Research Article

Pattern Dynamics of Nonlocal Delay SI Epidemic Model with the Growth of the Susceptible following Logistic Mode

Zun-Guang Guo,^{1,2} Jing Li³, Can Li,¹ Juan Liang¹ and Yiwei Yan¹

¹Department of Science, Taiyuan Institute of Technology, Taiyuan, Shanxi 030008, China

²Department of Computer Science and Technology, North University of China, Taiyuan, Shanxi 030051, China

³College of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan, Shanxi 030006, China

Correspondence should be addressed to Jing Li; jingli_2016@126.com

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In this paper, we investigate pattern dynamics of a nonlocal delay SI epidemic model with the growth of susceptible population following logistic mode. Applying the linear stability theory, the condition that the model generates Turing instability at the endemic steady state is analyzed; then, the exact Turing domain is found in the parameter space. Additionally, numerical results show that the time delay has key effect on the spatial distribution of the infected, that is, time delay induces the system to generate stripe patterns with different spatial structures and affects the average density of the infected. The numerical simulation is consistent with the theoretical results, which provides a reference for disease prevention and control.

1. Introduction

Research on infectious disease models can be traced back to the pioneering work of Kermack and Mckendrick [1, 2], and the traditional model is usually described by ordinary differential equations as a result of the spatially homogeneous assumption [3–10]. Considering the instantaneous change, researchers have established epidemic models via impulsive differential equations [11–15]. However, besides the change of time, the spatial distribution of populations has a significant effect on diseases transmission. As a matter of fact, spatial distribution of the population is not homogeneous, but depend on the spatial location and the surrounding environment factors, such as the amount of food, the number of natural enemies, and the number of the infected [16, 17]. Therefore, considering the space in epidemic models is more suitable for the spread process of infectious diseases. Recently, many reaction-diffusion models have been proposed to study their dynamical behaviors and study the corresponding control measures [18–29].

Since they can reach the current position from any other positions in the whole space when humans or animals are

moving in space, which is more reflective for the actual situation, therefore, the nonlocal delay of infectious disease has become a research focus in epidemiology. Wang and Wu [30] studied the Kermack–Mckendrick SIR model with nonlocal delay and discussed the dynamic properties of disease spatial diffusion. Pan [31] studied the existence of wave front solutions for a class of infectious disease models with nonlocal diffusion and time delay by constructing upper and lower solutions. Zhen et al. [32] also obtained the traveling wave solutions for a class of SIR epidemic models with spatiotemporal delays. Using the upper and lower solutions and its related monotone iterative techniques, Tang et al. [33] studied the sufficient conditions for the global asymptotic stability of the disease-free equilibrium of the bird system.

Most of the abovementioned studies focus on the traveling wave solutions and global stability of the nonlocal epidemic model. However, few literatures study the pattern formation of infectious disease models based on reaction-diffusion equation with nonlocal delay. Pattern dynamics can effectively characterize the spatial distribution of the infected, so as to provide decision-making guidance for the

government. In fact, in the natural environments with limited resources, the assumption of logistic growth is more consistent with the actual situation [34–37]; then, we assume the susceptible possess the logistic growth in the absence of

the infected. In addition, we take the standard incidence ratio to describe the infection between the susceptible and the infected. To this end, we propose the following model:

$$\begin{cases} \frac{\partial S}{\partial t} = D_S \Delta S + rS \left(1 - \frac{S}{K}\right) - \beta S \frac{\int_{-\infty}^t \int_{\Omega} Q(x-y, t-s) I(y, s) dy ds}{S+I} - dS - AS, \\ \frac{\partial I}{\partial t} = D_I \Delta I + \beta S \frac{\int_{-\infty}^t \int_{\Omega} Q(x-y, t-s) I(y, s) dy ds}{S+I} - \mu I - dI, \end{cases} \quad (1)$$

with initial boundary value condition

$$(\vec{n} \cdot \nabla) \begin{pmatrix} S \\ I \end{pmatrix} = 0, S(x, 0), I(x, 0) \text{ are given,} \quad (2)$$

and the descriptions of the symbols in model (1) are shown in Table 1. The kernel function $Q(x, t) = (1/4\pi t) \exp(-|x|^2/4t) (1/\tau) \exp(-(t/\tau))$ represents the weight from the other possible positions to the position (HTML translation failed) before time t . The nonlocal item depicts the cumulative number of the infected reaching x position at t time who starts from any y position in the entire space at s time.

$$\int_{-\infty}^t \int_{R^2} Q(x-y, t-s) I(y, s) ds dy. \quad (3)$$

In order to study the spatiotemporal dynamics of model (1), in Section 2, we first obtain the steady states of model (1), linearize the model at the endemic steady state, and then analyze the conditions of system generating Turing instability and parameter space of Turing pattern generated. In Section 3, different stripe patterns are shown for different time delays, which indicates that time delay affects the structures of Turing patterns. In Section 4, the summary and discussion are given.

2. Linear Analysis and Turing Patterns

In this section, we define a new variable $V(x, t)$ to replace the nonlocal term $\int_{-\infty}^t \int_{R^2} Q(x-y, t-s) I(y, s) dy ds$; then, system (1) can be transformed into 3-variable reaction-diffusion system:

$$\frac{\partial \mathbf{c}}{\partial t} = \mathbf{F}(\mathbf{c}) + D \Delta \mathbf{c}, \quad (4)$$

where

$$\mathbf{c} = \begin{pmatrix} S \\ I \\ V \end{pmatrix},$$

$$\mathbf{F}(\mathbf{c}) = \begin{pmatrix} rS \left(1 - \frac{S}{K}\right) - \frac{\beta S V}{S+I} - (d+A)S \\ \frac{\beta S V}{S+I} - (\mu+d)I \\ \frac{1}{\tau} (I - V) \end{pmatrix},$$

$$D = \begin{pmatrix} D_S & & \\ & D_I & \\ & & 1 \end{pmatrix}. \quad (5)$$

By studying system (4) without diffusion, so the ordinary differential equation is given:

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t) \left(1 - \frac{S(t)}{K}\right) - \frac{\beta S(t)V(t)}{S(t)+I(t)} - (d+A)S(t), \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)V(t)}{S(t)+I(t)} - (\mu+d)I(t), \\ \frac{dV(t)}{dt} &= \frac{1}{\tau} (I(t) - V(t)), \end{aligned} \quad (6)$$

system (6) has two constant steady states $E_0 = (S_0, I_0, V_0) = (K(r-A-d)/r, 0, 0)$ and $E_1^* = (S_1^*, I_1^*, V_1^*)$, where

$$\begin{aligned} S_1^* &= \frac{K(r+\mu-A-\beta)}{r}, \\ I_1^* &= \frac{K(r+\mu-A-\beta)(\beta-d-\mu)}{r(d+\mu)}, \\ V_1^* &= \frac{K(r+\mu-A-\beta)(\beta-d-\mu)}{r(d+\mu)}. \end{aligned} \quad (7)$$

According to the actually biological situation, the population number should be nonnegative. So, if $r > A + d$ holds, then $S_0 > 0$, that is, E_0 is disease-free steady states. In addition, if the condition

$$(H1) \quad r > A + \beta - \mu, \quad \beta > \mu + d, \quad (8)$$

TABLE 1: Symbol descriptions of model (1).

Symbol	Interpretations
$S(x, t)$	The density of the susceptible at position $x(x \in R^2)$ at time t
$I(x, t)$	The density of the infected at position $x(x \in R^2)$ at time t
A	The emigration rate of the susceptible
Ω	The diffusion domain
$\vec{n}$	Unit outward normal to $\partial\Omega$
$\partial\Omega$	The closed boundary of the diffusion domain Ω
Δ	Laplace operator $(\partial^2/\partial X^2) + (\partial^2/\partial Y^2)$
D_S	The diffusion rate of the susceptible
D_I	The diffusion rate of the infected
β	The infection rate
μ	The mortality rate due to infectious disease caused
d	The natural mortality
r	The intrinsic growth rate
K	The carrying capacity

is satisfied, then $S_1^* > 0$, $I_1^* > 0$, and $V_1^* > 0$, which shows that E_1^* is the endemic steady state.

The main focus of infectious diseases is to study the outbreak and prevalence, so as to provide theoretical guidance for the prevention and control of infectious diseases. Then, the following contents only study endemic steady state E_1^* .

Through setting

$$\mathbf{w} = \begin{pmatrix} S - S_1^* \\ I - I_1^* \\ V - V_1^* \end{pmatrix}, \quad (9)$$

we substitute (9) into (4), expand $\mathbf{F}(\mathbf{c})$ at the endemic steady state E_1^* based on Talyor expansion, and then obtain the following linear equation:

$$\frac{\partial \mathbf{w}}{\partial t} = D\Delta \mathbf{w} + \mathbf{J}\mathbf{w}, \quad \mathbf{J} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (10)$$

where

$$\begin{aligned} a_{11} &= \frac{A\beta + \beta^2 + \beta d - \beta r - d^2 - 2d\mu - \mu^2}{\beta}, \\ a_{12} &= \frac{(\beta - d - \mu)(d + \mu)}{\beta}, \\ a_{13} &= -d - \mu, \\ a_{21} &= \frac{(\beta - d - \mu)^2}{\beta}, \\ a_{22} &= -\frac{2\beta d + 2\beta\mu - d^2 - 2d\mu - \mu^2}{\beta}, \\ a_{23} &= d + \mu, \\ a_{31} &= 0, \\ a_{32} &= \frac{1}{\tau}, \\ a_{33} &= -\frac{1}{\tau}. \end{aligned} \quad (11)$$

In order to solve the solution of system (10) satisfying the boundary condition (2), we first define $\mathbf{W}(x)$ as the time-independent spatial eigenvalue problem, which satisfies the following system:

$$\Delta \mathbf{W} + k^2 \mathbf{W} = 0, \quad (\vec{n} \cdot \nabla) \mathbf{W} = 0, \quad \text{for } x \text{ on } \partial\Omega, \quad (12)$$

where k is the eigenvalue (i.e., wavenumber).

We now look for solutions $\mathbf{w}(x, t)$ of (10) in the form

$$\mathbf{w}(x, t) = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} e^{\lambda t + i \vec{k} \cdot x}, \quad (13)$$

where $\vec{k}$ is the wave vector with magnitude $k = |\vec{k}|$. By inserting (13) into (10), the characteristic equation matrix is obtained:

$$|J - \lambda E - D| = 0, \quad (14)$$

which is equivalent as follows:

$$\lambda^3 + a_2(k)\lambda^2 + a_1(k)\lambda + a_0(k) = 0, \quad (15)$$

with

$$\begin{aligned} a_2(k) &= (D_S + D_I + 1)k^2 - A - \beta + d + 2\mu + r + \frac{1}{\tau}, \\ a_1(k) &= (D_S + D_I + D_S D_I)k^4 \\ &\quad - \frac{1}{\beta\tau} \left((A\beta D_I \tau + \beta^2 D_I \tau - 2\beta d D_S \tau + \beta d D_I \tau - 2\beta D_S \mu \tau - \beta D_I r \tau \right. \\ &\quad \left. + d^2 D_S \tau - d^2 D_I \tau + 2d D_S \mu \tau - 2d D_I \mu \tau + D_S \mu^2 \tau - D_I \mu^2 \tau + A\beta \tau \right. \\ &\quad \left. + \beta^2 \tau - \beta d \tau - 2\beta \mu \tau - \beta r \tau - \beta D_S - \beta D_I) k^2 \right) - \frac{1}{\beta\tau} (2A\beta d \tau \end{aligned}$$

$$\begin{aligned}
& + 2A\beta\mu\tau - Ad^2\tau - 2A d\mu\tau - A\mu^2\tau + 3\beta^2 d\tau + 3\beta^2\mu\tau - 2\beta d^2\tau \\
& - 6\beta d\mu\tau - 2\beta dr\tau - 4\beta\mu^2\tau - 2\beta\mu r\tau + d^2\mu\tau + d^2 r\tau + 2 d\mu^2\tau \\
& + 2 d\mu r\tau + \mu^3\tau + \mu^2 r\tau + A\beta + \beta^2 - \beta\mu - \beta r), \\
a_0(k) = & D_S D_I k^6 - \frac{1}{\beta\tau} ((A\beta D_I \tau + \beta^2 D_I \tau - 2\beta d D_S \tau + \beta d D_I \tau - 2\beta D_S \mu \tau \\
& - \beta D_I r \tau + d^2 D_S \tau - d^2 D_I \tau + 2 d D_S \mu \tau - 2 d D_I \mu \tau + D_S \mu^2 \tau - D_I \mu^2 \tau \\
& - \beta D_S D_I) k^4) - \frac{1}{\beta\tau} ((2A\beta d\tau + 2A\beta\mu\tau - Ad^2\tau - 2A d\mu\tau \\
& - A\mu^2\tau + 3\beta^2 d\tau + 3\beta^2\mu\tau - 2\beta d^2\tau - 6\beta d\mu\tau - 2\beta dr\tau - 4\beta\mu^2\tau \\
& - 2\beta\mu r\tau + d^2\mu\tau + d^2 r\tau + 2 d\mu^2\tau + 2 d\mu r\tau + \mu^3\tau + \mu^2 r\tau \\
& + A\beta D_I + \beta^2 D_I - \beta d D_S + \beta d D_I - \beta D_S \mu - \beta D_I r + d^2 D_S \\
& - d^2 D_I + 2 d D_S \mu - 2 d D_I \mu + D_S \mu^2 - D_I \mu^2) k^2) \\
& - \frac{(d + \mu)(-d + \beta - \mu)(A + \beta - \mu - r)}{\beta\tau},
\end{aligned} \tag{16}$$

where $a_2(k) > 0$ for any k because of condition (H1) satisfied.

Moreover, one can obtain the corresponding characteristic equation for system (6) as follows:

$$\lambda^3 + a_2(0)\lambda^2 + a_1(0)\lambda + a_0(0) = 0, \tag{17}$$

where

$$\begin{aligned}
a_2(0) = & -A - \beta + d + 2\mu + r + \frac{1}{\tau}, \\
a_1(0) = & -\frac{1}{\beta\tau} (2A\beta d\tau + 2 d\mu r\tau + \mu^3\tau \\
& + \mu^2 r\tau + A\beta + \beta^2 - \beta\mu - \beta r \\
& + 2A\beta\mu\tau - Ad^2\tau - 2A d\mu\tau - A\mu^2\tau \\
& + 3\beta^2 d\tau + 3\beta^2\mu\tau - 2\beta d^2\tau \\
& - 6\beta d\mu\tau - 2\beta dr\tau - 4\beta\mu^2\tau - 2\beta\mu r\tau \\
& + d^2\mu\tau + d^2 r\tau + 2 d\mu^2\tau), \\
a_0(0) = & -\frac{(d + \mu)(\beta - d - \mu)(A + \beta - \mu - r)}{\beta\tau}.
\end{aligned} \tag{18}$$

On the basis of condition (H1), one can derive $a_2(0) > 0$ and $a_0(0) > 0$. According to Hurwitz criterion, the condition for the local asymptotic stability of E_1^* of system (6) is

$$(H2) a_2(0)a_1(0) - a_0(0) > 0. \tag{19}$$

Theorem 1. *The nonspatial system (6) gives rise to a Hopf bifurcation at endemic steady state E_1^* if only and if the condition*

$$(H3) a_2(0)a_1(0) - a_0(0) = 0, \tag{20}$$

is established.

Proof. It is obvious that $a_2(0) > 0$ and $a_0(0) > 0$ due to the existence of endemic equilibrium E_1^* (i.e., (H1)). According to the theorem of [38], when conditions $a_2(0) > 0$, $a_0(0) > 0$, and $a_2(0)a_1(0) - a_0(0) = 0$ hold, a Hopf bifurcation occurs for nonspatial system (6).

Next, we find the conditions under which system (4) generates Turing instability near endemic steady state E_1^* : E_1^* is locally asymptotic stability for system (6), but E_1^* loses stability for system (6) with diffusion (i.e., system (4)). Furthermore, we already know $a_2(k) > 0$ for any k , while the signs of $a_0(k)$ and $a_2(k)a_1(k) - a_0(k)$ are uncertain. Now, we need to look for the condition that E_1^* of system (4) becomes unstable. Obviously, Hurwitz criterion is not satisfied, that is, $a_0(k) < 0$ or $a_2(k)a_1(k) - a_0(k) < 0$; then, the corresponding instability conditions are given. Based on such two cases, we derive the following theorems with respect to Turing instability. $\square$

Theorem 2. *For system (4), if condition (H1) holds, and one of the following conditions*

$$(C1) \ h_1 < 0 \text{ and } H_2(z_1) < 0$$

$$(C2) \ h_1 > 0, \ h_2 < 0, \ (h_2)^2 - 3h_3h_1 > 0 \text{ and } H_2(z_1) < 0$$

is satisfied, then endemic steady state E_1^ is unstable for some k , where*

$$z_1 = \frac{-h_2 + \sqrt{(h_2)^2 - 3h_3h_1}}{3h_3}. \quad (21)$$

Proof. Let $z = k^2 > 0$ and $H_2(k^2) = a_0(k)$, then $H_2(z) = h_3z^3 + h_2z^2 + h_1z + h_0$, where

$$h_3 = D_S D_I > 0,$$

$$h_2 = -\frac{1}{\beta\tau} (A\beta D_I \tau + \beta^2 D_I \tau - 2\beta d D_S \tau + \beta d D_I \tau$$

$$- 2\beta D_S \mu \tau - \beta D_S D_I$$

$$- \beta D_I r \tau + d^2 D_S \tau - d^2 D_I \tau + 2 d D_S \mu \tau$$

$$- 2 d D_I \mu \tau + D_S \mu^2 \tau - D_I \mu^2 \tau),$$

$$h_1 = -\frac{1}{\beta\tau} (2A\beta d \tau + 2A\beta \mu \tau - A d^2 \tau - 2A d \mu \tau$$

$$- A \mu^2 \tau + 3\beta^2 d \tau + 3\beta^2 \mu \tau - 2\beta d^2 \tau - 6\beta d \mu \tau$$

$$- 2\beta d r \tau - 4\beta \mu^2 \tau$$

$$- 2\beta \mu r \tau + d^2 \mu \tau + d^2 r \tau + 2 d \mu^2 \tau + 2 d \mu r \tau + \mu^3 \tau + \mu^2 r \tau$$

$$+ A\beta D_I + \beta^2 D_I - \beta d D_S + \beta d D_I - \beta D_S \mu - \beta D_I r + d^2 D_S$$

$$- d^2 D_I + 2 d D_S \mu - 2 d D_I \mu + D_S \mu^2 - D_I \mu^2),$$

$$h_0 = -\frac{(d + \mu)(-d + \beta - \mu)(A + \beta - \mu - r)}{\beta\tau} > 0.$$

$$(22) \quad \text{where}$$

Considering the properties of cubic polynomials, the coefficient of the third order of $H_2(z)$ is greater than zero and $H_2(0) = h_0 > 0$. By solving the first-order derivative $H_2(z)$ about z , namely, $H_2'(z) = 3h_3z^2 + 2h_2z + h_1 = 0$, we can deduce

$$z_2 = \frac{-h_2 - \sqrt{(h_2)^2 - 3h_3h_1}}{3h_3} \leq z_1 = \frac{-h_2 + \sqrt{(h_2)^2 - 3h_3h_1}}{3h_3}. \quad (23)$$

(i) If condition $h_1 < 0$ holds, then one can obtain $z_2 < 0 < z_1$, which further shows that z_1 is the minimum point. Then, we can get $a_2(k) < 0$ for some k when $H_2(z_{2,\min}) < 0$ combining with $H_2(0) = h_0 > 0$ (see Figure 1(a)).

(ii) If conditions $h_1 > 0$, $h_2 < 0$, and $(h_2)^2 - 3h_3h_1 > 0$ hold, then we can get $0 < z_2 < z_1$, and z_2 and z_1 are maximum and minimum points, respectively. Furthermore, according to $H_2(z_1) < 0$, one can derive that $a_2(k) < 0$ for some k (see Figure 1(b)).

Obviously, Hurwitz criterion does not hold for Case (i) or Case (ii), and then the endemic steady state E_1^* is unstable for system (4).

Thus, the conditions of system (4) generating Turing instability is given:

$$\begin{cases} (H1), & (H2), \\ (C1) \text{ or, } & (C2). \end{cases} \quad (24)$$

In addition, we set $H_3(z) = a_2(k)a_1(k) - a_0(k)$ and $z = k^2 > 0$, that is,

$$H_3(z) = h_{q_3}z^3 + h_{q_2}z^2 + h_{q_1}z + h_{q_0}, \quad (25)$$

$$h_{q_3} = (D_I + 1)(D_S + 1)(D_S + D_I) > 0,$$

$$\begin{aligned} h_{q_2} = & -\frac{1}{\tau\beta} (2A\beta D_S D_I \tau + A\beta D_I^2 \tau + 2\beta^2 D_S D_I \tau + \beta^2 D_I^2 \tau - 2\beta d D_S^2 \tau \\ & - 2\beta d D_S D_I \tau + \beta d D_I^2 \tau - 2\beta D_S^2 \mu \tau - 4\beta D_S D_I \mu \tau - 2\beta D_S D_I r \tau \\ & - \beta D_I^2 r \tau + d^2 D_S^2 \tau - d^2 D_I^2 \tau + 2 d D_S^2 \mu \tau - 2 d D_I^2 \mu \tau + D_S^2 \mu^2 \tau \\ & - D_I^2 \mu^2 \tau + 2A\beta D_S \tau + 2A\beta D_I \tau + 2\beta^2 D_S \tau + 2\beta^2 D_I \tau - 2\beta d D_S \tau \\ & - 2\beta d D_I \tau - 4\beta D_S \mu \tau - 2\beta D_S r \tau - 4\beta D_I \mu \tau - 2\beta D_I r \tau + A\beta \tau \\ & + \beta^2 \tau - \beta d \tau - \beta D_S^2 - 2\beta D_S D_I - \beta D_I^2 - 2\beta \mu \tau - \beta r \tau - 2\beta D_S - 2\beta D_I), \end{aligned}$$

$$\begin{aligned} h_{q_1} = & \frac{1}{\tau^2\beta} (A^2 \beta D_I \tau^2 + 2A\beta^2 D_I \tau^2 - 4A\beta d D_S \tau^2 - 2A\beta d D_I \tau^2 - 4A\beta D_S \mu \tau^2 \\ & - 4A\beta D_I \mu \tau^2 - 2A\beta D_I r \tau^2 + 2Ad^2 D_S \tau^2 + 4Ad D_S \mu \tau^2 + 2Ad S \mu^2 \tau^2 \\ & + \beta^3 D_I \tau^2 - 5\beta^2 d D_S \tau^2 - 3\beta^2 d D_I \tau^2 - 5\beta^2 D_S \mu \tau^2 - 5\beta^2 D_I \mu \tau^2 \\ & - 2\beta^2 D_I r \tau^2 + 5\beta d^2 D_S \tau^2 + 14\beta d D_S \mu \tau^2 + 4\beta d D_S r \tau^2 + 2\beta d D_I \mu \tau^2 \end{aligned}$$

FIGURE 1: The diagrams of the cubic function $H_2(z)$ in Theorem 1: (a) case (i) and (b) case (ii).

$$\begin{aligned}
& + 2\beta dD_I r \tau^2 + 9\beta D_S \mu^2 \tau^2 + 4\beta D_S \mu r \tau^2 + 3\beta D_I \mu^2 \tau^2 + 4\beta D_I \mu r \tau^2 \\
& + \beta D_I r^2 \tau^2 - d^3 D_S \tau^2 + d^3 D_I \tau^2 - 5d^2 D_S \mu \tau^2 - 2d^2 D_S r \tau^2 \\
& + 3d^2 D_I \mu \tau^2 - 7d D_S \mu^2 \tau^2 - 4d D_S \mu r \tau^2 + 3d D_I \mu^2 \tau^2 - 3D_S \mu^3 \tau^2 \\
& - 2D_S \mu^2 r \tau^2 + D_I \mu^3 \tau^2 + A^2 \beta \tau^2 + 2A\beta^2 \tau^2 - 2A\beta d \tau^2 - 4A\beta \mu \tau^2 \\
& - 2A\beta r \tau^2 + \beta^3 \tau^2 - 2\beta^2 d \tau^2 - 4\beta^2 \mu \tau^2 - 2\beta^2 r \tau^2 + \beta d^2 \tau^2 \\
& + 4\beta d \mu \tau^2 + 2\beta d r \tau^2 + 4\beta \mu^2 \tau^2 + 4\beta \mu r \tau^2 + \beta r^2 \tau^2 - 2A\beta D_S \tau \\
& - 2A\beta D_I \tau - 2\beta^2 D_S \tau - 2\beta^2 D_I \tau + 2\beta d D_S \tau + \beta d D_I \tau + 4\beta D_S \mu \tau \\
& + 2\beta D_S r \tau + 3\beta D_I \mu \tau + 2\beta D_I r \tau - 2A\beta \tau - 2\beta^2 \tau + \beta d \tau \\
& + 3\beta \mu \tau + 2\beta r \tau + \beta D_S + \beta D_I), \\
h_{q_0} = & \frac{1}{\tau^2 \beta} (2A^2 \beta d \tau^2 + 2A^2 \beta \mu \tau^2 - A^2 d^2 \tau^2 - 2A^2 d \mu \tau^2 - A^2 \mu^2 \tau^2 \\
& + 5A\beta^2 d \tau^2 + 5A\beta^2 \mu \tau^2 - 5A\beta d^2 \tau^2 - 14A\beta d \mu \tau^2 - 4A\beta d r \tau^2 \\
& - 9A\beta \mu^2 \tau^2 - 4A\beta \mu r \tau^2 + A d^3 \tau^2 + 5A d^2 \mu \tau^2 + 2A d^2 r \tau^2 \\
& + 7A d \mu^2 \tau^2 + 4A d \mu r \tau^2 + 3A \mu^3 \tau^2 + 2A \mu^2 r \tau^2 + 3\beta^3 d \tau^2 \\
& + 3\beta^3 \mu \tau^2 - 5\beta^2 d^2 \tau^2 - 15\beta^2 d \mu \tau^2 - 5\beta^2 d r \tau^2 - 10\beta^2 \mu^2 \tau^2 \\
& - 5\beta^2 \mu r \tau^2 + 2\beta d^3 \tau^2 + 11\beta d^2 \mu \tau^2 + 5\beta d^2 r \tau^2 + 18\beta d \mu^2 \tau^2 \\
& + 14\beta d \mu r \tau^2 + 2\beta d r^2 \tau^2 + 9\beta \mu^3 \tau^2 + 9\beta \mu^2 r \tau^2 + 2\beta \mu r^2 \tau^2 \\
& - d^3 \mu \tau^2 - d^3 r \tau^2 - 4d^2 \mu^2 \tau^2 - 5d^2 \mu r \tau^2 - d^2 r^2 \tau^2 - 5d \mu^3 \tau^2 \\
& - 7d \mu^2 r \tau^2 - 2d \mu r^2 \tau^2 - 2\mu^4 \tau^2 - 3\mu^3 r \tau^2 - \mu^2 r^2 \tau^2 + A^2 \beta \tau \\
& + 2A\beta^2 \tau - 2A\beta d \tau - 4A\beta \mu \tau - 2A\beta r \tau + \beta^3 \tau - 3\beta^2 d \tau \\
& - 5\beta^2 \mu \tau - 2\beta^2 r \tau + \beta d^2 \tau + 4\beta d \mu \tau + 2\beta d r \tau + 4\beta \mu^2 \tau \\
& + 4\beta \mu r \tau + \beta r^2 \tau - A\beta - \beta^2 + \beta \mu + \beta r).
\end{aligned} \tag{26}$$

Through some analyses, we can give the following conclusion. $\square$

Theorem 3. For system (4), suppose that the following holds

$$(C3) \ h_{q_1} < 0, \ h_{q_0} > 0, \text{ and } H_3(z_3) < 0$$

$$(C4) \ h_{q_1} > 0, \ h_{q_0} > 0, \ h_{q_2} < 0, \ (h_{q_2})^2 - 3h_{q_3}h_{q_1} > 0, \text{ and } H_3(z_3) < 0$$

Then, endemic steady state E_1^* becomes unstable for some k , where

$$z_3 = \frac{-h_{q_2} + \sqrt{(h_{q_2})^2 - 3h_{q_3}h_{q_1}}}{3h_{q_3}}. \quad (27)$$

Given that the cubic polynomial functions $H_2(z)$ and $H_3(z)$ have similar structures and conditions, thus the proof process of Theorem 3 is similar to that of Theorem 2; then, one can easily obtain the above result.

Therefore, we derive another condition for system (4) to give rise to Turing instability:

$$\begin{cases} (H1), & (H2), \\ (C3)\text{or}, & (C4). \end{cases} \quad (28)$$

3. Numerical Results

In this section, we will carry out numerical simulations for system (4) with Neumann boundary conditions, which will be used to illustrate our theoretical analysis. For the convenience of numerical simulation, we select two-dimensional space area $\Omega = [0, a] \times [0, a]$ and set the same space step h in X and Y directions of the space area. Simultaneously, time step is fixed as Δt . Then, the mesh generation is given:

$$\begin{aligned} x_i &= ih, \quad i = 0, 1, 2, \dots, J, \\ y_j &= jh, \quad j = 0, 1, 2, \dots, J, \quad Jh = a, \\ t_n &= n\Delta t, \quad n = 0, 1, 2, \dots \end{aligned} \quad (29)$$

$$\mathbf{F}(\mathbf{c}) \text{ in system (4) is relabeled as } \mathbf{F}(\mathbf{c}) = \begin{pmatrix} f_1(S, I, V) \\ f_2(S, I, V) \\ f_3(S, I, V) \end{pmatrix},$$

and the following finite difference scheme is adopted for system (4):

$$\begin{cases} \frac{S_{i,j}^{n+1} - S_{i,j}^n}{\Delta t} = D_S \left(\frac{S_{i+1,j}^n - 2S_{i,j}^n + S_{i-1,j}^n}{h^2} + \frac{S_{i,j+1}^n - 2S_{i,j}^n + S_{i,j-1}^n}{h^2} \right) + f_1(S_{i,j}^n, I_{i,j}^n, V_{i,j}^n), \\ \frac{I_{i,j}^{n+1} - I_{i,j}^n}{\Delta t} = D_I \left(\frac{I_{i+1,j}^n - 2I_{i,j}^n + I_{i-1,j}^n}{h^2} + \frac{I_{i,j+1}^n - 2I_{i,j}^n + I_{i,j-1}^n}{h^2} \right) + f_2(S_{i,j}^n, I_{i,j}^n, V_{i,j}^n), \\ \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t} = \frac{V_{i+1,j}^n - 2V_{i,j}^n + V_{i-1,j}^n}{h^2} + \frac{V_{i,j+1}^n - 2V_{i,j}^n + V_{i,j-1}^n}{h^2} + f_3(S_{i,j}^n, I_{i,j}^n, V_{i,j}^n), \end{cases} \quad (30)$$

where $S_{i,j}^n$, $I_{i,j}^n$, and $V_{i,j}^n$ represent the approximate value of $S(x_i, y_j, t_n)$, $I(x_i, y_j, t_n)$, and $V(x_i, y_j, t_n)$, respectively. In the simulation process, the initial value is a small random disturbance at endemic steady state E_1^* , and the area range of individuals' activity is $\Omega = [0, 100] \times [0, 100]$. The space region is evenly divided into 100×100 grids, and the corresponding space step is $h = 1$. The time interval is $[0, 5000]$, and the time step is set to $\Delta t = 0.005$. The first-order difference is used for the first-order derivative of the time, and the second-order derivative of the space is discretized by the center difference method in process of numerical simulation. For infectious diseases, the spatial distribution of infected individuals is a key issue for the control of infectious disease. Turing pattern can visually describe the spatial distribution of infected individuals. The focus of our simulation is how nonlocal delay affects spatial distribution of infected individuals. We thus take parameter value as $K = 1000$, $d = 0.3$, $\mu = 0.2$, $r = 0.8$, $D_S = 1$, $D_I = 10$, $A = 0.4$, and $\beta = 0.58$; time

delay τ is selected as the control parameter. Figures 2 and 3 are the time sequence diagram of spatial distribution of the density of the infected for time delays $\tau = 0.1$ and $\tau = 0.6$, respectively. It can be seen from these two figures that spatial distribution of the density of the infected evolves with time, which finally presents a regular nonuniform macroscopic structure in limited time, namely, stripe patterns, which indicates the number of infected individuals gradually increases over time and eventually gathers together. Under different time delay parameters, the infected finally forms different spatial distribution structures in the first row of Figure 4, which are called strip patterns. These patterns are further presented in three-dimensional space in the second row of Figure 4, which directly reflects the change of the density of the infected with space. Figure 5 shows the relationship between the average density of the infected and time delay τ , that is, the average density of the infected decreases with the increase of time delay τ (τ is less than the

FIGURE 2: The spatial evolution of the density of the infected with time for time delay $\tau = 0.1$: (a) $t = 90000$; (b) $t = 200000$; (c) $t = 300000$; (d) $t = 1000000$.

FIGURE 3: Continued.

FIGURE 3: The spatial evolution of the density of the infected with time for time delay $\tau = 0.6$: (a) $t = 80000$; (b) $t = 180000$; (c) $t = 400000$; (d) $t = 1000000$.

FIGURE 4: The first row shows stripe patterns with three different spatial structures in two-dimensional space for different time delay parameters: (a and d) $\tau = 0.1$; (b and e) $\tau = 0.2$; (c and f) $\tau = 0.3$. These patterns are further presented in a three-dimensional space in the second row.

FIGURE 5: The relationship between the average density of the infected and time delay τ .

critical value), while the opposite case appears when τ is greater than the critical value.

4. Conclusions and Discussion

This paper investigates spatiotemporal dynamics of the SI epidemic model with nonlocal effect and the growth of the susceptible population following logistic mode, and we obtain two spatially homogeneous steady states including disease-free steady state and endemic steady state. We mainly analyze the dynamical behavior near endemic steady state-Turing instability. In general, the condition of system generating Turing instability is that the endemic steady state is stable in the system without diffusion (i.e., ordinary equation system), but becomes instability in the system with diffusion (i.e., reaction-diffusion system); then, the system finally gives rise to stable spatial structure, which is called Turing pattern. The conditions of system (4) generating Turing instability are (24) or (28) based on Hurwitz criterion, and the parameter space of the existence of Turing pattern is given by condition (24). While the corresponding parameter space is difficult to obtain because of the complexity condition (28), which will be carried out in the following work. Time delay widely exists in the process of disease transmission, such as incubation period and immune period. We thus study the influence of time delay on infectious diseases in numerical simulations, find that the average density of the infected firstly decreases, and then increase with the increase of time delay, which provides a theoretical support for disease control. At the same time, we simulate the evolution of spatial distribution of the density of the infected under different time delays, and the spatial distribution of the density of the infected finally forms stripe pattern with different spatial structure, which does not change with time. The high incidence area of infectious diseases is found, which also provides data information for prevention and control of infectious disease.

In the research process of pattern dynamics of system (4), we consider a special form of kernel function so that the nonlocal delay system could be transformed into general reaction-diffusion system; then, pattern dynamics can be followed. In fact, there are some other forms of kernel functions in the biological system, but different forms shall induce nonlocal delay system to convert into different differential system; thus, this is a very meaningful work to study the effects of different kernel functions on the pattern dynamics in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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