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Research Article

Time-Space Fractional Model for Complex Cylindrical Ion-Acoustic Waves in Ultrarelativistic Plasmas

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In this paper, the fractional order models are used to study the propagation of ion-acoustic waves in ultrarelativistic plasmas in nonplanar geometry (cylindrical). Firstly, according to the control equations, $(2+1)$-dimensional (2D) cylindrical Kadomtsev–Petviashvili (CKP) equation and 2D cylindrical-modified Kadomtsev–Petviashvili (CMKP) equation are derived by using multiscale analysis and reduced perturbation methods. Secondly, using the semi-inverse method and the fractional variation principle, the abovementioned equations are derived the time-space fractional equations (TSF-CKP and TSF-CMKP). Furthermore, based on the fractional order transformation, the 1-decay mode solution of the TSF-CKP equation is obtained by using the simplified homogeneous balance method, and using the generalized hyperbolic-function method, the exact analytic solution of TSF-CMKP equation is obtained. Finally, the effects of the phase speed λ, electron number density (through β_3) and the fractional order (α, β, ω) on the propagation of ion-acoustic waves in ultrarelativistic plasmas are analyzed.

1. Introduction

In recent years, plasma physics [1–4] has developed rapidly in the global environment, electromagnetic propagation, and especially in the astronomical environment. In many astronomical environments, matters exist in extremely dense conditions and are not found in the Earth's environment, and there is a strong interest in understanding the basic properties of matter under extreme dense conditions. In some interstellar dense objects (e.g., white dwarfs and neutron stars), the extreme conditions of matter are caused by the significant compression of the interstellar medium. One of these extreme conditions is the presence of high density degenerate substances in these compact objects. These objects are actually "relics of stars." They have stopped burning thermonuclear fuel and are no longer able to generate the thermal pressure needed to support the gravitational load of their own mass. These interstellar compact objects are greatly compressed, and their internal density becomes very high. Therefore, the nonthermal pressure is provided by degenerate fermion kinetic energy and particle interactions. This pressure is not sensitive to temperature, i.e., the pressure does not go down as the star cools.

Based on observations and theoretical analysis, there are two types of dense objects that resist gravitational collapse under the action of cold degenerate fermions/electron pressure. An example of the first category is a white dwarf supported by the pressure of degenerate electrons, the interior of which is close to a dense solid (ionic lattice surrounded by degenerate electrons, possibly other heavy particles or dust), and an example of the second category is a neutron star supported by the pressure of nuclear degeneracy and nuclear interaction, which is close to a giant atomic nucleus (mixture of nucleon and electron interactions, possibly other elementary particles and condensates). In such a compact object, the degenerate electron number density is very high. For example, in a white dwarf, the degenerate electron number density can reach the order of $10^{30}\,\mathrm{cm}^{-3}$, which is expected to be degenerated in a white dwarf of sufficient quality. Electrons become relativistic, making stars prone to gravitational collapse, and the declining electron density in neutron stars can even reach the order of 10^{36} cm^{-3}. In the white dwarf and neutron stars, the energy of an electron Fermi is comparable to the energy of an electron resting mass, and the speed of an electron can be comparable to the speed of light in a vacuum [5, 6].

Chandrasekhar [7–9] mathematically explains the equation of state of degenerate electrons in such interstellar dense objects, involving two limits, namely, nonrelativistic and ultrarelativistic limits. The degenerate electron equation of state of Chandrasekhar [7–9] is used viz. $P_e \propto n_e^{5/3}$ (nonrelativistic limit) and $P_e \propto n_e^{4/3}$ (ultrarelativistic limit), where P_e is the degenerate electron pressure and n_e is the degenerate electronic number density. We note that the degenerate electron pressure is only related to the electron number density and not to the electron temperature. These close interstellar objects provide us with a cosmic laboratory that studies the properties of matter (materials) and the fluctuations and instability in a state of extreme density degradation in such media. This has led to the study of linear and nonlinear ion-acoustic wave properties in ultrarelativistic plasmas.

Ion-acoustic waves [10–12] are one of the basic wave processes in plasma. A large number of theoretical and experimental studies have been conducted on ion-acoustic waves for a long time. In the early days, many researchers have extensively studied the nonlinear propagation of ion-acoustic waves in planar geometry [13, 14], such as the Korteweg–de Vries (KdV) equation [15] to describe the propagation of ion-acoustic waves in a one-dimensional plane geometry in plasma. However, for some practical situations such as astronomical environment, laboratory, and space plasma, there is a clear deficiency in one-dimensional planar geometry. Therefore, many scholars have begun to study the propagation of solitary waves in nonplanar cylindrical geometry. For example, Mamun [16] used modified Korteweg–de Vries (mKdV) equation to study the propagation of solitary waves in nonplanar geometry in dust plasma. Mushtaq [17] found that quantum correction and lateral perturbation in cylindrical geometry have effects on solitary wave propagation.

The calculus invented by Newton and Leibniz is a watershed between modern mathematics and classical mathematics. Fractional calculus [18, 19] is a theory of arbitrarily order differential and integral, and it is uniform with integer calculus and is a generalization of integer calculus. For centuries, the theoretical study of fractional calculus has been very limited, mainly in the field of mathematics. For example, Euler, Liouville, Riemann, and Caputo all contribute significantly to the field of fractional calculus. Most of them first introduce fractional order integrals and define fractional derivatives on this basis. However, after a series of improvements, the concept of fractional derivatives [20–22] is more suitable for the modeling of practical problems than the integer derivatives. Fractional differential equations [23–25] are also beginning to be used to solve problems in the fields of biological systems, thermal systems, and mechanical systems, and their practicality is getting stronger. Scholars pay great attention to the fractional calculus theory, especially the fractional differential equations abstracted from practical problems. Fractional calculus has attracted more and more scholars' research interest. Compared with the integer order model, the fractional order model can better describe the dynamic response of the actual system, improve the performance of the dynamic system, and solve practical problems. However, in the research of plasma physics, most of the models are established in integer order [26, 27]. This makes it necessary to establish a fractional order model to describe and study the propagation of solitary waves in plasma.

With the continuous development of the research on fractional differential equations [28–30], the solution of fractional differential equations becomes an important subject. The solution of fractional equations is the key means to solve and analyze the problem. At present, based on the method of solving integer differential equations [31–34], there are many methods for calculating the exact solutions and numerical solutions of fractional differential equations, such as (G/G')-expansion method [35, 36], the first integral method [37], $\exp(-\varphi(\xi))$ method [38], and Hirota bilinear method [31].

The rest of the paper is organized as follows. In Section 2, based on the control equations, the multiscale analysis and perturbation expansion method [39] are used to derive the CKP equation and CMKP equation of integer order. In Section 3, applying the semi-inverse method and the fractional variational principle [40], the integer order equations (CKP and CMKP) are, respectively, transformed into time-space fractional equations (TSF-CKP and TSF-CMKP). In Section 4, based on the fractional order transformation, the exact solutions of the TSF-CKP and TSF-CMKP equation are obtained by using the simplified homogeneous balance method [41] and the generalized hyperbolic-function method [42], respectively. In Section 5, the effects of the phase speed λ, electron number density (through β_3), temperature ratio σ_i, and fractional order value α, β, and ω on the ion-acoustic waves propagation in ultrarelativistic plasmas are studied.

2. Construct of Integer Order Models

We consider the nonlinear propagation of ion-acoustic waves in ultrarelativistic plasmas. In order to better deal with the problems studied, we made the necessary assumptions: the plasma model studied has no external magnetic field effect, i.e., unmagnetized and no collision effect, i.e., collisionless. The plasma studied in this paper consists of two parts: warm ions and ultrarelativistic degenerate electrons. At equilibrium, the electrical neutral condition is $n_{i0} = n_{e0} = n_0$, where n_{i0} is the undisturbed ion number density and n_{e0} is the undisturbed electron number density. Assuming that the electron fluid follows the equation of state in this form $P_e = hc/4\,(3/8\pi)^{1/3} n_e^{4/3}$, the nonlinear dynamics of the low frequency electrostatic ion-acoustic waves in such an ultrarelativistic degenerate dense plasma in 2D cylindrical geometry can be described by the following set of normalized equations:

$$\begin{cases} \frac{\partial n_i}{\partial t} + \frac{1}{r}\frac{\partial(rn_iu_i)}{\partial r} + \frac{1}{r}\frac{\partial(n_iv_i)}{\partial\theta} = 0, \\ \frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial r} + \frac{v_i}{r}\frac{\partial u_i}{\partial\theta} - \frac{v_i^2}{r} = -\frac{\partial\phi}{\partial r} - \sigma_i\frac{\partial n_i}{\partial r}, \\ \frac{\partial v_i}{\partial t} + u_i\frac{\partial v_i}{\partial r} + \frac{v_i}{r}\frac{\partial v_i}{\partial\theta} - \frac{u_iv_i}{r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{\sigma_i}{r}\frac{\partial n_i}{\partial\theta}, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = n_e - n_i, \\ n_e\left(\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) - \frac{3\beta_3}{4}\left(\frac{\partial n_e^{4/3}}{\partial r} + \frac{1}{r}\frac{\partial n_e^{4/3}}{\partial\theta}\right) = 0, \end{cases} \tag{1}$$

where $\sigma_i = 2T_i/T_{Fe}$ is the ratio of ion temperature to Fermi electron temperature, and $T_{Fe} = ((3\pi^2 n_{e0})^{2/3}\hbar^2)/2m_e k_B$ is the Fermi electron temperature. $\beta = Kn_{eo}^{1/3}/k_B T_{Fe}$, where $K = (3/4)\hbar c$, $\hbar$ is Planck constant divided by 2π, and c is the speed of light in a vacuum. The electron number density n_e and ion number density n_i are normalized by n_0, u_i, and n_i are, respectively, the velocity components of the ion-fluid in the direction of r and θ, and they are normalized by the ion-acoustic waves velocity C_i, and $C_i = (k_B T_{Fe}/m_i)^{1/2}$ where m_i is the ion mass. The electrostatic potential ϕ is normalized by $k_B T_{Fe}/e$, where e is the electronic charge. The spatial coordinates (r, θ) are normalized by the Debye radius $\lambda_D = (k_B T_{Fe}/4\pi n_{i0}e^2)^{1/2}$, the time t is normalized by the ion plasma period $\omega_{pi}^{-1} = (m_i/4\pi n_{i0}e^2)^{1/2}$, where k_B is the Boltzmann constant.

Equation (1) is a complex set of nonlinear equations. In order to study the motion of nonlinear ion-acoustic waves in ultrarelativistic degenerate plasmas with small amplitude, we use the reductive perturbation method to simplify the complex nonlinear equations into differential equations and retain the most important nonlinear part of the original equations. First, expand the independent variables as follows:

$$\begin{aligned} R &= \epsilon(r - \lambda t), \\ \Theta &= \epsilon^{-1}\theta, \\ T &= \epsilon^3 t, \end{aligned} \tag{2}$$

where ϵ is a small parameter characterizing the strength of nonlinearity.

According to equation (2), we have

$$\begin{aligned} \frac{\partial}{\partial t} &= \epsilon^3\frac{\partial}{\partial T} - \epsilon\lambda\frac{\partial}{\partial R}, \\ \frac{\partial}{\partial\theta} &= \epsilon^{-1}\frac{\partial}{\partial\Theta}, \\ \frac{\partial}{\partial r} &= \epsilon\frac{\partial}{\partial R}. \end{aligned} \tag{3}$$

The dependent variables n_e, u_i, u_i, v_i, and ϕ are expanded as follows:

$$\begin{cases} n_e = 1 + \epsilon^2 n_{e1} + \epsilon^4 n_{e2} + \cdots, \\ n_i = 1 + \epsilon^2 n_{i1} + \epsilon^4 n_{i2} + \cdots, \\ u_i = \epsilon^2 u_{i1} + \epsilon^4 u_{i2} + \cdots, \\ v_i = \epsilon^3 v_{i1} + \epsilon^5 v_{i2} + \cdots, \\ \phi = \epsilon^2\phi_1 + \epsilon^4\phi_2 + \cdots. \end{cases} \tag{4}$$

Substituting equations (3) and (4) into equation (1), we obtain the following equations:

$$\begin{cases} \epsilon^2\frac{\partial n_i}{\partial T} - \lambda\frac{\partial n_i}{\partial R} + \epsilon^2\frac{1}{\lambda T}n_iu_i + \frac{\partial(n_iu_i)}{\partial R} + \frac{\epsilon}{\lambda T}\frac{\partial(n_iv_i)}{\partial\Theta} = 0, \\ \epsilon^2\frac{\partial u_i}{\partial T} - \lambda\frac{\partial u_i}{\partial R} + u_i\frac{\partial u_i}{\partial R} + \frac{\epsilon}{\lambda T}v_i\frac{\partial u_i}{\partial\Theta} - \frac{\epsilon^2}{\lambda T}v_i^2 = -\frac{\partial\phi}{\partial R} - \sigma_i\frac{\partial n_i}{\partial R}, \\ \epsilon^2\frac{\partial v_i}{\partial T} - \lambda\frac{\partial v_i}{\partial R} + u_i\frac{\partial v_i}{\partial R} + \frac{\epsilon}{\lambda T}v_i\frac{\partial v_i}{\partial\Theta} - \frac{\epsilon^2}{\lambda T}v_iu_i = -\frac{\epsilon}{\lambda T}\frac{\partial\phi}{\partial\Theta} - \frac{\epsilon\sigma_i}{\lambda T}\frac{\partial n_i}{\partial\Theta}, \\ \frac{\epsilon^4}{\lambda T}\frac{\partial\phi}{\partial R} + \epsilon^2\frac{\partial^2\phi}{\partial R^2} + \frac{\epsilon^4}{(\lambda T)^2}\frac{\partial^2\phi}{\partial\Theta^2} = n_e - n_i, \\ n_e\left(\frac{\partial\phi}{\partial R} + \frac{\epsilon}{\lambda T}\frac{\partial\phi}{\partial\Theta}\right) - \beta_3 n_{\frac{1}{3}}\left(\frac{\partial n_e}{\partial R} + \frac{\epsilon}{\lambda T}\frac{\partial n_e}{\partial\Theta}\right) = 0. \end{cases} \tag{5}$$

According to the different power expansions of ϵ, we obtain

$$\epsilon^2: \begin{cases} -\lambda\frac{\partial n_{i1}}{\partial R} + \frac{\partial u_{i1}}{\partial R} = 0, \\ -\lambda\frac{\partial u_{i1}}{\partial R} = -\frac{\partial\phi_1}{\partial R} - \sigma_i\frac{\partial n_{i1}}{\partial R}, \\ n_{e1} - n_{i1} = 0, \\ \frac{\partial\phi_1}{\partial R} - \beta_3\frac{\partial n_{e1}}{\partial R} = 0, \end{cases} \tag{6}$$

$$\epsilon^3: \left\{ -\lambda\frac{\partial v_{i1}}{\partial R} = -\frac{1}{\lambda T}\frac{\partial\phi_1}{\partial\Theta} - \frac{\sigma_i}{\lambda T}\frac{\partial n_{i1}}{\partial\Theta}, \right. \tag{7}$$

$$\epsilon^4: \begin{cases} \frac{\partial n_{i1}}{\partial T} - \lambda\frac{\partial n_{i2}}{\partial R} + \frac{\partial u_{i2}}{\partial R} + \frac{\partial n_{i1}u_{i1}}{\partial R} + \frac{n_{i1}}{\lambda T} + \frac{1}{\lambda T}\frac{\partial v_{i1}}{\partial\Theta} = 0, \\ \frac{\partial u_{i1}}{\partial T} - \lambda\frac{\partial u_{i2}}{\partial R} + u_{i1}\frac{\partial u_{i1}}{\partial R} + \frac{\partial\phi_2}{\partial R} + \sigma_i\frac{\partial n_{i2}}{\partial R} = 0, \\ \frac{\partial^2\phi_1}{\partial R^2} = n_{e1} - n_{i1}, \\ n_{e1}\frac{\partial\phi_1}{\partial R} + \frac{\partial\phi_2}{\partial R} = \beta_3\frac{\partial n_{e2}}{\partial R}. \end{cases} \tag{8}$$

The following equations are obtained under the low-order approximation of ε:

$$\begin{cases} u_{i1} = \lambda n_{i1}, \\ -\lambda^2 n_{i1} = -\phi_1 - \sigma_i n_{i1}, \\ -\phi_1 = \left(\sigma_i - \lambda^2\right) n_{i1}, \\ n_{i1} = -\dfrac{1}{\sigma_i - \lambda^2}\phi_1, \\ u_{i1} = -\dfrac{\lambda}{\sigma_i - \lambda^2}\phi_1, \\ n_{e1} = \dfrac{\phi_1}{\beta}, \\ \beta_3 = \lambda^2 - \sigma_i, \\ \dfrac{\partial v_{i1}}{\partial R} = -\dfrac{1}{T\left(\sigma_i - \lambda^2\right)} \dfrac{\partial \phi_1}{\partial \theta}. \end{cases} \tag{9}$$

Under the high-order approximation of ε, we obtain the $(2+1)$ dimensional CKP equation, i.e.,

$$\frac{\partial}{\partial R}\left(\frac{\partial \phi_1}{\partial T} + A\phi_1 \frac{\partial \phi_1}{\partial R} + B\frac{\partial^3 \phi_1}{\partial R^3} + \frac{1}{2T}\phi_1\right) + \frac{1}{2\lambda T^2}\frac{\partial^2 \phi_1}{\partial \theta^2} = 0, \tag{10}$$

where

$$\begin{aligned} A &= \frac{2\lambda^2 + \sigma_i}{2\lambda\left(\lambda^2 - \sigma_i\right)}, \\ B &= \frac{\left(\sigma_i - \lambda^2\right)^2}{2\lambda}. \end{aligned} \tag{11}$$

Next, we use the same method to derive the $(2+1)$-dimensional CMKP equation. The independent expansion is the same as equation (2), and the new perturbation expansion is as follows:

$$\begin{cases} n_e = 1 + \epsilon n_{e1} + \epsilon^2 n_{e2} + \cdots, \\ n_i = 1 + \epsilon n_{i1} + \epsilon^2 n_{i2} + \cdots, \\ u_i = \epsilon u_{i1} + \epsilon^4 u_2 + \cdots, \\ v_i = \epsilon^2 v_{i1} + \epsilon^3 v_{i2} + \cdots, \\ \phi = \epsilon\phi_1 + \epsilon^2 \phi_2 + \cdots. \end{cases} \tag{12}$$

Substituting equations (3) and (12) into equation (1), we obtain the following equations:

$$\begin{cases} \epsilon^2 \dfrac{\partial n_i}{\partial T} - \lambda \dfrac{\partial n_i}{\partial R} + \epsilon^2 \dfrac{1}{\lambda T} n_i u_i + \dfrac{\partial\left(n_i u_i\right)}{\partial R} + \dfrac{\epsilon}{\lambda T}\dfrac{\partial\left(n_i v_i\right)}{\partial \Theta} = 0, \\ \epsilon^2 \dfrac{\partial u_i}{\partial T} - \lambda \dfrac{\partial u_i}{\partial R} + u_i \dfrac{\partial u_i}{\partial R} + \dfrac{\epsilon}{\lambda T} v_i \dfrac{\partial u_i}{\partial \Theta} - \dfrac{\epsilon^2}{\lambda T} v_i^2 = -\dfrac{\partial \phi}{\partial R} - \sigma_i \dfrac{\partial n_i}{\partial R}, \\ \epsilon^2 \dfrac{\partial v_i}{\partial T} - \lambda \dfrac{\partial v_i}{\partial R} + u_i \dfrac{\partial v_i}{\partial R} + \dfrac{\epsilon}{\lambda T} v_i \dfrac{\partial v_i}{\partial \Theta} - \dfrac{\epsilon^2}{\lambda T} v_i u_i = -\dfrac{\epsilon}{\lambda T}\dfrac{\partial \phi}{\partial \Theta} - \dfrac{\epsilon\sigma_i}{\lambda T}\dfrac{\partial n_i}{\partial \Theta}, \\ \dfrac{\epsilon^4}{\lambda T}\dfrac{\partial \phi}{\partial R} + \epsilon^2 \dfrac{\partial^2 \phi}{\partial R^2} + \dfrac{\epsilon^4}{(\lambda T)^2}\dfrac{\partial^2 \phi}{\partial \Theta^2} = n_e - n_i, \\ n_e\left(\dfrac{\partial \phi}{\partial R} + \dfrac{\epsilon}{\lambda T}\dfrac{\partial \phi}{\partial \Theta}\right) - \beta_3 n^{1/3}\left(\dfrac{\partial n_e}{\partial R} + \dfrac{\epsilon}{\lambda T}\dfrac{\partial n_e}{\partial \Theta}\right) = 0. \end{cases} \tag{13}$$

Expanding ϵ in equation (13) from high to low power, according to the lowest power of ϵ, we obtain

$$\epsilon: \begin{cases} -\lambda \dfrac{\partial n_{i1}}{\partial R} + \dfrac{\partial u_{i1}}{\partial R} = 0, \\ -\lambda \dfrac{\partial u_{i1}}{\partial R} = -\dfrac{\partial \phi_1}{\partial R} - \sigma_i \dfrac{\partial n_{i1}}{\partial R}, \\ n_{e1} - n_{i1} = 0, \\ \dfrac{\partial \phi_1}{\partial R} - \beta_3 \dfrac{\partial n_{e1}}{\partial R} = 0. \end{cases} \tag{14}$$

According to equation (14), we have

$$\begin{cases} n_{i1} = \dfrac{-\phi_1}{\sigma_i - \lambda^2}, \\ u_{i1} = \dfrac{-\lambda\phi_1}{\sigma_i - \lambda^2}, \\ n_{e1} = \dfrac{\phi_1}{\beta_3}, \lambda = \left(\beta + \sigma_i\right)^{1/2}. \end{cases} \tag{15}$$

Under the lower power expansion of ϵ, we obtain the following equation:

$$\epsilon^2: \begin{cases} -\lambda \dfrac{\partial n_{i2}}{\partial R} + \dfrac{\partial\left(n_{i1}u_{i1}\right)}{\partial R} + \dfrac{\partial u_{i2}}{\partial R} = 0, \\ -\lambda \dfrac{\partial u_{i2}}{\partial R} + u_{i1}\dfrac{\partial\left(u_{i1}\right)}{\partial R} = -\dfrac{\partial \phi_2}{\partial R} - \sigma_i \dfrac{\partial n_{i2}}{\partial R}, \\ -\lambda \dfrac{\partial v_{i2}}{\partial R} = -\dfrac{1}{\lambda T}\dfrac{\partial \phi_1}{\partial \Theta} - \dfrac{\sigma_i}{\lambda T}\dfrac{\partial n_{i1}}{\partial \Theta}, \\ n_{e2} - n_{i2} = 0, \\ n_{e1}\dfrac{\partial \phi_1}{\partial R} + \dfrac{\partial \phi_2}{\partial R} + \dfrac{1}{\lambda T}\dfrac{\partial \phi_1}{\partial \Theta} = \beta_3 \dfrac{\partial n_{e2}}{\partial R} + \dfrac{\beta_3}{\lambda T}\dfrac{\partial n_{e1}}{\partial \Theta}. \end{cases} \tag{16}$$

According to equation (16), we have

$$\begin{cases} n_{i2} = \dfrac{-2\lambda^2\left(\sigma_i - \lambda^2\right) - \lambda^2}{2\left(\sigma_i - \lambda^2\right)^2}\phi_1^2 - \dfrac{1}{\sigma_i - \lambda^2}\phi_2, \\ u_{i2} = \dfrac{-2\lambda^3\left(\sigma_i - \lambda^2\right) - \lambda^3 - 2\lambda\left(\sigma_i - \lambda^2\right)^2}{2\left(\sigma_i - \lambda^2\right)^2}\phi_1^2 - \dfrac{\lambda}{\sigma_i - \lambda^2}\phi_2, \\ n_{e2} = \dfrac{1}{2\beta_3^2}\phi_1^2 + \dfrac{1}{\beta_3}\phi_2, \\ \phi_2 = \dfrac{-2\lambda^2\left(\sigma_i - \lambda^2\right) - \sigma_i}{2\left(\sigma_i - \lambda^2\right)^2\left(1 + \sigma_i - \lambda^2\right)}\phi_1^2, \\ \sigma_i = 2\beta_3\lambda^2, \\ \dfrac{\partial v_{i1}}{\partial R} = -\dfrac{1}{T\left(\sigma_i - \lambda^2\right)}\dfrac{\partial \phi_1}{\partial \Theta}. \end{cases} \tag{17}$$

Under the highest power expansion of ϵ, we obtain

$$\epsilon^3: \begin{cases} \dfrac{\partial n_{i1}}{\partial T} - \lambda\dfrac{\partial n_{i3}}{\partial R} + \dfrac{1}{\lambda T}u_{i1} + \dfrac{\partial u_{i3}}{\partial R} + \dfrac{\partial\left(n_{i1}u_{i2}\right)}{\partial R} + \dfrac{\partial\left(n_{i2}u_{i1}\right)}{\partial R} \\ \quad + \dfrac{1}{\lambda T}\dfrac{\partial v_{i1}}{\partial \Theta} = 0, \\ \dfrac{\partial u_{i1}}{\partial T} - \lambda\dfrac{\partial u_{i3}}{\partial R} + u_{i1}\dfrac{\partial\left(u_{i2}\right)}{\partial R} + u_{i2}\dfrac{\partial\left(u_{i1}\right)}{\partial R} = -\dfrac{1}{\lambda T}\dfrac{\partial \phi_2}{\partial \Theta} \\ \quad - \dfrac{\sigma_i}{\lambda T}\dfrac{\partial n_{i2}}{\partial \Theta}, \\ \dfrac{\partial^2 \phi_1}{\partial R^2} = n_{e3} - n_{i3}, \\ \dfrac{\partial \phi_3}{\partial R} + n_{e1}\dfrac{\partial \phi_2}{\partial R} + n_{e2}\dfrac{\partial \phi_1}{\partial R} + \dfrac{1}{\lambda T}\dfrac{\partial \phi_2}{\partial \Theta} + \dfrac{n_{e1}}{\lambda T}\dfrac{\partial \phi_1}{\partial \Theta} \\ \quad = \beta_3\dfrac{\partial n_{e3}}{\partial R} + \dfrac{\beta_3}{\lambda T}\dfrac{\partial n_{e2}}{\partial \Theta}. \end{cases} \tag{18}$$

Substituting equations (17) and (18) into equation (16), we obtain the following differential equation:

$$\frac{\partial}{\partial R}\left(\frac{-2\lambda}{\sigma_i - \lambda^2}\frac{\partial \phi_1}{\partial T} + \frac{14\lambda^4 + 14\sigma_i\lambda^2 - \sigma_i^2}{2\left(\sigma_i - \lambda^2\right)^4}\phi_1^2\frac{\partial \phi_1}{\partial R} - \left(\sigma_i - \lambda^2\right)\frac{\partial^3 \phi_1}{\partial R^3} - \frac{\lambda}{T\left(\sigma_i - \lambda^2\right)}\phi_1 + \frac{2\lambda^2 + \sigma_i}{\left(\sigma_i - \lambda^2\right)^2}\frac{\partial \phi_1\phi_2}{\partial R}\right) - \frac{1}{T^2\left(\sigma_i - \lambda^2\right)}\frac{\partial^2 \phi_1}{\partial \Theta^2} = 0. \tag{19}$$

According to equations (17) and (19), we obtain the $(2+1)$-dimensional CMKP equation, i.e.,

$$\frac{\partial}{\partial R}\left(\frac{\partial \phi_1}{\partial T} + C\phi_1^2\frac{\partial \phi_1}{\partial R} + D\frac{\partial^3 \phi_1}{\partial R^3} + \frac{1}{2T}\phi_1\right) + \frac{1}{2\lambda T^2}\frac{\partial^2 \phi_1}{\partial \Theta^2} = 0, \tag{20}$$

where

$$C = \frac{\left(\sigma_i - \lambda^2\right)\left(2\lambda^4 + 8\sigma_i\lambda^2 - \sigma_i^2\right) + 14\lambda^4 + 8\sigma_i\lambda^2 - 4\sigma_i^2}{-4\lambda\left(\sigma_i - \lambda^2\right)^3\left(1 + \sigma_i - \lambda^2\right)},$$
$$D = \frac{\left(\sigma_i - \lambda^2\right)^2}{2\lambda}. \tag{21}$$

3. Derivation of Time-Space Fractional Cylindrical Equations

In Section 2, we derive a series of differential equations of integer order. However, with the development of scientific research, compared with the fractional model, the integer order model has obvious shortcomings in describing practical problems. Fractional calculus and fractal calculus have become the hotspots in the fields of mathematics, physics, and engineering. In order to further study the nonlinear propagation of ion-acoustic waves in ultra-relativistic plasmas plasma, in this section, we use the semi-inverse method and fractional variation principle to derive the space-time fractional order equations from equations (10) and (20).

Definition 1 (see [43]). Jumarie's modified Riemann–Liouville derivative of order α is defined as

$$D_\tau^\omega f(\tau) = \begin{cases} \dfrac{1}{\Gamma(1-\omega)}\displaystyle\int_0^\tau (\tau - T)^{-\omega-1}[f(T) - f(0)]dT & \omega < 0, \\ \dfrac{1}{\Gamma(1-\omega)}\displaystyle\int_0^\tau (\tau - T)^{-\omega}[f(T) - f(0)]dT & 0 < \omega < 1, \\ \left[f^{\omega-n}(\tau)\right]^n & n \le \omega < n+1, n \ge 1. \end{cases} \tag{22}$$

Some properties of the modified Riemann–Liouville derivative are as follows:

$$D_\tau^\omega \tau^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\omega)}\tau^{\gamma-\omega} \quad \gamma>0,$$
$$D_\tau^\omega (f(\tau)g(\tau)) = g(\tau)D_\tau^\omega f(\tau) + f(\tau)D_\tau^\omega g(\tau), \tag{23}$$
$$D_\tau^\omega f[g(\tau)] = f_g'[g(\tau)]D_\tau^\omega g(\tau) = f_g^\omega f[g(\tau)](g'(\tau))^\omega.$$

According to equation (20), we have

$$\left(\phi_T + a_2\phi^2\phi_R + a_3\phi_{RRR} + a_4\phi\right)_R + a_5\phi_{\Theta\Theta} = 0, \tag{24}$$

where the coefficients $a_i\,(i = 2,\ldots,5)$ refer to equation (20).

Equation (24) can be rewritten as follows:

$$\phi_T + a_2\phi^2\phi_R + a_3\phi_{RRR} + a_4\phi + D^{-1}(a_5\phi_{\Theta\Theta}) = 0, \tag{25}$$

where D^{-1} is the fractional integral of R and $a_4\phi$ is invariant in the process of deriving fractional order equations using the semi-inverse method and the fractional variational principle, so it is omitted in the following derivation process.

Assuming $\phi(T,R,\Theta) = P_R(T,R,\Theta)$, where $P_R(T,R,\Theta)$ is a potential function, and substituting it into equation (25), we obtain the potential equation as follows:

$$P_{RT} + a_2P_R^2P_{RR} + a_3P_{RRRR} + a_5P_{\Theta\Theta} = 0. \tag{26}$$

Then, the function of equation (26) can be written as

$$J(P) = \int_X dR\int_Y d\Theta\int_Z dT\Big[P\big(b_1P_{RT} + b_2a_2P_R^2P_{RR} + b_3a_3P_{RRRR} + b_4a_5P_{\Theta\Theta}\big)\Big], \tag{27}$$

where $b_i\,(i = 1,2,\ldots,4)$ are Lagrangian multipliers which can be obtained later.

Applying integration by parts to equation (27) and taking $P_R|_X = P_R|_Z = P_\Theta|_Y = P_{RRR}|_X = 0$, we have

$$J(P) = \int_X dR\int_Y d\Theta\int_Z dT\Big[-b_1P_RP_T - \frac{1}{3}b_2a_2(P_R)^4 + b_3a_3(P_{RR})^2 - b_4a_5(P_\Theta)^2\Big]. \tag{28}$$

Using the variation of equation (28), integrating each term by parts and applying the variation optimum condition, we obtain

$$F(R,\Theta,T,P,P_R,P_\Theta,P_T,P_{RR}) = \frac{\partial F}{\partial P} - \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial P_T}\right) - \frac{\partial}{\partial R}\left(\frac{\partial F}{\partial P_R}\right) + \frac{\partial^2}{\partial R^2}\left(\frac{\partial F}{\partial P_{RR}}\right) - \frac{\partial}{\partial\Theta}\left(\frac{\partial F}{\partial P_\Theta}\right)$$
$$= 2b_1P_{RT} + 4b_2a_2P_R^2P_{RR} + 2b_3a_3P_{RRRR} + 2b_4a_5P_{\Theta\Theta} = 0. \tag{29}$$

Equation (29) is equivalent to equation (26), by comparing the coefficients, we obtain the Lagrangian multiplier $b_i\,(i = 1,2,3,4,5)$, i.e.,

$$b_1 = \frac{1}{2},\quad b_2 = \frac{1}{4},\quad b_3 = \frac{1}{2},\quad b_4 = \frac{1}{2}. \tag{30}$$

According to equation (30), the Lagrangian form of equation (25) is as follows:

$$L(P_T,P_R,P_{RR},P_\Theta) = -\frac{1}{2}P_RP_T - \frac{1}{12}a_2(P_R)^4 + \frac{1}{2}a_3(P_{RR})^2 - \frac{1}{2}a_5(P_\Theta)^2. \tag{31}$$

Similarly, the Lagrangian form of the fractional form of equation (25) is given by

$$L_1\left(D_T^\omega P, D_R^\alpha P, D_\Theta^\beta P, D_R^{\alpha\alpha}P\right) = -\frac{1}{2}D_T^\omega PD_R^\alpha P - \frac{1}{12}a_2(D_R^\alpha P)^4 + \frac{1}{2}a_3(D_R^{\alpha\alpha}P)^2 - \frac{1}{2}a_5\left(D_\Theta^\beta P\right)^2, \tag{32}$$

where $D_R^{\alpha\alpha}P = D_R^\alpha(D_R^\alpha P)$.

Therefore, we obtain the function of the fractional form of equation (25):

$$J_{L_1}(P) = \int_X (dR)^\alpha\int_Y (d\Theta)^\beta\int_Z (dT)^\omega L_1\left(D_T^\omega P, D_R^\alpha P, D_\Theta^\beta P, D_R^{\alpha\alpha}P\right). \tag{33}$$

According to Agrawal's method [44], equation (33) changes to the following form:

$$\delta J_{L_1}(P) = \int_X (dR)^\alpha\int_Y (d\Theta)^\beta\int_Z (dT)^\omega\Bigg[\left(\frac{\partial L_1}{\partial D_T^\omega P}\right)\delta D_T^\omega P + \left(\frac{\partial L_1}{\partial D_R^\alpha P}\right)\delta D_R^\alpha P + \left(\frac{\partial L_1}{\partial D_R^{\alpha\alpha}P}\right)\delta D_R^{\alpha\alpha}P + \left(\frac{\partial L_1}{\partial D_\Theta^\beta P}\right)\delta D_\Theta^\beta P\Bigg]. \tag{34}$$

Applying the following fractional integration by parts [44],

$$\int_a^b (d\tau)^i f(x) D_x^i g(x) = \Gamma(1+i)\Big[g(x) f(x)\,|_a^b - \int_a^b (dx)^i g(x) D_x^i f(i) \Big], \quad f(x), g(x) \in [a,b], \tag{35}$$

we obtain the following equation:

$$\delta J_{L_1}(B) = \int_X (dR)^\alpha \int_Y (d\Theta)^\beta \int_Z (dT)^\omega \left[-D_T^\omega \left(\frac{\partial L_1}{\partial D_T^\omega P} \right) - D_R^\alpha \left(\frac{\partial L_1}{\partial D_R^\alpha P} \right) - D_\Theta^\beta \left(\frac{\partial L_1}{\partial D_\Theta^\beta P} \right) + D_R^{\alpha\alpha} \left(\frac{\partial L_1}{\partial D_R^{\alpha\alpha} P} \right) \right]. \tag{36}$$

Optimizing the variation equation (38) and taking $\delta J_F(B) = 0$, we obtain the Euler–Lagrange equation as

$$-D_T^\omega \left(\frac{\partial L_1}{\partial D_T^\omega P} \right) - D_R^\alpha \left(\frac{\partial L_1}{\partial D_R^\alpha P} \right) - D_\Theta^\beta \left(\frac{\partial L_1}{\partial D_\Theta^\beta P} \right) + D_R^{\alpha\alpha} \left(\frac{\partial L_1}{\partial D_R^{\alpha\alpha} P} \right) = 0. \tag{37}$$

Substituting equation (32) into equation (37), we have

$$D_T^\omega D_R^\alpha P + a_2 \left(D_R^\alpha P \right)^2 D_R^{\alpha\alpha} P + a_3 D_R^{\alpha\alpha\alpha\alpha} P + a_5 D_\Theta^{\beta\beta} P = 0. \tag{38}$$

To find the fractional derivative of the independent variable R on both sides of equation (38), we have

$$D_R^\alpha \left(D_T^\omega D_R^\alpha P + a_2 \left(D_R^\alpha P \right)^2 D_R^{\alpha\alpha} P + a_3 D_R^{\alpha\alpha\alpha\alpha} P \right) + a_5 D_R^\alpha D_\Theta^{\beta\beta} P = 0. \tag{39}$$

Letting $D_R^\alpha P = \phi$ and substituting it into equation (39), we obtain

$$D_R^\alpha \left(D_T^\omega \phi + a_2 \phi^2 D_R^\alpha \phi + a_3 D_R^{\alpha\alpha\alpha} \phi \right) + a_5 D_\Theta^{\beta\beta} \phi = 0. \tag{40}$$

Therefore, we obtained the TSF-CMKP equation as follows:

$$D_R^\alpha \left(D_T^\omega \phi + a_2 \phi^2 D_R^\alpha \phi + a_3 D_R^{\alpha\alpha\alpha} \phi + \frac{1}{2T} \phi \right) + \frac{1}{2\lambda T^2} D_\Theta^{\beta\beta} \phi = 0. \tag{41}$$

Similarly, using the semi-inverse method and fractional variation principle, the TSF-CKP equation can be derived as follows:

$$D_R^\alpha \left(D_T^\omega \phi + A \phi D_R^\alpha \phi + B D_R^{\alpha\alpha\alpha} \phi + \frac{1}{2T} \phi \right) + \frac{1}{2\lambda T^2} D_\Theta^{\beta\beta} \phi = 0. \tag{42}$$

Remark 1. When $\omega = \alpha = \beta = \gamma = 1$, equations (41) and (42) are the integer order equations. In some cases, we can think of integer order equations as a special case of fractional order equations. Therefore, the fractional model can better describe the nonlinear propagation of ion-acoustic waves in ultrarelativistic plasmas.

4. Solutions of TSF-CKP Equation and TSF-CMKP Equation

In order to better explain the physical phenomena represented by fractional differential equations, it is necessary to obtain exact or numerical solutions of fractional differential equations. In this section, first of all, fractional derivatives are transformed into classical derivatives by fractional transformations. Secondly, we obtain the 1-decay mode solution of TSF-CKP equation by using the simplified homogeneous balance method [41]. Finally, using generalized hyperbolic-function method [42], the exact analytic solution of TSF-CMKP equation is obtained.

4.1. 1-Decay Mode Solution for TSF-CKP Equation. The fractional transforms are introduced as follows [45]:

$$t = \frac{p_1 T^\omega}{\Gamma(1+\omega)}, \quad r = \frac{p_2 R^\alpha}{\Gamma(1+\alpha)}, \quad \theta = \frac{p_3 \Theta^\beta}{\Gamma(1+\beta)}, \tag{43}$$

where $p_i\,(i = 1, 2, 3, 4)$ are constants. Based on the above-mentioned transforms, we have

$$\frac{\partial^\omega \phi}{\partial T^\omega} = p_1 \frac{\partial \phi}{\partial t}, \quad \frac{\partial^\alpha \phi}{\partial R^\alpha} = p_2 \frac{\partial \phi}{\partial r}, \quad \frac{\partial^\beta \phi}{\partial \Theta^\beta} = p_3 \frac{\partial \phi}{\partial \theta}. \tag{44}$$

Substituting equation (44) into (41) and taking $\phi = u$, we obtain

$$\left(u_t + a_1 u u_r + a_3 u_{rrr} \right)_r + \frac{1}{2nt^{1/\omega}} u_r + \frac{1}{2\lambda n^2 t^{2/\omega}} u_{\theta\theta} = 0, \tag{45}$$

where $n = [\Gamma(1+\omega)]^{1/\omega}$.

We consider the homogeneous balance between uu_r and u_{rrr}, according to the simplified homogeneous balance method, which means that an undetermined function $\Omega(\phi)$ and its partial derivatives Ω_r are replaced by a logarithmic function $A(\ln\varphi)$ and its derivatives $A(\ln\varphi)_r$, respectively. Therefore, we suppose that the form of the solution of equation (45) is as follows:

$$u(r, t, \theta) = \varrho (\ln\varphi)_{rr} + u_0, \tag{46}$$

where ϱ is a constant, $\varphi(r, \theta, t)$ is an undetermined function, which can be obtained later, and u_0 is a particular solution of equation (45).

Substituting equation (46) into equation (45), since u_0 is a particular solution, i.e.,

$$\frac{\partial}{\partial r}\left(\frac{\partial u_0}{\partial t}+a_1u_0\frac{\partial u_0}{\partial r}+a_3\frac{\partial^3 u_0}{\partial r^3}\right)+\frac{1}{2nt^{1/\omega}}u_{0r}+\frac{1}{2\lambda n^2t^{2/\omega}}u_{0\theta\theta}=0, \tag{47}$$

we obtain

$$\begin{aligned}&\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial t}+a_1u\frac{\partial u}{\partial r}+a_3\frac{\partial^3 u}{\partial r^3}\right)+\frac{1}{2nt^{1/\omega}}u_r+\frac{1}{2\lambda n^2t^{2/\omega}}u_{\theta\theta}\\&=\varrho\frac{\partial^2}{\partial r^2}\left[(\ln\varphi)_{rt}+\frac{a_1A}{2}(\ln\varphi)_{rr}^2+a_2u_0(\ln\varphi)_{rr}+a_3(\ln\varphi)_{rrrr}+\frac{1}{2nt^{1/\omega}}(\ln\varphi)_r+\frac{1}{2\lambda n^2t^{2/\omega}}(\ln\varphi)_{\theta\theta}\right]\\&=\varrho\frac{\partial^2}{\partial r^2}\left[\frac{\varphi_{rt}+a_1u_0\varphi_{rr}+a_3\varphi_{rr}\varphi_r^2\varphi^5-3\varphi_{rr}^2\varphi^6-6\varphi_r^4\varphi^4}{\varphi}\right.\\&+\frac{-\varphi_r\varphi_t+a_1A/2\varphi_{rr}^2-a_2u_0\varphi_r^2-4a_3\varphi_{rrr}\varphi_r-3a_3\varphi_{rr}^2-1/2\lambda n^2t^{2/\omega}\varphi_\theta^2}{\varphi^2}\\&\left.+\frac{-a_1A\varphi_{rr}\varphi_r^2+12a_3A\varphi_{rr}\varphi_r^2}{\varphi^3}+\frac{a_1A/2\varphi_r^4-6a_3\varphi_r^4}{\varphi^4}\right].\end{aligned} \tag{48}$$

Setting the coefficient of φ^{-4} be zero, yields $\varrho=12a_3/a_1$, so equation (46) can be rewritten as follows:

$$u(r,t,\theta)=\varrho(\ln\varphi)_{rr}+u_0=\frac{12a_3}{a_1}(\ln\varphi)_{rr}+u_0. \tag{49}$$

Equations (48) and (49) can be simplified as follows:

$$\begin{aligned}&\left(u_t+a_1uu_r+a_3u_{rrr}\right)_r+\frac{1}{2nt^{1/\omega}}u_r+\frac{1}{2\lambda n^2t^{2/\omega}}u_{\theta\theta}\\&=\frac{12a_3}{a_1}\frac{\partial^2}{\partial r^2}\left\{\frac{1}{\varphi^2}\left[\varphi\left(\left(\varphi_t+a_2u_0\varphi_r+a_3\varphi_{rrr}+\frac{1}{2nt^{1/\omega}}\varphi\right)_r+\frac{1}{2n^2t^{2/\omega}}\varphi_{\theta\theta}\right)\right.\right.\\&\left.\left.-\left(\varphi_t\varphi_r+4a_3\varphi_r\varphi_{rrr}+a_3\varphi_{rr}^2-\frac{a_1A}{2}\varphi_{rr}^2+a_1u_0\varphi_r^2+\frac{1}{2n^2t^{2/\omega}}\varphi_\theta^2\right)\right]\right\}=0.\end{aligned} \tag{50}$$

The function $\varphi(r,\theta,t)$ satisfies the homogemeity equation, i.e.,

$$\begin{aligned}&\varphi\left[\left(\varphi_t+a_1u_0\varphi_r+a_3\varphi_{rrr}+\frac{1}{2nt^{1/\omega}}\varphi\right)_r+\frac{1}{2n^2t^{2/\omega}}\varphi_{\theta\theta}\right]\\&-\left(\varphi_t\varphi_r+4a_3\varphi_r\varphi_{rrr}+a_3\varphi_{rr}^2-\frac{a_1A}{2}\varphi_{rr}^2+a_1u_0\varphi_r^2+\frac{1}{2n^2t^{2/\omega}}\varphi_\theta^2\right)=0.\end{aligned} \tag{51}$$

Equations (49) and (51) are called the nonlinear transformation of equation (45), if $\varphi(r,\theta,t)$ is a solution of equation (51), substituting it into equation (49), we obtain the exact solution of equation (45). Next, using the nonlinear transformation, we will find the exact solutions of equation (45).

In view of the homogemeity of (51), we assume that the form of the solution of (51) is as follows:

$$\varphi(r,t,\theta)=1+e^{H(t)r-\int_0^t\left[a_3H[t]^3+a_6H[t]\right]\mathrm{d}t}, \tag{52}$$

where $a_6=a_1u_0-((1/2nt^{1/\omega})+a_2u_{0r})r$ and $B(t)$ can be obtained later.

Substituting equation (52) into equation (51), we obtain

$$e^{\eta}\left\{\left[H' + \left(\frac{1}{2nt^{1/\omega}} + a_1 u_{0r}\right)H\right](1 + Hr)\right\} + e^{2\eta}\left[H' + \left(\frac{1}{2nt^{1/\omega}} + a_1 u_{0r}\right)B\right] = 0. \tag{53}$$

According to equation (53), we obtain an ODE as follows:

$$H' + \left(\frac{1}{2nt^{1/\omega}} + a_1 u_0\right)H = 0. \tag{54}$$

Solving equation (54), we have

$$H(t) = ke^{(\omega/2n(\omega-1))t^{(\omega-1)/\omega} - \int a_2 u_{0r}\mathrm{d}t}, \tag{55}$$

where k is an arbitrary constant.

Substituting equation (55) into equation (52), we obtain a solution of equation (51):

$$\varphi(r,t,\theta) = 1 + e^{\left(ke^{(\omega/2n(\omega-1))t^{(\omega-1)/\omega} - \int a_2 u_{0r}\mathrm{d}t}\right)r - \int_0^t \left[a_3 k^3 e^{3\left(ke^{(\omega/2n(\omega-1))t^{(\omega-1)/\omega} - \int a_2 u_{0r}\mathrm{d}t}\right)} + a_4 ke^{ke^{(\omega/2n(\omega-1))t^{(\omega-1)/\omega} - \int a_2 u_{0r}\mathrm{d}t}}\right]\mathrm{d}t}. \tag{56}$$

Substituting equation (56) into equation (46), we obtain an exact 1-decay mode solution of the 2D time-space CKP equation as follows:

$$u(r,t,\theta) = \frac{12a_3 e^{\eta}\left(\eta_r^2 + \eta\eta_r^2 + \eta_{rr} + \eta\eta_{rr} - e^{\eta}\eta_r^2\right)}{a_1(1 + e^{\eta})^2} + u_0, \tag{57}$$

where

$$\begin{cases} \eta = \left(ke^{(\omega/2n(\omega-1))t^{(\omega-1)/\omega} - \int a_2 u_{0r}\mathrm{d}t}\right)r - \int_0^t \left[a_3 k^3 e^{3\left((\omega/2n(\omega-1))t^{(\omega-1)/\omega} - \int a_2 u_{0r}\mathrm{d}t\right)} + a_4 ke^{(\omega/2n(\omega-1))t^{(\omega-1)/\omega} - \int a_2 u_{0r}\mathrm{d}t}\right]\mathrm{d}t, \\ r = \dfrac{p_2 R^{\alpha}}{\Gamma(1+\alpha)}, \theta = \dfrac{p_3 \Theta^{\beta}}{\Gamma(1+\beta)}, t = \dfrac{p_1 T^{\omega}}{\Gamma(1+\omega)}. \end{cases} \tag{58}$$

4.2. Exact Analytic Solution for TSF-CMKP Equation. Similarly, based on the fractional transforms of equation (43), we have

$$\frac{\partial^{\omega}\phi}{\partial T^{\omega}} = p_1 \frac{\partial \phi}{\partial t}, \quad \frac{\partial^{\alpha}\phi}{\partial R^{\alpha}} = p_2 \frac{\partial \phi}{\partial r}, \quad \frac{\partial^{\beta}\phi}{\partial \Theta^{\beta}} = p_3 \frac{\partial \phi}{\partial \theta}. \tag{59}$$

Substituting equation (59) into equation (42), we obtain

$$\left(\phi_t + a_2\phi\phi_r + a_3\phi_{rrr}\right)_r + \frac{1}{2nt^{1/\omega}}\phi_r + \frac{1}{2\lambda n^2 t^{2/\omega}}\phi_{\theta\theta} = 0, \tag{60}$$

where $n = [\Gamma(1+\omega)]^{1/\omega}$.

Using the generalized hyperbolic-function method, we assume that solutions of equation (60) are the super position of different powers of the sech function, tanh function, and their combinations, i.e.,

$$\phi(r,\theta,t) = \sum_{l=0}^{L} \alpha_l(r,\theta,t)\tanh^l[\Psi(r,\theta,t)] + \sum_{j=0}^{J} \kappa_j(r,\theta,t)\mathrm{sech}[\Psi(r,\theta,t)]\tanh^j[\Psi(r,\theta,t)], \tag{61}$$

where α_l, κ_j, and $\Psi(r,\theta,t)$ are all differentiable functions, while $L = 1$ and $J = 0$ are determined via the balance of the highest-order contributions in equation (60). Substituting equation (61) into equation (60), along with the simplifications, we have

$$\Psi(r,\theta,t) = \beta_0(\theta,t) + r\beta_1(\theta,t), \quad \alpha_1 = \alpha_1(\theta,t), \kappa_0 = \kappa_0(\theta,t), \tag{62}$$

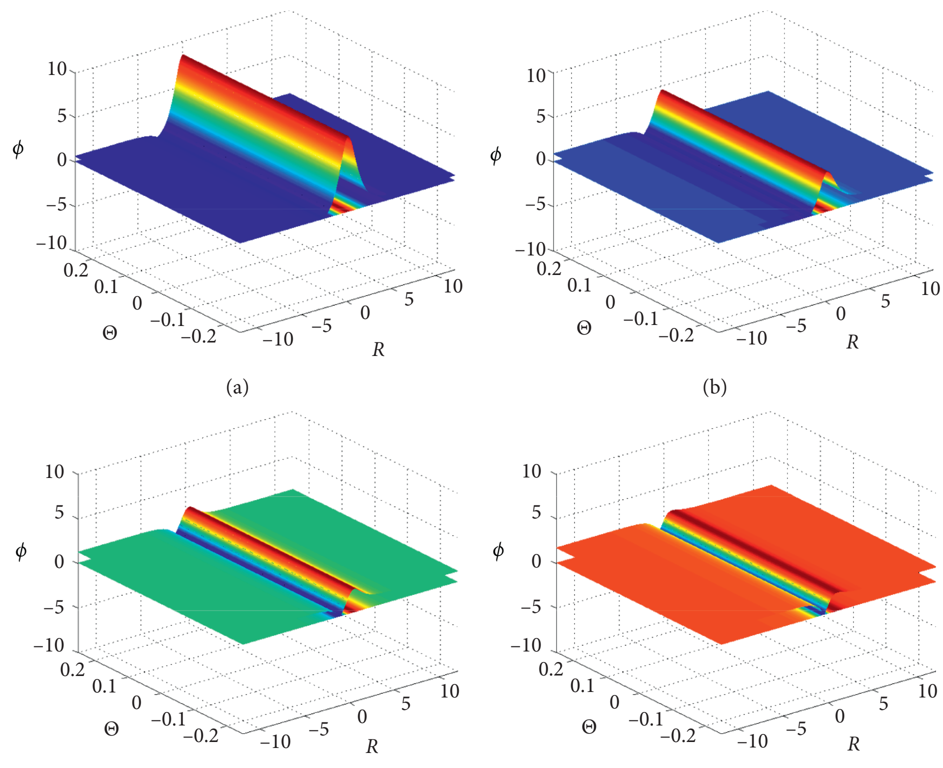

Figure 1: The values of the parameters are as follows. (a) $\beta_1 = 1.2$, $\beta_3 = 1.4$, $\omega = 1$, and $\varpi = 1$. (b) $\beta_1 = 1.2$, $\beta_3 = 1.6$, $\omega = 1$, and $\varpi = 1$. (c) $\beta_1 = 1.2$, $\beta_3 = 1.8$, $\omega = 1$, and $\varpi = 1$. (d) $\beta_1 = 1.2$, $\beta_3 = 2$, $\omega = 1$, and $\varpi = 1$.

where the functions β_0 and $\beta_1 \neq 0$ are both differentiable; supposing that $\alpha_1 \neq 0$ and $\kappa_0 \neq 0$ and equating to zero the coefficients of like powers of $\tanh \Psi$, $\operatorname{sech} \Psi$, etc., we obtain

$$\begin{cases} \alpha_1^2 = -\dfrac{a_3}{a_2}\beta_1^2, \\ \kappa_0^2 = -\dfrac{-a_3 + 4a_2a_3}{2a_2^2}\beta_1^2, \\ \alpha_0 = 0, \\ \beta_0 = \dfrac{n\lambda t^{1/\omega}\beta_1\theta^2}{2} - \dfrac{\lambda\beta_1\theta^2 t}{2}, \\ \beta_1 = \text{const} \neq 0, \end{cases} \tag{63}$$

$$\begin{cases} \alpha_1 = i\varpi\sqrt{\dfrac{a_3}{a_2}}\beta_1, & \varpi = \pm 1, \\ \kappa_0 = i\varpi\sqrt{\dfrac{-a_3 + 4a_2a_3}{2a_2^2}}\beta_1, & \varpi = \pm 1. \end{cases} \tag{64}$$

Using this algorithm, we obtain the exact analytical solution of equation (60):

$$\phi(r,\theta,t) = i\varpi\beta_1\left[\sqrt{\frac{-a_3 + 4a_2a_3}{2a_2^2}}\operatorname{sech}\left(\frac{n\lambda t^{1/\omega}\beta_1\theta^2}{2} - \frac{\lambda\beta_1\theta^2 t}{2} + r\beta_1\right) + \sqrt{\frac{a_3}{a_2}}\tanh\left(\frac{n\lambda t^{1/\omega}\beta_1\theta^2}{2} - \frac{\lambda\beta_1\theta^2 t}{2} + r\beta_1\right)\right], \tag{65}$$

where the independent variable (r, θ, t) satisfies the following transformation:

$$r = \frac{p_2 R^\alpha}{\Gamma(1+\alpha)}, \quad \theta = \frac{p_3 \Theta^\beta}{\Gamma(1+\beta)}, \quad t = \frac{p_1 T^\omega}{\Gamma(1+\omega)}. \tag{66}$$

5. The Property of the Ion-Acoustic Waves in Ultrarelativistic Plasmas

In this section, in order to further understand the propagation characteristics of ion-acoustic waves in astrophysical ultrarelativistic degenerate plasmas (especially in white dwarfs and neutron stars). Based on exact analytical solutions, we have studied the effects of the phase speed λ and electron number density (through β_3) on the propagation of

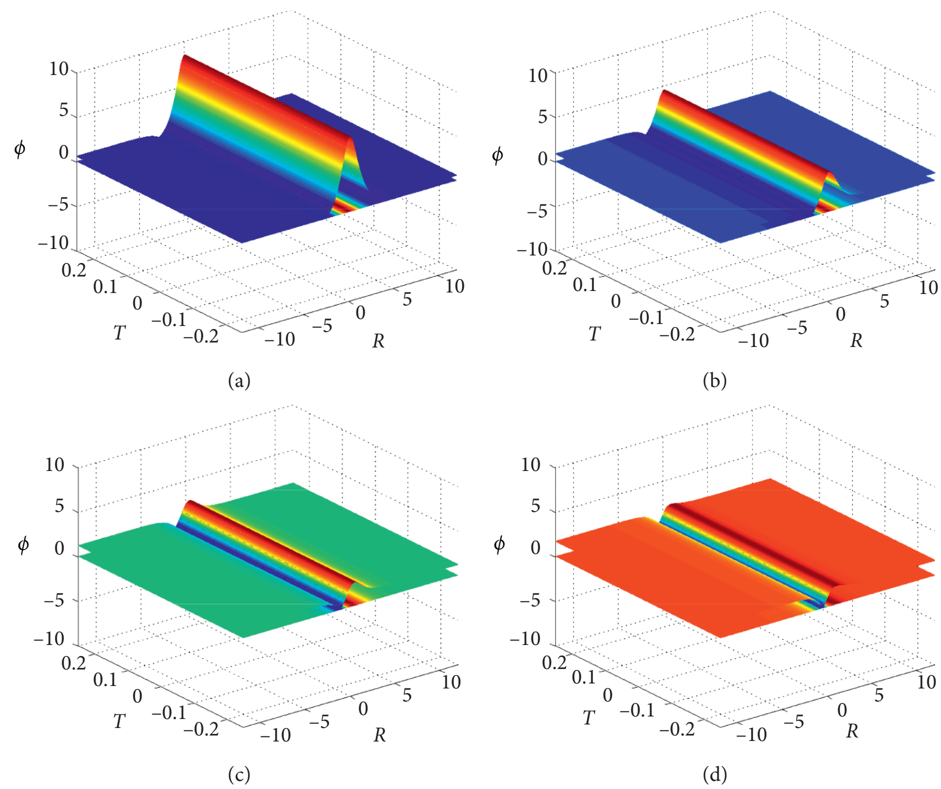

FIGURE 2: The values of the parameters are as follows. (a) $\beta_1 = 1.2$, $\beta_3 = 1.4$, $\omega = 1$, and $\varpi = 1$. (b) $\beta_1 = 1.2$, $\beta_3 = 1.6$, $\omega = 1$, and $\varpi = 1$. (c) $\beta_1 = 1.2$, $\beta_3 = 1.8$, $\omega = 1$, and $\varpi = 1$. (d) $\beta_1 = 1.2$, $\beta_3 = 2$, $\omega = 1$, and $\varpi = 1$.

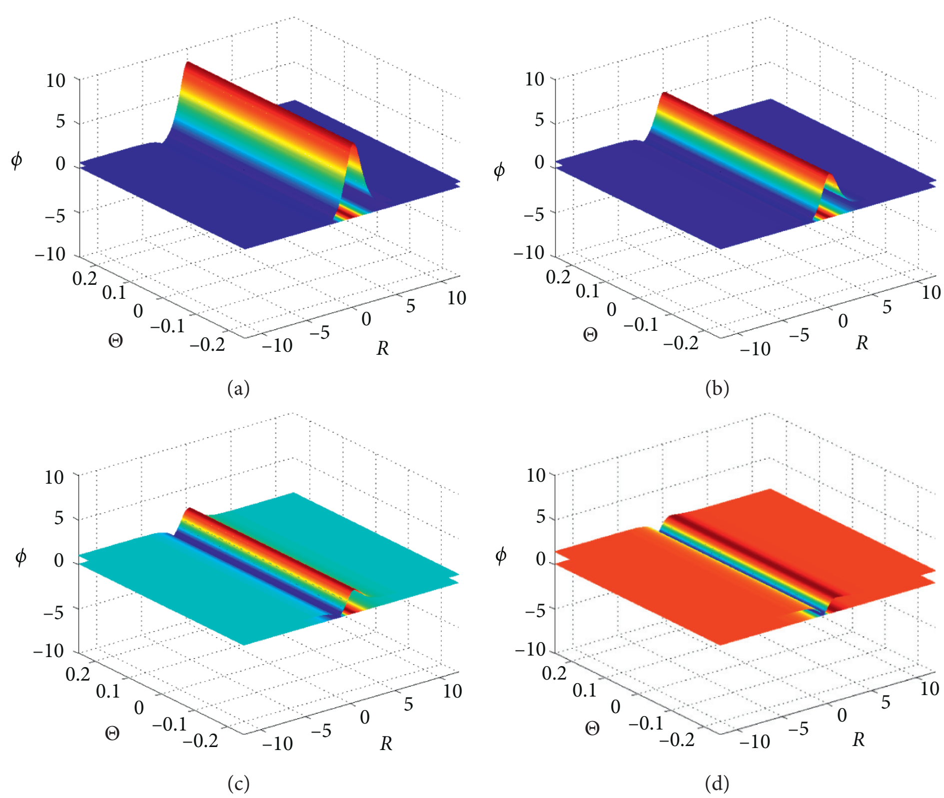

FIGURE 3: The values of the parameters are as follows. (a) $\beta_1 = 1.2$, $\lambda = 1.3$, $\omega = 1$, and $\varpi = 1$. (b) $\beta_1 = 1.2$, $\lambda = 1.2$, $\omega = 1$, and $\varpi = 1$. (c) $\beta_1 = 1.2$, $\lambda = 1.1$, $\omega = 1$, and $\varpi = 1$. (d) $\beta_1 = 1.2$, $\lambda = 1$, $\omega = 1$, and $\varpi = 1$.

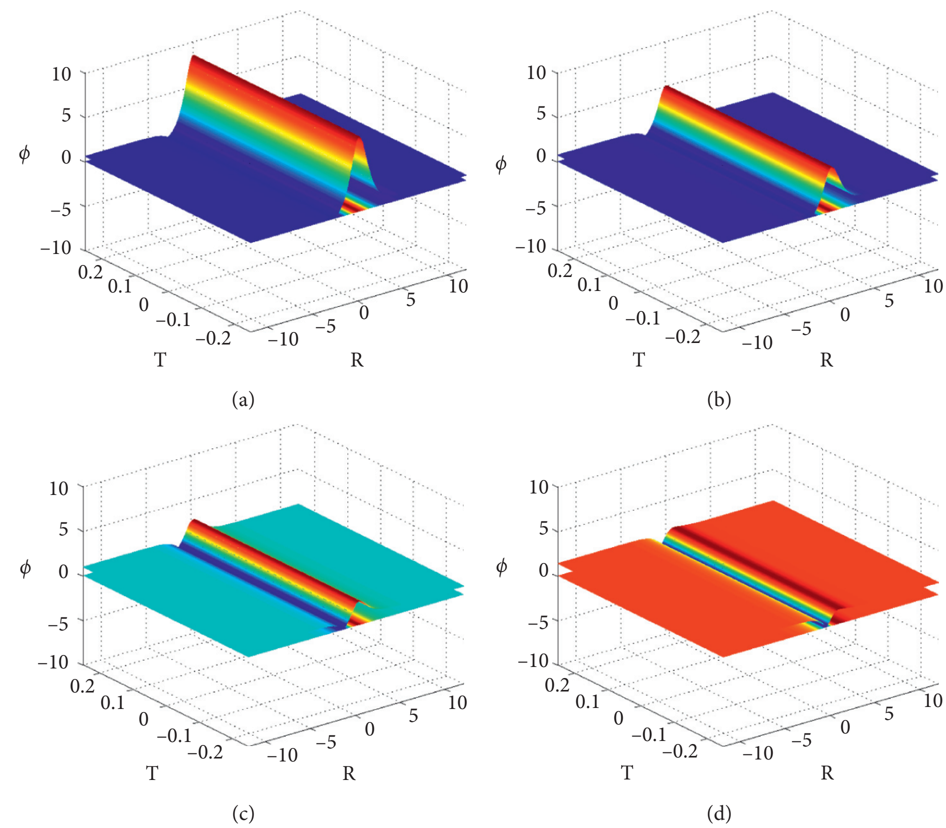

FIGURE 4: The values of the parameters are as follows. (a) $\beta_1 = 1.2$, $\lambda = 1.3$, $\omega = 1$, and $\varpi = 1$. (b) $\beta_1 = 1.2$, $\lambda = 1.2$, $\omega = 1$, and $\varpi = 1$. (c) $\beta_1 = 1.2$, $\lambda = 1.1$, $\omega = 1$, and $\varpi = 1$. (d) $\beta_1 = 1.2$, $\lambda = 1$, $\omega = 1$, and $\varpi = 1$.

ion-acoustic waves in astrophysical ultrarelativistic degenerate plasmas. In particular, due to the establishment of fractional order models, we studied the effect of fractional order (α, β, ω) on the propagation of ion-acoustic waves, which is rare in previous studies.

5.1. Study of the Effects of Electron Number Density (through β_3) on Ion-Acoustic Waves Propagation. First, we take the variable T as the determined value and draw the figures of the ion-acoustic waves under the axis of the independent variables R and Θ.

As shown in Figure 1, within a certain value range, as the value of β_3 increases, the peak of the ion-acoustic wave continuously decreases, and eventually the peak almost disappears, and only a valley exists. As the value of β_3 increases, the amplitude of the ion-acoustic wave continues to decrease, which indicates that the loudness of the ion-acoustic wave continues to decrease. Therefore, the plasma system makes the ion-acoustic waves damp with the increase of electron number density.

Then, we take the variable Θ as the determined value and draw the figures of the ion-acoustic waves under the axis of the independent variables R and T.

As shown in Figure 2, the ion-acoustic waves will undergo radial displacement over time, which is ignored by the one-dimensional model. With the increase of β_3, the trend of the peaks and troughs is the same as that shown in Figure 1. Similarly, the ion-acoustic waves will decay with the increase of β_3.

5.2. Study of the Effects of the Phase Speed λ on Ion-Acoustic Waves Propagation. As shown in Figures 3 and 4, it can be concluded that within a certain value range, as the value of the phase velocity λ decreases, the peak of the ion-acoustic waves continuously decreases, and finally the peak almost disappears, and only a trough exists, and the amplitude of ion-acoustic waves is decreasing. Although the specific change values are different, the trend of this change is similar to the effect of the electron number density on the propagation of ion-acoustic waves, which is precisely because of $\sigma_i = \lambda^2 - \beta_3$, so the phase speed also affects the propagation of ion-acoustic waves.

5.3. Study of the Effects of the Fractional Order Values on Ion-Acoustic Waves Propagation. Taking the variables T and Θ as definite values, Figure 5(a)–5(d) are, respectively, the figures of ion-acoustic waves when the fractional order value is 1, 1/2, 1/3, and1/4.

As shown in Figure 5, the properties of peaks, troughs, and amplitudes of ion-acoustic waves are different under different fractional orders, and Figure 5(b) with fractional order value is 1/2, which is more special.

Next, we focus on the properties of ion-acoustic waves when the fractional order is 1/2 and compare it with the case

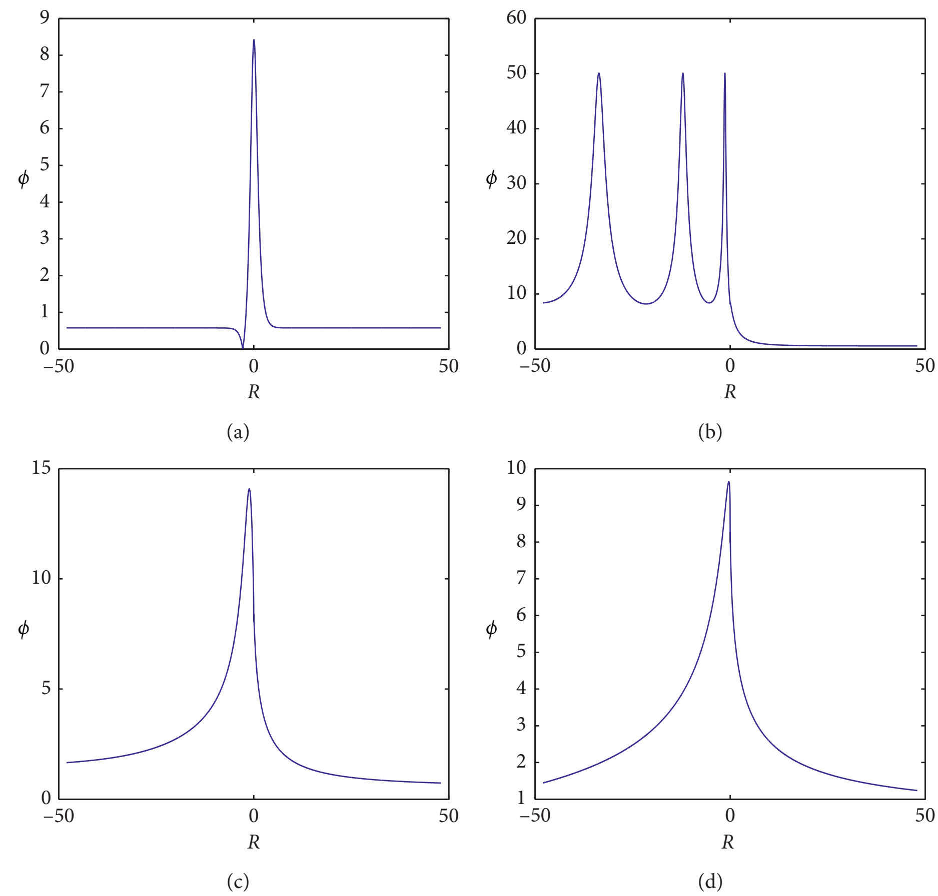

Figure 5: The values of the parameters are as follows. (a) $\beta_1 = 1.2$, $\beta_3 = 1.4$, $\lambda = 1.3$, $\omega = 1$, and $\varpi = 1$. (b) $\beta_1 = 1.2$, $\beta_3 = 1.4$, $\lambda = 1.3$, $\omega = 1$, and $\varpi = 1$. (c) $\beta_1 = 1.2$, $\beta_3 = 1.4$, $\lambda = 1.3$, $\omega = 1$, and $\varpi = 1$. (d) $\beta_1 = 1.2$, $\beta_3 = 1.4$, $\lambda = 1.3$, $\omega = 1$, and $\varpi = 1$.

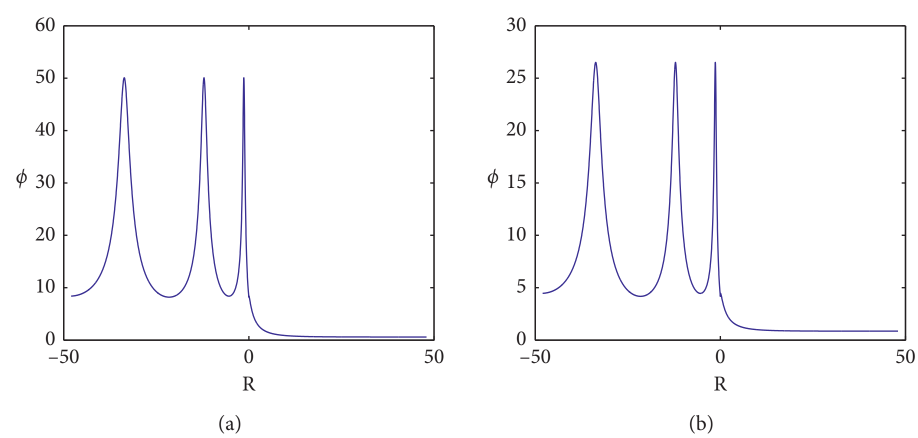

Figure 6: Continued.

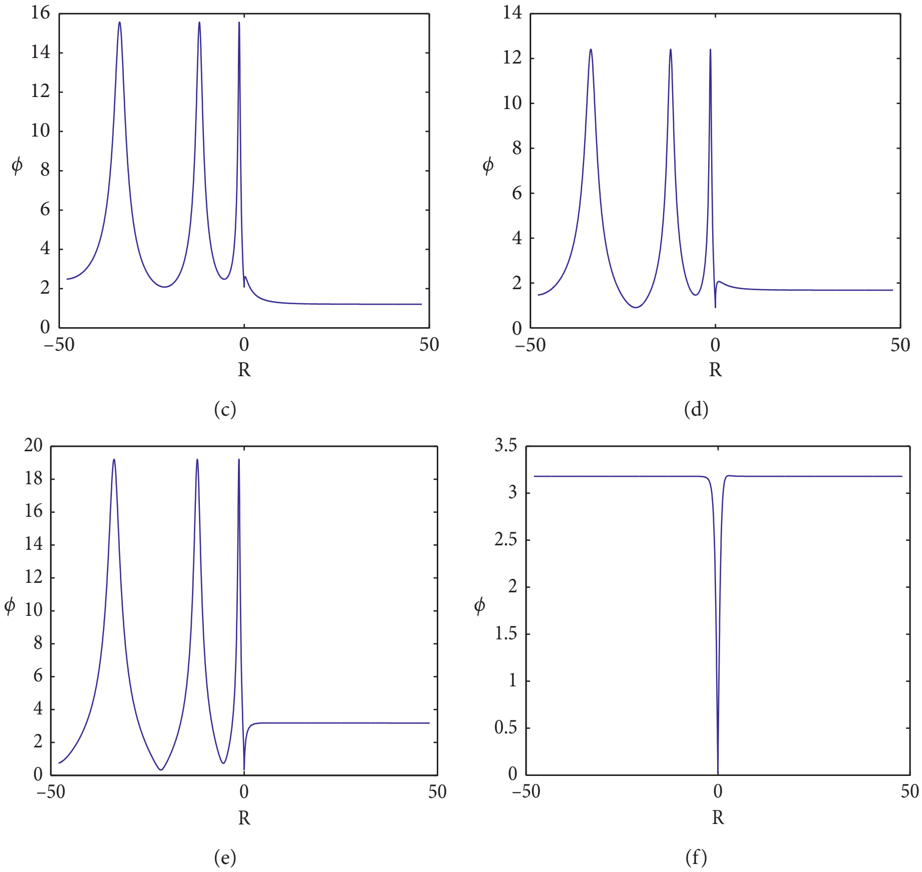

Figure 6: The values of the parameters are as follows. (a) $\beta_1 = 1.2$, $\beta_3 = 1.4$, $\omega = 1$, and $\varpi = 1$. (b) $\beta_1 = 1.2$, $\beta_3 = 1.6$, $\omega = 1$, and $\varpi = 1$. (c) $\beta_1 = 1.2$, $\beta_3 = 1.8$, $\omega = 1$, and $\varpi = 1$. (d) $\beta_1 = 1.2$, $\beta_3 = 2$, $\omega = 1$, and $\varpi = 1$. (e) $\beta_1 = 1.2$, $\beta_3 = 2.4$, $\omega = 1$, and $\varpi = 1$. (f) $\beta_1 = 1.2$, $\beta_3 = 2.4$, $\omega = 1$, and $\varpi = 1$.

of integer order. Figure 6(a)–6(e) are all the figures when the fractional order is 1/2, and Figure 6(f) is the figure when the fractional order is 1, i.e., the figure of integer order.

As shown in Figure 6, when the fractional order value is 1/2, as the β_3 decreases, the peak first increases and then decreases, the amplitude first increases and then decreases, and the trough gradually decreases. When $\beta_3 = 2$ and $\beta_3 = 2.4$, the peaks in the integer order figures gradually disappear and a deep trough appears, but the peaks in the fractional figures do not disappear but increase. This reflects the influence of the fractional order values on the ion-acoustic waves in ultra-relativistic plasmas, indicating that it is practical to study the fractional order models in ion-acoustic waves.

6. Conclusion

In this paper, the 2D CKP equation and the 2D CMKP equation in integer order are derived. Compared with the model in one-dimensional plane geometry, the study of models in nonplanar geometry is more in line with the actual situation of the laboratory, space environment, and so on. We extended these equations to the fractional order domain for the first time and obtained the TSF-CKP equation and the TSF-CMKP equation. Compared with the integer order model, the fractional order model can better describe the propagation of ion-acoustic waves in ultrarelativistic plasmas and solve practical problems. Based on the fractional order transformation, the 1-decay mode solution for the TSF-CKP equation is obtained by using the simplified homogeneous balance method, and using the generalized hyperbolic-function method, the exact analytic solution of TSF-CMKP equation is obtained. Next, we use the obtained exact solution to analyze the effects of the phase speed λ, electron number density (through β_3), and fractional order (α, β, ω) on the propagation of ion-acoustic waves in ultrarelativistic plasmas. In particular, the different effects of the fractional order of 1/2 and 1 on the propagation of ion-acoustic waves are analyzed. These results have potential value for studying the propagation characteristics of ion-acoustic waves in the ultrarelativistic plasmas.

Data Availability

There are no data in our manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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