

Research Article

Dynamics of a Cournot Duopoly Game with a Generalized Bounded Rationality

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In this paper, the dynamics of Cournot duopoly game with a generalized bounded rationality is considered. The fractional bounded rationality of the Cournot duopoly game is introduced. The conditions of local stability analysis of equilibrium points of the game are derived. The effect of fractional marginal profit on the game is investigated. The complex dynamics behaviors of the game are discussed by numerical computation when parameters are varied.

1. Introduction

Many scientists have created diverse variations of Cournot oligopoly games. Cournot duopoly game was the first oligopoly game [1]. Furth [2] studied existence and equilibrium stability in oligopoly games. Oligopoly game which contains two firms is called duopoly game; these two firms are in a competition and there is no collaboration among them. In order to maximize the profit, every firm takes action on the basis of its rivals reaction to compete with its rivals. After that, the modifications of these games turned into the core interest. Dana and Montrucchio [3] investigated complex dynamics in Cournot oligopoly games. Moreover, Puu [4, 5] studied the chaotic dynamics of Cournot duopoly games. The stability analysis of naive and bounded rationality oligopoly games has been discussed in [6]. Bischi and Naimzada [7] studied dynamics duopoly game based on bounded rationality. Agiza and Elsadany [8] investigated nonlinear dynamics occurring in heterogeneous duopoly game. The duopoly game based on altering heterogeneous players has been explored in [9]. Nonlinear Chinese cold-rolled steel market game has been examined in [10]. Nonlinear oligopoly games have been reviewed in [11]. The dynamics of a discrete duopoly game with players having adaptive expectations has been studied in [12]. The stability of Cournot duopoly game with

logarithmic price function has been investigated in [13]. Hommes [14] studied heterogeneous expectations and behavioral rationality in economic models. The isoelastic duopoly game with different expectations has been introduced in [15]. Fanti and Gori [16] investigated the differentiated competition duopoly game. Sarafopoulos [17] explored the dynamics of a nonlinear duopoly game with differentiated products. Askar and Al-khedhairi [18] discussed the influences of a cubic utility function on the stability of a nonlinear differentiated Cournot duopoly game. Tramontana and Elsadany [19] and Guirao et al. [20] studied oligopoly games while increasing the number of heterogeneous competitors. Dynamical heterogeneous duopoly games and their control are investigated in many other research studies [21–24]. The modified Puu duopoly game has been analyzed [25]. Fanti [26] investigated the dynamic banking duopoly game with capital regulations. An uncertainty Cournot duopoly game based on concave demand has been introduced in [25]. The impact of delay on Cournot duopoly game has been discussed in [27]. Elsadany [28] considered the Cournot duopoly game due to relative profit. Different investigations discussed for more realistic learning of firms structures of different strategies such as choices of firms and have demonstrated that the oligopoly games may tend to complex dynamics [29–32].

In recent years, the issue of incorporating game theory with complexity theory has been discussed by many authors [33–39]. Askar and Al-khedhairi [33] examined Cournot game that is constructed based on Cobb–Douglas preferences and, especially, analyzed its nonlinear dynamics. Tian et al. [34] investigated a dynamic duopoly Stackelberg model of competition on output with stochastic perturbations. Zhao et al. [35] extend the Cournot game to the case of multimarket with bounded rationality. Peng et al. [36] analyzed complex dynamics for Cournot-remanufacturing duopoly game based on bounded rationality. Cerboni Baiardi and Naimzada [37] considered the oligopoly model with rational and imitation rules. They found that the number of firms participating in the game, called a parameter of the game, has an ambiguous effect in influencing the stationary state stability property and double stability threshold has been observed. Al-khedhairi [38] introduced a fractional-order Cournot triopoly game and discussed the effects of the memory on the dynamics of the game. Furthermore, remanufacturing Cournot duopoly game based on a nonlinear utility function has been studied by Askar and Al-khedhairi [39].

The generalized bounded rationality is more applicable than the traditional one. The later ignores the memory of the production's previous prices adopted by production buyers. The traditional bounded rationality may be used to handle total amnesia of buyers, but the generalized one is suggested to remove that issue and takes into account the effect of memory. Memory is known to be an important factor in the economy. The fractional-order derivative is based on integration. Consequently, the fractional-order derivative is a nonlocal operator. Therefore, the fractional-order derivative is appropriate for representing complex systems like biological, economic, and social systems. The case of triopoly game with differentiated products based on generalized (fractional) bounded rationality is considered by Askar and Abouhawwash [40]. They showed that, for the firms to stay stable for a long time in the market, they should play with generalized bounded rationality rather than the traditional bounded rationality. The present paper constitutes a modification of the game introduced by [40]. The aim of this work is to present the generalized-order bounded rationality method. The Cournot duopoly games are more popular models describing the competition between firms and have been intensively studied in the literature. For this reason comes our contributions in this paper. We have adopted the generalized bounded rationality introduced in [40] to show that the chaotic behavior of such games persists under fractional bounded rationality for duopoly games. In addition, our proposed model can extend some models in the literature [7]. We investigate Cournot duopoly game based on fractional marginal profit. Our proposed game is described by generalized bounded rationality decisional learning and different marginal costs.

The paper is arranged as follows: We discuss Cournot duopoly game with generalized bounded rationality in Section 2. Section 3 analyzes the equilibrium point's stability. We have also performed numerical simulation to illustrate

complex dynamics, bifurcations, and chaos of the game in Section 4, and the arrived results are discussed in Section 5.

2. Model

We assume that there are two players, named $i = 1$ and 2 , producing the same products to be purchased in the market. Creation choices of the two firms happen at discrete time periods $t = 0, 1, 2, \dots$. We consider linear demand function in the market as follows:

$$p_t = f(Q_t) = a - bQ_t, \quad (1)$$

where $q_{i,t}$ is the quantity of firm i and a and b are non-negative parameters. Also, $Q_t = q_1(t) + q_2(t)$ is the total quantity in the market. We assume that the cost function in the linear form is

$$C_i(q_{i,t}) = c_i q_{i,t}, \quad i = 1, 2, \quad (2)$$

where the marginal costs are the positive parameters c_i . Hence, the profit of the firm i has the following form:

$$\Pi_{i,t}(q_{1,t}, q_{2,t}) = q_{i,t} (a - b(q_{i,t} + q_{j,t})) - c_i q_{i,t}, \quad i = 1, 2, \quad (3)$$

Equation (3) can be given as follows:

$$\Pi_{i,t}(q_{1,t}, q_{2,t}) = (a - c_i)q_{i,t} - b q_{i,t}^2 - b q_{i,t} q_{j,t}, \quad i, j = 1, 2 \quad i \neq j, \quad (4)$$

and the marginal profit of the firm i is

$$\Phi_i = \frac{\partial \Pi_{i,t}}{\partial q_{i,t}} = a - c_i - 2b q_{i,t} - b q_{j,t}, \quad i, j = 1, 2 \quad i \neq j. \quad (5)$$

Information in the game generally is deficient, so firms may utilize more complex strategies, for example, bounded rationality method. Firms with bounded rationality do not have the total information of the game; thus, the settling yield choices depend on a local estimate of the marginal profit $\partial \Pi_{i,t} / \partial q_{i,t}$. A firm, at each time period t , plans to increase its quantity produced $q_{i,t}$ at the period $(t + 1)$ if it has a positive marginal profit or decreases its quantity produced at the period $(t + 1)$ if the marginal profit is negative. When companies make use of this type of adjustments, they are to be rational players and the two-dimensional structure that defines the dynamics of the game's economic model is formed as follows:

$$\begin{cases} q_{1,t+1} = q_{1,t} + k(q_{1,t}) \frac{\partial \Pi_{1,t}(q_{1,t}, q_{2,t})}{\partial q_{1,t}}, \\ q_{2,t+1} = q_{2,t} + k(q_{2,t}) \frac{\partial \Pi_{2,t}(q_{1,t}, q_{2,t})}{\partial q_{2,t}}, \end{cases} \quad (6)$$

where $q_{i,t+1}$ is the quantity output of i th firm at time $(t + 1)$ and k represents a speed adjustment function. In the next section, we will discuss the fractional mechanism of the marginal profit.

2.1. Fractional-Order Marginal Profit. The generalized bounded rationality introduced here is a generalization of the traditional bounded rationality [7]. As we mentioned before, our aim of this work is to analyze the effect of fractional marginal profit in a duopoly game. To do so, we can write (5) as follows:

$$\begin{aligned} \frac{\partial^\beta \Pi_{i,t}(q_{1,t}, q_{2,t})}{\partial q_{i,t}^\beta} &= (a - c_i) \frac{\partial^\beta(q_{i,t})}{\partial q_{i,t}^\beta} - b \frac{\partial^\beta(q_{i,t}^2)}{\partial q_{i,t}^\beta} \\ &- bq_{j,t} \frac{\partial^\beta(q_{i,t})}{\partial q_{i,t}^\beta}, \quad i, j = 1, 2, i \neq j. \end{aligned} \quad (7)$$

To differentiate (6) where β is a fractional and $0 < \beta < 1$, we will use the following definition.

Definition 1. For $\beta \in \mathbb{R}^+$, let n be the nearest integer greater than β ; the Caputo fractional derivative of order $\beta > 0$ with $n - 1 < \beta < n$ of the power function $f(t) = t^p$ for $p \geq 0$ and $t > 0$ is given by

$$D^\beta t^p = \frac{\Gamma(p+1)}{\Gamma(p-\beta+1)} t^{p-\beta}, \quad (8)$$

where Γ is Euler's Gamma function. One can use the book Fractional Calculus such as Miller and Ross [41] for more information about fractional derivatives.

Consequently and using this definition, (6) is rewritten as follows:

$$\begin{aligned} \frac{\partial^\beta \Pi_{i,t}(q_{1,t}, q_{2,t})}{\partial q_{i,t}^\beta} &= \frac{(a - c_i)}{\Gamma(2-\beta)} q_{i,t}^{1-\beta} - \frac{bq_{j,t}}{\Gamma(2-\beta)} q_{i,t}^{1-\beta} \\ &- \frac{2b}{\Gamma(3-\beta)} q_{i,t}^{2-\beta}, \quad i, j = 1, 2, i \neq j. \end{aligned} \quad (9)$$

However, to increase the profit, both firms adopt gradient mechanism in which the duopolistic changes the quantity produced based on the direction of variation of profit, the same direction in case of positive variation and opposite direction in case of negative variation.

The fractional-order marginal profit method presented here is a generalization of the classical one. It thinks about the nearness of a memory of purchasers about the past costs of the generation. The customary limited objectivity can be utilized as a part of instance of all purchasers has aggregate amnesia. Along these lines, fractional derivatives are proposed to expel amnesia and consider the impact of memory. This implies that request can rely upon changes that may happen in prices amid a limited interim of time. In addition, the parameter β representing memory decay is added to describe the degree of memory decay throughout the time interval. Now, the generalized bounded rationality (the adjustment mechanism) takes the following form:

$$q_{i,t+1} = q_{i,t} + k \frac{\partial^\beta \Pi_{i,t}(q_{1,t}, q_{2,t})}{\partial q_{i,t}^\beta}, \quad 0 < \beta \leq 1, k > 0. \quad (10)$$

For simplicity, we take a constant relation for speed of adjustment function, where k is the speed of adjustment. From (9) and (10), we get the following two-dimensional nonlinear difference equation:

$$\begin{aligned} q_{i,t+1} &= q_{i,t} + k \left[\frac{(a - c_i)}{\Gamma(2-\beta)} q_{i,t}^{1-\beta} - \frac{bq_{j,t}}{\Gamma(2-\beta)} q_{i,t}^{1-\beta} \right. \\ &\left. - \frac{2b}{\Gamma(3-\beta)} q_{i,t}^{2-\beta} \right], \quad i, j = 1, 2, i \neq j. \end{aligned} \quad (11)$$

We will discuss the dynamics of the game (11) in the following sections.

3. Equilibrium and Stability

From (11), the duopoly dynamical system with generalized bounded rational firms has the following form:

$$\begin{cases} q_{1,t+1} = q_{1,t} + k \left\{ \frac{(a - c_1)}{\Gamma(2-\beta)} q_{1,t}^{1-\beta} - \frac{bq_{2,t}}{\Gamma(2-\beta)} q_{1,t}^{1-\beta} - \frac{2b}{\Gamma(3-\beta)} q_{1,t}^{2-\beta} \right\}, \\ q_{2,t+1} = q_{2,t} + k \left\{ \frac{(a - c_2)}{\Gamma(2-\beta)} q_{2,t}^{1-\beta} - \frac{bq_{1,t}}{\Gamma(2-\beta)} q_{2,t}^{1-\beta} - \frac{2b}{\Gamma(3-\beta)} q_{2,t}^{2-\beta} \right\}. \end{cases} \quad (12)$$

In order to explore the behavior of game (12), can define the fixed points of (12) as the solution of the following system:

$$\begin{cases} \frac{(a - c_1)}{\Gamma(2-\beta)} q_{1,t}^{1-\beta} - \frac{bq_{2,t}}{\Gamma(2-\beta)} q_{1,t}^{1-\beta} - \frac{2b}{\Gamma(3-\beta)} q_{1,t}^{2-\beta} = 0, \\ \frac{(a - c_2)}{\Gamma(2-\beta)} q_{2,t}^{1-\beta} - \frac{bq_{1,t}}{\Gamma(2-\beta)} q_{2,t}^{1-\beta} - \frac{2b}{\Gamma(3-\beta)} q_{2,t}^{2-\beta} = 0. \end{cases} \quad (13)$$

which is given by setting $q_{1,t+1} = q_{1,t}$ and $q_{2,t+1} = q_{2,t}$ in (12). System (13) has four fixed points:

$$\begin{aligned} E_1 &(0, 0), \\ E_2 &\left(\frac{(2-\beta)(a - c_1)}{2b}, 0 \right), \\ E_3 &\left(0, \frac{(2-\beta)(a - c_2)}{2b} \right), \end{aligned} \quad (14)$$

and $E_*(q_1^*, q_2^*)$, where

$$\begin{aligned} q_1^* &= \frac{(2-\beta)\{(2-\beta)(a - c_2) - 2(a - c_1)\}}{b[\beta^2 - 4\beta]}, \\ q_2^* &= \frac{(2-\beta)\{(2-\beta)(a - c_1) - 2(a - c_2)\}}{b[\beta^2 - 4\beta]}, \end{aligned} \quad (15)$$

which depends on the game parameters. The equilibria E_1 , E_2 , and E_3 are called the boundary equilibria [7]. E_2 and E_3 are nonnegative when

$$a > c_i, \quad i = 1, 2. \quad (16)$$

The equilibrium point E_* is the unique interior equilibrium point and has economic meaning (has nonnegative components) when

$$\begin{aligned} a &> \frac{2c_1 - (2-\beta)c_2}{\beta}, \\ a &> \frac{2c_2 - (2-\beta)c_1}{\beta}, \\ a &> c_i, \quad i = 1, 2, 0 < \beta < 1. \end{aligned} \quad (17)$$

In order to study the stability of the fixed points, we have to compute the Jacobian matrix of game (12) which is written as follows:

$$J(q_1, q_2) = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}, \quad (18)$$

where

$$\begin{aligned} \ell_{11} &= 1 + k \left[\frac{(a-c_1)}{\Gamma(1-\beta)} q_1^{-\beta} - \frac{bq_2}{\Gamma(1-\beta)} q_1^{-\beta} - \frac{2b}{\Gamma(2-\beta)} q_1^{1-\beta} \right], \\ \ell_{12} &= \frac{-kb}{\Gamma(2-\beta)} q_1^{1-\beta}, \\ \ell_{21} &= \frac{-kb}{\Gamma(2-\beta)} q_2^{1-\beta}, \text{ and} \\ \ell_{22} &= 1 + k \left[\frac{(a-c_2)}{\Gamma(1-\beta)} q_2^{-\beta} - \frac{bq_1}{\Gamma(1-\beta)} q_2^{-\beta} - \frac{2b}{\Gamma(2-\beta)} q_2^{1-\beta} \right]. \end{aligned} \quad (19)$$

The trivial equilibrium $E_1(0, 0)$ has no practical significance (no economic implications) because both the outputs of two firms are zero, so we exclude it from the analysis. The stability of equilibrium points E_2 , E_3 , and E_* will be determined by the eigenvalues of the Jacobian matrix computed at the corresponding equilibrium points.

Proposition 2. *The boundary equilibrium point $E_2(((2-\beta)(a-c_1)/2b), 0)$ of game (12) is stable if $k < 2^{(1-\beta)}(\Gamma(2-\beta)/(a-c_1))((2-\beta)(a-c_1)/b)^\beta$; otherwise, it is unstable.*

Proof. Jacobian matrix (18) at $E_2(((2-\beta)(a-c_1)/2b), 0)$ reads

$$J(E_2) = \begin{bmatrix} \Omega_1 & \Omega_2 \\ 0 & 1 \end{bmatrix}, \quad (20)$$

where

$$\begin{aligned} \Omega_1 &= 1 - \frac{k(a-c_1)}{\Gamma(2-\beta)} \left(\frac{(2-\beta)(a-c_1)}{2b} \right)^{-\beta}, \\ \Omega_2 &= -\frac{kb}{\Gamma(2-\beta)} \left(\frac{(2-\beta)(a-c_1)}{2b} \right)^{1-\beta}. \end{aligned} \quad (21)$$

The trace of the $J(E_2)$ is given by

$$\text{Tr}J(E_2) = 2 - \frac{k(a-c_1)}{\Gamma(2-\beta)} \left(\frac{(2-\beta)(a-c_1)}{2b} \right)^{-\beta}. \quad (22)$$

The determinant of the $J(E_2)$ is

$$\text{Det}J(E_2) = 1 - \frac{k(a-c_1)}{\Gamma(2-\beta)} \left(\frac{(2-\beta)(a-c_1)}{2b} \right)^{-\beta}. \quad (23)$$

Depending on the Jury conditions (Puu [42]), the E_2 is stable if and only if

$$\begin{aligned} 1 - \text{Tr}J(E_2) + \text{Det}J(E_2) &> 0, \\ 1 + \text{Tr}J(E_2) + \text{Det}J(E_2) &> 0, \\ 1 - |\text{Det}J(E_2)| &> 0. \end{aligned} \quad (24)$$

Substituting $\text{Tr}J(E_2)$ and $\text{Det}J(E_2)$ into the above inequalities, the first and third conditions are satisfied. The second conditions becomes

$$2 - \frac{k(a-c_1)}{\Gamma(2-\beta)} \left(\frac{(2-\beta)(a-c_1)}{2b} \right)^{-\beta} > 0. \quad (25)$$

Therefore, the equilibrium point E_2 is stable under the following condition:

$$k < 2^{(1-\beta)} \frac{\Gamma(2-\beta)}{(a-c_1)} \left(\frac{(2-\beta)(a-c_1)}{b} \right)^\beta. \quad (26)$$

This completes the proof.

By a similar argument as the proof of Proposition 2, we can prove the following proposition.

Proposition 4. *The boundary equilibrium point $E_3(0, ((2-\beta)(a-c_2)/2b))$ of game (12) is stable if $k < 2^{(1-\beta)}(\Gamma(2-\beta)/(a-c_2))((2-\beta)(a-c_2)/b)^\beta$; otherwise, it is unstable.*

Now, we discuss the local stability of the interior equilibrium point $E_*(q_1^*, q_2^*)$, linearizing game (12) at E_* . We can easily get its Jacobian matrix as follows:

$$J(E_*) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \quad (27)$$

where

$$\begin{aligned} v_{11} &= 1 + k \left[\frac{(a-c_1)}{\Gamma(1-\beta)} q_1^{*(-\beta)} - \frac{bq_2^*}{\Gamma(1-\beta)} q_1^{*(-\beta)} - \frac{2b}{\Gamma(2-\beta)} q_1^{*(1-\beta)} \right], \\ v_{12} &= \frac{-kb}{\Gamma(2-\beta)} q_1^{*(1-\beta)}, \\ v_{21} &= \frac{-kb}{\Gamma(2-\beta)} q_2^{*(1-\beta)}, \\ v_{22} &= 1 + k \left[\frac{(a-c_2)}{\Gamma(1-\beta)} q_2^{*(-\beta)} - \frac{bq_1^*}{\Gamma(1-\beta)} q_2^{*(-\beta)} - \frac{2b}{\Gamma(2-\beta)} q_2^{*(1-\beta)} \right], \end{aligned} \quad (28)$$

where q_1^* and q_2^* are defined in (14).

The characteristics equation of $J(E_*)$ is given by

$$P(\lambda) = \lambda^2 - \text{Tra}(J(E_*))\lambda + \text{Det}(J(E_*)) = 0, \quad (29)$$

where $\text{Tra}(J(E_*))$ and $\text{Det}(J(E_*))$ are the trace and determinant of the Jacobian matrix, respectively:

$$\begin{aligned} \text{Tra}(J(E_*)) &= 2 + k \left[\frac{(a-c_1)}{\Gamma(1-\beta)} q_1^{*(-\beta)} - \frac{bq_2^*}{\Gamma(1-\beta)} q_1^{*(-\beta)} - \frac{2b}{\Gamma(2-\beta)} q_1^{*(1-\beta)} \right] + k \left[\frac{(a-c_2)}{\Gamma(1-\beta)} q_2^{*(-\beta)} - \frac{bq_1^*}{\Gamma(1-\beta)} q_2^{*(-\beta)} - \frac{2b}{\Gamma(2-\beta)} q_2^{*(1-\beta)} \right], \\ \text{Det}(J(E_*)) &= \left[1 + k \left[\frac{(a-c_1)}{\Gamma(1-\beta)} q_1^{*(-\beta)} - \frac{bq_2^*}{\Gamma(1-\beta)} q_1^{*(-\beta)} - \frac{2b}{\Gamma(2-\beta)} q_1^{*(1-\beta)} \right] \right] \times \left[1 + k \left[\frac{(a-c_2)}{\Gamma(1-\beta)} q_2^{*(-\beta)} - \frac{bq_1^*}{\Gamma(1-\beta)} q_2^{*(-\beta)} - \frac{2b}{\Gamma(2-\beta)} q_2^{*(1-\beta)} \right] \right] \\ &\quad - \frac{k^2 b^2}{(\Gamma(2-\beta))^2} q_1^{*(1-\beta)} q_2^{*(1-\beta)}. \end{aligned} \quad (30)$$

The roots of characteristic equation (19) are inside the unit disk when Jury's conditions (Puu [42]) are satisfied. Then, the interior equilibrium point E_* is asymptotically stable if and only if

$$\begin{cases} 1 + \text{Tra}(J(E_*)) + \text{Det}(J(E_*)) > 0, \\ 1 - \text{Tra}(J(E_*)) + \text{Det}(J(E_*)) > 0, \\ 1 - \text{Det}(J(E_*)) > 0. \end{cases} \quad (31)$$

By using the equations in (20), the stability conditions become

$$\begin{cases} 3 + k(u_1 - u_2 - u_3) + k(u_4 - u_5 - u_6) + [1 + k(u_1 - u_2 - u_3)][1 + k(u_4 - u_5 - u_6)] - u_7 > 0, \\ -1 - k(u_1 - u_2 - u_3) - k(u_4 - u_5 - u_6) + [1 + k(u_1 - u_2 - u_3)][1 + k(u_4 - u_5 - u_6)] - u_7 > 0, \\ 1 - [1 + k(u_1 - u_2 - u_3)][1 + k(u_4 - u_5 - u_6)] + u_7 > 0, \end{cases} \quad (32)$$

where

$$\begin{aligned} u_1 &= \frac{(a-c_1)}{\Gamma(1-\beta)} q_1^{*(-\beta)}, \\ u_2 &= \frac{bq_2^*}{\Gamma(1-\beta)} q_1^{*(-\beta)}, \\ u_3 &= \frac{2b}{\Gamma(2-\beta)} q_1^{*(1-\beta)}, \\ u_4 &= \frac{(a-c_2)}{\Gamma(1-\beta)} q_2^{*(-\beta)}, \\ u_5 &= \frac{bq_1^*}{\Gamma(1-\beta)} q_2^{*(-\beta)}, \\ u_6 &= \frac{2b}{\Gamma(2-\beta)} q_2^{*(1-\beta)}, \\ u_7 &= \frac{k^2 b^2}{(\Gamma(2-\beta))^2} q_1^{*(1-\beta)} q_2^{*(1-\beta)}. \end{aligned} \quad (33)$$

The second and third conditions of (23) are explicitly always met, while the first condition is violated. Hence, the interior equilibrium point E_* is locally stable if and only if

$$k(u_4 - u_5 - u_6) < \frac{4 + k(2u_1 - 2u_2 - 2u_3 + u_4 - u_5 - u_6) - u_7}{-1 - k(u_1 - u_2 - u_3)}. \quad (34)$$

4. Numerical Simulation

Currently, some numerical simulations are performed to have more insights into the stability of our game (12) and confirm the results obtained above. Such simulations contain bifurcation diagrams, phase portrait, and the maximal Lyapunov exponents (MLEs), to further investigate the unpredictable behavior of the game. We will study the impact of the game parameters on dynamics of game (12), the speed of adjustment parameter k , the generalized bounded rationality parameter β , and the maximum price in the market a , and these are discussed in the following.

More specifically, we illustrate the stabilizing effect of the generalized bounded rationality on the dynamics of game (12). To study this effect, we choose the fractional parameter $0 < \beta < 1$ and parameter k as bifurcation parameters (varied parameters) and other game parameters as fixed parameters, otherwise stated. Let us take the parameters by the following values $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $\beta = 0.9$. The initial state of game (12) is $(0.3, 0.5)$. Figure 1(a) shows the bifurcation diagram of game (12) with respect to k ; it is clear that the equilibrium points become locally stable when the

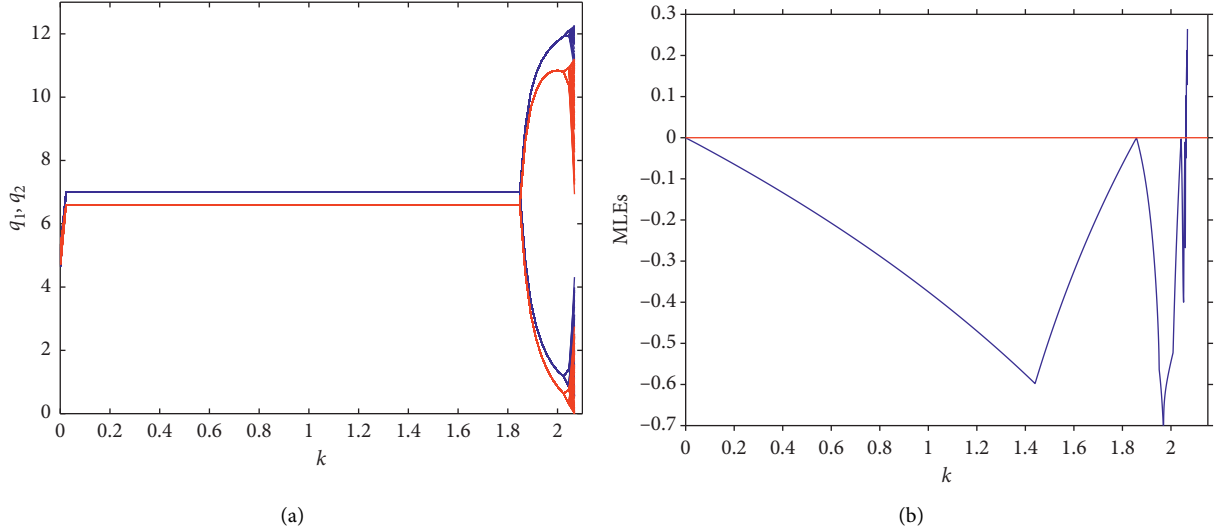


FIGURE 1: Bifurcation diagram and MLEs for game (12) with respect to control parameter k at $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $\beta = 0.9$.

parameter approaches $k = 1.8$ where the appearance of period-doubling bifurcation exists. Therefore, any increase above this point makes the system enter the chaotic region. It is known that the positive Lyapunov exponent is a good indicator for chaos. The corresponding maximal Lyapunov exponents are plotted in Figure 1(b). Obviously, the period-doubling bifurcation arises as k reaches the value of $k = 1.8$. After that, the Nash equilibrium point loses its stability as k increases.

A strange attractor can be seen in Figure 2, when the dynamics of the game becomes very complicated. Bifurcation diagram of game (12) as a function of k , with $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $\beta = 0.6$, is displayed in Figure 3(a). The MLEs plot corresponding to Figure 3(a) of game (12) is shown in Figure 3(b). We see that the interior equilibrium changes from stable to unstable and loses its stability via flip bifurcation. Consequently, game (12) shows irregular and unpredictable behaviors in the interval $k \in (1.3, 1.45)$. We find a rise in change speed k playing a destabilizing role. When $k = 1.45$, the phase portrait is displayed in Figure 4.

We have chosen values for the parameter β close to 1 in Figure 1(a) (which means the memory adopted by buyers is close to the current state of the market where traditional bounded rationality may be used). It is also clear that the interval of stability is better than that when we use values of memory far from the current state of the market as shown in Figure 3(a), where we take $\beta = 0.6$.

The bifurcation diagrams and associated MLEs graphs for two different values of β are given in Figures 5(a) and 5(b) and Figures 6(a) and 6(b), respectively. The bifurcation diagram of game (12) is shown in Figure 5(a) at $\beta = 0.5$, and the corresponding maximal Lyapunov exponents are plotted in Figure 5(b). Also, a bifurcation diagram of game (12) with respect to k is given in Figure 6(a) at $\beta = 0.2$, and the corresponding maximal Lyapunov exponents are plotted in Figure 6(b). The bifurcation diagrams show that increasing values of the parameter k may destabilize the

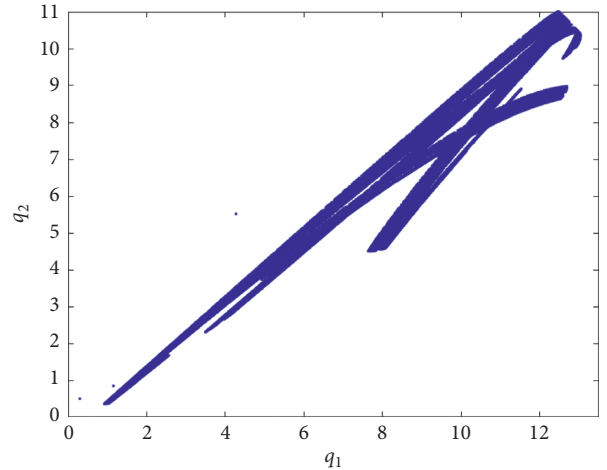


FIGURE 2: Strange attractor of the chaotic state of duopoly game (12), with the parameter values $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, $\beta = 0.9$, and $k = 2.1$.

interior equilibrium point through flip bifurcation. This means as k goes far from the current state of the market, the equilibrium point becomes unstable, and then, we claim that the memory effect of parameter should be in a range close to 1 where traditional bounded rationality may be used. After the occurrence of the bifurcation, period doubling exists and describes the long-run behavior of the game. Chaotic attractors exist after the accumulation of a period-doubling cascade; i.e., the dynamics of the game will become more and more confused. It is observed from Figures 1–6 that increasing the parameter k and fixing the generalized-order bounded rationality parameter a destabilize game (12) and chaotic behavior occurs. It is shown that game (12) is stabilized only for a relatively small value of the parameter k . A faster adjustment speed is disadvantageous for the game to keeping the stability of game (12).

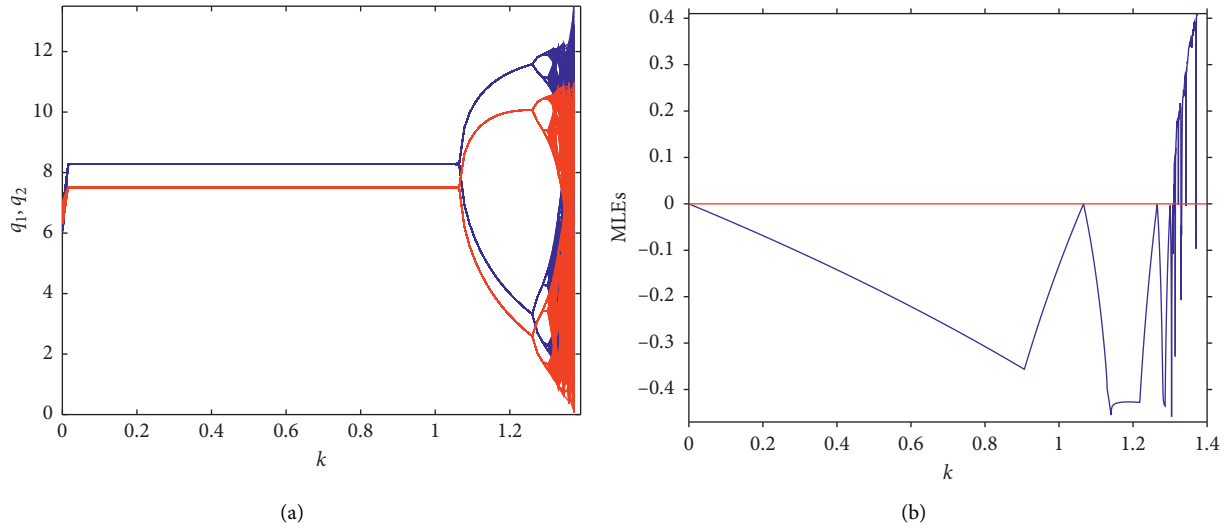


FIGURE 3: Bifurcation diagram and MLEs for game (12) with respect to control parameter k at $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $\beta = 0.6$.

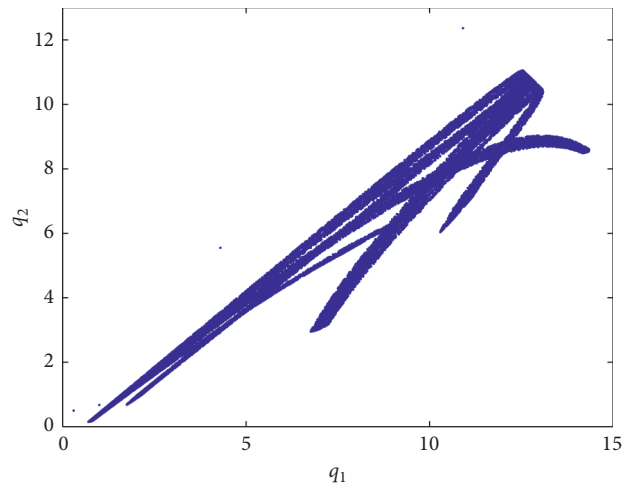


FIGURE 4: Phase portraits of the chaotic attractor of game (12) are shown using parameters values $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, $\beta = 0.6$, and $k = 1.4$.

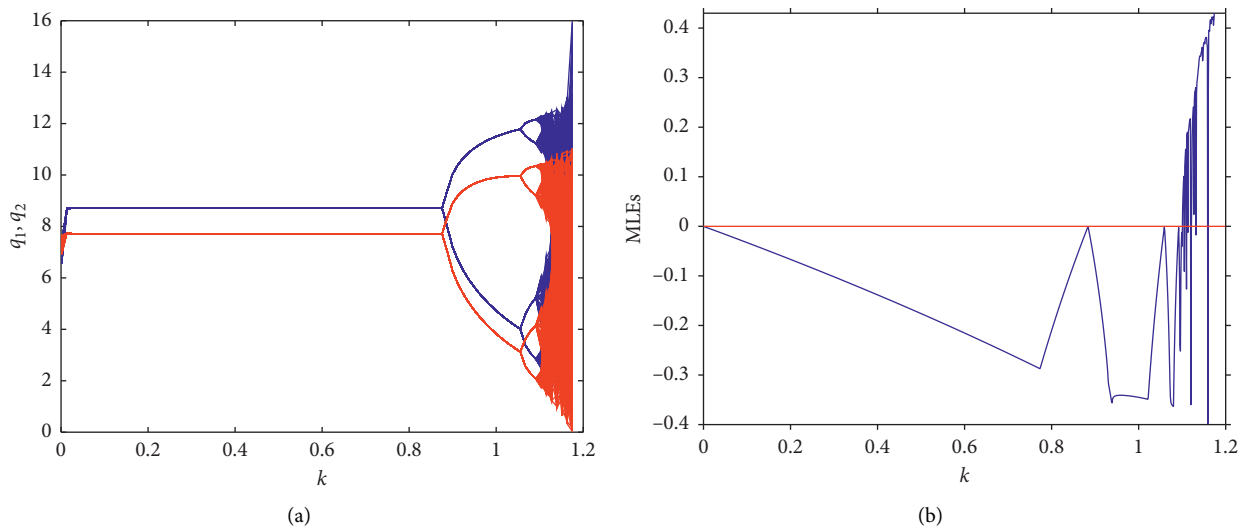


FIGURE 5: Bifurcation diagram and MLEs for game (12) as a function of k at $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $\beta = 0.5$.

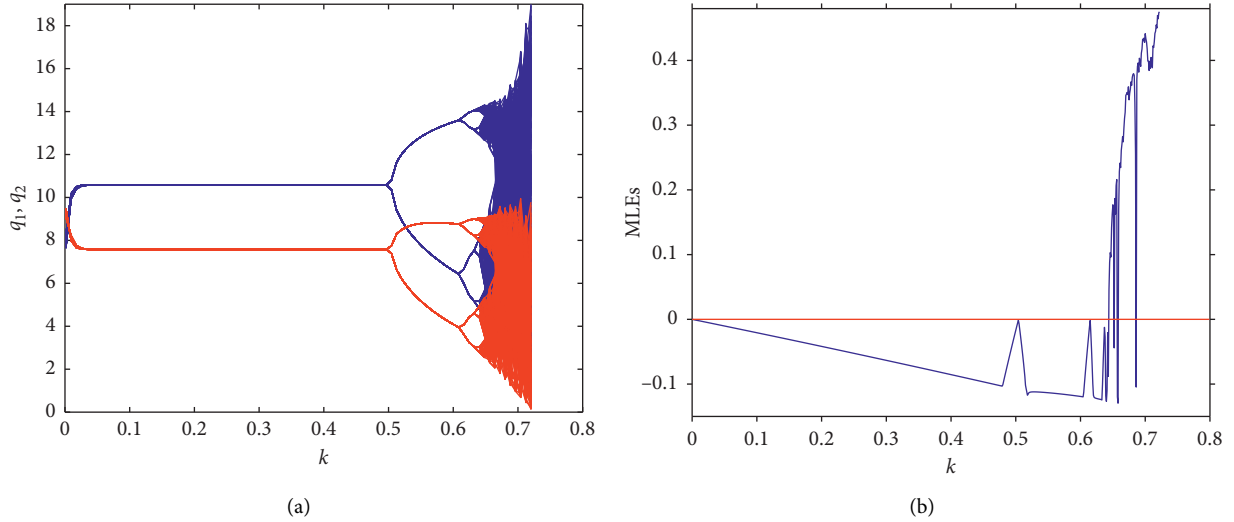


FIGURE 6: The bifurcation diagram and MLEs for game (12) with respect to control parameter k at $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $\beta = 0.2$.

In Figure 7, we illustrate the impact of the parameter a on the dynamics of game (12). One can deduce that the interior equilibrium changes from stable to unstable, leading to increasingly complex attractors as a increases. From the above analysis, a high level of the speed of adjustment k and the maximum price in the market a lead to instability of the game.

4.1. Effect of the Generalized Bounded Rationality Method on the Dynamics of the Game. To study the effect of the generalized bounded rationality method on the dynamics of game (12), under variations of the parameter β , we will analyze the dynamics of game (12). We have plotted the bifurcation diagrams of game (12) with respect to the parameter β for different values of the parameter k . The bifurcation diagram as a function of β when $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $k = 0.6$ is displayed in Figure 8. As can be seen in Figure 8, game (12) loses insatiability and enters the stability region with increase of β .

The bifurcation diagram of game (12) with respect to β , with $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $k = 0.9$, is given in Figure 9. It is observed that the size of chaotic attractor, or in other words, the amplitude of chaotic fluctuation in quantity outputs, decreases as β increases. When the parameter has values $a = 6$, $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, and $k = 1.2$, the bifurcation diagram as a function of β is plotted in Figure 10. This bifurcation diagram describes that bifurcation of the backward flip occurs at $\beta = 0.65$ from the interior equilibrium point. It demonstrates that firms have a better chance of achieving the equilibrium point with an increase of β with various adjustment speed values. Therefore, it can be observed that the stability chance of interior equilibrium point when $\beta < 0.65$ is less than the one for $\beta > 0.65$ and that there is an optimal value corresponding to the most probability of stability in certain cases for $\beta \in (0.65, 1)$. Thus, it is shown that the generalized bounded rationality parameter β has an effect on the dynamics of the game. The stabilization can be

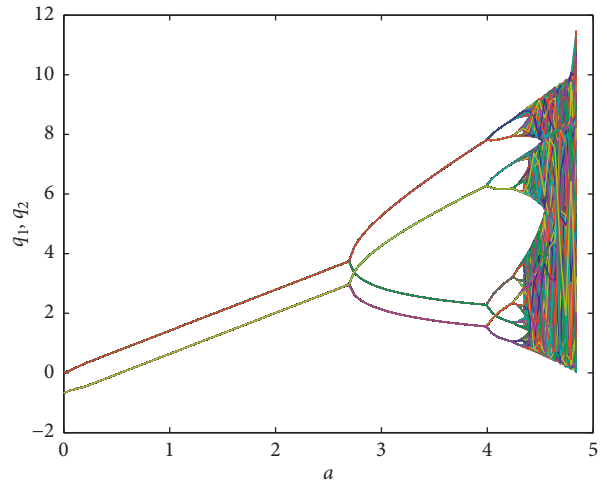


FIGURE 7: Bifurcation diagrams for game (12) as a function of a obtained at $b = 0.3$, $c_1 = 0.2$, $c_2 = 0.3$, $k = 1.5$, and $\beta = 0.5$.

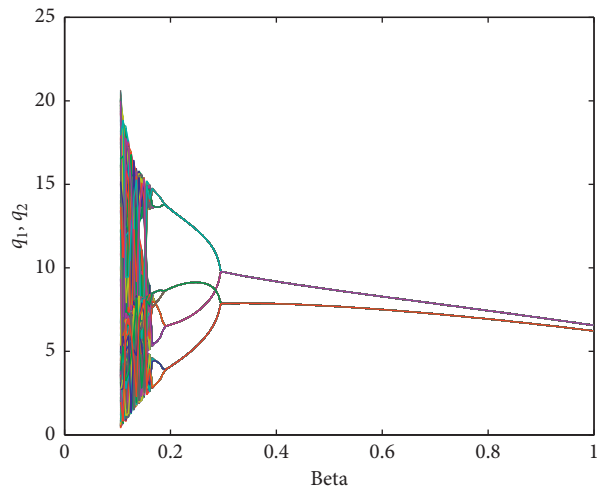


FIGURE 8: Bifurcation diagram for game (12) as a function of control parameter β when $k = 0.6$.

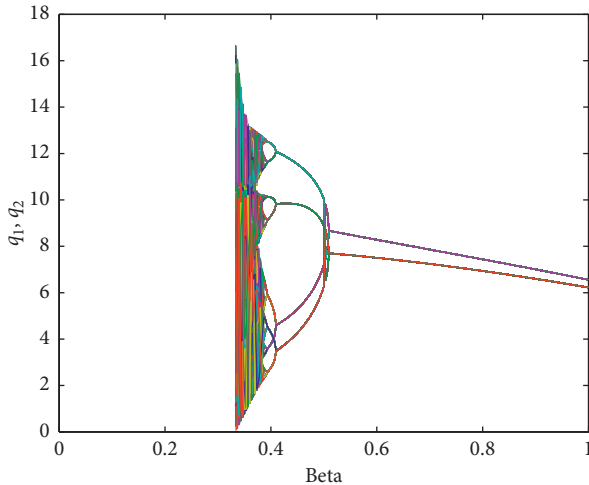


FIGURE 9: Bifurcation diagram for game (12) as a function of control parameter β when $k = 0.9$.

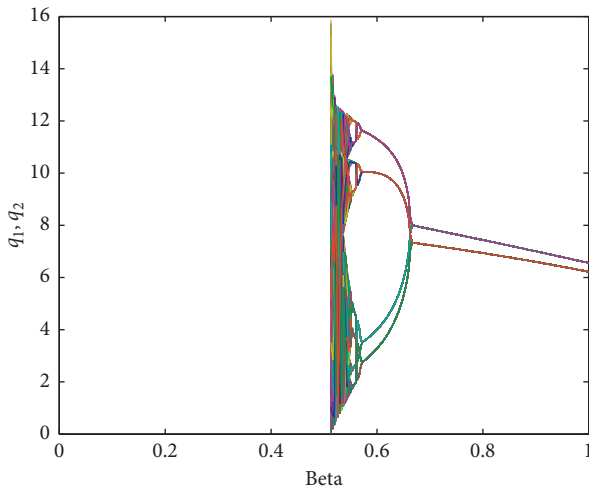


FIGURE 10: Bifurcation diagram for game (12) as a function of control parameter β when $k = 1.2$.

achieved if the generalized bounded rationality parameter takes values close to 1 where traditional bounded rationality may be used.

5. Conclusion

This paper has presented Cournot duopoly game based on generalized bounded rationality. The generalized bounded rationality method has been presented to study the dynamics of the Cournot duopoly game. The Cournot duopoly game with the fractional marginal profit approach has been analyzed on the stability of equilibria, bifurcation, and chaotic behaviors. Our motivation is to show the effect of buyer's memory when it becomes close to the current state of the market. The numerical results have investigated the dynamic behavior of duopoly game with generalized bounded rationality for different values of the memory parameter β . The basic properties of the game have been analyzed by meaning of bifurcation diagrams, the maximal Lyapunov exponents,

and phase portraits. Memory is a key economic factor. The effect of the generalized bounded rationality method has shown that it has a stabilizing effect on the dynamics of the game. This stabilization can be achieved if generalized bounded rationality parameter β takes values close to 1 where traditional bounded rationality may be used. Our obtained results have given interesting results regarding the memory effect on the game stabilization.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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References

- [1] A. Cournot, *Researches into the Mathematical Principles of the Theory of Wealth*, Macmillan, New York, NY, USA, 1897.
- [2] D. Furth, "Stability and instability in oligopoly," *Journal of Economic Theory*, vol. 40, no. 2, pp. 197–228, 1986.
- [3] R.-A. Dana and L. Montrucchio, "Dynamic complexity in duopoly games," *Journal of Economic Theory*, vol. 40, no. 1, pp. 40–56, 1986.
- [4] T. Puu, "Complex dynamics with three oligopolists," *Chaos Solitons Fractals*, vol. 12, pp. 207–581, 1996.
- [5] T. Puu, "On the stability of Cournot equilibrium when the number of competitors increases," *Journal of Economic Behavior & Organization*, vol. 66, pp. 445–456, 2007.
- [6] L. C. Corchon and A. Mas-Colell, "On the stability of best reply and gradient systems with applications to imperfectly competitive models," *Economics Letters*, vol. 51, no. 1, pp. 59–65, 1996.
- [7] G. Bischi and A. Naimzada, "Global analysis of a duopoly game with bounded rationality," *Advance Dynamic Games and Applications*, vol. 5, pp. 361–385, 1999.
- [8] H. N. Agiza and A. A. Elsadany, "Chaotic dynamics in nonlinear duopoly game with heterogeneous players," *Applied Mathematics and Computation*, vol. 149, no. 3, pp. 843–860, 2004.
- [9] T. Dubiel-Teleszynski, "Nonlinear dynamics in a heterogeneous duopoly game with adjusting players and diseconomies of scale," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 296–308, 2011.
- [10] Z. Sun and J. Ma, "Complexity of triopoly price game in Chinese cold rolled steel market," *Nonlinear Dynamics*, vol. 67, no. 3, pp. 2001–2008, 2012.
- [11] G. I. Bischi, C. Chiarella, M. Kopel, and F. Szidarovszky, *Nonlinear Oligopolies: Stability and Bifurcations*, Springer, Berlin, Germany, 2009.
- [12] M. Bai and Y. Gao, "Chaos control on a duopoly game with homogeneous strategy," *Discrete Dynamics in Nature and Society*, vol. 2016, Article ID 7418252, 7 pages, 2016.

- [13] S. S. Askar, A. M. Alshamrani, and K. Alnowibet, "The arising of cooperation in cournot duopoly games," *Applied Mathematics and Computation*, vol. 273, pp. 535–542, 2016.
- [14] C. Hommes, *Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems*, Cambridge University Press, Cambridge, UK, 2013.
- [15] F. Tramontana, "Heterogeneous duopoly with isoelastic demand function," *Economic Modelling*, vol. 27, no. 1, pp. 350–357, 2010.
- [16] L. Fanti and L. Gori, "The dynamics of a differentiated duopoly with quantity competition," *Economic Modelling*, vol. 29, no. 2, pp. 421–427, 2012.
- [17] G. Sarafopoulos, "On the dynamics of a duopoly game with differentiated goods," *Procedia Economics and Finance*, vol. 19, pp. 146–153, 2015.
- [18] S. S. Askar and A. Al-khedhairi, "Analysis of nonlinear duopoly games with product differentiation: stability, global dynamics, and control," *Discrete Dynamics in Nature and Society*, vol. 2017, Article ID 2585708, 13 pages, 2017.
- [19] F. Tramontana and A. E. A. Elsadany, "Heterogeneous triopoly game with isoelastic demand function," *Nonlinear Dynamics*, vol. 68, no. 1-2, pp. 187–193, 2012.
- [20] J. L. G. a. Guirao, M. Lampart, and G. H. Zhang, "On the dynamics of a 4D local Cournot model," *Applied Mathematics & Information Sciences*, vol. 7, no. 3, pp. 857–865, 2013.
- [21] J.-g. Du, Y.-q. Fan, Z.-h. Sheng, and Y.-z. Hou, "Dynamics analysis and chaos control of a duopoly game with heterogeneous players and output limiter," *Economic Modelling*, vol. 33, pp. 507–516, 2013.
- [22] J. Ma and Z. Guo, "The parameter basin and complex of dynamic game with estimation and two-stage consideration," *Applied Mathematics and Computation*, vol. 248, pp. 131–142, 2014.
- [23] L. Zhao and J. Zhang, "Analysis of a duopoly game with heterogeneous players participating in carbon emission trading," *Nonlinear Analysis: Modelling and Control*, vol. 19, no. 1, pp. 118–131, 2014.
- [24] W. Yu and Y. Yu, "The complexion of dynamic duopoly game with horizontal differentiated products," *Economic Modelling*, vol. 41, pp. 289–297, 2014.
- [25] H. N. Agiza, A. A. Elsadany, and M. M. El-Dessoky, "On a new Cournot duopoly game," *Jason Chaos*, vol. 5, p. 487803, 2013.
- [26] L. Fanti, "The dynamics of a banking duopoly with capital regulations," *Economic Modelling*, vol. 37, pp. 340–349, 2014.
- [27] A. A. Elsadany and A. E. Matouk, "Dynamic Cournot duopoly game with delay," *Journal of Complex Systems*, vol. 2014, pp. 384843–7, 2014.
- [28] A. A. Elsadany, "Dynamics of a Cournot duopoly game with bounded rationality based on relative profit maximization," *Applied Mathematics and Computation*, vol. 294, pp. 253–263, 2017.
- [29] L. Fanti, L. Gori, C. Mammana, and E. Michetti, "Local and global dynamics in a duopoly with price competition and market share delegation," *Chaos, Solitons & Fractals*, vol. 69, pp. 253–270, 2014.
- [30] L. Fanti, L. Gori, and M. Sodini, "Nonlinear dynamics in a Cournot duopoly with isoelastic demand," *Mathematics and Computers in Simulation*, vol. 108, pp. 129–143, 2015.
- [31] F. Cavalli, A. Naimzada, and F. Tramontana, "Nonlinear dynamics and global analysis of a heterogeneous Cournot duopoly with a local monopolistic approach versus a gradient rule with endogenous reactivity," *Communications in Nonlinear Science and Numerical Simulation*, vol. 23, no. 1-3, pp. 245–262, 2015.
- [32] Z. Ding, Q. Li, S. Jiang, and X. Wang, "Dynamics in a Cournot investment game with heterogeneous players," *Applied Mathematics and Computation*, vol. 256, pp. 939–950, 2015.
- [33] S. S. Askar and A. Al-khedhairi, "Cournot duopoly games: models and investigations," *Mathematics*, vol. 7, no. 11, p. 1079, 2019.
- [34] B. Tian, Y. Zhang, and J. Li, "Stochastic perturbations for a duopoly Stackelberg model," *Physica A: Statistical Mechanics and Its Applications*, vol. 545, p. 123792, 2019.
- [35] L. Zhao, J. Du, and Q. Wang, "Nonlinear analysis and chaos control of the complex dynamics of multi-market Cournot game with bounded rationality," *Mathematics and Computers in Simulation*, vol. 162, pp. 45–57, 2019.
- [36] Y. Peng, Q. Lu, Y. Xiao, and X. Wu, "Complex dynamics analysis for a remanufacturing duopoly model with nonlinear cost," *Physica A: Statistical Mechanics and Its Applications*, vol. 514, pp. 658–670, 2019.
- [37] L. Cerboni Baiardi and A. K. Naimzada, "An evolutionary Cournot oligopoly model with imitators and perfect foresight best responders," *Metroeconomica*, vol. 70, no. 3, pp. 458–475, 2019.
- [38] A. Al-khedhairi, "Dynamical analysis of fractional-order differentiated Cournot triopoly game," *Journal of Vibration and Control*, 2020.
- [39] S. S. Askar and A. Al-Khedhairi, "The dynamics of a business game: a 2D-piecewise smooth nonlinear map," *Physica A: Statistical Mechanics and Its Applications*, vol. 537, p. 122766, 2020.
- [40] S. S. Askar and M. Abouhawwash, "Quantity and price competition in a differentiated triopoly: static and dynamic investigations," *Nonlinear Dynamics*, vol. 91, no. 3, pp. 1963–1975, 2018.
- [41] R. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, Hoboken, NJ, USA, 1993.
- [42] T. Puu, *Attractors, Bifurcations and Chaos: Nonlinear Phenomena in Economica*, Springer, Berlin, Germany, 2000.