

## Research Article

# Two-Round Diagnosability Measures for Multiprocessor Systems

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In a multiprocessor system, as a key measure index for evaluating its reliability, diagnosability has attracted lots of attentions. Traditional diagnosability and conditional diagnosability have already been widely discussed. However, the existing diagnosability measures are not sufficiently comprehensive to address a large number of faulty nodes in a system. This article introduces a novel concept of diagnosability, called two-round diagnosability, which means that all faulty nodes can be identified by at most a one-round replacement (repairing the faulty nodes). The characterization of two-round  $t$ -diagnosable systems is provided; moreover, several important properties are also presented. Based on the abovementioned theories, for the  $n$ -dimensional hypercube ( $Q_n$ ), we show that its two-round diagnosability is  $(n^2 + n/2)$ , which is  $(n + 1/2)$  times its classic diagnosability. Furthermore, a fault diagnosis algorithm is proposed to identify each node in the system under the PMC model. For  $Q_n$ , we prove that the proposed algorithm is the time complexity of  $O(n2^n)$ .

## 1. Introduction

With the growth of the large scale integration technology, a huge number of multiprocessors are integrated to a multiprocessor computer system. It is not difficult to predict that, in such a system, some faulty processors (nodes) will be produced. To make sure that the system works properly, the designers should consider the problem that the system needs to have the ability to diagnose itself faulty processors such that they can be repaired or replaced with the new fault-free processors. In dealing with the problem of fault diagnosis for multiprocessor systems, two approaches are used: one is the system-level approach and another is the logic-circuit-level approach. Since the system-level approach is helpful for user-transparent reconfiguration, automatic, and recovery in the multiprocessor system while the logic-circuit approach is not, the designers prefer to design the system into a system-level fault-diagnosis system. In 1967, Preparata et al. proposed an automatic diagnosis procedure in multiprocessor systems, which is known as the first system-level diagnosis approach [1, 2]. This model proposed by Preparata et al. [1] is called the Preparata, Metzger, and Chien (PMC) model. In theory, a digraph  $G = (V, E)$  can usually be

used to denote a PMC model, where for two processors  $i$  and  $j$ ,  $(i, j) \in E$  if and only if processor  $u$  is tested processor  $i$ . For each testing edge  $(i, j)$ , we can use 1 or 0 to denote their test result  $\sigma(i, j)$ , where  $\sigma(i, j) = 1$  implies that  $i$  judges  $j$  to be faulty and  $\sigma(i, j) = 0$  implies that  $i$  judges  $j$  to be fault-free.

There are several fundamental system-level diagnosis strategies:  $t$ -diagnosis,  $t/s$ -diagnosis ( $s \geq t$ ) and conditional  $t$ -diagnosis. Suppose that a system  $S$  has at most  $t$  faulty nodes, if each node in the system can be diagnosed correctly as either fault-free or faulty, then the system is called a  $t$ -diagnosable system; some research results on a  $t$ -diagnosable system can be found in [1–4], etc. In theory, the  $t$ -diagnosis approach is a key measurement for the reliability of considered network system. Besides, the  $t$ -diagnosis approach is desirable to be applied to the areas being related to network control, for example, in the research on reinforcement learning and adaptive optimization, we know that the neural network is often used to represent actor network and chosen as a optimal control policy [5]. Before an adaptive optimal controller is designed, it is necessary and important to test whether the nodes (neurons) in the neural network are fault-free or faulty by the  $t$ -diagnosis approach. For a system having at most  $t$  faulty nodes, if it can

determine a set with the size  $s$  ( $s \leq t$ ) that contains all its faulty nodes, then it is  $t/s$ -diagnosable. Numerous studies have been reported on a  $t/s$ -diagnosable system, such as [6–12].

In a system, denoted by  $G = (V, E)$ , with at most  $t$  faulty nodes, a subset  $V_i \subset V$  with  $|V_i| \leq t$  is called a conditional faulty set if there does not exist a node  $v$  such that  $N(v) \subset V_i$ , where  $N(v)$  is a set consisting of all  $v$ 's neighbors. If for any two conditional faulty sets  $V_1$  and  $V_2$  with  $V_1 \neq V_2$ ,  $\sigma(V_1) \cap \sigma(V_2) = \emptyset$ , where  $\sigma(V_i)$  is the set of syndromes produced by  $V_i$ , then the system is called conditionally  $t$ -diagnosable. Lots of efforts are made to study the conditionally  $t$ -diagnosable system, see [13–22].

It is worth mentioning that the above three strategies are one-round diagnosis strategies, whose diagnosabilities are usually not too large. For instance,  $Q_n$  is shown to be  $n$ -diagnosable and its  $t/t$ -diagnosability and conditional diagnosability are  $n$  and  $(2n-2)/(2n-2)$ , respectively, based on the PMC model. However, when the size of the faulty node set is larger than the diagnosability of the above diagnosis strategies, the above diagnosable systems can do little for the diagnosis. Therefore, to address the issue that a system has a large number of faulty nodes, Chen et al. introduced a novel strategy, called by  $t/k$ -diagnosis, for the star network [23], where  $1 \leq k \leq t$ . A  $t/k$ -diagnosable system guarantees to identify at least  $k$  faulty nodes only if as long as the size of the set consisting of faulty nodes in it does not exceed  $t$ . Although a  $t/k$ -diagnosis has a large diagnosability, it takes much longer to repair faulty nodes, which leads to low efficiency. Therefore, it provides strong motivation for the study of a diagnosis strategy that can reach a balance between improving the diagnosability and being highly efficient. This paper presents a novel diagnosis approach, called by two-round  $t$ -diagnosis. Using the two-round  $t$ -diagnosis approach, the system can guarantee that each faulty node can be diagnosed by at most a one-round replacement (repairing the faulty nodes).

A simple introduction on this paper's remainder is as follows. Some related notations and definitions are presented as the preliminaries in Section 2. In Section 3, two-round  $t$ -diagnosable systems are characterized and several important properties are also presented. In Section 4, the properties of a two-round  $t$ -diagnosable system is applied for computing the two-round  $t$ -diagnosability of  $Q_n$ . In Section 5, a fast two-round diagnosis algorithm, whose time complexity is  $O(n2^n)$ , is proposed for  $Q_n$ . Section 6 draws the conclusions.

## 2. Preliminaries

In the section, we introduce some necessary notations and definitions that are frequently used in the rest of the paper.

Under the PMC model, for a system given by graph  $G = (V, E)$ , let  $\Gamma x = \{y \mid (x, y) \in E \text{ and } x, y \in V\}$  and  $\Gamma^{-1}x = \{y \mid (y, x) \in E \text{ and } x, y \in V\}$ . Similarly, for any subset  $X \subset V$ ,  $\Gamma X = \cup_{x \in X} \Gamma x - X$  and  $\Gamma^{-1}X = \cup_{x \in X} \Gamma^{-1}x - X$ . In particular, if  $G$  is undirected, then  $\Gamma x = \Gamma x^{-1} = N(x)$  and  $\Gamma X = \Gamma X^{-1} = N(X)$ , where  $X \subset V$ .

**Definition 1.** Suppose that  $G = (V, E)$  is a graph and has  $k$  connected components, say  $C_1, C_2, \dots, C_k$ . Then,  $C_{\text{sub}}(G) = \{C_1, C_2, \dots, C_k\}$  is called a connected subgraph set of  $G$ . In particular, if  $G$  is connected, then  $C_{\text{sub}}(G) = \{G\}$ .

Additionally, let  $\max\{C_{\text{sub}}(G)\} = C_i$ , where  $|V(C_i)| \geq |V(C_j)|$  for each  $C_j \in C_{\text{sub}}(G)$ .

**Definition 2.** Let  $G = (V, E)$  denote a graph and  $X \subset V$ . Then,  $G[X] = (X, \bar{E})$  is called the induced subgraph by  $X$ , where  $\bar{E} = \{(u, v) \mid (u, v) \in E, u, v \in X\}$ .

**Definition 3.** Let  $G = (V, E)$  denote a graph, let  $\text{Card}_k(C_{\text{sub}}(G)) = \{C \mid C \in C_{\text{sub}}(G) \text{ and } |V(C)| = k\}$  be the set of connected subgraphs with  $k$  nodes in  $G$ .

For instance, for graph  $G$  shown in Figure 1,  $C_{\text{sub}}(G) = \{C_1, C_2\}$ , where  $C_1 = G[\{v_2, v_3, v_6\}]$ ,  $C_2 = G[\{v_1, v_4, v_5, v_7\}]$ ,  $\text{Card}_3(C_{\text{sub}}(G)) = G[\{v_2, v_3, v_6\}]$ , and  $\text{Card}_4(C_{\text{sub}}(G)) = G[\{v_1, v_4, v_5, v_7\}]$ .

**Definition 4.** Let  $G = (V, E)$  denote a system,  $M \subset V$ . For a given syndrome  $\sigma$ , if the following conditions are satisfied, then  $M \subset V$  is said to be an allowable fault set (AFS):

- (1)  $\sigma(x, y) = 0$ , for any  $x, y \in V - M$
- (2)  $\sigma(x, y) = 1$ , for any  $x \in V - M$  and  $y \in M$

**Lemma 1.** Suppose that  $G = (V, E)$  is a system and  $\sigma$  is a given syndrome. Let  $F_1, F_2 \subseteq V$  be two AFSs, then  $F_1 \cup F_2$  is also an AFS.

*Proof.* To the contrary, assume that  $F_1 \cup F_2$  is not an AFS; then, there exists at least one condition in Definition 4, which is not true.

If condition (1) is not true, then  $\exists x, y \in V - (F_1 \cup F_2)$  such that  $\sigma(x, y) = 1$ , a condition to  $F_1$  and  $F_2$  are two AFS.

If condition (2) is not true, then  $\exists x \in V - (F_1 \cup F_2)$  and  $y \in F_1 \cup F_2$ , where  $(x, y) \in E$  such that  $\sigma(x, y) = 0$ . If  $y \in F_1$ , then  $F_1$  is not an AFS; if  $y \in F_2$ , then  $F_2$  is not an AFS, which contradicts the hypothesis.  $\square$

**Lemma 2.** For a given syndrome  $\sigma$  on the system  $G = (V, E)$ , suppose that  $G$  has  $k$  AFSs for  $\sigma$ , say  $F_1, F_2, \dots, F_k \subseteq V$ , then,  $\cap_{i=1}^k F_i$  is the fault set of the system.

*Proof.* Let  $F_c = \cap_{i=1}^k F_i$ . Let  $F$  be the maximal fault set, which exactly consists of all faulty nodes. We will show that  $F$  is an AFS for  $\sigma$ . To this end, let  $(u, v) \in E$ .  $u, v \in V - F$  implies that  $u$  and  $v$  are fault-free nodes, and then  $\sigma(u, v) = 0$ . Hence, condition (1) holds for  $F$ . For condition (2),  $u \in V - F$  and  $v \in F$  implies that  $u$  is a fault-free node and  $v$  is a faulty node and then that  $\sigma(x, y) = 1$ . Hence, condition (2) holds for  $F$ . Then,  $F$  is an AFS for  $\sigma$ . According to the assumption, we have that  $F \in \{F_1, F_2, \dots, F_k\}$ . So,  $F_c = \cap_{i=1}^k F_i \subseteq F$ .  $\square$

**Definition 5.** Let  $G = (V, E)$  be a system,  $t$  an integer,  $X \subseteq V$  with  $0 < |X| < t$ . Suppose that  $X_1, X_2 \subseteq V - X$  with  $|X_1 \cup X| \leq t$  and  $|X_2 \cup X| \leq t$  are two different subsets.  $(X_1, X_2)$  is called a pair of distinguishable subsets of  $V - X$  if there exists an edge from  $X$  to  $X_1 \Delta X_2$ .

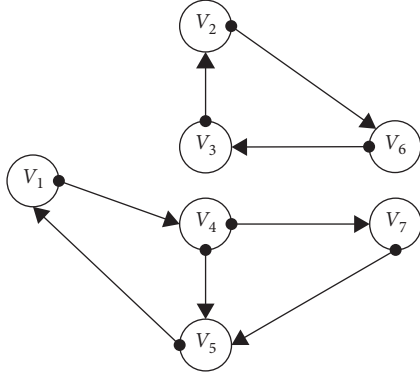


FIGURE 1: A graph with 7 nodes.

According to Definition 5, the following results are true. In the system  $G = (V, E)$ , if  $X \subseteq V$  with  $|X| < k$ , any two subsets  $X_1 \subseteq V - X$  with  $|X_1 \cup X| \leq t$ ,  $X_2 \subseteq V - X$  with  $|X_2 \cup X| \leq t$ , and  $(X_1, X_2)$  is a pair of distinguishable subsets in  $V - X$ , then the system can determine the fault set, provided that  $G$  has less than  $k$  faulty nodes and all faulty nodes in  $X$  have been repaired or replaced with additional fault-free nodes.

**Lemma 3.** *Suppose that the undirected graph  $G = (V, E)$  has less than  $t$  faulty nodes and  $G' \subset G$  with  $|V(G')| \geq t + 1$  is connected. If each result in  $G'$  is 0, then  $G'$  does not have faulty nodes.*

*Proof.* To the contrary, let  $G' \subset G$  with  $|V(G')| \geq t + 1$  be connected, in which each result in it is 0 and there is a faulty node, say  $u$ . Then, it is clear that each node in  $N_{G'}(u)$  is faulty. Similarly, each node in  $N_{G'}(N_{G'}(u))$  is 1 faulty. As a result, each node  $v \in V(G')$  is faulty. Note that  $|V(G')| \geq t + 1$ ; this implies that the number of fault nodes in  $G'$  exceeds  $t$ , a contradiction. Therefore, each node in  $V(G')$  is fault-free.  $\square$

### 3. Two-Round $t$ -Diagnosable Systems

At the beginning of the section, the concept of a two-round  $t$ -diagnosable system is presented as follows.

**Definition 6.** A system is two-round  $t$ -diagnosable if, for given syndrome  $\sigma$ , after repairing or replacing the faulty nodes identified by one-round diagnosis, the system can diagnose the remaining faulty nodes without replacement, provided that the system has less than  $t + 1$  faulty nodes.

According to Definition 6, we can obtain the following necessary conditions for a two-round  $t$ -diagnosable system.

**Theorem 1.** *Let  $G = (V, E)$  represent a system. Then,  $G$  is two-round  $t$ -diagnosable if and only if for any a subset  $Y \subseteq V$  with  $|Y| \leq t$  and any two distinct subsets  $Y_1, Y_2 \subseteq V - Y$ , where  $|Y \cup Y_1| \leq t$ ,  $|Y \cup Y_2| \leq t$ , and  $Y_1 \cap Y_2 = \emptyset$ , there exists at least an edge from  $V - Y_1 - Y_2$  to  $Y_1 \Delta Y_2$ .*

*Proof.*

Necessity: since for any  $Y \subseteq V$  with  $|Y| = t$ , the result is trivial, next, we show that the result is true for the case of

$|Y| < t$ . To the contrary, suppose that there exists a subset  $Y \subseteq V$ , where  $|Y| < t$  and distinct subsets  $Y_1, Y_2 \subseteq V - Y$  with  $|Y \cup Y_1| \leq t$ ,  $|Y \cup Y_2| \leq t$ ,  $Y_1 \cap Y_2 = \emptyset$ , such that there are no edges from  $V - Y_1 - Y_2$  to  $Y_1 \Delta Y_2$ . Without loss of generality, suppose that each node in  $Y$  is faulty and  $G$  has more than  $|Y|$  faulty nodes. Define a syndrome  $\sigma$  as follows. Let  $x, y \in V$  with  $(x, y) \in E$ :

- (1) If  $x, y \in V - Y - Y_1 - Y_2$ , then  $\sigma(x, y) = 0$
- (2) If  $x \in V - Y - Y_1 - Y_2$  and  $y \in Y$ , then  $\sigma(x, y) = 1$
- (3) If  $x \in Y_1 - Y_2$  and  $y \in Y_2 - Y_1$ , then  $\sigma(x, y) = 1$
- (4) If  $x \in Y_2 - Y_1$  and  $y \in Y_1 - Y_2$ , then  $\sigma(x, y) = 1$
- (5) The remaining test results are arbitrary

Both  $(Y \cup Y_1)$  and  $(Y \cup Y_2)$  are allowable sets for syndrome  $\sigma$ . Suppose that  $F_1, F_2, \dots, F_k \subseteq V$  are all the allowable fault sets for  $\sigma$ . Let  $F_c = \bigcap_{i=1}^k F_i$ ;  $F_c$  is a fault set of  $G$  and  $F_c \subseteq Y$ , which implies that the fault set  $Y_1$  identified by the first-round diagnosis is a subset of  $Y$ . Let  $G'$  denote the system after replacing the nodes of  $Y_1$ , and  $\sigma$  is a syndrome obtained by performing a test task on  $G'$ . Since there are no edges from  $V - Y_1 - Y_2$  to  $Y_1 \Delta Y_2$ , then  $\sigma(x, y) = \sigma(x, y) = 1$  for  $x \in Y_1 - Y_2$ ,  $y \in Y_2 - Y_1$ , and  $\sigma(x, y) = \sigma(x, y) = 1$  for  $x \in Y_2 - Y_1$ ,  $y \in Y_1 - Y_2$ . Hence, both  $Y_1$  and  $Y_2$  are AFSs for  $\sigma$ , which is a contradiction to the hypothesis that  $G = (V, E)$  is two-round  $t$ -diagnosable.

Sufficiency: for a syndrome  $\sigma$ , let  $F_c$  be the intersection of all AFSs for  $\sigma$ . According to Lemma 2,  $F_c$  is a fault set, where  $|F_c| \leq t$ . If  $|F_c| = t$ , then each system has been diagnosed by syndrome  $\sigma$ , which implies that  $G$  is two-round  $t$ -diagnosable. If  $|F_c| < t$ , then let  $G'$  denote the system for which all the nodes in  $F_c$  are repaired or replaced with additional fault-free nodes from  $G$ . Then, the number of faulty nodes in  $G'$  will not be more than  $t - |F_c|$ , and these faulty nodes belong to  $V - F_c$ . Let  $\sigma$  denote a syndrome obtained by performing the test task to  $G'$ . We claim that the fault set  $Y_1 \subseteq V - F_c$  of  $G'$  where  $|Y_1| \leq t - |F_c|$  can be determined by  $\sigma$ . In contrast, there exists another nonempty allowable subset  $Y_2 \subseteq V - F_c$  of  $G'$ , where  $|Y_2| \leq t - |F_c|$  for  $\sigma$ , and we derive a contradiction. Consider the following situations.

Case 1: there is an edge  $(x, y)$  from  $V - F_c - Y_1 - Y_2$  to  $Y_1 \Delta Y_2$ . Without loss of generality, let  $y \in Y_1$ . Since  $Y_2$  is an allowable subset of  $V - F_c$  for  $\sigma$ ,  $\sigma(x, y) = 0$ . On the contrary, since  $Y_1$  is a fault subset of  $V - F_c$  for  $\sigma$ ,  $\sigma(x, y) = 1$  is a contradiction.

Case 2: there are no edges from  $V - F_c - Y_1 - Y_2$  to  $Y_1 \Delta Y_2$ . According to this assumption, there exists an edge  $(x, y)$  from  $F_c$  to  $Y_1 \Delta Y_2$ . Without loss of generality, suppose that  $y \in Y_1 - Y_2$ . Since  $Y_2$  is an allowable set for  $\sigma$ ,  $\sigma(x, y) = 0$ . On the contrary, since  $Y_1$  is a fault set of  $G'$ ,  $\sigma(x, y) = 1$  is a contradiction. Hence,  $G$  is two-round  $t$ -diagnosable.

According to the proof of Theorem 1, the two corollaries described as follows are obvious.  $\square$

**Corollary 1.** *A system  $G = (V, E)$  is two-round  $t$ -diagnosable if for any subset  $Y \subseteq V$  with  $|Y| < t$ , and for any distinct disjoint*

subsets  $Y_1, Y_2 \subseteq V - Y$ , where  $|Y \cup Y_1| \leq t$  and  $|Y \cup Y_2| \leq t$ , and  $(Y_1, Y_2)$  is a pair of distinguishable subsets of  $V - Y$ .

**Definition 7.** Let  $S$  be a network system. The maximum nonnegative integer  $t$  that guarantees  $S$  to be two-round  $t$ -diagnosable is called the two-round diagnosability of  $S$ .

For convenience, it is necessary to introduce a notation  $\Gamma_2^{-1}v = \Gamma^{-1}v \cup \Gamma^{-1}(\Gamma^{-1}v) - \{v\}$  for the coming corollary, where  $G = (V, E)$  is a system,  $v \in V$ .

**Corollary 2.** For the system  $G = (V, E)$ , let  $\alpha = \min\{|\Gamma_2^{-1}v| \mid v \in V\} + 1$ . Then, the system is not two-round  $\alpha$ -diagnosable.

*Proof.* Let  $v \in V$  be a node such that  $\alpha = |\Gamma_2^{-1}v| + 1$ . Consider the case such that  $F = \Gamma_2^{-1}v \cup \{v\}$  is a fault set that consists of exactly all faulty nodes in the system. Note that  $|F| = \alpha$ . Define a syndrome  $\sigma$  as follows. Let  $x, y \in V$  with  $(x, y) \in E$  (see Figure 2):

- (i) If  $x, y \in V - \Gamma_2^{-1}v - \{v\}$ , then  $\sigma(x, y) = 0$
- (ii) If  $x \in V - \Gamma_2^{-1}v - \{v\}$  and  $y \in \Gamma^{-1}(\Gamma^{-1}v) - \{v\}$ , then  $\sigma(x, y) = 1$  (before replacement)
- (iii) If  $x \in \Gamma^{-1}(\Gamma^{-1}v) - \{v\}$  and  $y \in V - \Gamma_2^{-1}v - \{v\}$ , then  $\sigma(x, y) = 0$
- (iv) If  $x \in \Gamma^{-1}(\Gamma^{-1}v) - \{v\}$  and  $y \in \Gamma^{-1}v$ , then  $\sigma(x, y) = 1$
- (v) If  $x \in \Gamma^{-1}v$ , then  $\sigma(v, x) = 1$
- (vi) If  $x \in \Gamma^{-1}(\Gamma^{-1}v) - \{v\}$ , then  $\sigma(v, x) = 0$
- (vii) If  $x \in \Gamma^{-1}v$ , then  $\sigma(v, x) = 1$
- (viii) If  $x \in \Gamma^{-1}(\Gamma^{-1}v) - \{v\}$ , then  $\sigma(v, x) = 1$  (before replacement)
- (ix) The other test results are arbitrary

For  $\sigma$ , the nodes of subset  $\Gamma^{-1}(\Gamma^{-1}v) - \{v\}$  can be identified correctly as faulty, which implies that the nodes of subset  $\Gamma^{-1}v \cup \{v\}$  cannot be identified by syndrome  $\sigma$ . After the faulty nodes of subset  $\Gamma^{-1}(\Gamma^{-1}v) - \{v\}$  are repaired, there exist no edges from fault-free nodes to  $v$ , and the test results from  $v$  to faulty (fault-free) nodes are 1 (0), which implies that we cannot judge the state of node  $v$ . Therefore, to identify the state of node  $v$ , we need a second replacement. So, the system is not two-round  $\alpha$ -diagnosable.  $\square$

#### 4. Two-Round Diagnosability of Hypercube Networks

$Q_n$  is a regular graph with  $2^n$  nodes and  $n2^n$  edges. Each node in  $Q_n$  can be denoted by an  $n$ -bit binary string.  $(x, y) \in E(Q_n)$  if and only if there is exactly different one bit position between  $x$  and  $y$ . Figure 3 is an illustration of a 4-dimensional hypercube network  $Q_4$ .

**Lemma 4** (see [17]). Let  $X \subset V(Q_n)$  with  $|Y| = s$  and  $0 > s \geq n + 1$ . Then, the size of the neighbor set of  $Y$  is more than  $(sn - s(s + 1)/2)$ .

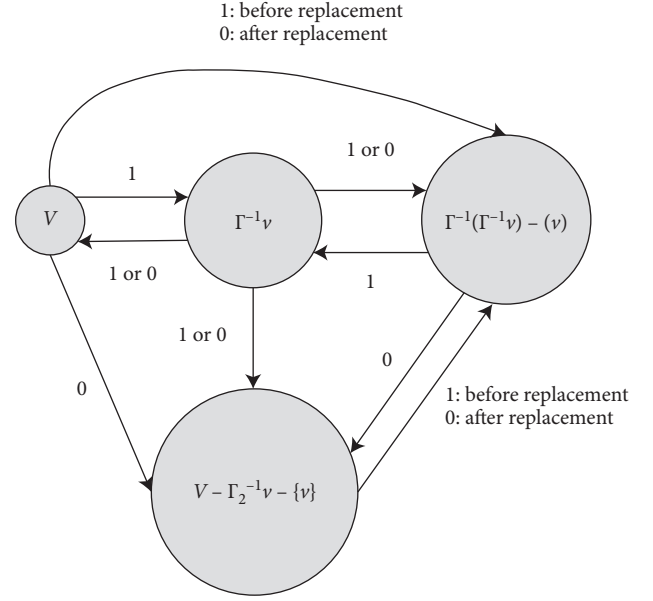


FIGURE 2: A syndrome of Corollary 2.

**Lemma 5.** For  $Q_n$  ( $n \geq 5$ ), let  $S = \{v_i \mid v_i \in V(Q_n), 1 \leq i \leq n, |\cap_{i=1}^n N(v_i)| = 1\}$ . Then,  $|N(S)| = (n^2 - n/2) + 1$ .

*Proof.* We use  $\text{add}(v)$  to denote  $v'$  address. Suppose that  $\cap_{i=1}^n N(v_i) = \{v_0\}$  and  $\text{add}(v_0) = a_1 a_2 \cdots a_n$ . According to the definition of  $Q_n$ , without loss of generality, assume that  $\text{add}(v_1) = \bar{a}_1 a_2 \cdots a_n$ ,  $\text{add}(v_2) = a_1 \bar{a}_2 \cdots a_n, \dots, \text{add}(v_n) = a_1 a_2 \cdots \bar{a}_n$ . For some  $i$ , we have that  $N(v_i) = \{\bar{a}_1 a_2 \cdots \bar{a}_i \cdots a_n, a_1 \bar{a}_2 a_3 \cdots \bar{a}_i \cdots a_n, \dots, a_1 a_2 \cdots \bar{a}_i \cdots \bar{a}_n\}$ . Then,  $N(v_i) \cap N(v_j) = \{v_0, u\}$ , where  $\text{add}(u) = a_1 a_2 \cdots \bar{a}_i \cdots \bar{a}_j \cdots a_n$  and  $1 \leq i, j \leq n, i \neq j$ . So,  $|N(S)| = n(n-1) - C_n^2 + 1 = n^2 - (n(n+1)/2) + 1 = (n^2 - n/2) + 1$ .

According to Lemmas 4 and 5, for an  $n$ -dimensional hypercube and a subset  $S \subseteq V$ , where  $|S| = n + 1$ , if  $\exists v \in S$  with  $N(v) \subseteq S$ , then  $|N(S)| = (n^2 - n/2)$ .  $\square$

**Lemma 6** (see [19]). Suppose that  $Q_n$  is modelled by a graph  $G(V, E)$  ( $n \geq 5$ ). Let  $S \subset V$  with  $n \leq |S| \leq 2(n-1) - 1$ . If  $G[V - S]$  is disconnected and  $C_{\text{sub}}(G[V - S]) = \{C_1, C_2, \dots, C_m\}$ , then the following conditions hold:

- (i)  $\sum_{i=0}^1 i |C_{\text{sub}}(G[V - S])| \leq 1$
- (ii) There is unique  $C_i \in C_{\text{sub}}(G[V - S])$ , where  $|V(C_i)| \geq 2$

**Lemma 7** (see [24]). Suppose that  $Q_5$  is modelled by a graph  $G(V, E)$ . Let  $S \subset V$  with  $\leq 5k - (k(k+1)/2)$  ( $1 \leq k \leq 4$ ). If  $G[V - S]$  is disconnected and  $C_{\text{sub}}(G[V - S]) = \{C_1, C_2, \dots, C_m\}$ , then the following conditions hold:

- (i)  $\sum_{i=0}^{k-1} i |C_{\text{sub}}(G[V - S])| \leq k - 1$
- (ii) There is unique  $C_i \in C_{\text{sub}}(G[V - S])$ , where  $|V(C_i)| \geq k$

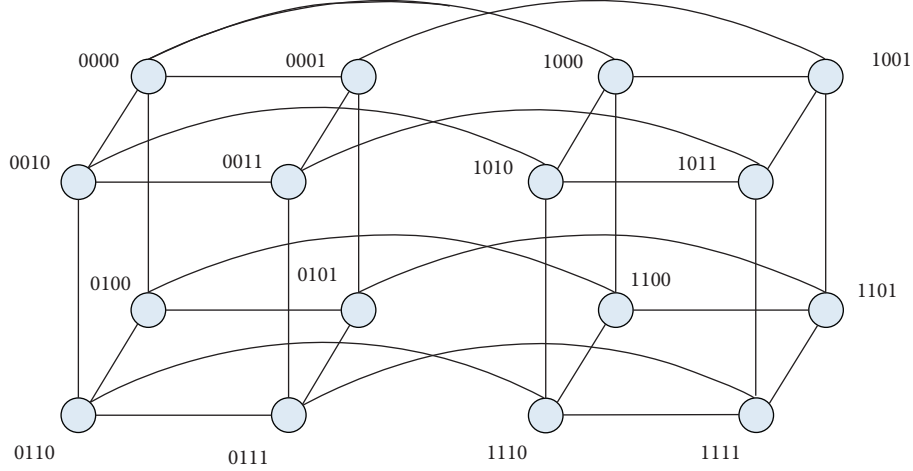


FIGURE 3: A 4-dimensional hypercube.

**Lemma 8** Let  $x > 0$  and  $y > 0$  are two integers. Then,  $(x + y) \leq 1 + xy$ .

*Proof.* Consider the function  $g(x, y) = (1 + xy) - (x + y)$ . It is obvious that  $g(x, y) = (x - 1)(y - 1)$ . Since  $x \geq 1$  and  $y \geq 1$ , then  $g(x, y) \geq 0$ , which implies that  $(x + y) \leq 1 + xy$ .  $\square$

**Lemma 9** (see [24]). Suppose that  $Q_n$  is modelled by a graph  $G(V, E)$  ( $n \geq 5$ ). Let  $S \subset V$  with  $|S| \leq kn - (k(k+1)/2)$  ( $1 \leq k \leq n-1$ ). If  $G[V-S]$  is disconnected and  $C_{sub}(G[V-S]) = \{C_1, C_2, \dots, C_m\}$ , then the following conditions are true:

- (i)  $\sum_{i=0}^{k-1} i |Card_i(C_{sub}(G[V-S]))| \leq k - 1$
- (ii) There is unique  $C_i \in C_{sub}(G[V-S])$ , where  $|V(C_i)| \geq k$

**Lemma 10** (see [24]). Suppose that  $Q_5$  is modelled by a graph  $G(V, E)$ . Let  $S \subset V$  with  $|S| \leq 9$ . If  $G[V-S]$  is disconnected and  $C_{sub}(G[V-S]) = \{C_1, C_2, \dots, C_m\}$ , then the following conditions are true:

- (i)  $\sum_{i=0}^{k-1} i |Card_i(C_{sub}(G[V-S]))| \leq 4$
- (ii) There is unique  $C_i \in C_{sub}(G[V-S])$ , where  $|V(C_i)| \geq 5$

**Lemma 11** (see [24, 25]). Suppose that  $Q_n$  is modelled by a graph  $G(V, E)$  ( $n \geq 5$ ). Let  $S \subset V$  with  $|S| \leq n^2 - (n(n+1)/2) - 1$ . If  $G[V-S]$  is disconnected and  $C_{sub}(G[V-S]) = \{C_1, C_2, \dots, C_m\}$ , then the two conditions as follows are true:

- (i)  $\sum_{i=0}^{n-1} i |Card_i(C_{sub}(G[V-S]))| \leq n - 2$
- (ii) There is unique  $C_i \in C_{sub}(G[V-S])$ , where  $|V(C_i)| \geq n$

With the above preliminaries, we shall discuss the two-round diagnosability of  $Q_n$ .

**Theorem 2.** An  $n$ -dimensional ( $n \geq 6$ ) hypercube given by  $G = (V, E)$  is not two-round  $(n^2 + n/2) + 1$ -diagnosable.

*Proof.* Note that, for each node  $v \in V$ , we have  $|\Gamma_2^{-1}v \cup \{v\}| = (n^2 + n/2) + 1$  by Lemma 5. According to Corollary 2, it is easily determined that an  $n$ -dimensional hypercube is not two-round  $(n^2 + n/2) + 1$ -diagnosable.  $\square$

**Theorem 3.** An  $n$ -dimensional ( $n \geq 6$ ) hypercube given by  $G = (V, E)$  is two-round  $(n^2 + n/2)$ -diagnosable.

*Proof.* Let  $X \subseteq V$  be a subset of  $V$ , where  $|X| < (n^2 + n/2)$ . According to Theorem 1, we will show that, for any two distinct subsets  $S_1, S_2 \subseteq V - X$ , where  $|X \cup S_1| \leq (n^2 + n/2)$ ,  $|X \cup S_2| \leq (n^2 + n/2)$ , and  $S_1 \cap S_2 = \emptyset$ , there exists at least an edge from  $V - S_1 - S_2$  to  $S_1 \Delta S_2$ . Without loss of generality, let  $S_1 - S_2 \neq \emptyset$ .

Now, consider the two situations:

Case 1:  $|X| \geq (n^2 - n/2)$ .

Note that  $|S_1| < (n^2 + n/2) - |X| \leq (n^2 + n/2) - (n^2 - n/2) = n$  and  $|S_2| < (n^2 + n/2) - |X| \leq (n^2 + n/2) - (n^2 - n/2) = n$ , which implies that  $|(S_1 - S_2)| < n$ . Let  $k = |(S_1 - S_2)|$ ; then,  $1 \leq k < n$ . According to Lemma 4, we have that  $N(S_1 - S_2) \geq kn - (k(k+1)/2) + 1$ . Since  $f(k) = kn - (k(k+1)/2) + 1$  is an increasing function for  $k \in [1, n - (1/2)]$ ,  $f(k) \geq f(1) = n$ , which implies that  $|S_1 - S_2| = f(k) > n > |S_2|$ . Hence, there exists at least an edge from  $V - S_1 - S_2$  to  $S_1 - S_2$ .

Case 2:  $|X| < (n^2 - n/2)$ .

To the contrary, assume that  $\exists S_1, S_2 \subseteq V - X$  with  $|X \cup S_1| \leq (n^2 + n/2)$ ,  $|X \cup S_2| \leq (n^2 + n/2)$ , and  $S_1 \cap S_2 = \emptyset$ , but that there are no edges from  $V - S_1 - S_2$  to  $S_1 \Delta S_2$ . Next, we derive a contradiction. According to Lemma 11, we know that the system can be divided into three parts by a subset  $X$  (shown in Figure 4). Note that  $A$  is the largest component of  $G[V - X]$ , where  $|A| \geq 2^n - ((n^2 - n/2) - 1) - (n - 1) = 2^n + 2 - (n^2 + n/2)$ , and  $B$  is the union of the remaining components of  $G[V - X]$ , where  $|B| \leq n - 1$ . Since  $|B| \leq n - 1$  and  $Q_n$  is a

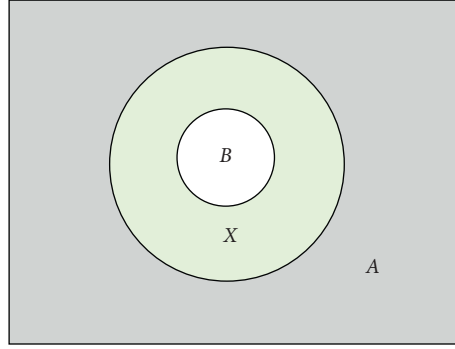


FIGURE 4: System divided into  $A$ ,  $B$ , and  $X$  by subset  $X$ .

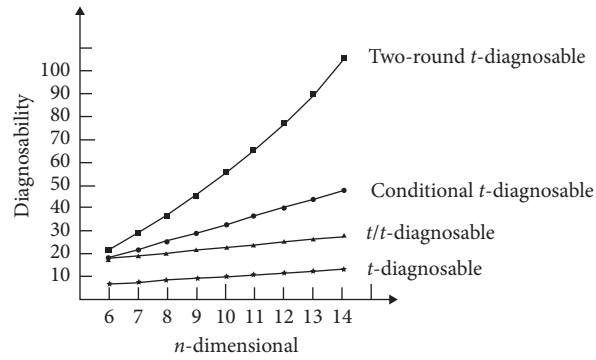


FIGURE 5: Comparison of the diagnosabilities of  $Q_n$  between the traditional diagnosis approach and the two-round diagnosis approach.

$n$ -regular graph,  $S_1 \cap B = \phi$ . A similar argument can be used to obtain that  $S_2 \cap B = \phi$ . Hence,  $S_1 \subseteq A$ , and  $S_2 \subseteq A$ .

Note that  $|A - S_1 \cup S_2| = |A| - |S_1| - |S_2| \geq 2^n - |X| - |S_1| - |S_2| - (n-1) \geq 2^n - (3/2)(n^2 + n/2) - (n-1) > 0$  ( $n \geq 6$ ), and  $A$  is a component of  $V - S$ . This property implies that  $N(S_1 \Delta S_2) \cap A \neq \phi$ , which is a contradiction.

In summary, we conclude that  $Q_n$  ( $n \geq 6$ ) is two-round  $(n^2 + n/2)$ -diagnosable.  $\square$

As we know,  $Q_n$  is  $n$ -diagnosable,  $(2n - 2/2n - 2)$ -diagnosable, and conditionally  $4n - 7$ -diagnosable. A previous study has shown that  $Q_n$  is also two-round  $(n^2 + n/2)$ -diagnosable. Figure 5 gives an intuitive comparison between these diagnosabilities for  $Q_n$ .

## 5. A Fault Diagnosis Algorithm of Two Round $t$ -Diagnosable Hypercubes

In Section 4, we observed that an  $n$ -dimensional hypercube ( $Q_n$ ) was two-round  $(n^2 + n/2)$ -diagnosable. Then, identifying all faults with at most a one-round replacement remains an open question. This section presents a fast identification algorithm to address this issue (Algorithm 1). The completeness of the identification algorithm is demonstrated, provided that the system has less than  $t + 1$  faulty nodes ( $t \leq (n^2 - n/2) - 1$ ). The fast identification algorithm is described in detail in Algorithm 2.

Algorithm 1 is applied to each node of  $Q_n$  with at most  $t$  faulty nodes ( $t \leq (n^2 - n/2) - 1$ ). According to Lemmas 9 and 11, the unique set  $S$  can be output by Algorithm DFS, where  $|S| \geq t + 1$ .

**Theorem 4.** *The time complexity of the fast identification algorithm is  $O(N \log_2 N)$ , where  $N = 2^n$ .*

*Proof.* For the sake of convenience, let  $F$  denote a set exactly consisting of faulty nodes in the system,  $L$  the largest component of the induced subgraph by  $V - F$ , and  $S$  a set consisting of all remaining small components in the induced subgraph by  $V - F$ . When  $u_i \in L$ , Step 1 takes an amount of time equal to  $O(N \log_2 N)$ . When the  $(u_i) \in S \cup F$ , Step 1 takes an amount of time equal to  $O((n^2 + n/2) + N \log_2 N) = O(N \log_2 N)$ . Hence, Step 1 takes an amount of time equal to  $O(N \log_2 N)$ . Step 2 and Step 3 take an amount of time equal to  $O(N)$ . So, the total time for Fast Identification is  $O(N \log_2 N)$ .

According to Lemmas 9 and 11, the completeness of the fast identification algorithm is obvious, provided that  $Q_n$  has no more than  $(n^2 - n/2) - 1$  faulty nodes. Note that  $Q_n$  is two-round  $(n^2 + n/2)$ -diagnosable. Then, there is a question of whether our algorithm is suitable for the scenario in which there are  $t((n^2 - n/2) \leq t \leq (n^2 + n/2))$  faults in the system. We perform a simulation to evaluate the system; in the following simulation, we assume that  $Q_n$  has  $(n^2 + n/2)$

**Input:**

- (i) An undirected graph  $G = (V, E)$  with  $|V| = N$  representing a network system of interest and a network node  $x \in V$ . Let  $S = \{x\}$ .

**Output:**

A subset  $S \subseteq V$ .

- (1) DFS( $v$ ):  
 for each  $u \in N(x)$ ,  
 if  $\sigma(y, x) = \sigma(x, y) = 0$ ,  
 $S = S \cup \{y\}$  and DFS( $y$ ).
- (2) Output the node set  $S$ .

ALGORITHM 1: Depth-first search.

**Input:**

An undirected graph  $G = (V, E)$  with  $|V| = 2^n$ , a positive integer  $t$  ( $t \leq (n^2 - n/2) - 1$ ) and a syndrome  $\sigma$ .

**(i) Output:**

A fault set  $A$ , a fault-free node set  $B$  and a second-round fault set  $C$ , where  $A \cup B \cup C = V$ .

- (1) Let  $S_i = \emptyset$  ( $1 \leq i \leq 2^n$ ) and  $B = A = C = \emptyset$ .  
 For any  $u_i \in V - \cup_{j=1}^i S_j$  ( $1 \leq i \leq 2^n$ ), perform DAS( $u_i$ );  $S_i$  DAS( $u_i$ ).  
 while  $|S_i| \geq t + 1$ , let  $B = B \cup S_i$ , and  $A = A \cup N(B)$ .
- (2) While  $V = B \cup A$ , output  $B$  and  $A$ . Otherwise, go to Step 3.
- (3) While  $|A| \geq t$ , let  $B = V - A$ , output  $B$  and  $A$ .  
 Otherwise, repair the faulty nodes in  $A$  and execute  
 Best ( $A, B, C$ ) until  $B \cup A \cup C = V$ . Additionally, output subsets  $B$ ,  $A$ , and  $C$ .  
 Best ( $A, B, C$ ):  
 For any  $v \in V - A - B$ , if  $\sigma(u, v) = 1$  ( $u \in A \cup B$ ), then  $C = C \cup \{v\}$ ; otherwise,  $B = B \cup \{v\}$ .

ALGORITHM 2: Fast identification.

TABLE 1: Number of faulty nodes identified by the two-round algorithm applied to an  $n$ -dimensional hypercube.

| $n$               | 7  | 8  | 9  | 10 | 11 | 12 |
|-------------------|----|----|----|----|----|----|
| Faults            | 28 | 36 | 45 | 66 | 78 | 91 |
| Identified faults | 28 | 36 | 45 | 66 | 78 | 91 |

faulty nodes, and each node of the system may be faulty with the same probability. We run our algorithm 1000 times. Table 1 gives the corresponding experimental results.

The simulation shows that our algorithm is suitable for  $Q_n$ , provided that it has no more than  $(n^2 + n/2)$  faulty nodes.  $\square$

## 6. Conclusion

In this article, we introduce a novel diagnosis strategy called the two-round diagnosis strategy that implies that each node can be determined by at most a one-round replacement. A necessary and sufficient condition of the system being two-round  $t$ -diagnosable is presented. Additionally, several important properties of this system are described. Using the theory of a two-round  $t$ -diagnosable system, we show that  $Q_n$  is two-round  $(n^2 + n/2)$ -diagnosable. Compared to the traditional diagnosis strategy, the two-round diagnosability of  $Q_n$  is  $(n + 1/2)$  times as large as  $n$ , the classic diagnosability of  $Q_n$ . Furthermore, an  $O(n2^n)$  algorithm is provided

to identify faulty nodes for  $Q_n$ . The conditionally  $t$ -diagnosable network systems are a class of typical nonlinear systems, in which the state (syndrome) of a node impacted these nodes in its surrounding area. Recent years, there are some studies to analyze nonlinear systems by using online policy iterative optimization algorithms [20, 26]. The combination of these algorithms and our algorithm will be a try to obtain an optimal fault set for considered conditionally  $t$ -diagnosable network system; this is one of our studies in the future.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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