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Research Article

The Topological Entropy of Cyclic Permutation Maps and Some Chaotic Properties on Their MPE sets

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In this paper, we study some chaotic properties of s-dimensional dynamical system of the form $\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1}))$, where $a_k \in H_k$ for any $k \in \{1, 2, \ldots, s\}$, $s \geq 2$ is an integer, and H_k is a compact subinterval of the real line $\mathbb{R} = (-\infty, +\infty)$ for any $k \in \{1, 2, \ldots, s\}$. Particularly, a necessary and sufficient condition for a cyclic permutation map $\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1}))$ to be LY-chaotic or h-chaotic or RT-chaotic or D-chaotic is obtained. Moreover, the LY-chaoticity, h-chaoticity, RT-chaoticity, and D-chaoticity of such a cyclic permutation map is explored. Also, we proved that the topological entropy $h(\Psi)$ of such a cyclic permutation map is the same as the topological entropy of each of the following maps: $g_j \circ g_{j-1} \circ \cdots \circ g_1 l \circ g_s \circ g_{s-1} \circ \cdots \circ g_{j+1}$, if $j = 1, \ldots, s-1$and $g_s \circ g_{s-1} \circ \cdots \circ g_1$, and that Ψ is sensitive if and only if at least one of the coordinates maps of Ψ^s is sensitive.

1. Introduction

Let $f: Y \longrightarrow X$ and $g: X \longrightarrow Y$ be continuous maps on the compact subintervals X and Y of the real line $\mathbb{R}$, and let $\Phi: X \times Y \longrightarrow X \times Y$ be a continuous map defined by $\Phi(x, y) = (f(y), g(x))$ for any $(x, y) \in X \times Y$. These maps have been proposed to give a mathematical description of competition in a duopolistic market, called Cournot duopoly (see [1]). This is the reason why the above map Φ is called a Cournot map, and the above two maps f and g are called reaction functions (that is, the maps f and g give laws to organize the production of some firms which are competitors in a market).

From [1, 2], we know that there are the so-called Markov perfect equilibria (MPE henceforth) processes, where the two players move alternatively such that each of them chooses the best reply to the previous action of another player. This occurs if the phase point (x_t, y_t) belongs alternatively to the graphs of the reaction curves $y = g(x)$ and $x = f(y)$. This condition can be satisfied if the initial condition is a point of a reaction curve. That is, $y_0 = g(x_0)$ (player 1 moves first) or $x_0 = f(y_0)$ (player 2 moves first). This follows from the fact that the union of the graphs of the two reaction function is trapping for Φ.

Probably, the first paper which gives the concept of chaos in a mathematically rigorous way is that of Li and Yorke [3]. Since then many different rigorous notions of chaos have been proposed. Each of these concepts tries to describe some kind of unpredictability in the evolution of the system. The notion of Li–Yorke sensitivity (LY-sensitivity) was presented for the first time by Akin and Kolyada in [4]. Moreover, they introduced the notion of spatiotemporal chaos. A very important generalization is distributional chaos, proposed by Schweizer and Smítal [5], mainly because it is equivalent to positive topological entropy and some other concepts of chaos when restricted to some spaces (see [5, 6]). It is noted that this equivalence does not transfer to higher dimensions, e.g., positive topological entropy does not imply

distributional chaos in the case of triangular maps of the unit square [7] (the same happens when the dimension is zero [8]). In [9], Wang et al. introduced the concept of distributional chaos in a sequence and showed that it is equivalent to Li–Yorke chaos (LY-chaos) for continuous maps of the interval. During the last years, many researchers paid attention to the chaotic behavior of Cournot maps (see [1, 2, 10–17]).

From [18], we know that if X is an infinite metric space, then if a continuous map f is transitive and has dense periodic points then it has sensitive dependence on initial conditions, which means that the third assumption in definition of chaos in sense of Devaney (D-chaos) is not necessary. By definition, it is clear that if a continuous map f is D-chaotic, then it is chaotic in sense of Ruelle–Takens (RT-chaotic). It was proved that the converse is false (see [13]). It is well known that D-chaotic maps are LY-chaotic (see [19]) and that maps with positive topological entropy (h-chaotic) are also LY-chaotic maps (see [20]). However, there exist LY-chaotic interval maps with zero topological entropy (see [21]). And from Theorem 1.2 in [13], we know that there exist LY-chaotic maps which are not D-chaotic. For interval continuous maps, J. S. Canovas and M. Ruiz Marin had the particular cases established in Theorem 1.2 which is from [13].

Let $W_1 = \{(f(y), y): y \in Y\}$ and $W\ 2 = \{\ (\ x, g(x)): x \in X\}$ The set $W_{12} = W_1 \cup W_2$ represents the union of the graphs of the two reaction function and is trapping for Φ, i.e., $\Phi(W_{12}) \subseteq W_{12}$. Canovas and Ruiz Marín called the set W_{12} a MPE set for Φ (see [13]). Moreover, they discussed and studied several chaotic properties (e.g., Devaney chaos, RT chaos, topological chaos, and Li–Yorke chaos) of Cournot maps and showed that it is not true that any chaotic property they considered satisfies the condition that Φ is chaotic if and only if $\Phi|_{W_{12}}$ is chaotic. Recently, Lu and Zhu further investigated the dynamical properties of Cournot maps, and more precisely, for the maps $\Phi|_{W_{12}}$, $\Phi^2|_{W_1}$, and $\Phi^2|_{W_2}$ (see [15]). In [22], Linero Bas and Soler Lopez defined a cyclically permuted direct product map and studied the topics of (totally) topological transitivity and (weakly) topological mixing for cyclically permuted direct product maps by exploring the relationship between the dynamics of F and that of the compositions $f_{\sigma(i)} \circ \cdots \circ f_{\sigma^n(i)}$, where $i \in \{1, \ldots, n\}$, $F(x_1, x_2, \ldots, x_n) = (f_{\sigma(1)}(x_{\sigma 1}), f_{\sigma(2)}(x_{\sigma(2)}), \ldots, f_{\sigma(n)}(x_{\sigma(n)}))$ (is called to be cyclically permuted direct product maps), which is defined from the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ into itself, where $X_1, X_2, \ldots, X_n$ are general topological spaces, each map $f_{\sigma_i(i)}: X_{\sigma(i)} \longrightarrow X_i$ is continuous, $i \in \{1, 2, \ldots, n\}$, and σ is a cyclic permutation of $\{1, 2, \ldots, n\}$, $n > 1$. In [23], Linero Bas and Soler Lopez established some results on transitivity for cyclically permuted direct product maps of the Cartesian product I^n, where $I = [0, 1]$. Particularly, it was shown that, for $n \geq 3$, the transitivity of this map F is equivalent to the total transitivity, and if $n = 2$, they obtained a splitting result for transitive maps. Also, they extended well-known properties of transitivity from interval maps to cyclically permuted direct product maps. To do it, they used the strong link between the map F and the compositions $\{\phi_j = f_{\sigma(j)} \circ \cdots \circ f_{\sigma^n(j)}, j = 1, \ldots, n\}$. For cyclically permuted direct product maps, if $n = 2$ and $X_1 = X_2 = [0, 1]$, these maps appears associated with certain economical model so called Cournot duopoly (see [2, 11–13], etc.). Even one can find them in age-structured population models, as in [24], where it is analyzed the Leslie model. For study on population models, we refer the reader to [24] and the references therein.

For chaotic maps, there have been many applications. Since Li and Yorke [3] introduced the term of chaos in 1975, chaotic dynamical systems were highly discussed and investigated in the literature (see [18, 25, 26] and the references therein) as they are very good examples of problems coming from the theory of topological dynamics and model and many phenomena from biology, physics, chemistry, engineering, and social sciences. Recently, some new 1D and 2D chaotic systems with complex chaos performance have been developed (see [27–35] and the references therein).

Motivated by [13, 15, 22, 23], we will deal with the dynamical properties of cyclic permutation maps:

$$\begin{aligned} \Psi\colon \Pi H_j &\longrightarrow \Pi H_j, \quad \psi(x_1, \ldots, x_s) \\ &= (g_s(x_s), g_1(x_1), g_2(x_2), \ldots, g_{s-1}(x_{s-1})), \end{aligned} \tag{1}$$

defined from a Cartesian product $\Pi H_j = H_1 \times \cdots \times H_s$ into itself, where each H_i is a compact subinterval of the real line, and $g_j\colon H_j \longrightarrow H_{j+1}$, $j = 1, \ldots, s-1$.

$g_s\colon H_S \longrightarrow H_1$. Clearly, cyclic permutation map is a cyclically permuted direct product map. Since Ψ^S is a direct product of interval maps, $\Psi^s = \phi_1 \times \phi_2 \times \cdots \times \phi_s$, with $\phi_j = g_{j-1} \circ \cdots \circ g_1 \circ g_s \circ \cdots \circ g_j$ for any $j = 2, \cdots, s$ and $\phi_1 = g_s \circ g_{s-1} \circ \cdots \circ g_1$, it makes sense to analyze the dynamics of Ψ in terms of these compositions ϕ_j. In particular, a necessary and sufficient condition for a cyclic permutation map $\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1}))$ to be LY-chaotic or h-chaotic or RT-chaotic or D-chaotic is given. Moreover, the LY-chaoticity, h-chaoticity, RT-chaoticity, and D-chaoticity of such a cyclic permutation map is discussed. More precisely, for a cyclic permutation map,

$$\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1})), \tag{2}$$

where

$$(a_1, a_2, \ldots, a_s) \in H_1 \times H_2 \times \cdots \times H_S. \tag{3}$$

We obtain the following results:

(1) Ψ is LY-chaotic if and only if φ_j is LY-chaotic for any $j = 1, \ldots, s$

(2) Ψ is h-chaotic if and only if φ_j is h-chaotic for any $j = 1, \ldots, s$

(3) If Ψ is RT-chaotic, then φ_j is RT-chaotic for any $j = 1, \ldots, s$

(4) If Ψ is D-chaotic, then φ_j is D-chaotic for any $j = 1, \ldots, s$

(5) Ψ is topologically transitive if and only if Ψ is D-chaotic if and only if Ψ is RT-chaotic

(6) If Ψ is topologically transitive, then it is h-chaotic

(7) If Ψ is D-chaotic, then $\Psi|_{W_{12,\ldots,s}}$ is D-chaotic

(8) If Ψ is RT-chaotic, then so is $\Psi|_{W_{12,\ldots,s}}$

(9) $\Psi|_{W_{12,\ldots,s}}$ is h-chaotic if and only if so is Ψ

(10) $\Psi|_{W_{12,\ldots,s}}$ is LY-chaotic if and only if so is Ψ

Our results extend some existing ones on two-dimensional dynamical systems. Also, it is shown that the topological entropy $h(\Psi)$ of such a cyclic permutation map is the same as the topological entropy of each of the following maps: φ_j, $j = 1, \ldots, s$, and that it is sensitive if and only if at least one of the following maps is sensitive: φ_j, $j = 1, \ldots, s$.

The interest of studying continuous cyclic permutation maps is as follows. Firstly, they are s-dimensional maps whose dynamical behavior is close to that of one-dimensional maps (see [22, 23, 36, 37] and the references therein), where $s \geq 2$ is any given integer, and their study could give us information about the dynamics of general s-dimensional maps. Secondly, these maps are closely related with the model of an economic process called Cournot duopoly (see [22–24, 38]), the age-structured population models (see [24] and the references therein), and the cyclically permuted direct product maps (see [22, 23] and the references therein). Thirdly, the chaotic dynamics is widely used in nonlinear control, synchronization communication, and many other applications (see [27–35] and the references therein).

2. Preliminaries

Let X be a metric space with metric d.

A dynamical system (X, f) or a continuous map $f: X \longrightarrow X$ is said to be

(1) Transitive if for every pair of nonempty open sets U and V of X, there exists a positive integer n such that $f^n(U) \cap V \neq \varphi$.

(2) Mixing if for every pair of nonempty open sets U and V of X, there exists a positive integer N such that $f^n(U) \cap V \neq \varphi$ for every integer $n \geq N$.

(3) Sensitive if there is an $\varepsilon > 0$ such that whenever U is a nonempty open set of X, there exist points $x, y \in U$ such that $d(f^n(x), f^n(y)) > \varepsilon$ for some positive integer n, where ε is called a sensitive constant of f.

(4) Chaotic in the sense of Ruelle–Takens (or RT-chaotic, for short) (see [39]) if it is both transitive and sensitive.

(5) Chaotic in the sense of Li–Yorke (or LY-chaotic, for short) if there is an uncountable set $D \subset X/\mathrm{Per}(f)$ such that for any $a, b \in D$ with $a \neq b$:

$$\begin{aligned} &\liminf_{k \longrightarrow +\infty} d\left(f^k(a), f^k(b)\right) = 0, \\ &\limsup_{k \longrightarrow +\infty} d\left(f^k(a), f^k(b)\right) > 0, \end{aligned} \tag{4}$$

where $\mathrm{Per}(f)$ is the set of periodic points of f and a point $x \in X$ is called a periodic point of f if there is an integer $k > 0$ such that $f^k(x) = x$.

(6) Chaotic in the sense of Devaney (or D-chaotic, for short) if f is transitive and sensitive with $\overline{\mathrm{Per}(f)} = X$, $\overline{A}$ denotes the closure of the set A.

A subset $D \subset X$ is said to be an (n, ε)-separated if for any $a, b \in D$ with $a \neq b$ there is $i \in \{0, 1, \ldots, n-1\}$ with $d(f^i(a), f^i(b)) > \varepsilon$. Let $s_n(\varepsilon, f)$ be the maximal possible cardinality of an (n, ε)-separated set. The topological entropy of f (see [25, 40]) is defined as

$$h(f) = \lim_{\varepsilon \longrightarrow 0} \limsup_{n \longrightarrow +\infty} \left(\frac{1}{n}\right) \log s_n(\varepsilon, f). \tag{5}$$

Recall that $h(f^n) = nh(f)$ for any integer $n \geq 0$. A dynamical system (X, f) or a map $f: X \longrightarrow X$ is h-chaotic if $h(f) > 0$.

Let H_k be a compact interval of the real line $\mathbb{R} = (-\infty, +\infty)$ for any $k \in \{1, 2, \ldots, s\}$ and $H_1 \times H_2 \times \cdots \times H_S$ endowed with the product metric σ which is defined by

$$\sigma((a_1, a_2, \ldots, a_s), (b_1, b_2, \ldots, b_s)) = \max_{1 \leq k \leq s} \left\{\left|b_k - a_k\right|\right\}, \tag{6}$$

for any

$$(a_1, a_2, \ldots, a_s), (b_1, b_2, \ldots, b_s) \in H_1 \times H_2 \times \cdots \times H_s. \tag{7}$$

For any continuous map $f: X \longrightarrow X$ on a metric space X and any $x \in X$, the set of limit points of the sequence $(f^m(x))_{m \geq 0}^{\infty}$ is the ω-limit set of x which is written by $\omega_f(x)$. Then, if X has no isolated points, then the map f is transitive if there exists $x \in X$ with $\omega_f(x) = X$. Write $\mathrm{Tr}(f) = \left\{x \in X: \omega_f(x) = X\right\}$. Then, $\mathrm{Tr}(f)$ is the set of transitive points of f.

3. Main Results

3.1. Relation between Some Chaotic Properties of a Continuous Cyclic Permutation Map and the Corresponding Properties of Every Coordinate Map. It is known in the frame of general topological spaces (and for general cyclic permutations) that if Ψ is transitive (even mixing or weakly mixing or totally transitive) then the associate compositions φ_j $(j = 1, 2, \ldots, s)$ so are. This was done in [22]. For completeness, we give the proof of Lemma 1.

We need the following two lemmas.

Lemma 1. *For a continuous cyclic permutation map,*

$$\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1})), \tag{8}$$

where

$$(a_1, a_2, \ldots, a_s) \in H_1 \times H_2 \times \cdots \times H_S. \tag{9}$$

If it is topologically transitive, then every coordinate map of Ψ^s is topologically transitive.

Proof. As the proof of the result in Lemma 1 is similar for any $s \geq 3$, for simplicity we only prove Lemma 1 for $s = 3$. Suppose that Ψ is topologically transitive and let

$$(a_1, a_2, a_3) \in H_1 \times H_2 \times .H_3, \tag{10}$$

with

$$\omega_\Psi(a_1, a_2, a_3) = H_1 \times H_2 \times .H_3, \tag{11}$$

where $\omega_\Psi(a_1, a_2, a_3)$ is the ω-limit set of (a_1, a_2, a_3) under Ψ. By (3) in [36], hypothesis, the definition,

$$\begin{aligned}
\Psi^{3n}(x, y, z) &= (g_3 \circ g_2 \circ g_1(x), g_1 \circ g_3 \circ g_2(y), g_2 \circ g_1 \circ g_3(z)),\\
\Psi^{3n+1} &= (g_3 \circ g_2 \circ g_1 \circ g_3(z), g_1 \circ g_3 \circ g_2 \circ g_1(x), g_2 \circ g_1 \circ g_3 \circ g_2(y)),\\
\Psi^{3n+2} &= (g_3 \circ g_2 \circ g_1 \circ g_3 \circ g_2(y), g_1 \circ g_3 \circ g_2 \circ g_1 \circ g_3(z), g_2 \circ g_1 \circ g_3 \circ g_2 \circ g_1(x)).
\end{aligned} \tag{12}$$

One can easily prove that

$$\begin{aligned}
H_1 \times H_2 \times .H_3 &= \omega_\Psi(a_1, a_2, \ldots, a_s),\\
&\subset \left(\omega_{g_3 \circ g_2 \circ g_1}(a_1) \times \omega_{g_1 \circ g_3 \circ g_2}(a_2) \times \omega_{g_2 \circ g_1 \circ g_3}(a_3)\right),\\
&\cup \left(\omega_{g_3 \circ g_2 \circ g_1}(g_3(a_3)) \times \omega_{g_1 \circ g_3 \circ g_2}(g_1(a_1)) \times \omega_{g_2 \circ g_1 \circ g_3}(g_2(a_2))\right),\\
&\cup \left(\omega_{g_3 \circ g_2 \circ g_1}(g_3(g_2(a_2)))\right) \times \omega_{g_1 \circ g_3 \circ g_2}(g_1(g_3(a_3))) \times \omega_{g_2 \circ g_1 \circ g_3}(g_2(g_1(a_1))),\\
&\subset H_1 \times H_2 \times .H_3.
\end{aligned} \tag{13}$$

Now, we show that if

$$\omega_{g_3 \circ g_2 \circ g_1}(a_1) = H_1, \tag{14}$$

then

$$\begin{aligned}
\omega_{g_1 \circ g_3 \circ g_2}(g_1(a_1)) &= H_2,\\
\omega_{g_2 \circ g_1 \circ g_3}(g_2(g_1(a_1))) &= H_3.
\end{aligned} \tag{15}$$

As Ψ is topologically transitive, g_i is surjective for any $i \in \{1, 2, 3\}$. So, for any $y_2 \in H_2$ there is $y_1 \in H_1$ with $g_1(y_1) = y_2$. Let $\{m_i\}_{i=1}^{\infty}$ be a sequence of positive integers with

$$\lim_{i \longrightarrow \infty} (g_3 \circ g_2 \circ g_1)^{m_i}(a_1) = y_1. \tag{16}$$

By the continuity of g_1,

$$\lim_{i \longrightarrow \infty} g_1((g_3 \circ g_2 \circ g_1)^{m_i}(a_1)) = g_1(y_1), \tag{17}$$

That is, $\lim_{i \longrightarrow \infty} (g_3 \circ g_2 \circ g_1)^{m_i}(g_1(a_1)) = y_2$. This implies that

$$\omega_{g_1 \circ g_3 \circ g_2}(g_1(a_1)) = H_2. \tag{18}$$

Similarly, one can prove

$$\omega_{g_2 \circ g_1 \circ g_3}(g_2(g_1(a_1))) = H_3. \tag{19}$$

By the similar argument, we obtain that if

$$\omega_{g_1 \circ g_3 \circ g_2}(a_2) = H_2, \tag{20}$$

then

$$\begin{aligned}
\omega_{g_2 \circ g_1 \circ g_3}(g_2(a_2)) &= H_3,\\
\omega_{g_3 \circ g_2 \circ g_1}(g_3(g_2(a_2))) &= H_1,
\end{aligned} \tag{21}$$

and that if

$$\omega_{g_2 \circ g_1 \circ g_3}(a_3) = H_3, \tag{22}$$

then

$$\begin{aligned}
\omega_{g_3 \circ g_2 \circ g_1}(g_3(a_3)) &= H_1,\\
\omega_{g_1 \circ g_3 \circ g_2}(g_1(g_3(a_3))) &= H_2.
\end{aligned} \tag{23}$$

By the above argument and the transitivity of Ψ, it follows that the case $\omega_{g_3 \circ g_2 \circ g_1}(a_1) \neq H_1, \omega_{g_1 \circ g_3 \circ g_2}(a_2) \neq H_2$, and $\omega_{g_2 \circ g_1 \circ g_3}(a_3) \neq H_3$ cannot happen. Thus, by the above argument, we have

$$\begin{aligned}
\omega_{g_3 \circ g_2 \circ g_1}(a_1) &= H_1,\\
\omega_{g_1 \circ g_3 \circ g_2}(a_2) &= H_2,\\
\omega_{g_2 \circ g_1 \circ g_3}(a_3) &= H_3.
\end{aligned} \tag{24}$$

This means that $g_3 \circ g_2 \circ g_1$, $g_1 \circ g_3 \circ g_2$, and $g_2 \circ g_1 \circ g_3$ are topologically transitive.

However, it is not true that the transitivity of ϕ_j implies the transitivity of Ψ. In fact, we can construct examples of maps Ψ which are not transitive but the compositions ϕ_j are transitive. Let us show an example. □

Example 1. Let $\Psi = (y, z, f(x)): I^n \longrightarrow I^n$, $n = 3$, with $f: I \longrightarrow I$ transitive but not totally transitive, that is, $f^2|_I$ is not transitive (here, $I = [0, 1]$ is the unit interval, but we can translate the example to arbitrary sets $H_1 \times H_2 \times H_3$). Since $f^2|_I$ is not transitive, we can decompose the unit interval into $I = I_1 \cup I_2$, being I_1 and I_2 compact intervals such that $I_1 \cap I_2 = \{u\}$, with $f(u) = u$ and such that $f(I_1) = I_2$, $f(I_2) = I_1$, and $f^2|_I$ mixing, $j = 1, 2$(for details, consult, for instance, [26]). In this case, $\Psi^6(x, y, z) = (f^2(x), f^2(y), f^2(z))$, and taking into account that $f^2|_{I_j} = I_j$, we find that $\Psi^6(I_1 \times I_1 \times I_1) \subseteq I_1 \times I_1 \times I_1 \subseteq I^3$. Consequently, Ψ^6 is not transitive. Due to the fact that $n \geq 3$, according to [23] (see Corollary B), we have that Ψ is transitive if and only if Ψ is totally transitive; since Ψ^6 is not transitive, we deduce that Ψ is not transitive although its associated composition $\phi_j = f$ is. Nevertheless, in the two-dimensional case, when $\Psi(x, y) = (f(y), g(x))$, and $H_1 = H_2 = I$, the transitivity of $f \circ g$ implies the transitivity of Ψ, as it was proved in Lemma 2.1 in [13] (Canovas-Marin).

We note that the following lemma holds in the general setting of topological spaces: to this end, consult Lemma 7 in [23]. So, the proof of Lemma 2 is omitted here.

Lemma 2. *For a topologically transitive cyclic permutation map,*

$$\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1})), \tag{25}$$

where

$$(a_1, a_2, \ldots, a_s) \in H_1 \times H_2 \times \cdots \times H_S, \tag{26}$$

$\overline{Per(\phi_j)} = H_j$ *for any* $j = 1, \ldots, s$. *Then, one has*

$$\overline{\mathrm{Per}(\Psi)} = H_1 \times H_2 \times \cdots \times H_S. \tag{27}$$

Theorem 1. *For a cyclic permutation map,*

$$\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1})), \tag{28}$$

where

$$(a_1, a_2, \ldots, a_s) \in H_1 \times H_2 \times \cdots \times H_S. \tag{29}$$

The following hold:

(1) Ψ is LY-chaotic if and only if φ_j is LY-chaotic for any $j = 1, \ldots, s$

(2) Ψ is h-chaotic if and only if φ_j is h-chaotic for any $j = 1, \ldots, s$

(3) If Ψ is RT-chaotic, then φ_j is RT-chaotic for any $j = 1, \ldots, s$

(4) If Ψ is D-chaotic, then φ_j is D-chaotic for any $j = 1, \ldots, s$

(5) If φ_j is RT-chaotic for any $j = 1, \ldots, s$ and satisfies that there is an transitive point of $h_j \in H_j$ for any $j = 1, \ldots, s$ such that

$$\begin{aligned}
\omega_\Psi(h_1, h_2, \ldots, h_s) = {} & \left[\omega_{\varphi_1}(h_1) \times \omega_{\varphi_2}(h_2) \times \ldots \times \omega_{\varphi_s}(h_s)\right], \\
& \cup \left[\omega_{\varphi_1}(g_s(h_s)) \times \omega_{\phi_2}(g_1(h_1)) \times \cdots \times \omega_{\varphi_s}(g_{s-1}(h_{s-1}))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1}(h_{s-1})) \times \omega_{\varphi_2}(g_1 \circ g_s(h_s)) \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ (g_{s-2}(h_{s-2})))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ g_{s-2}(h_{s-2})) \times \omega_{\varphi_2}(g_1 \circ g_s \circ g_{s-1}(h_{s-1})) \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ (g_{s-3}(h_{s-3})))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ \cdots \circ g_3(h_3)) \times \omega_{\varphi_2}(g_1 \circ g_s \circ g_{s-1} \circ \cdots \circ g_4(h_4)) \times \cdots \times \omega_{\varphi_{s-2}}\right. \\
& \quad \left. \cdot (g_{s-3} \circ g_{s-4} \circ \cdots \circ g_2 \circ g_1 \circ g_s(h_s)) \times \omega_{\varphi_{s-1}}(g_{s-2} \circ \cdots \circ g_1(h_1)) \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ \cdots \circ g_2(h_2))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ g_2(h_2)) \times \omega_{\varphi_2}(g_1 \circ g_s \circ \cdots \circ g_3(h_3)) \times \cdots \times \omega_{\varphi_{s-1}}(g_{s-2} \circ g_1 \circ g_s(h_s))\right. \\
& \quad \left. \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ g_1(h_1))\right].
\end{aligned} \tag{30}$$

then Ψ is RT-chaotic.

(6) If φ_j is D-chaotic for any $j = 1, \ldots, s$ and satisfies that there is an transitive point of $h_j \in H_j$ for any $j = 1, \ldots, s$ such that

$$\begin{aligned}
\omega_\Psi(h_1, h_2, \ldots, h_s) = {} & \left[\omega_{\varphi_1}(h_1) \times \omega_{\varphi_2}(h_2) \times \cdots \times \omega_{\varphi_s}(h_s)\right], \\
& \cup \left[\omega_{\varphi_1}(g_s(h_s)) \times \omega_{\varphi_2}(g_1(h_1)) \times \cdots \times \omega_{\varphi_s}(g_{s-1}(h_{s-1}))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1}(h_{s-1})) \times \omega_{\varphi_2}(g_1 \circ g_s(h_s)) \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2}(h_{s-2}))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ g_{s-2}(h_{s-2})) \times \omega_{\varphi_2}(g_1 \circ g_s \circ g_{s-1}(h_{s-1})) \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ g_{s-3}(h_{s-3}))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ \cdots \circ g_3(h_3)) \times \omega_{\varphi_2}(g_1 \circ g_s \circ g_{s-1} \circ \cdots \circ g_4(h_4)) \times \cdots \times \omega_{\varphi_{s-2}}\right. \\
& \quad \left. \cdot (g_{s-3} \circ g_{s-4} \circ \cdots \circ g_2 \circ g_1 \circ g_s(h_s)) \times \omega_{\varphi_{s-1}}(g_{s-2} \circ \cdots \circ g_1(h_1)) \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ \cdots \circ g_2(h_2))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ g_2(h_2)) \times \omega_{\varphi_2}(g_1 \circ g_s \circ \cdots \circ g_3(h_3)) \times \cdots \times \omega_{\varphi_{s-1}}(g_{s-2} \circ g_1 \circ g_s(h_s)) \times \cdots \times \omega_{\varphi_s}\right. \\
& \quad \left. \cdot (g_{s-1} \circ g_{s-2} \circ g_1(h_1))\right].
\end{aligned} \tag{31}$$

Then, Ψ is D-chaotic.

(7) If ϕ_j is RT-chaotic for any $j = 1, \ldots, s$ and satisfies that, for any $j = 1, \ldots, s$ and any $h_j \in H_j$,

$$\begin{aligned}
\omega_\Psi(h_1, h_2, \ldots, h_s) \subset {} & \left[\omega_{\varphi_1}(h_1) \times \omega_{\varphi_2}(h_2) \times \cdots \times \omega_{\varphi_s}(h_s)\right], \\
& \cup \left[\omega_{\varphi_1}(g_s(h_s)) \times \omega_{\varphi_2}(g_1(h_1)) \times \cdots \times \omega_{\varphi_s}(g_{s-1}(h_{s-1}))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1}(h_{s-1})) \times \omega_{\varphi_2}(g_1 \circ g_s(h_s)) \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2}(h_{s-2}))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ g_{s-2}(h_{s-2})) \times \omega_{\varphi_2}(g_1 \circ g_s \circ g_{s-1}(h_{s-1})) \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ g_{s-3}(h_{s-3}))\right], \\
& \cup \cdots, \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ \cdots \circ g_3(h_3)) \times \omega_{\varphi_2}(g_1 \circ g_s \circ g_{s-1} \circ \cdots \circ g_4(h_4)) \times \cdots \times \omega_{\varphi_{s-2}}\right. \\
& \quad \left. \cdot (g_{s-3} \circ g_{s-4} \circ \cdots \circ g_2 \circ g_1 \circ g_s(h_s)) \times \omega_{\varphi_{s-1}}(g_{s-2} \circ \cdots \circ g_1(h_1)) \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ \cdots \circ g_2(h_2))\right], \\
& \cup \left[\omega_{\varphi_1}(g_s \circ g_{s-1} \circ g_2(h_2)) \times \omega_{\varphi_2}(g_1 \circ g_s \circ \cdots \circ g_3(h_3)) \times \cdots \times \omega_{\varphi_{s-1}}\right. \\
& \quad \left. \cdot (g_{s-2} \circ g_1 \circ g_s(h_s)) \times \cdots \times \omega_{\varphi_s}(g_{s-1} \circ g_{s-2} \circ g_1(h_1))\right],
\end{aligned} \tag{32}$$

where $A \subset B$ means that $A \subseteq B$ and $A \neq B$; then, Ψ is not RT-chaotic.

(8) If φ_j is D-chaotic for any $j = 1, \ldots, s$ and satisfies that, for any $j = 1, \ldots, s$ and any $h_j \in H_j$,

$$\begin{aligned}\omega_\Psi(h_1,h_2,\ldots,h_s) \subset & \left[\omega_{\varphi_1}(h_1)\times\omega_{\varphi_2}(h_2)\times\cdots\times\omega_{\varphi_s}(h_s)\right],\\ & \cup\left[\omega_{\varphi_1}(g_s(h_s))\times\omega_{\varphi_2}(g_1(h_1))\times\cdots\times\omega_{\varphi_s}(g_{s-1}(h_{s-1}))\right],\\ & \cup\left[\omega_{\varphi_1}(g_s\circ g_{s-1}(h_{s-1}))\times\omega_{\varphi_2}(g_1\circ g_s(h_s))\times\cdots\times\omega_{\varphi_s}(g_{s-1}\circ g_{s-2}(h_{s-2}))\right],\\ & \cup\left[\omega_{\varphi_1}(g_s\circ g_{s-1}\circ g_{s-2}(h_{s-2}))\times\omega_{\varphi_2}(g_1\circ g_s\circ g_{s-1}(h_{s-1}))\times\cdots\times\omega_{\varphi_s}(g_{s-1}\circ g_{s-2}\circ g_{s-3}(h_{s-3}))\right],\\ & \cup\cdots,\\ & \cup\left[\omega_{\varphi_1}(g_s\circ g_{s-1}\circ\cdots\circ g_3(h_3))\times\omega_{\varphi_2}(g_1\circ g_s\circ g_{s-1}\circ\cdots\circ g_4(h_4))\times\cdots\times\omega_{\varphi_{s-2}}\right.\\ & \quad\left.\cdot(g_{s-3}\circ g_{s-4}\circ\cdots\circ g_2\circ g_1\circ g_s(h_s))\times\omega_{\varphi_{s-1}}(g_{s-2}\circ\cdots\circ g_1(h_1))\times\omega_{\varphi_s}(g_{s-1}\circ g_{s-2}\circ\cdots\circ g_2(h_2))\right],\\ & \cup\left[\omega_{\varphi_1}(g_s\circ g_{s-1}\circ g_2(h_2))\times\omega_{\varphi_2}(g_1\circ g_s\circ\cdots\circ g_3(h_3))\times\cdots\times\omega_{\varphi_{s-1}}(g_{s-2}\circ g_1\circ g_s(h_s))\times\cdots\times\omega_{\varphi_s}\right.\\ & \quad\left.\cdot(g_{s-1}\circ g_{s-2}\circ g_1(h_1))\right],\end{aligned} \tag{33}$$

where $A \subset B$ means that $A \subseteq B$ and $A \neq B$, then Ψ is not D-chaotic.

Proof. By Propositions 3.1 and 3.2 in [15], the definition, hypothesis, and

$$\Psi^s = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_s. \tag{34}$$

One can easily verify that statement (1) is true. By [41], the definition, and $\Psi^s = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_s$, we can easily prove that (2) is true. Now, we show that statement (4) is true. Suppose that Ψ is D-chaotic, that is, it is topologically transitive and sensitive such that

$$\overline{\mathrm{Per}(\Psi)} = \overline{\mathrm{Per}(\Psi^s)} = H_1 \times H_2 \times \cdots \times H_S. \tag{35}$$

As

$$\Psi^s = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_s, \tag{36}$$

and $\mathrm{Per}(\Psi^s) = \mathrm{Per}(\Psi)$, by Lemma 1, Theorem 1.2 in [13, 42] if Ψ is D-chaotic then so is ϕ_j for any $j = 1, \ldots, s$.

Next, we show that statement (3) is also true. Suppose that Ψ is RT-chaotic. Then, it is transitive and sensitive. By Lemmas 1 and 2, one has that ϕ_j is topologically transitive for any $j = 1, \ldots, s$ such that $\overline{\mathrm{Per}(\phi_j)} = H_j$ for any $j = 1, \ldots, s$. By Theorem 1.2 in [13], ϕ_j is sensitive for any $j = 1, \ldots, s$, which implies that ϕ_j is RT-chaotic for any $j = 1, \ldots, s$.

Finally, by the proof of Lemma 2.1 in [13], Lemma 1 in [43], the definitions, and hypothesis, statements (5), (6), (7), and (8) are true. □

Theorem 2. *For a cyclic permutation map,*

$$\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1})), \tag{37}$$

where

$$(a_1, a_2, \ldots, a_s) \in H_1 \times H_2 \times \cdots \times H_S, \tag{38}$$

the following holds:

(1) Ψ is topologically transitive if and only if Ψ is D-chaotic if and only if Ψ is RT-chaotic

(2) If Ψ is topologically transitive then it is h-chaotic

Proof 3

(1) If Ψ is D-chaotic, then by definition it is RT-chaotic. If Ψ is RT-chaotic, then it is topologically transitive. Therefore, it is enough to prove that if Ψ is topologically transitive, then it is D-chaotic. Now, we let Ψ be topologically transitive. By Lemma 2,

$$\overline{\mathrm{Per}(\Psi)} = \overline{\mathrm{Per}(\Psi^s)} = H_1 \times H_2 \times \cdots \times H_S, \tag{39}$$

which means that Ψ is sensitive. So, Ψ is D-chaotic.

(2) Suppose that Ψ is topologically transitive. By Lemma 1, ϕ_j is topologically transitive for any $j = 1, \ldots, s$. As

$$\Psi^s = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_s, \tag{40}$$

by Theorem 1.2 in [13] $h(\Psi^s) > 0$. This implies that $h(\Psi) > 0$. □

3.2. Chaos on MPE Set for a Permutation Map. For a permutation map,

$$\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1})), \tag{41}$$

where

$$(a_1, a_2, \ldots, a_s) \in H_1 \times H_2 \times \cdots \times H_S. \tag{42}$$

Let

$$\begin{aligned} W_s &= \{(g_s(a_s), g_1 \circ g_s(a_s), g_2 \circ g_1 \circ g_s(a_s), \ldots, g_{s-2} \circ \cdots \\ &\quad \circ g_1 \circ g_s(a_s), a_s) \colon a_s \in H_s\}, \\ W_{s-1} &= \{(g_s(\mathrm{b}_s), g_1 \circ g_s(b_s), g_2 \circ g_1 \circ g_s(b_s), \ldots, g_{s-3} \circ \cdots \\ &\quad \circ g_1 \circ g_s(b_s), a_{s-1}, b_s) \colon a_{s-1} \in H_{s-1}\}, \\ &\vdots \\ W_1 &= \{(a_1, g_1(a_1), \ldots, g_{s-1} \circ \cdots \circ g_1(a_1)) : a_1 \in H_1\}, \end{aligned} \tag{43}$$

where $b_s = g_{s-1}(a_{s-1})$. Let $W_{12,\ldots,s} = W_1 \cup W_2 \cup \ldots \cup W_S$ be the MPE set for Ψ.

Theorem 3. *For a permutation map,*

$$\Psi(a_1, a_2, \ldots, a_s) = (g_s(a_s), g_1(a_1), \ldots, g_{s-1}(a_{s-1})), \tag{44}$$

where

$$(a_1, a_2, \ldots, a_s) \in H_1 \times H_2 \times \cdots \times H_S, \tag{45}$$

the following hold:

(1) If Ψ is D-chaotic, then $\Psi|_{W_{12,\ldots,s}}$ is D-chaotic

(2) If Ψ is RT-chaotic, then so is $\Psi|_{W_{12,\ldots,s}}$

(3) $\Psi|_{W_{12,\ldots,s}}$ is h-chaotic if and only if so is Ψ

(4) $\Psi|_{W_{12,\ldots,s}}$ is LY-chaotic if and only if so is Ψ

Proof 4. As the proof of the result in Theorem 3 is similar for any $s \geq 3$, for simplicity, we only prove Theorem 3 for $s = 3$. First, we claim that if Ψ is topologically transitive then so is $\Psi|_{W_{123}}$. To prove this, we let $(a_1, a_2, a_3) \in H_1 \times H_2 \times H_3$ with $\omega_\Psi(a_1, a_2, a_3) \in H_1 \times H_2 \times H_3$. By the proof of Lemma 1, one can easily see that $a_1 \in \mathrm{Tr}(g_3 \circ g_2 \circ g_1)$ or $a_2 \in \mathrm{Tr}(g_1 \circ g_3 \circ g_2)$ or $a_3 \in \mathrm{Tr}(g_2 \circ g_1 \circ g_3)$. By the proof of Lemma 1, if $a_1 \in \mathrm{Tr}(g_3 \circ g_2 \circ g_1)$, then $g_1(a_1) \in \mathrm{Tr}(g_1 \circ g_3 \circ g_2)$ and $g_2 \circ g_1(a_1) \in \mathrm{Tr}(g_2 \circ g_1 \circ g_3)$. Fix $a \in H_1$. Then, $(a, g_1(a), g_2 \circ g_1(a)) \in W_1$. By hypothesis and the definition, there is a sequence $\{m_i\}_{i=1}^{\infty}$ of positive integers with

$$\lim_{i \longrightarrow \infty} \Psi^{m_i}(a, g_1(a), g_2 \circ g_1(a)) = (a, g_1(a), g_2 \circ g_1(a)). \tag{46}$$

Fix $\mathrm{b} \in H_2$. Then, $(\mathrm{g}_3 \circ g_2(b), b, g_2(b)) \in W_2$. As g_1 is onto, there is $a \in H_1$ with $g_1(a) = b$. As $g_1(a_1) \in \mathrm{Tr}(g_1 \circ g_3 \circ g_2)$,

$$\begin{aligned} &\lim_{i \longrightarrow \infty} \Psi^{m_i+1}(a_1, g_1(a_1), g_2 \circ g_1(a_1)), \\ &\quad = \lim_{i \longrightarrow \infty} \Psi^{m_i}(g_3 \circ g_2 \circ g_1(a_1), g_1(a_1), g_2 \circ g_1(a_1)), \\ &\quad = (g_3 \circ g_2(b), b, g_s(b)). \end{aligned} \tag{47}$$

Fix $\mathrm{c} \in H_3$. Then, $(\mathrm{g}_3(c), g_1 \circ g_3(c), c) \in W_3$. As g_1 and g_2 is onto, there is $a \in H_1$ with $g_2 \circ g_1(a) = c$. As $g_2 \circ g_1(a_1) \in \mathrm{Tr}(g_2 \circ g_1 \circ g_3)$,

$$\begin{aligned} &\lim_{i \longrightarrow \infty} \Psi^{m_i+2}(a_1, g_1(a_1), g_2 \circ g_1(a_1)), \\ &\quad = \lim_{i \longrightarrow \infty} \Psi^{m_i}(g_3 \circ g_2 \circ g_1(a_1), \mathrm{g}_1 \circ g_3 \circ g_2 \circ g_1(a_1), \\ &\quad\quad g_2 \circ g_1(a_1)), \\ &\quad = (g_3(c), g_1 \circ g_3(c), c). \end{aligned} \tag{48}$$

(1) Suppose that Ψ is D-chaotic. By the above argument, $\Psi|_{W_{123}}$ is topologically transitive. As $\Psi(W_{123}) \subseteq W_{123}$, by hypothesis and the definition $\Psi|_{W_{123}}$ is D-chaotic.

(2) Suppose that Ψ is RT-chaotic, which implies that it is topologically transitive. By the above argument, $\Psi|_{W_{123}}$ is topologically transitive. Let $a \in H_1$. Then, $(a, g_1(a), g_2 \circ g_1(a)) \in W_1$. By Theorem 1, there exists $\delta > 0$ such that, for any open neighborhood V of a, there exist $b \in V$ and an integer $m \geq 0$ with

$$\left|(g_3 \circ g_2 \circ g_1)^m(a) - (g_3 \circ g_2 \circ g_1)^m(b)\right| > \delta. \tag{49}$$

This means that

$$\sigma\left(\Psi^{3m}(a, g_1(a), g_2 \circ g_1(a)), \Psi^{3m}(b, g_1(b), g_2 \circ g_1(b))\right) > \delta. \tag{50}$$

Similarly, the condition can be proved for any $(g_3 \circ g_2(a), a, g_2(a)) \in W_2$ and any $(g_3(a), g_1 \circ g_3(a), a) \in W_3$. By the definition, $\Psi|_{W_{123}}$ is RT-chaotic.

(3) Clearly, $h(\Psi|_{W_{123}}) > 0$ implies that $h(\Psi) > 0$. Suppose that $h(\Psi) > 0$. Consider $\Psi^3|_{W_{123}}$. It is easy to see that W_1 is trapping for Ψ^3 and that

$$\begin{aligned} h\left(\Psi^3|_{W_1}\right) &= h(g_3 \circ g_2 \circ g_1) + h\left(g_1 \circ g_3 \circ g_2|_{g_1(H_1)}\right) \\ &\quad + h\left(g_2 \circ g_1 \circ g_3|_{g_2 \circ g_1(H_1)}\right). \end{aligned} \tag{51}$$

As

$$\begin{aligned} h(\Psi) &= h(g_3 \circ g_2 \circ g_1) = h\left(g_1 \circ g_3 \circ g_2|_{g_1(H_1)}\right) \\ &= h\left(g_2 \circ g_1 \circ g_3|_{g_2 \circ g_1(H_1)}\right) > 0, \end{aligned} \tag{52}$$

$h(\Psi^3|_{W_{123}}) > 0$, which implies that $h(\Psi|_{W_{123}}) > 0$.

(4) Obviously, by the definition, if $\Psi|_{W_{123}}$ is LY-chaotic then so is Ψ. Suppose that Ψ is LY-chaotic. By Theorem 1, $g_3 \circ g_2 \circ g_1$, $g_1 \circ g_3 \circ g_2$, and $g_2 \circ g_1 \circ g_3$ are LY-chaotic. Let D be an uncountable Li–Yorke chaotic set for $g_3 \circ g_2 \circ g_1$ and let

$$B = \{(a, g_1(a), g_2 \circ g_1(a)) \colon a \in D\}. \tag{53}$$

Then, $B \subset W_1$ is uncountable. Now, we will prove that B is a Li–Yorke chaotic set for $\Psi^3|_{W_1}$. For any

$$(a, g_1(a), g_2 \circ g_1(a)), (b, g_1(b), g_2 \circ g_1(b)) \in B, \tag{54}$$

with $a \neq b$, one has that

$$\limsup_{m\longrightarrow\infty}\sigma\left(\Psi^{3m}(a,g_1(a),g_2\circ g_1(a)),\Psi^{3m}(b,g_1(b),g_2\circ g_1(b))\right),$$
$$\geq \limsup_{m\longrightarrow\infty}\left|(g_3\circ g_2\circ g_1)^m(a)-(g_3\circ g_2\circ g_1)^m(b)\right|>0. \tag{55}$$

Since

$$\liminf_{m\longrightarrow\infty}\left|(g_3\circ g_2\circ g_1)^m(a)-(g_3\circ g_2\circ g_1)^m(b)\right|=0, \tag{56}$$

there is an increasing sequence $\{m_i\}_{i=1}^{\infty}$ of positive integers with

$$\lim_{\mathrm{i}\longrightarrow\infty}\left|(g_3\circ g_2\circ g_1)^{m_{\mathrm{i}}}(\mathrm{g}_1(a))-(g_3\circ g_2\circ g_1)^{m_{\mathrm{i}}}(g_1(b))\right|=0. \tag{57}$$

As g_1 and g_2 are uniformly continuous,

$$\lim_{\mathrm{i}\longrightarrow\infty}\left|(g_3\circ g_2\circ g_1)^{m_{\mathrm{i}}}(\mathrm{g}_1(a))-(g_3\circ g_2\circ g_1)^{m_{\mathrm{i}}}(g_1(b))\right|=0, \tag{58}$$

$$\lim_{\mathrm{i}\longrightarrow\infty}\left|(g_3\circ g_2\circ g_1)^{m_{\mathrm{i}}}(g_2\circ \mathrm{g}_1(a))-(g_3\circ g_2\circ g_1)^{m_{\mathrm{i}}}(g_2\circ g_1(b))\right|=0, \tag{59}$$

which mean that

$$\begin{aligned}&\liminf_{m\longrightarrow\infty}\sigma\left(\Psi^{3m}(a,g_1(a),g_2\circ g_1(a)),\Psi^{3m}(b,g_1(b),g_2\circ g_1(b))\right),\\&\leq \liminf_{i\longrightarrow\infty}\sigma\left(\Psi^{3m_i}(a,g_1(a),g_2\circ g_1(a)),\Psi^{3m_i}(b,g_1(b),g_2\circ g_1(b))\right),\\&\leq \lim_{i\longrightarrow\infty}\Big(\left|(g_3\circ g_2\circ g_1)^{m_i}(a)-(g_3\circ g_2\circ g_1)^{m_i}(b)\right|+\left|(g_3\circ g_2\circ g_1)^{m_i}\right.\\&\quad\cdot(g_1(a))-(g_3\circ g_2\circ g_1)^{m_i}(g_1(b))\Big|+\left|(g_3\circ g_2\circ g_1)^{m_i}(g_2\circ g_1(a))\right.\\&\quad\left.-(g_3\circ g_2\circ g_1)^{m_i}(g_2\circ g_1(b))\right|\Big)=0.\end{aligned} \tag{60}$$

Consequently, B is a Li–Yorke chaotic set for $\Psi^3|_{W_1}$, which implies that $\Psi^3|_{W_1}$ is LY-chaotic. Thus, $\Psi|_{W_{123}}$ is LY-chaotic. □

3.3. The Topological Entropy and Sensitivity of a Continuous Cyclic Permutation Map. For the properties $h(\Psi^n)=nh(\Psi)$ and $h(F\times G)=h(F)+h(G)$, we refer the reader to [41]. And for the property

$$\begin{aligned}&\mathrm{h}\left(g_j\circ g_{j-1}\circ\cdots\circ g_1\circ g_s\circ g_{s-1}\circ\cdots\circ g_{j+1}\right)\\&\quad=h\left(g_s\circ g_{s-1}\circ\cdots\circ g_2\circ g_1\right),\end{aligned} \tag{61}$$

we refer the reader to [44], where it is proved, in the general setting of continuous selfmaps $f_1,\ldots,f_m$ of a compact topological space X that

$$h\left(f_m\circ\cdots\circ f_2\circ f_1\right)=h\left(f_{j-1}\circ\cdots\circ f_2\circ f_1\circ f_m\circ\cdots\circ f_j\right),\quad j=2,\ldots,m. \tag{62}$$

Moreover, the topological entropy of this kind of maps has been already computed in [45]. However, for completeness, we give Theorem 4 and its proof here.

Theorem 4. *For a continuous cyclic permutation map,*

$$\Psi(a_1,a_2,\ldots,a_{\mathrm{s}})=\left(g_s(a_s),g_1(a_1),\ldots,g_{s-1}(a_{s-1})\right), \tag{63}$$

where

$$(a_1,a_2,\ldots,a_s)\in H_1\times H_2\times\cdots\times H_S, \tag{64}$$

$h(\Psi)=h(\phi_j)$ *for any* $j=1,\ldots,s$.

Proof. As

$$\Psi^{\mathrm{s}}=\varphi_1\times\varphi_2\times\cdots\times\varphi_s. \tag{65}$$

by [38] and Proposition 5.2 in [2], Theorem 4 holds.

For the sensitive properties of product maps or semiflows, we refer the reader to [43, 46]. However, for completeness, we give Theorem.5 and its proof here. □

Theorem 5. *For a continuous cyclic permutation map,*

$$\Psi(a_1,a_2,\ldots,a_{\mathrm{s}})=\left(g_s(a_s),g_1(a_1),\ldots,g_{s-1}(a_{s-1})\right), \tag{66}$$

where

$$(a_1,a_2,\ldots,a_s)\in H_1\times H_2\times\cdots\times H_S, \tag{67}$$

and it is sensitive if and only if at least one of the following maps is sensitive: ϕ_j, $j=1,\ldots,s$.

Proof. As

$$\Psi^{\mathrm{s}}=\varphi_1\times\varphi_2\times\cdots\times\varphi_s, \tag{68}$$

by Lemma 1 in [43] and Theorem 1 in [47], Theorem 5 holds. □

3.4. Discussion on Applications. It is clear that a continuous cyclic permutation map defined by

$$\Psi(a_1,a_2,\ldots,a_{\mathrm{s}})=\left(g_s(a_s),g_1(a_1),\ldots,g_{s-1}(a_{s-1})\right), \tag{69}$$

for any

$$(a_1,a_2,\ldots,a_s)\in H_1\times H_2\times\cdots\times H_S, \tag{70}$$

in this paper is a cyclically permuted direct product (see [22, 23] and references therein). For cyclically permuted direct products, there have been many important results and applications (see [22, 23] and references therein).

When $s=2$ and $H_1=H_2=[0,1]$, this type of continuous cyclic permutation map appears as certain economical model so called Cournot duopoly (see [1, 2, 10–17, 38, 42] and references therein). For many important results and applications of Cournot duopoly games, we refer the reader to [1, 2, 10–17, 38, 42] and the references therein.

From [31], we know that one can find continuous cyclic permutation maps in age-structured population models, as in [24], where it is analyzed as the Leslie model:

$$y_1(n+1) = y_N g(y_N), y_2(n+1) = y_1(n), \ldots, y_N(n+1) = y_{N-1}(n), \tag{71}$$

where g is a C^1-map and each variable $y_i(n)$, $i = 1, \ldots, N, n = 0, 1, 2, \ldots$, determines the population size of the j-age class in the nth period, being $y_i(0)$ the initial population. To study some dynamical behavior of this kind of model is equivalent to explore the same behavior of the c.p.d.p. map $F(y_1, y_2, \ldots, y_N) = (y_N g(y_N), y_1, \ldots, y_{N-1})$ (which is also a continuous cyclic permutation map), where $\sigma(i) = (i-1)(\mathrm{mod} N)$, $i = 1, \ldots, N$. For study, results and applications of population models, we refer the reader to [24] and the references therein.

Because the chaotic maps have the excellent properties of unpredictability, ergodicity, and sensitivity to their parameters and initial values, they are widely used in security applications. In [35], Zhou et al. introduced a simple and effective chaotic system by a combination of two existing one-dimension (1D) chaotic maps which are called seed maps. By simulations and performance evaluations, it was shown that the proposed system can produce lots of 1D chaotic maps having larger chaotic ranges and better chaotic behaviors compared with their seed maps. To explore its applications in multimedia security, a novel image encryption algorithm is presented. By using a same set of security keys, this algorithm can generate a completely different encrypted image each time when it is used to the same original image. By experiments and security analysis it was shown that the algorithm has excellent performance in image encryption and various attacks. In [27], Hua et al. introduced a new two-dimensional Sine Logistic modulation map (2D-SLMM) which is given by the Logistic and Sine maps. When it is compared with existing chaotic maps, we can find that it has the better chaotic range, better ergodicity, hyperchaotic property, and relatively low implementation cost. Also, to investigate its applications, they proposed a chaotic magic transform (CMT) to efficiently change the image pixel positions. Combining 2D-SLMM with CMT, they further introduced a new image encryption algorithm. Simulation results and security analysis show that this algorithm can present images with low time complexity and a high security level as well as to resist various attacks. In [32], Wu et al. Noonan introduced a number of Sudoku-associated matrix element representations besides the conventional representation by matrix row-column pair. Particularly, they are representations via Sudoku matrix row-digit pair, digit-row pair, column-digit pair, digit-column pair, block-digit pair, and digit-block pair. So, one can secretly represent matrix elements by a Sudoku matrix and develop new Sudoku associated 2D parametric bijections. To show the effectiveness and randomness of thes bijections, they introduced a simple and effective Sudoku Associated Image Scrambler by 2D Sudoku-associated bijections for image scrambling without bandwidth expansion. By simulations and comparisons, it was demonstrated that the proposed method can outperform some state-of-the-art methods. In [28], Hua and Zhou gave a two-dimensional Logistic-adjusted-Sine map (2D-LASM). By performance evaluations, it was shown that this map has better ergodicity and unpredictability, and a wider chaotic range than many the existing chaotic maps. By this map, they further designed a 2D-LASM-based image encryption scheme (LAS-IES). The principle of diffusion and confusion are strictly finished, and a mechanism of adding random values to plain image is established to enhance the security level of cipher image. By simulation results and security analysis, it was shown that the LAS-IES can efficiently encrypt different kinds of images into random-like ones which have strong ability of resisting various security attacks. In [29], Hua et al. gave a sine-transform-based chaotic system (STBCS) which generates one-dimensional (1-D) chaotic maps. This chaotic system performs a sine transform to the combination of the outputs of the two existing chaotic maps (seed maps). Users have the flexibility to pick any existing 1D chaotic maps as seed maps in STBCS to generate lots of new chaotic maps. The complex chaotic behavior of STBCS is verified by using the principle of Lypunov exponent. To show the usability and effectiveness of STBCS, they presented three new chaotic maps as examples. By theoretical analysis, it was shown that these chaotic maps have complex dynamics properties and robust chaos. Performance evaluations show that they have better chaotic ranges, better complexity, and unpredictability, compared with chaotic maps generated by other methods and the corresponding seed maps. Also, to explain the simplicity of STBCS in hardware implementation, they simulated the three new chaotic maps by the field-programmable gate array (FPGA). In [30], Hua et al. presented a two-dimensional (2D) sine chaotification system (2D-SCS). Such a system can not only significantly enhance the complexity of 2D chaotic maps but also greatly extend their chaotic ranges. As examples, they applied 2D-SCS to two existing 2D chaotic maps to get two enhanced chaotic maps. By performance evaluations, it was shown that these two enhanced chaotic maps have robust chaotic behaviors in much larger chaotic ranges than the existing 2D chaotic maps. Also, a microcontroller-based experiment platform was designed to implement these enhanced chaotic maps in hardware devices. Furthermore, to discuss the application of 2D-SCS, these two enhanced chaotic maps are used to design a pseudorandom number generator. By experiment results, one can see that these enhanced chaotic maps are able to produce better random sequences than the existing 2D and several state-of-the-art one-dimensional (1D) chaotic maps. In [31], Hua et al. Zhou gave a two-dimensional (2D) modular chaotification system (2D-MCS) to improve the chaos complexity of any 2D chaotic map. Since the modular operation is a bounded transform, the chaotic maps which are improved by 2D-MCS can generate chaotic behaviors in wide parameter ranges while the existing chaotic maps cannot. Three improved chaotic maps were given as typical examples to verify the effectiveness of 2D-MCS. The chaotic properties of one example of 2D-MCS are mathematically analyzed by using Lyapunov exponent. By performance evaluations, it was shown that these improved chaotic maps have continuous and large chaotic ranges, and their outputs are distributed more uniformly than the outputs of the

existing 2D chaotic maps. To explain the application of 2D-MCS, they applied the improved chaotic maps of 2D-MCS to secure communication. By the simulation results, it was shown that these improved chaotic maps exhibit better performance than a few existing and newly developed chaotic maps in terms of resisting different channel noise.

In the future, we will further discuss and explore some properties and applications of continuous cyclic permutation maps [48].

Data Availability

The data used to support the findings are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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