-time NN control laws have been provided for nonlinear systems with full-state constraints. Based on a modified DSC method and adaptive NN control, together with the BLFs, the fast finite-time control performance of the closed-loop nonlinear systems can be ensured, while the full-state constraints are never violated. Theoretical proofs and experimental simulation show that all the internal signals in the closed-loop system are uniformly bounded, and the tracking error signals can converge into compact sets in a finite time with sufficient accuracy, respectively. To extend this control scheme to solve the finite-time tracking control problem for some more complicated systems, such as MIMO nonlinear systems, switched nonlinear systems are also the direction of our future efforts.

Data Availability

This paper is a theoretical study and no data were used to support this study.

Conflicts of Interest

The authors declare that they do not have any financial or nonfinancial conflicts of interest.

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