Research Article

Metric Dimension, Minimal Doubly Resolving Sets, and the Strong Metric Dimension for Jellyfish Graph and Cocktail Party Graph

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Let Γ be a simple connected undirected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The metric dimension of a graph Γ is the least number of vertices in a set with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex. For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of vertices in a graph Γ and a vertex v of Γ , the metric representation of v with respect to W is the k -vector $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. If every pair of distinct vertices of Γ have different metric representations, then the ordered set W is called a resolving set of Γ . It is known that the problem of computing this invariant is NP-hard. In this paper, we consider the problem of determining the cardinality $\psi(\Gamma)$ of minimal doubly resolving sets of Γ and the strong metric dimension for the jellyfish graph $JFG(n, m)$ and the cocktail party graph $CP(k+1)$.

1. Introduction

In this paper, we consider finite, simple, and connected graphs. The vertex and edge sets of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. For $u, v \in V(\Gamma)$, the length of a shortest path from u to v is called the distance between u and v and is denoted by $d_\Gamma(u, v)$, or simply $d(u, v)$. The adjacency and nonadjacency relations are denoted by $\sim$ and $\nsim$, respectively. The size of the largest clique in the graph Γ is denoted by $\omega(\Gamma)$ and the size of the largest independent sets of vertices by $\alpha(\Gamma)$. A vertex $x \in V(\Gamma)$ is said to resolve a pair $u, v \in V(\Gamma)$, if $d_\Gamma(u, x) \neq d_\Gamma(v, x)$. For an ordered subset $W = \{w_1, w_2, \dots, w_k\}$ of vertices in a connected graph Γ and a vertex v of Γ , the metric representation of v with respect to W is the k -vector $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. If every pair of distinct vertices of Γ have different metric representations, then the ordered set W is called a resolving set of Γ . Indeed, the set W is called a resolving set for Γ if $r(u|W) = r(v|W)$ which implies that $u = v$ for all pairs u, v of vertices of Γ . If the set W is as small as possible, then it is called a metric basis of the graph Γ . We recall that the metric dimension of Γ , denoted by $\beta(\Gamma)$, is defined as the minimum cardinality of a resolving set for Γ . If $\beta(\Gamma) = k$,

then Γ is said to be k -dimensional. Chartrand et al. [1] determined the bounds of the metric dimensions for any connected graphs and determined the metric dimensions of some well-known families of graphs such as trees, paths, and complete graphs. Bounds on $\beta(\Gamma)$ are presented in terms of the order and the diameter of Γ . All connected graphs of order n having metric dimension 1, $n-1$, or $n-2$ are determined. The concept of resolving the set and that of metric dimension date back to the 1950s. They were defined by Bluementhal [2] in the context of metric space. These notions were introduced to graph theory by Harary and Melter [3] and Slater [4] in the 1970s. For more results related to these concepts see [5–10]. These concepts have different applications in the areas of network discovery and verification [11], robot navigation [12], chemistry [1], and combinatorial optimization [13]. Notice for each connected graph Γ and each ordered set $W = \{w_1, w_2, \dots, w_k\}$ of vertices of Γ that the i th coordinate of $r(w_i|W)$ is 0 and that the i th coordinate of all other vertex representations is positive. Thus, certainly $r(u|W) = r(v|W)$ implies that $u = v$ for $u \in W$. Therefore, when testing whether an ordered subset W of $V(\Gamma)$ is a resolving set for Γ , we need only be concerned with the vertices of $V(\Gamma) - W$.

Cáceres et al. [14] define the notion of a doubly resolving set as follows. Vertices x, y of the graph Γ of order at least 2 are said to doubly resolve vertices u, v of Γ if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A set $Z = \{z_1, z_2, \dots, z_l\}$ of vertices of Γ is a doubly resolving set of Γ if every two distinct vertices of Γ are doubly resolved by some two vertices of Z . The minimal doubly resolving set is a doubly resolving set with minimum cardinality. The cardinality of the minimum doubly resolving set is denoted by $\psi(\Gamma)$. The minimal doubly resolving sets for Hamming and prism graphs have been obtained in [15,16], respectively. Other researchers in [17] determined the minimal doubly resolving sets for the necklace graph. Since if x, y doubly resolve u, v , then $d(u, x) - d(v, x) \neq 0$ or $d(u, y) - d(v, y) \neq 0$, and hence x or y resolve u, v . Therefore, a doubly resolving set is also a resolving set, and $\beta(\Gamma) \leq \psi(\Gamma)$.

The strong metric dimension problem was introduced by Sebö and Tannier [13] and further investigated by Oellermann and Peters-Fransen [18]. Recently, the strong metric dimension of distance hereditary graphs has been studied by May and Oellermann [19]. For more results related to this concept see [20]. A vertex w strongly resolves two vertices u and v if u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. A set $N = \{n_1, n_2, \dots, n_m\}$ of vertices of Γ is a strong resolving set of Γ if every two distinct vertices of Γ are strongly resolved by some vertex of N . A strong metric basis of Γ is a strong resolving set of the minimum cardinality. Now, the strong metric dimension of Γ , denoted by $\text{sdim}(\Gamma)$, is defined as the cardinality of its strong metric basis. It is easy to see that if a vertex w strongly resolves vertices u and v , then w also resolves these vertices. Hence, every strong resolving set is a resolving set, and $\beta(\Gamma) \leq \text{sdim}(\Gamma)$.

All three previously defined problems are NP-hard in general case. The proofs of NP-hardness are given for the metric dimension problem in [12], for the minimal doubly resolving set problem in [21], and for the strong metric dimension problem in [18]. Intrinsic metrics on a graph have become of interest, as generally discussed in [5,16,22–25], for some classes of graphs. In this paper, we consider the problem of determining the cardinality $\psi(\Gamma)$ of minimal doubly resolving sets of Γ and the strong metric dimension for the jellyfish graph $\text{JFG}(n, m)$ and the cocktail party graph $\text{CP}(k + 1)$. In fact, we find the minimum cardinality of the resolving set, doubly resolving set, and strong resolving set of the jellyfish graph $\text{JFG}(n, m)$. In particular, we show that the minimum cardinality of the resolving set, doubly resolving set, and strong resolving set of the cocktail party graph $\text{CP}(k + 1)$ is $k + 1$.

2. Definitions and Preliminaries

Definition 1 (see [26]). Let Γ be a graph with automorphism group $\text{Aut}(\Gamma)$. We say that Γ is a vertex transitive graph if, for any vertices x, y of Γ , there is some φ in $\text{Aut}(\Gamma)$, such that $\varphi(x) = y$. Also, we say that Γ is symmetric if, for all vertices u, v, x, y of Γ such that u and v are adjacent, and also, x and y are adjacent, there is an automorphism φ such that $\varphi(u) = x$

and $\varphi(v) = y$. Finally, we say that Γ is a distance transitive if, for all vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$, there is an automorphism φ such that $\varphi(u) = x$ and $\varphi(v) = y$.

Definition 2 (see [26]). Let G be a finite group and Ω a subset of G that is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = \text{Cay}(G, \Omega)$ is a graph whose vertex set and edge set are defined as follows:

$$\begin{aligned} V(\Gamma) &= G; \\ E(\Gamma) &= \{\{x, y\} \mid x^{-1}y \in \Omega\}. \end{aligned} \quad (1)$$

Remark 1 (see [26]). Let Γ be a graph. It is clear that we have a hierarchy of the conditions as follows:

$$\text{distance transitive} \Rightarrow \text{symmetric} \Rightarrow \text{vertex transitive}. \quad (2)$$

Proposition 1 (see [27]). Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 4$), where $S_1 = \{1, n - 1\}, \dots, S_k = S_{k-1} \cup \{k, n - k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lfloor n/2 \rfloor - 1$. Then, $\chi(\Gamma) = \omega(\Gamma) = k + 1$ if and only if $k + 1 \mid n$, where the chromatic number $\chi(\Gamma)$ of Γ is the minimum number k such that Γ is k colorable.

Definition 3 (see [23]). Let Γ be a graph, and let $W = \{w_1, \dots, w_k\} \subseteq V(\Gamma)$. For each vertex $v \in V(\Gamma)$, the adjacency representation of v with respect to W is the k -vector:

$$\hat{r}(v \mid W) = (a_\Gamma(v, w_1), \dots, a_\Gamma(v, w_k)), \quad (3)$$

where

$$a_\Gamma(v, w_i) = \begin{cases} 0, & \text{if } v = w_i, \\ 1, & \text{if } v \sim w_i, \\ 2, & \text{if } v \not\sim w_i. \end{cases} \quad (4)$$

The set W is an adjacency resolving set for Γ if the vectors $\hat{r}(v \mid W)$ for $v \in V(\Gamma)$ are distinct. The minimum cardinality of an adjacency resolving set is the adjacency dimension of Γ , denoted by $\hat{\beta}(\Gamma)$. An adjacency resolving set of cardinality $\hat{\beta}(\Gamma)$ is an adjacency basis of Γ .

Lemma 1 (see [23]). Let Γ be a graph of order n .

- (1) If $\text{diam}(\Gamma) = 2$, then $\hat{\beta}(\Gamma) = \beta(\Gamma)$.
- (2) If Γ is connected, then $\hat{\beta}(\Gamma) = \beta(\Gamma)$.
- (3) $1 \leq \hat{\beta}(\Gamma) \leq n - 1$.

3. Main Results

3.1. Metric Dimension, Minimal Doubly Resolving Sets, and the Strong Metric Dimension of the Jellyfish Graph $\text{JFG}(n, m)$.

For an integer $n \geq 3$, the n -cycle is the graph C_n with vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ or simply $\{1, 2, \dots, n\}$ and edge set $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n\}$, where $v_{n+1} = v_1$. An interesting family of graphs of order $nm + n$ is defined as follows. Let Γ be a graph with a vertex set $V_1 \cup V_2$, where $V_1 = V(C_n)$ and $V_2 = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$, and let $A_{ij} = \cup_{j=1}^m v_{ij}$, $1 \leq i \leq n$.

Suppose that every vertex $i \in V_1$ is adjacent to vertices $v_{i1}, v_{i2}, \dots, v_{im} \in A_{ij}$ and $\deg(v_{ij}) = 1$ for every vertex $v_{ij} \in A_{ij}$, then the resulting graph is called the jellyfish graph $JFG(n, m)$ with parameters m and n . In particular, if n is an even integer, then the jellyfish graph $JFG(n, m)$ is a bipartite graph. In this paper, we consider the problem of determining the cardinality $\psi(JFG(n, m))$ of minimal doubly resolving sets of the jellyfish graph $JFG(n, m)$. First, we find the metric dimension of the jellyfish graph $JFG(n, m)$, in fact we prove that if $n \geq 3$ and $m \geq 2$, then the metric dimension of the jellyfish graph $JFG(n, m)$ is $nm - n$. Also, we consider the problem of determining the cardinality $\psi(JFG(n, m))$ of minimal doubly resolving sets of $JFG(n, m)$ and the strong metric dimension of the jellyfish graph $JFG(n, m)$. Moreover, we find an adjacency dimension of the jellyfish graph $JFG(n, m)$.

Theorem 1. *Let n, m be integers such that $n \geq 3$ and $m \geq 2$. Then, the metric dimension of the jellyfish graph $JFG(n, m)$ is $nm - n$.*

Proof. Let $V(JFG(n, m)) = V_1 \cup V_2$, where $V_1 = V(C_n) = \{1, 2, \dots, n\}$ and $V_2 = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$, and let $A_{ij} = \cup_{j=1}^m v_{ij}$, $1 \leq i \leq n$. Suppose that every vertex $i \in V_1$ is adjacent to vertices $v_{i1}, v_{i2}, \dots, v_{im} \in A_{ij}$. We can show that the diameter of the jellyfish graph $JFG(n, m)$ is $\lceil n/2 \rceil + 2$. In the following cases, we show that the metric dimension of the jellyfish graph $JFG(n, m)$ is $nm - n$.

Case 1. Let W be an ordered subset of V_1 in the jellyfish graph $JFG(n, m)$ such that $|W| \leq n$. It is easy to prove that if $|W| \leq n$, then W is not a resolving set of the jellyfish graph $JFG(n, m)$. In particular, if $|W| = n$, then we show that W is not a resolving set of the jellyfish graph $JFG(n, m)$. We may assume without loss of generality that an ordered subset is $W = \{1, 2, \dots, n\}$. Hence, $V(JFG(n, m)) - W = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$. Therefore, the metric representation of the vertices $v_{11}, v_{12}, \dots, v_{1m} \in A_{1j}$ with respect to W is the same as the n -vector. Thus, W is not a resolving set of the jellyfish graph $JFG(n, m)$.

Case 2. Let W be an ordered subset of V_2 in the jellyfish graph $JFG(n, m)$ such that $W = \{A_{2j}, A_{3j}, \dots, A_{nj}\}$. Hence, $V(JFG(n, m)) - W = \{1, 2, \dots, n, A_{1j}\}$. We know that $|W| = nm - m$. Therefore, the metric representation of the vertices $v_{11}, v_{12}, \dots, v_{1m} \in A_{1j}$ with respect to W is the same as the $nm - m$ -vector. Thus, W is not a resolving set of jellyfish graph $JFG(n, m)$.

Case 3. Let W be an ordered subset of V_2 in the jellyfish graph $JFG(n, m)$ such that $W = \{A_{1j}, A_{2j}, A_{3j}, \dots, A_{nj} - \{v_{n1}, v_{n2}\}\}$. Hence, $V(JFG(n, m)) - W = \{1, 2, \dots, n, v_{n1}, v_{n2}\}$. We know that $|W| = nm - 2$. Therefore, the metric representation of the vertices $v_{n1}, v_{n2} \in A_{nj}$ with respect to W is the same as the $nm - 2$ -vector. Thus, W is not a resolving set of the jellyfish graph $JFG(n, m)$.

Case 4. Let W be an ordered subset of V_2 in the jellyfish graph $JFG(n, m)$ such that $|W| = nm - 1$. We show that

W is a resolving set of the jellyfish graph $JFG(n, m)$. We may assume without loss of generality that an ordered subset is $W = \{A_{1j}, A_{2j}, \dots, A_{nj} - v_{nm}\}$. Hence, $V(JFG(n, m)) - W = \{1, 2, \dots, n, v_{nm}\}$. We can show that all the vertices $1, 2, \dots, n, v_{nm} \in V(JFG(n, m)) - W$ have different representations with respect to W . For every $k \in V(JFG(n, m)) - W$, $1 \leq k \leq n$ and $v_{ij} \in A_{ij}$, $1 \leq i \leq n$, and $1 \leq j \leq m$, if $k = i$, then we have $d(k, v_{ij}) = 1$, otherwise $d(k, v_{ij}) > 1$. Also, for the vertex $v_{nm} \in V(JFG(n, m)) - W$ with $v_{nm} \neq v_{ij} \in A_{ij}$, $1 \leq i \leq n$, and $1 \leq j \leq m$, if $i = n$, then we have $d(v_{nm}, v_{ij}) = 2$, otherwise $d(v_{nm}, v_{ij}) > 2$. Therefore, all the vertices $1, 2, \dots, n, v_{nm} \in V(JFG(n, m)) - W$ have different representations with respect to W . This implies that W is a resolving set of the jellyfish graph $JFG(n, m)$.

Case 5. Let W be an ordered subset of V_2 in the jellyfish graph $JFG(n, m)$ such that $W = \{A_{1j} - v_{1m}, A_{2j} - v_{2m}, \dots, A_{nj} - v_{nm}\}$. Hence, $V(JFG(n, m)) - W = \{1, 2, \dots, n, v_{1m}, v_{2m}, \dots, v_{nm}\}$. We know that $|W| = nm - n$. In a similar fashion which is done in Case 4, we can show that all the vertices $1, 2, \dots, n, v_{1m}, v_{2m}, \dots, v_{nm} \in V(JFG(n, m)) - W$ have different representations with respect to W . This implies that W is a resolving set of the jellyfish graph $JFG(n, m)$.

Case 6. In particular, let W be an ordered subset of V_2 in the jellyfish graph $JFG(n, m)$ such that $|W| = nm$. We show that W is a resolving set of the jellyfish graph $JFG(n, m)$. We may assume without loss of generality that an ordered subset is $W = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$. Hence, $V(JFG(n, m)) - W = \{1, 2, \dots, n\}$. We can show that all the vertices $1, 2, \dots, n \in V(JFG(n, m)) - W$ have different representations with respect to W . This implies that W is a resolving set of the jellyfish graph $JFG(n, m)$.

From the above cases, we conclude that the minimum cardinality of a resolving set of the jellyfish graph $JFG(n, m)$ is $nm - n$. $\square$

Lemma 2. *Let n and m be integers such that $n \geq 3$ and $m \geq 2$. Then, the subset $Z = \{A_{1j} - v_{1m}, A_{2j} - v_{2m}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is not a doubly resolving set of the jellyfish graph $JFG(n, m)$.*

Proof. We know that the ordered subset $Z = \{A_{1j} - v_{1m}, A_{2j} - v_{2m}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is a resolving set of the jellyfish graph $JFG(n, m)$ of size $nm - n$. Also by Theorem 1, the metric dimension of the jellyfish graph $JFG(n, m)$ is $\beta(JFG(n, m)) = nm - n$. Moreover, $B(JFG(n, m)) \leq \psi(JFG(n, m))$. We show that the subset $Z = \{A_{1j} - v_{1m}, A_{2j} - v_{2m}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is not a doubly resolving set of the jellyfish graph $JFG(n, m)$. If $u = v_{im}$ and $v = i$ and $1 \leq i \leq n$, then for every $x, y \in Z$, we have $d(u, x) - d(u, y) = d(v, x) - d(v, y)$. $\square$

Lemma 3. *Let n and m be integers such that $n \geq 3$ and $m \geq 2$. Then, the subset $Z = \{A_{1j}, A_{2j}, \dots, A_{nj} - v_{nm}\}$ of vertices in*

the jellyfish graph $JFG(n, m)$ is not a doubly resolving set of the jellyfish graph $JFG(n, m)$.

Proof. We show that the subset $Z = \{A_{1j}, A_{2j}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is not a doubly resolving set of the jellyfish graph $JFG(n, m)$. If $u = v_{nm}$ and $v = n$, then for every $x, y \in Z$, we have $d(u, x) - d(u, y) = d(v, x) - d(v, y)$. $\square$

Theorem 2. Let n and m be integers such that $n \geq 3$ and $m \geq 2$. Then, the cardinality of the minimum doubly resolving set of the jellyfish graph $JFG(n, m)$ is nm .

Proof. Let $V(JFG(n, m)) = V_1 \cup V_2$, where $V_1 = V(C_n) = \{1, 2, \dots, n\}$ and $V_2 = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$, and let $A_{ij} = \cup_{j=1}^m v_{ij}$, $1 \leq i \leq n$. Suppose that every vertex $i \in V_1$ is adjacent to vertices $v_{i1}, v_{i2}, \dots, v_{im} \in A_{ij}$. We know that the ordered subset $v_{i1}, v_{i2}, \dots, v_{im} \in A_{ij}$ of vertices in the jellyfish graph $JFG(n, m)$ is a resolving set of the jellyfish graph $JFG(n, m)$ of size nm . Also by Theorem 1, the metric dimension of the jellyfish graph $JFG(n, m)$ is $JFG(n, m)$. Moreover, $B(JFG(n, m)) \leq \psi(JFG(n, m))$. We show that the subset $Z = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is a doubly resolving set of the jellyfish graph $JFG(n, m)$. It is sufficient to show that for two vertices u and v of the jellyfish graph $JFG(n, m)$ there are vertices $x, y \in Z$ such that $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. Consider two vertices u and v of the jellyfish graph $JFG(n, m)$. Then, we have the following:

Case 1. Let $u \notin Z$ and $v \notin Z$. Hence, $u, v \in V_1 = V(C_n) = \{1, 2, \dots, n\}$. We can assume without loss of generality that $u = i$ and $v = j$, $1 \leq i, j \leq n$, and $i \neq j$. Therefore, if $x = v_{i1}$ and $y = v_{j1}$, and then we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, x) - d(u, y) < 0$ and $d(v, x) - d(v, y) > 0$.

Case 2. Let $u \in Z$ and $v \in Z$. Hence, $u, v \in V_2 = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$. Therefore, if $x = u$ and $y = v$, then we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, x) - d(u, y) < 0$ and $d(v, x) - d(v, y) > 0$.

Case 3. Finally, let $u \notin Z$ and $v \in Z$. Hence, $u \in V_1 = V(C_n) = \{1, 2, \dots, n\}$ and $v \in V_2 = \{A_{1j}, A_{2j}, \dots, A_{nj}\}$. We can assume without loss of generality that $u = k$, $1 \leq k \leq n$, and $v = v_{11} \in A_{11}$. Therefore, if $x = v_{k2}$ and $y = v_{11}$, then we have $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$ because $d(u, x) - d(u, y) \leq 0$ and $d(v, x) - d(v, y) > 0$.

Thus, by Lemmas 2 and 3 and the above cases, we conclude that the cardinality of the minimum doubly resolving set of the jellyfish graph $JFG(n, m)$ is nm . $\square$

Lemma 4. Let n and m be integers such that $n \geq 3$ and $m \geq 2$. Then, the subset $N = \{A_{1j} - v_{1m}, A_{2j} - v_{2m}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is not a strong resolving set of the jellyfish graph $JFG(n, m)$.

Proof. Let $M = V_2 - N = \{v_{1m}, v_{2m}, \dots, v_{nm}\}$, where V_2 is the set which is defined already. It is not hard to see that for every two distinct vertices $u, v \in M$, there is not a vertex $w \in N$ such that u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. So, the subset $N = \{A_{1j} - v_{1m}, A_{2j} - v_{2m}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is not a strong resolving set of the jellyfish graph $JFG(n, m)$. We conclude that if N is a strong resolving set of the jellyfish graph $JFG(n, m)$, then $|N| \geq nm - 1$ because $|M|$ must be less than 2. $\square$

Theorem 3. Let n and m be integers such that $n \geq 3$ and $m \geq 2$. Then, the strong metric dimension of the jellyfish graph $JFG(n, m)$ is $nm - 1$.

Proof. By Lemma 4, we know that if N is a strong resolving set of the jellyfish graph $JFG(n, m)$, then $|N| \geq nm - 1$. We show that the subset $N = \{A_{1j}, A_{2j}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is a strong resolving set of the jellyfish graph $JFG(n, m)$. It is sufficient to prove that every two distinct vertices $u, v \in V(JFG(n, m)) - N = \{1, 2, \dots, n, v_{nm}\}$ are strongly resolved by a vertex $w \in N$. In the following cases, we show that the strong metric dimension of the jellyfish graph $JFG(n, m)$ is $nm - 1$.

Case 1. Let u and v be two distinct vertices in $V(JFG(n, m)) - N$ such that $u, v \in V_1 = V(C_n) = \{1, 2, \dots, n\}$. So, there is $i, j \in V_1$ such that $u = i$ and $v = j$. Therefore, i and j will be strongly resolved by some $v_{ij} \in A_{ij}$ because i and v_{ij} are adjacent, and hence i belongs to a shortest $v_{ij} - j$ path.

Case 2. Now, let u and v be two distinct vertices in $V(JFG(n, m)) - N$ such that $u \in V_1 = \{1, 2, \dots, n\}$ and $v = v_{nm}$. Without loss of generality we may assume $u = i$, where $i \in V_1$. Therefore, i and v_{nm} will be strongly resolved by some $v_{ij} \in A_{ij}$ because i and v_{ij} are adjacent, and hence i belongs to a shortest $v_{ij} - v_{nm}$ path.

From the above cases, we conclude that the minimum cardinality of a strong metric dimension of the jellyfish graph $JFG(n, m)$ is $nm - 1$. $\square$

Lemma 5. Let n and m be integers such that $n \geq 3$ and $m \geq 2$. Then, an ordered subset $W = \{A_{1j} - v_{1m}, A_{2j} - v_{2m}, \dots, A_{nj} - v_{nm}\}$ of vertices in the jellyfish graph $JFG(n, m)$ is not the adjacency resolving set of the jellyfish graph $JFG(n, m)$.

Proof. Let $\Gamma = JFG(n, m)$ and $M = V_2 - W = \{v_{1m}, v_{2m}, \dots, v_{nm}\}$, where V_2 is the set which is defined already. Thus, the adjacency representation of the vertices $v_{1m}, v_{2m}, \dots, v_{nm} \in V(JFG(n, m)) - W$ with respect to W is the $nm - n$ -vector $\hat{r}(v_{1m} | W) = \hat{r}(v_{2m} | W) = \dots = \hat{r}(v_{nm} | W) = (2, 2, \dots, 2)$. For every vertex $w \in W$, we have $a_\Gamma(w, v_{1m}) = a_\Gamma(w, v_{1m}) = \dots = a_\Gamma(w, v_{nm}) = 2$. We conclude that if W is an adjacency resolving set of the jellyfish graph $JFG(n, m)$, then $|W| \geq nm - 1$ because $|M|$ must be less than 2. $\square$

Theorem 4. Let n and m be integers such that $n \geq 3$ and $m \geq 2$. Then, the adjacency dimension of the jellyfish graph $\text{JFG}(n, m)$ is $nm - 1$.

Proof. By Lemma 5, we know that if W is an adjacency resolving set of the jellyfish graph $\text{JFG}(n, m)$, then $|W| \geq nm - 1$. Now, let $\Gamma = \text{JFG}(n, m)$ and W be an ordered subset of V_2 in the jellyfish graph $\text{JFG}(n, m)$ such that $|W| = nm - 1$. We show that W is an adjacency resolving set of the jellyfish graph $\text{JFG}(n, m)$. We may assume without loss of generality that an ordered subset is $W = \{A_{1j}, A_{2j}, \dots, A_{nj} - v_{nm}\}$. Hence, $V(\text{JFG}(n, m)) - W = \{1, 2, \dots, n, v_{nm}\}$. We can show that all the vertices $1, 2, \dots, n, v_{nm} \in V(\text{JFG}(n, m)) - W$ have different adjacency representations with respect to W . For every $k \in V(\text{JFG}(n, m)) - W$, $1 \leq k \leq n$ and $v_{ij} \in A_{ij}$, $1 \leq i \leq n$, and $1 \leq j \leq m$, if $k = i$, then we have $a_\Gamma(k, v_{ij}) = 1$, otherwise $a_\Gamma(k, v_{ij}) = 2$. Also, for the vertex $v_{nm} \in V(\text{JFG}(n, m)) - W$ with $v_{nm} \neq v_{ij} \in A_{ij}$, $1 \leq i \leq n$, and $1 \leq j \leq m$, we have $a_\Gamma(v_{nm}, v_{ij}) = 2$. Therefore, all the vertices $1, 2, \dots, n, v_{nm} \in V(\text{JFG}(n, m)) - W$ have different adjacency representations with respect to W . This implies that W is an adjacency resolving set of the jellyfish graph $\text{JFG}(n, m)$. We now conclude that the minimum cardinality of the adjacency resolving set of the jellyfish graph $\text{JFG}(n, m)$ is $nm - 1$. $\square$

3.2. Metric Dimension, Minimal Doubly Resolving Sets, and the Strong Metric Dimension of the Cocktail Party Graph $\text{CP}(k+1)$. Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic additive group $\mathbb{Z}_n$, where $S_1 = \{1, n-1\}$, $\dots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lfloor n/2 \rfloor - 1$. If n is an even integer and $k = (n/2) - 1$, then we can show that $\text{Cay}(\mathbb{Z}_n, S_k)$ is obtained from the complete graph $K_{2(k+1)}$ by deleting a perfect matching, and hence $\text{Cay}(\mathbb{Z}_n, S_k)$ is isomorphic to the cocktail party graph $\text{CP}(k+1)$. Moreover, we can show that $\text{Cay}(\mathbb{D}_{2n}, \Omega)$, where

$$\mathbb{D}_{2n} = \langle a, b \mid a^n = b^2 = 1, ba = a^{n-1}b \rangle, \quad (5)$$

is the dihedral group of order $2n$ and $\Omega = \{a, a^2, \dots, a^{n-1}, ab, a^2b, \dots, a^{n-1}b\}$ is the inverse closed subset of $\mathbb{D}_{2n} - \{1\}$, is isomorphic to the cocktail party graph $\text{CP}(n)$. In this section, we consider the problem of determining the cardinality $\psi(\Gamma)$ of minimal doubly resolving sets of Γ . First, we show that if n is an even integer and $k = (n/2) - 1$, then the metric dimension of Γ is $k+1$. Also, we prove that if n is an even integer and $k = (n/2) - 1$, then every minimal resolving set of Γ is also a doubly resolving set, and, consequently, $\psi(\Gamma)$ is equal to the metric dimension of $\beta(\Gamma)$, which is known from the literature. Moreover, we find an explicit expression for the strong metric dimension of Γ .

Theorem 5. Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 8$), where $S_1 = \{1, n-1\}$, $\dots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \lfloor n/2 \rfloor - 1$. If n is an even integer and $k = (n/2) - 1$, then the metric dimension of Γ is $k+1$.

Proof. Let $V(\Gamma) = \{1, \dots, n\}$ be the vertex set of Γ . By proof of Proposition 3.4 in [27], we know that the diameter of Γ is 2. Hence, for all the vertices $x, y \in V(\Gamma)$, the length of a shortest path from x to y is $d(x, y) = 1$ or 2. Also, we know that the size of the largest clique in the graph Γ is $k+1$. Now, let W be an ordered subset of vertices in the graph Γ such that W is a clique in Γ . Indeed, for every $x, y \in W$, we have $d_\Gamma(x, y) = 1$. In the following cases, we show that the metric dimension of Γ is $k+1$.

Case 1. It is easy to see that if $|W| \leq k$, then W is not a resolving set of Γ . In particular, if $|W| = k$, then we show that W is not a resolving set of Γ . We may assume without loss of generality that an ordered subset of vertices in the graph Γ is $W = \{1, 2, 3, \dots, k\}$. Hence, $V(\Gamma) - W = \{k+1, k+2, \dots, n\}$. On the contrary, by proof of Proposition 3.2 in [27], we know that for every vertex x in Γ , there is exactly one y in Γ such that $x^{-1}y = k+1$, that is, $d(x, y) = 2$. Hence, there are vertices $k+2, k+3, \dots, n-1$ in $V(\Gamma) - W$ such that $d(1, k+2) = 2, d(2, k+3) = 2, \dots, d(k, n-1) = 2$.

Therefore, the metric representations of the vertices $k+2, k+3, \dots, n-1 \in V(\Gamma) - W$ with respect to W are the k -vectors $r(k+2|W) = (2, 1, 1, \dots, 1), r(k+3|W) = (1, 2, 1, \dots, 1), \dots, r(n|W) = (1, 1, 1, \dots, 2)$. Moreover, the metric representation of the vertices $k+1, n \in V(\Gamma) - W$ with respect to W is the k -vector $r(k+1|W) = r(n|W) = (1, 1, 1, \dots, 1)$. Thus, W is not a resolving set of Γ because the metric representation of the vertices $k+1, n$ is the same as the k -vector.

Case 2. Let W be a clique in the graph Γ such that $x \in W$ and $|W| = k$. We know that there is exactly one y in $V(\Gamma) - W$ such that $x^{-1}y = k+1$, that is, $d(x, y) = 2$. In the following, we show that an ordered subset $(W \cup y)$ of vertices in the graph Γ is not a resolving set of Γ . In this case, we may assume without loss of generality that an ordered subset of vertices in the graph Γ is $W = \{1, 2, \dots, k\}$, and let $x = 1$ and $y = k+2$. Therefore, the metric representation of the vertices $k+1, n \in V(\Gamma) - (W \cup y)$ with respect to $(W \cup y)$ is the $k+1$ -vector $r(k+1|W) = r(n|W) = (1, 1, 1, \dots, 1)$. Thus, $(W \cup y)$ is not a resolving set of Γ .

Case 3. Now, let W be a clique in the graph Γ such that $|W| = k+1$. We show that W is a resolving set of Γ . We may assume without loss of generality that an ordered subset of vertices in the graph Γ is $W = \{1, 2, 3, \dots, k, k+1\}$. Hence, $V(\Gamma) - W = \{k+2, k+3, \dots, n\}$. Therefore, the metric representations of the vertices $k+2, k+3, \dots, n \in V(\Gamma) - W$ with respect to W are the $(k+1)$ -vectors $r(k+2|W) = (2, 1, 1, \dots, 1), r(k+3|W) = (1, 2, 1, \dots, 1), \dots, r(n|W) = (1, 1, 1, \dots, 2)$. Thus, all the vertices of $V(\Gamma) - W$ have different representations with respect to W . This implies that W is a resolving set of Γ . Moreover, the metric dimension of $\beta(\Gamma) \leq k+1$ because Γ is a vertex transitive graph.

Case 4. In particular, let W be a clique in the graph Γ such that $|W| = k+1$. We show that for each

$x \in V(\Gamma) - W$, an ordered subset $(W \cup x)$ of vertices in the graph Γ is also a resolving set of Γ . We may assume without loss of generality that an ordered subset of vertices in the graph Γ is $W = \{1, 2, 3, \dots, k, k+1\}$, and let $x = k+2$. So, $(W \cup x) = \{1, 2, 3, \dots, k+1, k+2\}$. Hence, $V(\Gamma) - (W \cup x) = \{k+3, k+4, \dots, n\}$. Therefore, the metric representations of the vertices $k+3, k+4, \dots, n \in V(\Gamma) - (W \cup x)$, with respect to $(W \cup x)$, are the $k+2$ -vectors $r(k+3|W) = (1, 2, 1, \dots, 1, 1), r(k+4|W) = (1, 2, 1, \dots, 1, 1), \dots, r(n|W) = (1, 2, 1, \dots, 1, 1)$. This implies that $(W \cup x)$ is also a resolving set of Γ .

From the above cases, we conclude that the minimum cardinality of a resolving set of Γ is $k+1$. Moreover, it is well known that every Cayley graph is a vertex transitive. Hence, the cardinality of every minimal resolving set in Γ is $k+1$. $\square$

Theorem 6. Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 8$), where $S_1 = \{1, n-1\}, \dots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}, 1 \leq k \leq \lfloor n/2 \rfloor - 1$. If n is an even integer and $k = (n/2) - 1$, then the cardinality of the minimum doubly resolving set of Γ is $k+1$.

Proof. In this theorem, let $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$, where $v_i = i$ for $1 \leq i \leq n$. We know that the ordered subset $N = \{1, 2, 3, \dots, k, k+1\}$ of vertices in the graph Γ is a resolving set for Γ of size $k+1$. Also, by Theorem 5, the metric dimension of Γ is $\beta(\Gamma) = k+1$. Moreover, $B(\Gamma) \leq \psi(\Gamma)$. We show that the subset $Z = \{1, 2, 3, \dots, k, k+1\}$ of vertices in the graph Γ is a doubly resolving set of Γ . It is sufficient to show that for two vertices v_i and v_j of Γ , there are vertices $x, y \in Z$ such that $d(v_i, x) - d(v_i, y) \neq d(v_j, x) - d(v_j, y)$. Consider two vertices v_i and v_j of Γ . We may assume that $i < j$. In the following cases, we show that the cardinality of the minimum doubly resolving set of Γ is $k+1$.

Case 1. If $1 \leq i < j \leq k+1$, then $v_i, v_j \in Z$. So, $d(v_i, v_j) = 1$. We can assume that $x = v_i \in Z$ and $y = v_j \in Z$. Hence, we have $d(v_i, x) - d(v_i, y) \neq d(v_j, x) - d(v_j, y)$ because $d(v_i, x) - d(v_i, y) < 0$ and $d(v_j, x) - d(v_j, y) > 0$.

Case 2. Let $1 \leq i \leq k+1 < j \leq n$. Hence, $v_i \in Z$ and $v_j \notin Z$. Moreover, we know that $d(v_i, v_j) = 1$ or 2 . In the following, let $d(v_i, v_j) = 1$. We may assume that $v_i = 1$ and $v_j = n$. Hence, by taking $x = v_i \in Z$ and $y = k+1 \in Z$, we have $0 = 0 - 0 = d(v_i, x) - d(v_i, y) \neq d(v_j, x) - d(v_j, y) = 1 - 2 = -1$. Thus, the vertices x and y of Z doubly resolve v_i, v_j . Now, let $d(v_i, v_j) = 2$. We may assume that $v_i = 1$ and $v_j = k+2$. Hence, by taking $x = 1 \in Z$ and $y = 2 \in Z$, we have $-1 = 0 - 1 = d(v_i, x) - d(v_i, y) \neq d(v_j, x) - d(v_j, y) = 2 - 1 = 1$. Thus, the vertices x and y of Z doubly resolve v_i, v_j .

Case 3. Finally, let $1 < k+1 < i < j \leq n$. Hence, $v_i \notin Z$ and $v_j \notin Z$. Also, we know that $d(v_i, v_j) = 1$. On the contrary, by proof of Proposition 3.2 in [27], we know that for every vertex $v_i \in V(\Gamma) - Z$, there is exactly one

$x \in Z$ such that $d(v_i, x) = 2$. Also, for every vertex $v_j \in V(\Gamma) - Z$, there is exactly one $y \in Z$ such that $d(v_j, y) = 2$. Hence, $1 = 2 - 1 = d(v_i, x) - d(v_i, y) \neq d(v_j, x) - d(v_j, y) = 1 - 2 = -1$. Thus, the vertices x and y of Z doubly resolve v_i, v_j .

From the above cases, we conclude that the minimum cardinality of a doubly resolving set of Γ is $k+1$. $\square$

Theorem 7. Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 8$), where $S_1 = \{1, n-1\}, \dots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}, 1 \leq k \leq \lfloor n/2 \rfloor - 1$. If n is an even integer and $k = (n/2) - 1$, then the strong metric dimension of Γ is $k+1$.

Proof. In this theorem, let $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$, where $v_i = i$ for $1 \leq i \leq n$. We know that the ordered subset $N = \{1, 2, 3, \dots, k, k+1\}$ of vertices in the graph Γ is a resolving set for Γ of size $k+1$. Also, by Theorem 5, the metric dimension of Γ is $\beta(\Gamma) = k+1$. Moreover, $B(\Gamma) \leq \text{sdim}(\Gamma)$. We show that the subset $N = \{1, 2, 3, \dots, k, k+1\}$ of vertices in the graph Γ is a strong resolving set of Γ . Consider two vertices v_i and v_j of Γ . Assume that $i < j$. It is sufficient to prove that there exists a vertex $w \in N$ such that v_i belongs to a shortest $v_j - w$ path or v_j belongs to a shortest $v_i - w$ path. Let $1 < k+1 < i < j \leq n$. Hence, $v_i \notin N$ and $v_j \notin N$. Moreover, we have $d(v_i, v_j) = 1$. On the contrary, by proof of Proposition 3.2 in [27], we know that for every vertex $v_i \in V(\Gamma) - N$, there is exactly one $w \in N$ such that $v_i^{-1}w = k+1$, indeed $d(v_i, w) = 2$. Hence, $d(v_j, w) = 1$. So, $d(v_i, w) = d(v_i, v_j) + d(v_j, w)$, that is, vertex v_j belongs to a shortest $v_i - w$ path, and hence w strongly resolves vertices v_i and v_j . Also, for $v_i \in N$ or $v_j \in N$, vertex v_i or vertex v_j obviously strongly resolves pair v_i, v_j . Therefore, N is a strong resolving set. Thus, the minimum cardinality of a strong resolving set of Γ is $k+1$. $\square$

Conclusion 1. Let $\text{CP}(n) \cong \text{Cay}(\mathbb{D}_{2n}, \Omega)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n}$, where Ω is the inverse closed subset of $\mathbb{D}_{2n} - \{1\}$ which is defined already. Then, the minimum cardinality of resolving set, doubly resolving set, and strong resolving set of $\text{CP}(n)$ is n .

4. Conclusion

In this paper, we determined the minimum cardinality of the resolving set, doubly resolving set, and strong resolving set of the jellyfish graph $\text{JFG}(n, m)$. In particular, it has been shown that the minimum cardinality of the resolving set, doubly resolving set, and strong resolving set of the cocktail party graph $\text{CP}(k+1)$ is $k+1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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