Hindawi
Complexity
Volume 2020, Article ID 9720653, 13 pages
https://doi.org/10.1155/2020/9720653

Research Article

Two New Conjugate Gradient Methods for Unconstrained Optimization

Meixing Liu,[1,2] **Guodong Ma**,[2] **and Jianghua Yin**[2,3]

[1]*Department of Mathematics, Shanghai University, Shanghai 200444, China*
[2]*Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin, Guangxi 537000, China*
[3]*College of Mathematics and Computer Science, Guangxi Science & Technology Normal University, Laibin, Guangxi 546199, China*

Correspondence should be addressed to Guodong Ma; mgd2006@163.com

Received 14 November 2019; Accepted 9 March 2020; Published 22 April 2020

Academic Editor: Quanmin Zhu

The conjugate gradient method is very effective in solving large-scale unconstrained optimal problems. In this paper, on the basis of the conjugate parameter of the conjugate descent (CD) method and the second inequality in the strong Wolfe line search, two new conjugate parameters are devised. Using the strong Wolfe line search to obtain the step lengths, two modified conjugate gradient methods are proposed for general unconstrained optimization. Under the standard assumptions, the two presented methods are proved to be sufficient descent and globally convergent. Finally, preliminary numerical results are reported to show that the proposed methods are promising.

1. Introduction

The conjugate gradient method (CGM for short) plays an important role in obtaining the numerical solution of the optimal control problem for nonlinear dynamic systems and other mathematical models [1, 2]. In this paper, we study the nonlinear CGM for the following unconstrained optimization problem:

$$\min\{f(x) \mid x \in R^n\}, \tag{1}$$

where $f: R^n \longrightarrow R$ is smooth and its gradient $g(x) = \nabla f(x)$ is available.

The iterates of the classic CGMs for solving problem (1) are generated by $x_{k+1} = x_k + \alpha_k d_k$. First, the step length $\alpha_k > 0$ is usually yielded by a suitable inexact line search along the search direction d_k, such as the Wolfe line search

$$\begin{cases} f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^{\mathrm{T}} d_k, \\ g(x_k + \alpha_k d_k)^{\mathrm{T}} d_k \ge \sigma g_k^{\mathrm{T}} d_k, \end{cases} \tag{2}$$

or the strong Wolfe line search

$$\begin{cases} f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^{\mathrm{T}} d_k, \\ \left| g(x_k + \alpha_k d_k)^{\mathrm{T}} d_k \right| \le \sigma \left| g_k^{\mathrm{T}} d_k \right|, \end{cases} \tag{3}$$

where parameters δ and σ satisfy $0 < \delta < \sigma < 1$. Second, the search direction d_k is computed by

$$d_k = \begin{cases} -g_k, & k = 1, \\ -g_k + \beta_k d_{k-1}, & k \ge 2, \end{cases} \tag{4}$$

where $g_k = g(x_k)$ and β_k is the conjugate parameter. To the best of our knowledge, different choices for the scalar β_k lead to different CGMs works. In particular, the well-known formulas for β_k include the Hestenes–Stiefel (HS) [3], Fletcher–Reeves (FR) [4], Polak–Ribiere (PRP) [5], Conjugate Descent (CD) [6], Liu–Story (LS) [7], and Dai–Yuan (DY) [8] formulas, and they are specified as follows:

$$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T(g_k - g_{k-1})},$$

$$\beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2},$$

$$\beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{\|g_{k-1}\|^2},$$

$$\beta_k^{CD} = \frac{\|g_k\|^2}{-d_{k-1}^T g_{k-1}}, \tag{5}$$

$$\beta_k^{LS} = \frac{g_k^T(g_k - g_{k-1})}{-d_{k-1}^T g_{k-1}},$$

$$\beta_k^{DY} = \frac{\|g_k\|^2}{d_{k-1}^T(g_k - g_{k-1})},$$

where $\|\cdot\|$ stands for the Euclidean norm. A great number of CGMs with good convergence properties and effective numerical performance are deuterogenic by the six methods above, see e.g., [9–15]. As we know, Jiang et al. [12] studied a CGM work called JMJ method, where the parameter β_k is correspondingly specified by

$$\beta_k^{JMJ} = \begin{cases} \dfrac{g_k^T\left(g_k - \left(\|g_k\|/\|d_{k-1}\|\right)d_{k-1}\right)}{d_{k-1}^T(g_k - g_{k-1})}, & \text{if } g_k^T d_{k-1} > 0, \\ \dfrac{g_k^T\left(g_k + \left(\|g_k\|/\|d_{k-1}\|\right)d_{k-1}\right)}{d_{k-1}^T(g_k - g_{k-1})}, & \text{otherwise.} \end{cases} \tag{6}$$

This further can be integrated and rewritten as $\beta_k^{JMJ} = (\|g_k\|^2 - (\|g_k\|/\|d_{k-1}\|)|g_k^T d_{k-1}|)/d_{k-1}^T(g_k - g_{k-1})$. It is obvious that the abovementioned formula reduces to the DY formula under the exact line search. Moreover, the JMJ method keeps the descent property at each iteration and converges globally for general nonconvex functions under the Wolfe line search.

In this paper, we focus our attention on the ideas of the JMJ and CD methods as well as the strong Wolfe line search. In particular, by the second inequality of the strong Wolfe line search, it follows that $|g_k^T d_{k-1}|/(-d_{k-1}^T g_{k-1}) \le \sigma$ if $d_{k-1}^T g_{k-1} < 0$. Thus, based on the formula β_k^{JMJ}, making full use of the characteristics of the JMJ and CD methods, and to ensure that our proposed methods possess nice convergent properties and to improve the numerical performance, two new formulas for β_k are constructed in this paper. The first one is generated by replacing the term $\|g_k\|/\|d_{k-1}\|$ in β_k^{JMJ} with the CD formula β_k^{CD}, namely,

$$\beta_k^{LMYCD1} = \frac{\|g_k\|^2 - \beta_k^{CD}|g_k^T d_{k-1}|}{d_{k-1}^T(g_k - g_{k-1})}. \tag{7}$$

On the contrary, replacing the denominator in β_k^{LMYCD1} with $\|g_{k-1}\|^2$, the second one is presented with

$$\beta_k^{LMYCD2} = \frac{\|g_k\|^2 - \beta_k^{CD}|g_k^T d_{k-1}|}{\|g_{k-1}\|^2}. \tag{8}$$

From (7) and (8), it is not difficult to know that the former formula β_k^{LMYCD1} reduces to the DY formula when the exact line search is used, and accordingly, in the same condition, the later one β_k^{LMYCD2} reduces to the FR formula.

The rest of the paper is organized as follows. In Section 2, two modified methods and the sufficient descent properties are presented. Global convergence properties of the proposed methods are analyzed in Section 3. Some numerical results are reported in Section 4. Finally, we draw a conclusion in Section 5.

2. Methods and Sufficient Descent Properties

In this section, we first describe the details of the two proposed methods, which for convenience are called the LMYCD1 and LMYCD2 methods in Algorithm 1 and Algorithm 2, respectively.

2.1. Sufficient Descent Condition. If there exists a constant $c > 0$ such that $g_k^T d_k \le -c\|g_k\|^2$, $\forall k \ge 1$, then, we say that the search direction d_k of the method satisfies the sufficient descent condition, which is often used to analyze the convergence properties of CGMs for this kind of problem (1) under inexact line search, see e.g., [10–14].

The next lemmas show that the search directions yielded by the two proposed methods always satisfy the sufficient descent condition.

Lemma 1. *Suppose that d_k is generated by the LMYCD1 method and $0 < \sigma < 1$. Then,*

$$g_k^T d_k \le -\frac{1}{\sigma + 1}\|g_k\|^2. \tag{9}$$

Proof. We prove (9) by induction. For $k = 1$, it is easy to know from $0 < \sigma < 1$ that $g_1^T d_1 = -\|g_1\|^2 < -(1/(\sigma+1))\|g_1\|^2$. Suppose that (9) is satisfied for $k-1$, namely, $g_{k-1}^T d_{k-1} \le -(1/(\sigma+1))\|g_{k-1}\|^2$, then obviously $\beta_k^{CD} > 0$ holds by its definition. Furthermore, from the strong Wolfe line search, we have

$$0 < -(1-\sigma)g_{k-1}^T d_{k-1} \le d_{k-1}^T(g_k - g_{k-1}) \le -(1+\sigma)g_{k-1}^T d_{k-1}, \tag{10}$$

and $0 \le \beta_k^{CD}|g_k^T d_{k-1}| \le \beta_k^{CD}(-\sigma d_{k-1}^T g_{k-1}) = \sigma\|g_k\|^2$. This, together with β_k^{LMYCD1} and relation (10), further implies that

$$0 < \frac{(1-\sigma)\|g_k\|^2}{d_{k-1}^T(g_k - g_{k-1})} \le \beta_k^{LMYCD1} \le \frac{\|g_k\|^2}{d_{k-1}^T(g_k - g_{k-1})}. \tag{11}$$

Now, we prove that (9) holds for k via the following three cases:

Initialization. Given constants $\epsilon > 0$ and $0 < \delta < \sigma < 1$, as well as $x_1 \in R^n$. Let $d_1 = -g_1$, $k := 1$.
Step 1. If $\|g_k\| \le \epsilon$, then stop. Otherwise, go to Step 2.
Step 2. Determine a step length α_k by the strong Wolfe line search (3).
Step 3. Let $x_{k+1} := x_k + \alpha_k d_k$, compute $g_{k+1} = g(x_{k+1})$ and $\beta_{k+1} = \beta_{k+1}^{\mathrm{LMYCD1}}$.
Step 4. Compute $d_{k+1} = -g_{k+1} + \beta_{k+1} d_k$. Set $k := k + 1$, and go back to Step 1.

Algorithm 1: (LMYCD1 method).

Initialization. Given constants $\epsilon > 0$, $0 < \delta < \sigma < 1/2$ and $x_1 \in R^n$. Let $d_1 = -g_1$, $k := 1$.
Step 1 and **Step 2** are the same as the step 1 and step 2 of LMYCD1 method.
Step 3. Let $x_{k+1} := x_k + \alpha_k d_k$, compute $g_{k+1} = g(x_{k+1})$ and $\beta_{k+1} = \beta_{k+1}^{\mathrm{LMYCD2}}$.
Step 4 is the same as the step 4 of LMYCD1 method.

Algorithm 2: (LMYCD2 method).

(i) If $g_k^{\mathrm{T}} d_{k-1} = 0$, then, by the definition of d_k in Algorithm 1, we have

$$g_k^{\mathrm{T}} d_k = -\|g_k\|^2 + \beta_k^{\mathrm{LMYCD1}} g_k^{\mathrm{T}} d_{k-1} = -\|g_k\|^2 < -\frac{1}{\sigma+1}\|g_k\|^2. \tag{12}$$

(ii) If $g_k^{\mathrm{T}} d_{k-1} > 0$, then, similarly from definitions of d_k and $\beta_k^{\mathrm{LMYCD1}}$, taking $\beta_k^{\mathrm{CD}} > 0$, $g_{k-1}^{\mathrm{T}} d_{k-1} < 0$ and (10), we obtain

$$\begin{aligned}
g_k^{\mathrm{T}} d_k &= -\|g_k\|^2 + \beta_k^{\mathrm{LMYCD1}} g_k^{\mathrm{T}} d_{k-1} \\
&= -\|g_k\|^2 + \frac{\|g_k\|^2 - \beta_k^{\mathrm{CD}} |g_k^{\mathrm{T}} d_{k-1}|}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})} g_k^{\mathrm{T}} d_{k-1} \\
&= \frac{\|g_k\|^2 g_{k-1}^{\mathrm{T}} d_{k-1} - \beta_k^{\mathrm{CD}} (g_k^{\mathrm{T}} d_{k-1})^2}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})} \\
&< \frac{\|g_k\|^2 g_{k-1}^{\mathrm{T}} d_{k-1}}{d_{k-1}^{\mathrm{T}}(g_k - g_{k-1})} \le \frac{\|g_k\|^2 g_{k-1}^{\mathrm{T}} d_{k-1}}{-(1+\sigma) g_{k-1}^{\mathrm{T}} d_{k-1}} = -\frac{1}{\sigma+1}\|g_k\|^2.
\end{aligned} \tag{13}$$

(iii) If $g_k^{\mathrm{T}} d_{k-1} < 0$, then, using d_k and $\beta_k^{\mathrm{LMYCD1}} > 0$, we also obtain

$$g_k^{\mathrm{T}} d_k = -\|g_k\|^2 + \beta_k^{\mathrm{LMYCD1}} g_k^{\mathrm{T}} d_{k-1} < -\|g_k\|^2 < -\frac{1}{\sigma+1}\|g_k\|^2. \tag{14}$$

Therefore, the assertion is satisfied for all $k \ge 1$. □

Lemma 2. *Let the search direction d_k be yielded by the LMYCD2 method and $0 < \sigma < 1/2$. Then,*

$$-\frac{1}{1-\sigma} \le \frac{g_k^{\mathrm{T}} d_k}{\|g_k\|^2} \le -\frac{1-2\sigma}{1-\sigma}. \tag{15}$$

Moreover, relation $0 < \beta_k^{LMYCD2} \le \beta_k^{FR}$ holds.

Proof. For $k = 1$, one has $g_1^{\mathrm{T}} d_1 / \|g_1\|^2 = -1$, so (15) clearly holds. Suppose that relation (15) is satisfied for $k-1$. Now, we continue to prove that (15) holds for k. By the strong Wolfe line search and $g_{k-1}^{\mathrm{T}} d_{k-1} < 0$, it is clear that $0 \le |g_k^{\mathrm{T}} d_{k-1}| \le -\sigma g_{k-1}^{\mathrm{T}} d_{k-1}$, and hence

$$0 < 1 - \sigma \le 1 - \frac{|g_k^{\mathrm{T}} d_{k-1}|}{-g_{k-1}^{\mathrm{T}} d_{k-1}} \le 1. \tag{16}$$

Thus, using $\beta_k^{\mathrm{LMYCD2}}$, we obtain

$$\begin{aligned}
0 &< \frac{(1-\sigma)\|g_k\|^2}{\|g_{k-1}\|^2} \le \beta_k^{\mathrm{LMYCD2}} \\
&= \frac{\|g_k\|^2 \left(1 - |g_k^{\mathrm{T}} d_{k-1}| / (-g_{k-1}^{\mathrm{T}} d_{k-1})\right)}{\|g_k\|^2} \le \frac{\|g_k\|^2}{\|g_{k-1}\|^2} = \beta_k^{\mathrm{FR}}.
\end{aligned} \tag{17}$$

Furthermore, recalling d_k in Algorithm 2, β_k^{CD} and $\beta_k^{\mathrm{LMYCD2}}$, one has

$$\begin{aligned}
\frac{g_k^{\mathrm{T}} d_k}{\|g_k\|^2} &= -1 + \frac{\|g_k\|^2 - \left(\|g_k\|^2 / -d_{k-1}^{\mathrm{T}} g_{k-1}\right) |g_k^{\mathrm{T}} d_{k-1}|}{\|g_{k-1}\|^2} \frac{g_k^{\mathrm{T}} d_{k-1}}{\|g_k\|^2} \\
&= -1 + \left(1 - \frac{|g_k^{\mathrm{T}} d_{k-1}|}{-d_{k-1}^{\mathrm{T}} g_{k-1}}\right) \frac{g_k^{\mathrm{T}} d_{k-1}}{\|g_{k-1}\|^2}.
\end{aligned} \tag{18}$$

This together with the right-hand side of (16) further shows that

$$-1 - \frac{|g_k^{\mathrm{T}} d_{k-1}|}{\|g_{k-1}\|^2} \le \frac{g_k^{\mathrm{T}} d_k}{\|g_k\|^2} \le -1 + \frac{|g_k^{\mathrm{T}} d_{k-1}|}{\|g_{k-1}\|^2}. \tag{19}$$

Next, based on the strong Wolfe line search, from the left-hand side of (19) and assertion (15) for $k-1$, we have

$$\frac{g_k^T d_k}{\|g_k\|^2} \geq -1 + \sigma \frac{g_{k-1}^T d_{k-1}}{\|g_{k-1}\|^2} \geq -1 - \frac{\sigma}{1-\sigma} = -\frac{1}{1-\sigma}. \quad (20)$$

Similarly, from the right-hand side of (19), we also obtain

$$\frac{g_k^T d_k}{\|g_k\|^2} \leq -1 - \sigma \frac{g_{k-1}^T d_{k-1}}{\|g_{k-1}\|^2} \leq -1 + \frac{\sigma}{1-\sigma} = -\frac{(1-2\sigma)}{(1-\sigma)}. \quad (21)$$

Thus, the proof is completed. □

3. Convergence Results

Throughout this paper, we make the following elementary assumptions for the objective function:

(H1) The level set $\Lambda = \{x \in R^n \mid f(x) \leq f(x_1)\}$ is bounded

(H2) In a neighborhood U of Λ, $f(x)$ is differentiable and its gradient $g(x)$ is Lipschitz continuous, namely, there exists a constant $L > 0$ such that $\|g(x) - g(y)\| \leq L\|x - y\|, \forall x, y \in U$

To proceed, the well-known Zoutendijk condition [16] is reviewed in the following.

3.1. Zoutendijk Condition. Suppose that assumptions (H1)–(H2) hold, the search direction d_k is a descent direction and the step length α_k satisfies the Wolfe line search condition, then we have $\sum_{k=1}^{\infty} (g_k^T d_k)^2/\|d_k\|^2 < \infty$. In particular, if the sufficient descent condition is satisfied, then $\sum_{k=1}^{\infty} \|g_k\|^4/\|d_k\|^2 < +\infty$.

Now, before establishing the global convergence of the LMYCD1 method, we show that the LMYCD1 method has similar properties to that of the DY method, which is very important to analyze the global convergence property of the method.

Lemma 3. *Let the sequence $\{x_k\}$ be generated by the LMYCD1 method, then $0 < \beta_k^{LMYCD1} \leq g_k^T d_k / g_{k-1}^T d_{k-1}$ always hold for $k \geq 1$.*

Proof. From relation (11), it is clear that $\beta_k^{\text{LMYCD1}} > 0$. If $g_k^T d_{k-1} = 0$, then using relation (10), one has $\beta_k^{\text{LMYCD1}} = \|g_k\|^2/(-g_{k-1}^T d_{k-1}) = g_k^T d_k / g_{k-1}^T d_{k-1}$. In the case that $g_k^T d_{k-1} \neq 0$, we prove $\beta_k^{\text{LMYCD1}} \leq g_k^T d_k / g_{k-1}^T d_{k-1}$ by the following two cases:

(i) If $g_k^T d_{k-1} > 0$, from relation (13), we have $g_k^T d_k < \|g_k\|^2 g_{k-1}^T d_{k-1} / d_{k-1}^T (g_k - g_{k-1})$. Then, dividing this inequality by the negative term $g_{k-1}^T d_{k-1}$, it follows that $\|g_k\|^2 / d_{k-1}^T (g_k - g_{k-1}) < g_k^T d_k / g_{k-1}^T d_{k-1}$, then combining with (11), we have $\beta_k^{\text{LMYCD1}} < (g_k^T d_k / g_{k-1}^T d_{k-1})$.

(ii) If $g_k^T d_{k-1} < 0$, from definitions of d_k and β_k^{LMYCD1}, by the strong Wolfe line search, we have

$$\begin{aligned} g_k^T d_k &= -\|g_k\|^2 + \beta_k^{\text{LMYCD1}} g_k^T d_{k-1} \\ &= \frac{\|g_k\|^2 g_{k-1}^T d_{k-1} - \beta_k^{\text{CD}} |g_k^T d_{k-1}| g_k^T d_{k-1}}{d_{k-1}^T (g_k - g_{k-1})} \\ &\leq \frac{\|g_k\|^2 g_{k-1}^T d_{k-1} - \sigma \beta_k^{\text{CD}} |g_k^T d_{k-1}| g_{k-1}^T d_{k-1}}{d_{k-1}^T (g_k - g_{k-1})} \\ &= \frac{\left(\|g_k\|^2 - \sigma \beta_k^{\text{CD}} |g_k^T d_{k-1}|\right) g_{k-1}^T d_{k-1}}{d_{k-1}^T (g_k - g_{k-1})}, \end{aligned} \quad (22)$$

and hence $g_k^T d_k / g_{k-1}^T d_{k-1} \geq (\|g_k\|^2 - \sigma \beta_k^{\text{CD}} |g_k^T d_{k-1}|) / d_{k-1}^T (g_k - g_{k-1})$ since $g_{k-1}^T d_{k-1} < 0$. Again, from $0 < \sigma < 1$, we have

$$\beta_k^{\text{LMYCD1}} = \frac{\|g_k\|^2 - \beta_k^{\text{CD}} |g_k^T d_{k-1}|}{d_{k-1}^T (g_k - g_{k-1})} \leq \frac{\|g_k\|^2 - \sigma \beta_k^{\text{CD}} |g_k^T d_{k-1}|}{d_{k-1}^T (g_k - g_{k-1})}. \quad (23)$$

This implies that $\beta_k^{\text{LMYCD1}} \leq g_k^T d_k / g_{k-1}^T d_{k-1}$, and the proof is completed. □

Subsequently, based on Lemma 1 and Lemma 3, we can prove the global convergence of the LMYCD1 method.

Theorem 1. *Suppose that Assumptions (H1)–(H2) hold. Let the sequence $\{x_k\}$ be generated by the LMYCD1 method, then the LMYCD1 method is globally convergent by the way of $\liminf_{k \to \infty} \|g_k\| = 0$.*

Proof. By contradiction, we suppose that the conclusion is not true, then there exists a constant $\gamma > 0$ such that $\|g_k\|^2 \geq \gamma, \forall k \geq 1$. Since $d_k + g_k = \beta_k^{\text{LMYCD1}} d_{k-1}$, it follows from Lemma 3 that

$$\begin{aligned} \|d_k\|^2 &= \left(\beta_k^{\text{LMYCD1}}\right)^2 \|d_{k-1}\|^2 - 2 g_k^T d_k - \|g_k\|^2 \\ &\leq \left(\frac{g_k^T d_k}{g_{k-1}^T d_{k-1}}\right)^2 \|d_{k-1}\|^2 - 2 g_k^T d_k - \|g_k\|^2. \end{aligned} \quad (24)$$

Dividing this by $(g_k^T d_k)^2$, it is easy to get that

$$\begin{aligned} \frac{\|d_k\|^2}{(g_k^T d_k)^2} &\leq \frac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} - \frac{2}{g_k^T d_k} - \frac{\|g_k\|^2}{(g_k^T d_k)^2} \\ &= \frac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} - \left(\frac{1}{\|g_k\|} + \frac{\|g_k\|}{g_k^T d_k}\right)^2 + \frac{1}{\|g_k\|^2} \\ &\leq \frac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{\|g_k\|^2}. \end{aligned} \quad (25)$$

Notice that $\|d_1\|^2/(g_1^T d_1)^2 = 1/\|g_1\|^2$, using the abovementioned formula, we have

Table 1: Numerical results for the first group methods.

No.	Problems Name/n	LMYCD1 Itr/NF/NG/Tcpu	hDY Itr/NF/NG/Tcpu	NHS Itr/NF/NG/Tcpu	JMJ Itr/NF/NG/Tcpu	MDL1 Itr/NF/NG/Tcpu	MDL2 Itr/NF/NG/Tcpu
1	cosine 200	16/168/98/0.035	103/1624/853/0.140	23/266/141/0.037	23/267/146/0.023	21/298/154/0.022	21/263/140/0.020
2	cosine 1000	19/198/103/0.037	NaN/NaN/NaN/NaN	25/362/180/0.086	24/327/168/0.075	21/264/151/0.051	24/329/166/0.062
3	cosine 10000	20/198/109/0.333	NaN/NaN/NaN/NaN	25/396/196/0.586	27/393/219/0.634	20/201/106/0.284	22/165/84/0.251
4	cosine 100000	20/198/109/2.425	NaN/NaN/NaN/NaN	55/1352/651/16.743	27/365/193/4.301	21/164/94/2.135	25/202/110/2.653
5	dixmaanj 3000	712/8825/4742/24.677	753/10019/5290/27.130	819/10004/5376/27.108	588/8189/4271/21.668	NaN/NaN/NaN/NaN	750/9567/5102/25.574
6	dixmaank 3000	558/8089/4042/21.231	706/9188/4868/24.756	869/11340/6028/30.712	533/6368/3414/17.636	1146/17072/8887/45.486	792/9774/5205/26.502
7	dixmaanl 3000	533/6689/3598/18.359	513/7035/3766/19.059	836/10716/5808/29.109	466/5473/2934/14.674	679/11438/5517/29.298	478/5633/3035/15.046
8	dixon3dq 20	200/2293/1258/0.157	171/2270/1204/0.117	236/2750/1492/0.134	213/2865/1493/0.134	189/2710/1434/0.138	172/2280/1184/0.122
9	dixon3dq 100	1013/12988/6889/0.665	1088/13497/7144/0.655	NaN/NaN/NaN/NaN	975/12564/6605/0.603	1168/16229/8469/0.788	1265/16115/8600/0.792
10	dqrtic 10	12/29/32/0.002	12/29/32/0.002	12/29/32/0.002	12/29/32/0.003	11/28/30/0.002	12/29/32/0.002
11	dqdrtic 10000	60/695/345/0.421	100/1409/717/0.692	61/723/395/0.385	90/1231/666/0.605	74/841/438/0.494	77/1206/632/0.568
12	dqdrtic 100000	68/713/392/2.755	73/954/491/3.383	86/970/527/3.722	80/996/517/3.817	NaN/NaN/NaN/NaN	69/1068/527/3.768
13	dqdrtic 1000000	58/618/358/25.828	87/1409/727/54.533	110/2431/1025/89.848	107/1590/815/63.473	83/1038/534/41.895	68/916/463/38.205
14	edensch 50	29/275/160/0.019	48/697/341/0.043	27/213/128/0.021	31/303/169/0.020	43/660/331/0.041	31/310/160/0.024
15	edensch 1000	37/531/270/0.268	37/467/254/0.260	33/435/239/0.277	28/274/146/0.138	37/410/225/0.240	37/564/281/0.295
16	edensch 10000	33/370/214/1.936	51/815/412/3.951	35/441/250/2.182	36/483/265/2.438	42/607/313/2.902	30/312/151/1.647
17	fletchcr 110	36/491/251/0.024	39/364/205/0.018	49/581/320/0.028	49/680/313/0.033	44/580/304/0.031	NaN/NaN/NaN/NaN
18	freuroth 5	121/1831/964/0.074	NaN/NaN/NaN/NaN	335/7995/3349/0.436	145/2101/1145/0.101	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
19	genrose 1400	196/2510/1309/0.266	204/2531/1337/0.342	210/2563/1370/0.348	NaN/NaN/NaN/NaN	147/2332/1195/0.246	175/2517/1319/0.337
20	genrose 500000	97/1198/625/37.852	98/1320/693/41.850	215/2405/1318/78.872	158/1644/909/53.339	112/1162/621/40.218	108/1185/648/38.429
21	genrose 1000000	122/1639/881/105.302	156/2310/1194/144.932	154/2264/1202/140.419	NaN/NaN/NaN/NaN	133/1882/989/117.067	NaN/NaN/NaN/NaN
22	genrose 1500000	127/1299/702/127.953	151/1740/932/170.937	186/2209/1182/209.864	NaN/NaN/NaN/NaN	101/1438/751/132.774	106/1259/685/118.715
23	himmelbg 50	8/21/28/0.003	8/23/30/0.002	8/23/30/0.002	8/23/30/0.002	8/22/29/0.002	8/23/30/0.002
24	himmelbg 5000	9/25/33/0.025	9/26/34/0.026	9/26/34/0.026	9/26/34/0.033	9/25/33/0.032	9/26/34/0.036
25	himmelbg 20000	9/25/33/0.112	9/26/34/0.112	9/26/34/0.108	9/26/34/0.115	9/25/33/0.097	9/26/34/0.122
26	himmelbg 500000	10/28/37/2.707	10/29/38/2.792	10/29/38/2.970	10/29/38/3.071	10/28/37/2.961	10/29/38/2.881
27	mccormak 2	18/167/85/0.010	NaN/NaN/NaN/NaN	18/189/96/0.012	NaN/NaN/NaN/NaN	28/232/127/0.017	NaN/NaN/NaN/NaN
28	nondquar 5	70/725/406/0.070	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
29	nonscomp 2	34/445/227/0.026	43/682/343/0.030	61/1032/475/0.051	53/940/497/0.043	47/833/413/0.036	NaN/NaN/NaN/NaN
30	penalty1 10	9/10/13/0.006	6/10/10/0.001	6/10/10/0.001	6/10/10/0.001	6/9/9/0.001	6/10/10/0.001
31	penalty1 500	7/22/17/0.032	6/22/16/0.037	5/22/15/0.031	5/22/15/0.032	5/21/14/0.027	5/22/15/0.027
32	penalty1 1500	8/24/19/0.232	11/24/22/0.274	10/24/21/0.262	10/24/21/0.241	10/23/20/0.226	10/24/21/0.256
33	penalty1 2500	10/26/22/0.708	10/26/22/0.739	10/26/22/0.768	10/26/22/0.744	10/25/21/0.701	10/26/22/0.768

Table 1: Continued.

No.	Problems Name/n	LMYCD1 Itr/NF/NG/Tcpu	hDY Itr/NF/NG/Tcpu	NHS Itr/NF/NG/Tcpu	JMJ Itr/NF/NG/Tcpu	MDL1 Itr/NF/NG/Tcpu	MDL2 Itr/NF/NG/Tcpu
34	penalty1 10000	10/30/24/12.546	11/30/25/12.838	11/30/25/12.769	11/30/25/12.932	11/29/24/12.570	11/30/25/12.808
35	power1 4	34/307/171/0.014	45/776/398/0.040	34/343/191/0.014	49/697/380/0.027	82/1971/655/0.073	34/405/201/0.021
36	quartc 50	14/40/40/0.005	15/40/41/0.004	15/40/41/0.004	15/40/41/0.008	16/40/42/0.007	15/40/41/0.004
37	quartc 300	16/51/48/0.011	18/51/50/0.013	18/51/50/0.013	18/51/50/0.012	21/119/80/0.029	18/51/50/0.012
38	quartc 450	18/52/50/0.019	19/53/52/0.022	19/53/52/0.018	19/53/52/0.019	33/164/110/0.050	19/53/52/0.020
39	sine 1200	22/181/105/0.053	NaN/NaN/NaN/NaN	21/296/164/0.062	25/363/195/0.065	31/491/255/0.087	17/171/95/0.033
40	sine 50000	19/179/88/1.320	NaN/NaN/NaN/NaN	29/552/277/4.156	19/171/95/1.344	26/358/174/2.532	42/937/438/6.981
41	sine 1000000	19/239/114/31.308	51/1197/560/178.174	754/23822/9561/3300.832	NaN/NaN/NaN/NaN	27/299/171/41.517	NaN/NaN/NaN/NaN
42	tridia 5	58/669/354/0.033	69/887/447/0.041	66/687/354/0.037	51/546/284/0.026	53/669/341/0.030	50/597/314/0.031
43	tridia 100	397/4862/2584/0.280	429/5999/3112/0.389	405/5843/3057/0.350	328/4146/2239/0.207	453/7170/3542/0.465	353/4357/2305/0.309
44	tridia 150	480/6130/3248/0.331	528/7801/4061/0.537	540/6307/3390/0.337	463/5441/2963/0.348	481/7390/3906/0.507	552/7319/3783/0.427
45	tridia 1150	1428/19793/9820/2.478	1595/21705/11472/2.759	NaN/NaN/NaN/NaN	1247/15946/8435/1.875	NaN/NaN/NaN/NaN	1894/26206/13791/2.954
46	tridia 5000	1944/29689/13110/12.150	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
47	bdexp 10000	10/23/32/0.176	12/24/35/0.210	12/24/35/0.203	12/24/35/0.207	12/23/34/0.200	12/24/35/0.197
48	bdexp 100000	10/23/32/1.414	12/24/35/1.668	12/24/35/1.611	12/24/35/1.674	12/23/34/1.633	12/24/35/1.587
49	bdexp 1000000	11/27/37/16.882	13/26/38/17.484	13/26/38/17.751	13/26/38/17.919	13/25/37/17.563	13/26/38/17.759
50	beale 2	41/349/195/0.032	52/658/349/0.046	49/578/289/0.050	44/773/372/0.053	NaN/NaN/NaN/NaN	28/365/189/0.020
51	biggsb1 10	45/486/261/0.028	78/1240/626/0.058	50/575/318/0.025	62/931/489/0.042	60/721/390/0.034	52/796/408/0.038
52	biggsb1 50	205/2667/1419/0.119	213/2723/1468/0.131	379/4896/2556/0.316	196/2531/1355/0.191	244/3237/1670/0.191	199/2645/1352/0.115
53	bv 4	36/353/203/0.037	57/781/407/0.044	45/682/361/0.049	42/509/264/0.039	58/1084/487/0.072	53/802/409/0.054
54	bv 500	100/1371/714/2.532	267/3700/1928/6.837	148/2007/1067/3.837	253/3310/1727/6.161	115/2026/970/3.662	337/4193/2179/7.834
55	bv 1000	22/272/151/1.317	41/617/315/2.804	24/269/138/1.273	40/561/291/2.672	17/232/118/1.067	40/616/312/2.830
56	diagonal1 10	28/245/123/0.025	28/239/151/0.023	29/275/159/0.013	27/366/184/0.017	42/783/389/0.039	29/400/206/0.023
57	diagonal2 50	49/426/238/0.023	58/769/401/0.039	52/518/280/0.033	49/343/216/0.018	62/775/434/0.065	52/648/345/0.048
58	diagonal2 200	114/1235/697/0.135	107/1653/889/0.128	113/1482/799/0.116	98/1141/610/0.090	111/1257/717/0.087	99/1311/741/0.083
59	diagonal3 10	31/215/118/0.014	38/440/214/0.020	32/374/188/0.021	36/560/292/0.025	28/207/117/0.012	32/306/158/0.017
60	diagonal3 50	66/588/329/0.050	NaN/NaN/NaN/NaN	69/1158/574/0.087	75/1027/529/0.085	NaN/NaN/NaN/NaN	73/739/438/0.046
61	exdenschnb 100000	15/71/44/0.360	3/4/4/0.021	15/75/45/0.356	16/138/79/0.616	21/205/107/0.918	15/108/66/0.491
62	exdenschnf 5000	21/115/70/0.083	36/496/249/0.274	30/407/214/0.211	24/179/96/0.091	38/633/310/0.388	28/308/174/0.186
63	exdenschnf 50000	23/148/88/0.860	36/496/249/2.362	31/408/216/2.017	25/180/98/0.932	40/698/329/3.275	29/309/176/1.572
64	exdenschnf 100000	23/148/87/1.574	37/496/251/4.593	31/408/216/3.835	26/213/116/1.992	41/700/321/6.019	29/309/172/2.878
65	genquartic 50000	21/123/79/0.411	48/664/334/1.991	23/305/165/0.954	29/442/228/1.419	18/140/74/0.413	28/372/202/1.187

Table 1: Continued.

No.	Problems Name/n	LMYCD1 Itr/NF/NG/Tcpu	hDY Itr/NF/NG/Tcpu	NHS Itr/NF/NG/Tcpu	JMJ Itr/NF/NG/Tcpu	MDL1 Itr/NF/NG/Tcpu	MDL2 Itr/NF/NG/Tcpu
66	genquartic 100000	29/327/172/2.023	67/858/482/5.113	24/280/134/1.684	24/215/117/1.324	22/268/135/1.546	27/277/151/1.605
67	ie 3	12/99/56/0.011	12/65/36/0.005	12/132/68/0.009	12/132/70/0.008	13/101/59/0.006	11/99/56/0.012
68	ie 10	12/99/59/0.010	12/65/41/0.008	12/132/66/0.019	12/132/63/0.016	13/103/54/0.012	11/99/47/0.010
69	ie 50	13/99/57/0.140	12/65/41/0.101	12/132/75/0.187	12/132/76/0.185	15/166/85/0.197	11/99/56/0.116
70	ie 150	13/99/60/1.252	13/65/43/0.870	12/132/74/1.550	12/132/76/1.509	15/166/81/1.874	12/99/55/1.145
71	ie 200	13/99/60/2.073	13/65/43/1.516	13/132/73/2.618	13/132/75/2.685	15/166/91/3.305	12/99/58/2.059
72	ie 800	13/99/56/31.879	13/65/45/23.171	13/132/74/40.877	13/132/77/41.227	16/166/88/51.090	12/99/58/32.305
73	ie 1000	14/99/57/51.186	13/65/45/35.255	13/132/74/65.817	13/132/80/65.400	16/166/89/81.337	13/99/56/50.446
74	ie 2000	14/99/57/203.034	13/65/44/144.064	13/132/71/251.499	14/165/93/320.752	16/166/86/319.077	13/99/58/202.787
75	lin 50	2/2/2/0.006	2/2/2/0.002	2/2/2/0.002	2/2/2/0.003	2/2/2/0.002	2/2/2/0.002
76	lin 100	2/2/2/0.004	2/2/2/0.004	2/2/2/0.004	2/2/2/0.006	2/2/2/0.006	2/2/2/0.007
77	lin 300	2/2/2/0.016	2/2/2/0.014	2/2/2/0.016	2/2/2/0.016	2/2/2/0.012	2/2/2/0.023
78	lin 500	2/2/2/0.022	2/2/2/0.022	2/2/2/0.022	2/2/2/0.026	2/2/2/0.025	2/2/2/0.036
79	lin 1000	9/5/9/8.799	3/5/3/3.406	11/5/11/10.082	11/5/11/10.456	11/6/11/10.495	11/5/11/10.403
80	lin 2000	14/129/45/1399.218	3/32/16/340.750	12/33/25/497.186	12/33/25/501.923	12/34/25/510.829	12/33/25/508.992
81	lin 3000	7/9/7/645.228	10/9/10/828.686	10/9/10/831.544	10/9/10/825.478	10/10/10/850.265	10/9/10/819.908
82	pen1 50	195/2534/1365/0.222	109/1676/850/0.123	202/3459/1695/0.253	374/3067/1653/0.245	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
83	raydan1 100	90/1064/555/0.055	91/1122/661/0.058	99/1283/709/0.069	86/1168/616/0.065	98/1466/700/0.088	90/733/439/0.039
84	raydan1 550	211/2880/1399/0.220	230/3337/1658/0.263	221/3130/1638/0.228	213/2560/1313/0.208	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
85	raydan1 850	353/5182/2719/0.479	427/6444/3162/0.617	426/6577/3383/0.540	NaN/NaN/NaN/NaN	331/5592/2799/0.492	292/4371/2299/0.443
86	raydan1 1550	406/5155/2639/0.608	539/8573/4322/1.068	515/7443/3835/0.928	411/5995/3045/0.730	NaN/NaN/NaN/NaN	561/8552/4439/1.044
87	raydan2 8000	16/258/126/0.216	17/269/151/0.191	15/226/127/0.186	16/289/157/0.250	15/225/134/0.192	16/258/144/0.215
88	raydan2 10000	17/324/158/0.343	NaN/NaN/NaN/NaN	14/194/113/0.225	14/194/113/0.216	15/225/129/0.237	14/194/113/0.218
89	trid 40	40/438/242/0.054	89/1164/627/0.155	41/566/308/0.062	39/335/184/0.031	51/851/431/0.070	42/470/275/0.043
90	trid 150	85/1024/521/0.250	97/1372/703/0.376	91/1003/530/0.288	79/757/403/0.184	83/814/442/0.206	86/1100/599/0.285
91	trid 200	84/1033/553/0.362	92/1216/640/0.359	97/1480/767/0.466	88/1044/533/0.305	94/1791/855/0.509	93/1173/642/0.390
92	trid 400	86/971/534/1.345	112/1479/771/1.978	100/1264/677/1.717	89/885/499/1.248	92/1382/717/1.844	101/1207/639/1.629
93	trid 900	75/947/459/3.721	95/1377/712/5.347	86/1096/564/4.310	97/1181/620/4.736	164/3703/1426/13.257	76/994/546/3.986
94	trid 1500	86/807/438/7.548	88/1168/614/10.417	93/1347/727/12.063	82/982/512/8.789	101/1462/737/12.781	77/771/429/7.194
95	trid 2000	75/828/446/12.334	94/1344/668/19.059	91/842/461/12.558	84/950/530/14.230	100/1365/711/19.965	84/811/419/11.853
96	trid 2500	89/973/529/21.609	106/1285/681/28.453	97/1151/599/25.250	84/848/466/19.354	130/2409/1039/49.331	111/1719/891/39.398
97	trid 5000	83/1099/555/90.581	107/1555/791/124.931	95/1227/658/100.214	84/1023/569/84.970	106/1540/776/123.150	85/1038/572/85.971
98	vardim 2	10/11/13/0.006	12/11/15/0.002	12/11/15/0.001	12/11/15/0.001	12/10/14/0.001	12/11/15/0.001
99	vardim 8	9/30/19/0.002	11/30/21/0.002	11/30/21/0.002	11/30/21/0.004	11/29/20/0.004	11/30/21/0.003
100	watson 2	14/75/44/0.012	15/136/79/0.014	18/175/107/0.025	14/109/64/0.016	17/144/83/0.012	19/146/82/0.013

Table 2: Numerical results for the second group methods.

No.	Problems Name/n	LMYCD2 Itr/NF/NG/Tcpu	NPRP Itr/NF/NG/Tcpu	JMJ Itr/NF/NG/Tcpu	MDL3 Itr/NF/NG/Tcpu	MDL4 Itr/NF/NG/Tcpu
1	dixmaana 3000	11/99/60/0.320	14/133/76/0.335	15/232/127/0.717	14/165/100/0.424	15/166/88/0.480
2	dixmaanb 3000	8/8/11/0.037	7/8/10/0.031	7/8/10/0.033	7/8/10/0.047	7/8/10/0.044
3	dixmaanc 3000	8/10/12/0.046	8/10/12/0.043	8/10/12/0.042	8/10/12/0.044	8/10/12/0.042
4	dixmaand 3000	10/12/15/0.060	9/12/14/0.062	9/12/14/0.096	9/12/14/0.070	9/12/14/0.061
5	dixon3dq 5	68/1458/731/0.074	72/1681/797/0.076	127/3187/1608/0.132	NaN/NaN/NaN/NaN	148/3488/1696/0.143
6	dixon3dq 20	224/4410/2262/0.225	219/5179/2654/0.230	318/7695/3847/0.377	265/6633/3359/0.281	295/7536/3776/0.424
7	dixon3dq 40	245/4791/2482/0.265	846/20327/10219/0.936	510/12183/6010/0.488	559/13660/6924/0.597	494/12236/6183/0.565
8	dqrtic 5	8/24/26/0.003	7/20/21/0.002	7/20/21/0.002	7/24/25/0.001	7/24/25/0.002
9	dqrtic 50	13/68/51/0.006	15/143/89/0.010	22/317/177/0.036	16/143/98/0.010	16/142/86/0.010
10	dqrtic 100	15/74/57/0.008	21/305/173/0.034	20/312/174/0.039	23/408/215/0.039	22/303/168/0.043
11	dqdrtic 10000	64/1478/694/0.695	99/2453/1192/1.243	165/4184/2065/2.130	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
12	dqdrtic 20000	108/2547/1218/2.358	95/2065/1022/1.882	270/6666/3372/6.113	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
13	dqdrtic 30000	69/1587/794/1.940	94/2074/1000/2.469	244/5875/2919/7.362	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
14	dqdrtic 100000	107/2563/1232/9.380	81/1746/863/6.361	154/3576/1787/12.649	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
15	dqdrtic 500000	87/1837/926/36.856	103/2266/1173/46.490	218/5514/2774/112.655	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
16	dqdrtic 800000	85/1889/856/59.337	64/1507/716/49.550	186/4379/2200/141.928	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
17	edensch 5	45/1033/536/0.070	51/1190/584/0.070	41/925/463/0.051	38/833/396/0.038	38/863/435/0.044
18	edensch 50	40/782/380/0.069	38/833/401/0.053	47/1034/556/0.064	43/931/467/0.058	53/1223/608/0.080
19	edensch 1000	43/849/449/0.436	44/935/494/0.470	52/1105/575/0.583	NaN/NaN/NaN/NaN	50/1180/570/0.592
20	fletcbv3 5	8/220/134/0.027	10/221/126/0.009	11/285/147/0.012	9/225/136/0.009	10/283/142/0.014
21	fletchcr 5	66/1439/715/0.067	73/1728/791/0.064	76/1875/916/0.072	NaN/NaN/NaN/NaN	152/4088/2046/0.184
22	fletchcr 50	68/1359/738/0.064	NaN/NaN/NaN/NaN	122/3233/1547/0.186	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
23	fletchcr 100	78/1762/920/0.096	72/1500/716/0.094	56/1447/704/0.064	102/2460/1229/0.109	NaN/NaN/NaN/NaN
24	himmelbg 50	5/21/25/0.010	6/23/28/0.001	6/23/28/0.002	6/23/28/0.002	6/23/28/0.002
25	himmelbg 2000	6/26/31/0.013	6/23/28/0.010	6/23/28/0.009	6/23/28/0.008	6/23/28/0.010
26	himmelbg 15000	6/26/31/0.078	7/28/34/0.085	7/28/34/0.090	7/28/34/0.095	7/28/34/0.092
27	himmelbg 20000	6/26/31/0.098	7/28/34/0.112	7/28/34/0.101	7/28/34/0.094	7/28/34/0.096
28	himmelbg 30000	6/26/31/0.138	7/28/34/0.169	7/28/34/0.171	7/28/34/0.166	7/28/34/0.159
29	himmelbg 500000	6/26/31/2.252	7/28/34/2.493	7/28/34/2.519	7/28/34/2.544	7/28/34/2.572
30	liarwhd 5	35/733/351/0.037	61/1368/708/0.070	84/1961/971/0.110	98/2487/1264/0.134	69/1741/877/0.103
31	liarwhd 40	51/1121/552/0.051	54/1124/578/0.051	96/2329/1152/0.126	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
32	liarwhd 150	259/7393/3614/0.455	138/3644/1756/0.245	414/10287/5102/0.579	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
33	liarwhd 250	186/5054/2460/0.383	NaN/NaN/NaN/NaN	254/6022/3047/0.404	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
34	liarwhd 550	201/4502/2328/0.447	160/4340/2127/0.400	331/8852/4412/0.870	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
35	liarwhd 850	240/5864/2800/0.562	NaN/NaN/NaN/NaN	449/12111/6085/1.253	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
36	liarwhd 2300	873/24662/12176/3.795	NaN/NaN/NaN/NaN	882/25260/12572/4.251	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
37	mccormak 2	20/420/201/0.019	25/517/257/0.020	19/386/181/0.015	25/545/275/0.021	22/482/215/0.018
38	nonscomp 5	724/17084/8691/0.916	NaN/NaN/NaN/NaN	456/11568/5816/0.598	285/7081/3493/0.324	NaN/NaN/NaN/NaN

Table 2: Continued.

No.	Problems Name/n	LMYCD2 Itr/NF/NG/Tcpu	NPRP Itr/NF/NG/Tcpu	JMJ Itr/NF/NG/Tcpu	MDL3 Itr/NF/NG/Tcpu	MDL4 Itr/NF/NG/Tcpu
39	penalty1 50	1233/39330/14712/4.998	1828/58058/23959/7.366	1828/58058/23940/6.867	1828/58058/24001/6.467	1828/58058/24020/7.298
40	penalty1 100	1700/54044/21008/10.701	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
41	penalty1 2500	10/159/87/3.184	651/20314/10086/398.089	651/20314/10086/386.162	651/20314/10086/395.460	651/20314/10086/388.435
42	power1 4	46/860/433/0.036	60/1301/650/0.049	69/1588/816/0.076	79/1846/932/0.079	48/1089/553/0.050
43	power1 10	106/2007/1047/0.088	117/2787/1344/0.117	131/3061/1526/0.132	131/3162/1558/0.123	169/4132/2048/0.153
44	quartc 50	13/68/51/0.008	15/143/89/0.010	22/317/177/0.039	16/143/98/0.022	16/142/86/0.017
45	quartc 100	15/74/57/0.008	21/305/173/0.041	20/312/174/0.050	23/408/215/0.033	22/303/168/0.026
46	quartc 350	15/83/61/0.019	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
47	quartc 450	29/427/209/0.093	34/482/259/0.152	27/299/159/0.070	34/570/305/0.138	39/672/358/0.151
48	tridia 2	43/932/474/0.057	66/1580/790/0.062	67/1560/789/0.060	60/1521/747/0.081	55/1447/696/0.062
49	tridia 10	94/1741/903/0.110	107/2639/1302/0.159	125/2848/1423/0.120	NaN/NaN/NaN/NaN	147/3800/1940/0.155
50	band 3	44/1112/547/0.056	15/100/55/0.006	68/1839/916/0.113	103/2955/1482/0.150	95/2699/1310/0.156
51	band 200	52/1080/540/1.946	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN	NaN/NaN/NaN/NaN
52	bdexp 40	6/18/23/0.006	6/18/23/0.001	6/18/23/0.001	6/18/23/0.001	6/18/23/0.002
53	bdexp 100	6/19/24/0.002	7/22/28/0.002	7/22/28/0.003	7/22/28/0.003	7/22/28/0.002
54	bdexp 80000	7/23/29/1.047	8/26/33/1.162	8/26/33/1.184	8/26/33/1.211	8/26/33/1.117
55	bdexp 100000	7/23/29/1.294	8/26/33/1.420	8/26/33/1.461	8/26/33/1.473	8/26/33/1.489
56	bdexp 150000	7/23/29/1.898	8/26/33/2.114	8/26/33/2.196	8/26/33/2.130	8/26/33/2.137
57	bdexp 500000	7/23/29/6.325	8/26/33/7.023	8/26/33/7.239	8/26/33/7.340	8/26/33/7.407
58	biggsb1 5	38/735/382/0.031	78/1795/911/0.068	85/1980/1010/0.087	42/1007/497/0.039	80/1991/1004/0.078
59	biggsb1 35	231/4623/2331/0.283	191/4598/2337/0.184	353/8190/4155/0.385	329/8469/4176/0.355	280/7123/3669/0.347
60	bv 4	57/1140/574/0.065	66/1622/769/0.085	77/1726/899/0.096	75/1631/853/0.090	91/2076/1077/0.120
61	bv 800	34/704/350/2.273	52/1200/595/3.867	51/1220/644/4.078	50/1275/646/4.103	74/1798/900/5.769
62	bv 1000	19/370/184/1.624	16/356/179/1.579	45/1061/538/4.735	50/1275/665/5.684	42/995/492/4.393
63	bv 2000	3/34/14/0.457	10/222/108/3.075	6/126/62/1.866	5/64/34/0.908	8/188/89/2.599
64	diagonal1 5	28/621/314/0.026	261/8131/4025/0.540	261/8131/4004/0.338	261/8131/4007/0.390	261/8131/4011/0.479
65	diagonal3 10	43/833/424/0.036	180/5424/2703/0.248	180/5424/2753/0.235	182/5487/2701/0.291	58/1349/666/0.095
66	exdenschnf 60	27/494/251/0.027	NaN/NaN/NaN/NaN	22/330/172/0.015	42/921/430/0.038	595/18392/9084/0.905
67	exdenschnf 1000	28/528/262/0.057	NaN/NaN/NaN/NaN	25/426/213/0.044	43/925/458/0.095	599/18519/9115/2.072
68	exdenschnf 6000	28/530/260/0.342	NaN/NaN/NaN/NaN	24/365/183/0.223	390/11922/5923/7.245	601/18552/9140/11.358
69	exdenschnf 10000	28/532/273/0.522	NaN/NaN/NaN/NaN	25/397/210/0.452	26/433/226/0.453	599/18495/9176/19.247
70	exdenschnf 100000	32/631/324/5.064	NaN/NaN/NaN/NaN	27/463/244/3.952	27/441/231/3.796	601/18545/9144/144.960
71	gauss 3	18/404/202/0.037	12/232/110/0.024	27/589/279/0.060	28/691/353/0.049	NaN/NaN/NaN/NaN
72	genquartic 50	27/563/284/0.033	NaN/NaN/NaN/NaN	30/561/270/0.023	35/771/375/0.031	27/523/242/0.021
73	genquartic 100	23/337/178/0.015	NaN/NaN/NaN/NaN	44/972/479/0.041	39/839/447/0.036	24/459/245/0.020
74	lin 5	2/2/2/0.002	2/2/2/0.001	2/2/2/0.002	2/2/2/0.002	2/2/2/0.002
75	lin 50	2/2/2/0.002	2/2/2/0.001	2/2/2/0.001	2/2/2/0.002	2/2/2/0.002
76	lin 100	2/2/2/0.005	2/2/2/0.004	2/2/2/0.003	2/2/2/0.004	2/2/2/0.004
77	lin 200	2/2/2/0.007	2/2/2/0.007	2/2/2/0.007	2/2/2/0.008	2/2/2/0.007
78	lin 500	2/2/2/0.021	2/2/2/0.023	2/2/2/0.022	2/2/2/0.021	2/2/2/0.021
79	lin 600	6/65/27/3.352	7/97/20/4.558	7/97/20/4.527	7/97/20/4.534	7/97/20/4.511
80	lin 750	2/4/2/0.617	2/4/2/0.583	2/4/2/0.552	2/4/2/0.603	2/4/2/0.591
81	pen2 10000	1/1/2/1.408	1/1/2/0.584	1/1/2/0.550	1/1/2/0.586	1/1/2/0.556
82	pen2 15000	1/1/2/3.110	1/1/2/3.683	1/1/2/2.083	1/1/2/1.249	1/1/2/1.312
83	pen2 20000	1/1/2/4.952	1/1/2/9.870	1/1/2/4.941	1/1/2/8.111	1/1/2/8.627

Table 2: Continued.

No.	Problems Name/n	LMYCD2 Itr/NF/NG/Tcpu	NPRP Itr/NF/NG/Tcpu	JMJ Itr/NF/NG/Tcpu	MDL3 Itr/NF/NG/Tcpu	MDL4 Itr/NF/NG/Tcpu
84	raydan2 2	28/669/328/2.864	130/3942/1958/0.311	130/3942/1958/0.162	130/3942/1958/0.301	130/3942/1958/0.171
85	raydan2 50	28/669/328/0.032	130/3942/1958/0.167	130/3942/1958/0.163	130/3942/1958/0.165	130/3942/1958/0.165
86	raydan2 500	29/669/329/0.048	130/3942/1958/0.366	130/3942/1958/0.360	130/3942/1958/0.377	130/3942/1958/0.311
87	raydan2 1000	29/669/329/0.075	131/3942/1959/0.509	131/3942/1959/0.498	131/3942/1959/0.605	131/3942/1959/0.528
88	raydan2 5000	29/669/329/0.479	131/3942/1959/2.392	131/3942/1959/2.376	131/3942/1959/2.521	131/3942/1959/2.373
89	raydan2 8000	29/669/329/0.750	131/3942/1959/3.662	131/3942/1959/3.759	131/3942/1959/3.784	131/3942/1959/3.751
90	raydan2 10000	29/669/329/0.683	131/3942/1959/4.793	131/3942/1959/4.801	131/3942/1959/4.822	131/3942/1959/4.823
91	raydan2 20000	29/669/330/1.181	131/3942/1959/8.871	131/3942/1959/8.928	131/3942/1959/8.911	131/3942/1959/8.855
92	raydan2 100000	30/669/336/5.506	131/3942/1959/ 40.716	131/3942/1959/ 40.606	131/3942/1959/ 40.753	131/3942/1959/ 40.791
93	raydan2 1000000	30/669/319/54.549	131/3942/1966/ 405.365	131/3942/1966/ 409.842	131/3942/1966/ 414.512	131/3942/1966/ 414.930
94	vardim 2	6/45/23/0.016	NaN/NaN/NaN/ NaN	NaN/NaN/NaN/ NaN	NaN/NaN/NaN/ NaN	NaN/NaN/NaN/ NaN
95	vardim 8	7/96/49/0.010	1685/53792/6599/ 2.492	1685/53792/6600/ 2.695	1685/53792/6600/ 2.551	1685/53792/6600/ 2.507
96	watson 4	107/2152/1072/0.250	115/2723/1368/0.409	135/3144/1554/0.375	NaN/NaN/NaN/ NaN	122/2824/1397/0.357

$$\frac{\|d_k\|^2}{(g_k^T d_k)^2} \le \frac{\|d_{k-1}\|^2}{(g_{k-1}^T d_{k-1})^2} + \frac{1}{\|g_k\|^2} \le \frac{\|d_{k-2}\|^2}{(g_{k-2}^T d_{k-2})^2} + \frac{1}{\|g_{k-1}\|^2} + \frac{1}{\|g_k\|^2} \le \cdots \le \sum_{i=1}^{k} \frac{1}{\|g_i\|^2} \le \frac{k}{\gamma}. \tag{26}$$

Thus, $(g_k^T d_k)^2/\|d_k\|^2 \ge \gamma/k$, which further implies that $\sum_{k=1}^{\infty} (g_k^T d_k)^2/\|d_k\|^2 = \infty$. It contradicts the Zoutendijk condition, and it follows that $\lim\inf_{k\longrightarrow\infty}\|g_k\| = 0$. □

Next, based on Lemma 2 and the Zoutendijk condition, we also obtain the global convergence for the LMYCD2 method.

Theorem 2. *Suppose that Assumptions (H1)–(H2) hold. Let the sequence $\{x_k\}$ be generated by the LMYCD2 method, then the LMYCD2 method is globally convergent by the way of* $\lim\inf_{k\longrightarrow\infty}\|g_k\| = 0$.

Proof. Suppose by contradiction that assertion is not true, then there exists a constant $\gamma > 0$ such that $\|g_k\| \ge \gamma, \forall k \ge 1$. Squaring both sides of d_k, one gets $\|d_k\|^2 = \|g_k\|^2 - 2\beta_k^{LMYCD2} g_k^T d_{k-1} + (\beta_k^{LMYCD2})^2 \|d_{k-1}\|^2$. From Lemma 2 and the strong Wolfe line search, we have

$$-2\beta_k^{LMYCD2} g_k^T d_{k-1} \le -2\beta_k^{LMYCD2} |g_k^T d_{k-1}| \le 2\beta_k^{FR} |g_k^T d_{k-1}| \le 2\frac{\|g_k\|^2}{\|g_{k-1}\|^2}\left(\sigma|g_{k-1}^T d_{k-1}|\right) \le \frac{2\sigma\|g_k\|^2}{1-\sigma}. \tag{27}$$

Thus, we obtain $\|d_k\|^2 \le ((1+\sigma)/(1-\sigma))\|g_k\|^2 + (\|g_k\|^4/\|g_{k-1}\|^4)\|d_{k-1}\|^2$. Again, dividing this inequality by $\|g_k\|^4$, we have $(\|d_k\|^2/\|g_k\|^4) \le ((1+\sigma)/(1-\sigma))(1/\|g_k\|^2) + \|d_{k-1}\|^2/\|g_{k-1}\|^4$. Taking $\|d_1\|^2 = \|g_1\|^2$ and $\|g_k\| \ge \gamma$, utilizing the abovementioned formula, one has

$$\frac{\|d_k\|^2}{\|g_k\|^4} \le \left(\frac{1+\sigma}{1-\sigma}\right)\sum_{i=1}^{k}\frac{1}{\|g_i\|^2} \le \frac{1+\sigma}{1-\sigma}\frac{k}{\gamma^2}. \tag{28}$$

This implies that $\|g_k\|^4/\|d_k\|^2 \ge ((1-\sigma)\gamma^2/(1+\sigma))(1/k)$; thus, $\sum_{k=1}^{\infty}\|g_k\|^4/\|g_k\|^2 = \infty$, which also contradicts the Zoutendijk condition. Therefore, we can conclude that the LMYCD2 method is globally convergent by the way of $\lim\inf_{k\longrightarrow\infty}\|g_k\| = 0$. □

4. Numerical Results

In this section, we study the computational efficiency of the LMYCD1 and LMYCD2 methods, in contrast to the NHS [11], NPRP [11], JMJ [12] methods, the hybrid Dai-Yuan (hDY) method with $\beta_k = \max\{0, \min\{\beta_k^{HS}, \beta_k^{DY}\}\}$ [10], and a family of modified Dai-Liao CGMs [17] including MDL1, MDL2, MDL3, and MDL4. The 171 test problems in all are the unconstrained problems with dimensions ranging from 2 to 1500000, in which some are obtained from the CUTEr library [18] including test examples 1–46 in Table 1 and 1–49 in Table2and others from [19,20]. Moreover, the step length α_k is yielded by using the strong Wolfe line search. All the considered methods were coded in Matlab R2016a and ran on a PC with 3.6 GHz CPU processor and 8 GB RAM and Windows 10 operating system. We terminated the iteration when one of the following conditions was satisfied: (i) $\|g_k\| \le 10^{-6}$ and (ii) the total number of iterations Itr > 2000. If condition (ii) occurs, the method is deemed to fail for solving the corresponding test problem and denote it by "NaN."

Table 1 lists the detailed numerical results of the first group methods, which contain the LMYCD1, hDY, NHS, JMJ, MDL1, and MDL2 methods. Here, parameters $\sigma = 0.25$

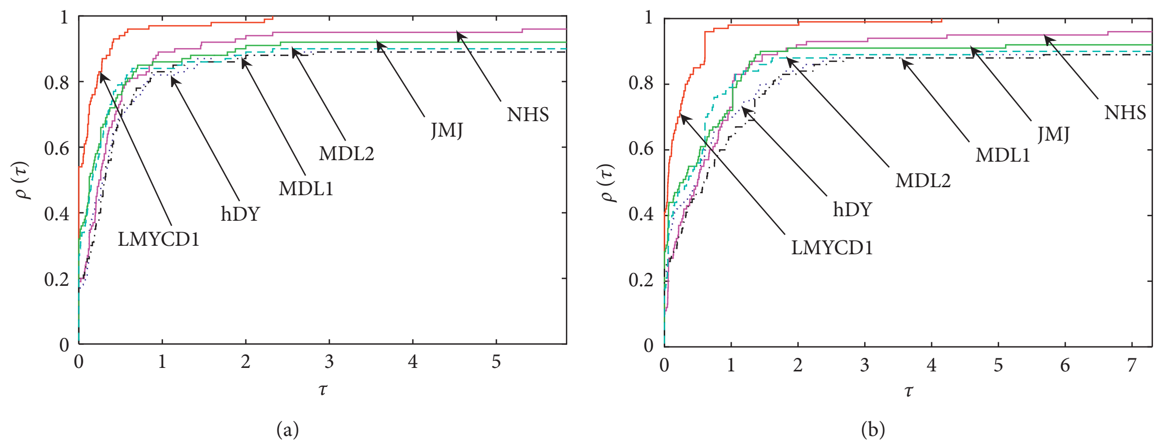

Figure 1: Performance profiles of Itr (a) and NF (b) for the first group methods.

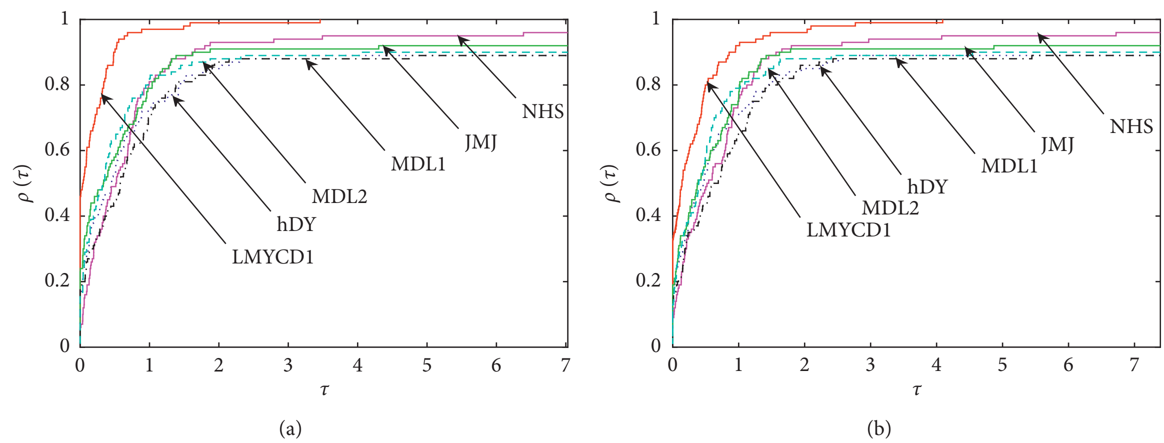

Figure 2: Performance profiles of NG (a) and Tcpu (b) for the first group methods.

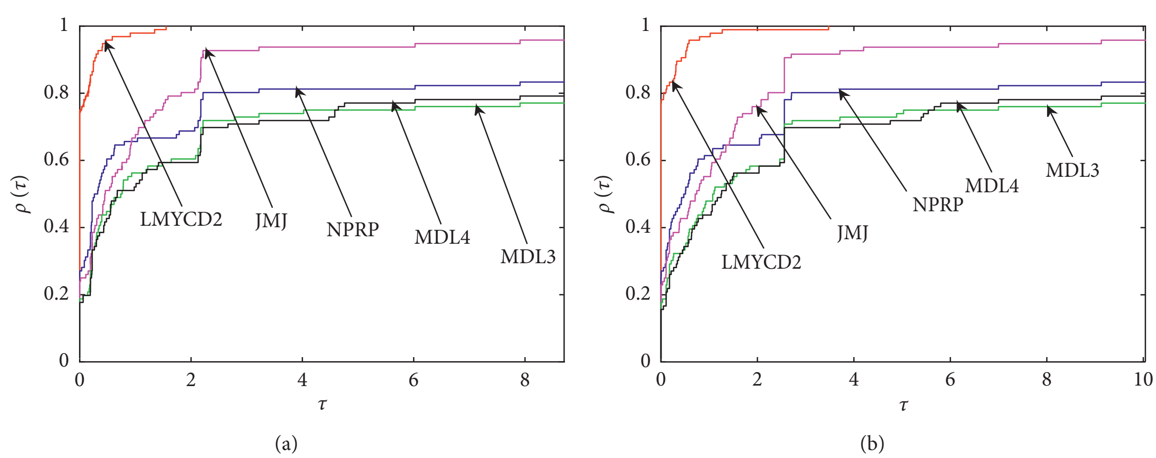

Figure 3: Performance profiles of Itr (a) and NF (b) for the second group methods.

and $\delta = 0.1$. Notice that "n/Itr/NF/NG/Tcpu" denote the dimension of the test problems, the total number of iterations, function evaluations, gradient evaluations, and the CPU time in seconds, respectively. For the second group methods consisting of the LMYCD2, NPRP, JMJ, MDL3, and MDL4 methods, the corresponding numerical results are reported in Table 2. Here, parameters $\sigma = 0.1$ and $\delta = 0.001$.

In addition, we make use of the performance profiles of Dolan and Moré in [21] to compare the performance of the

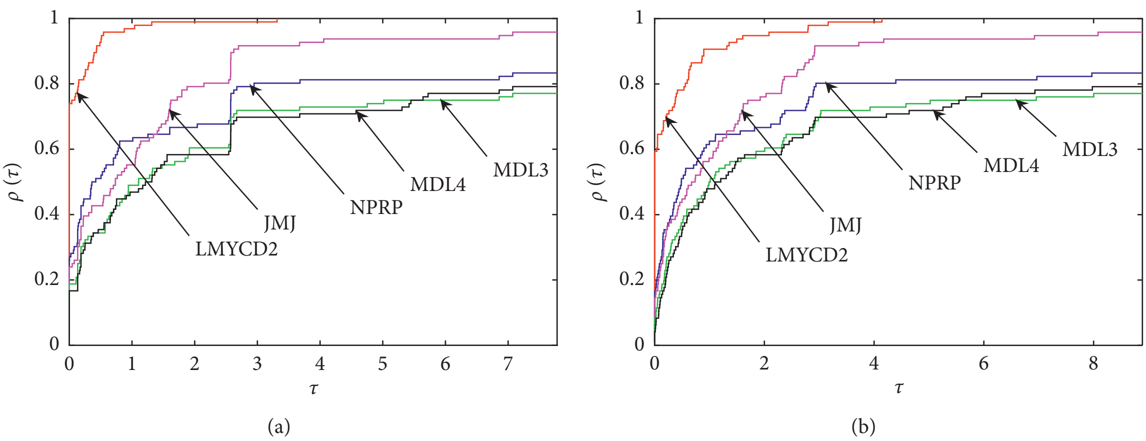

Figure 4: Performance profiles of NG (a) and Tcpu (b) for the second group methods.

tested methods listed above, and readers refer to this literature for details about the introduction of the performance profiles. It is worth noting that the left side of each performance profile figure indicates the percentage of the test problems, which is the fastest among these methods, whereas the right side gives the percentage of the test problems that are successfully solved by each method. The top curve means that the corresponding method implements best in contrast to other methods.

Figures 1 and 2 illustrate the performance profiles for the LMYCD1, hDY, NHS, JMJ, MDL1, and MDL2 methods, by the total number of iterations, function evaluations, gradient evaluations, and the CPU time (s), respectively. In Figures 3 and 4, the performance profiles of the LMYCD2, NPRP, JMJ, MDL3, and MDL4 methods are described.

Observing from all Figures 1–4, the LMYCD1 and LMYCD2 methods, are competitive and both of them outperform the tested methods in each group, with respect to the characteristics Itr, NF, NG, and Tcpu, respectively. In addition, the two proposed methods in this paper ultimately solve 100% of the respective test problems successfully. All numerical results show that the efficiency of the LMYCD1 and LMYCD2 methods is encouraging.

5. Conclusion

In this paper, we construct two new formulas for setting parameter β_k by using substantially the information of the JMJ and CD formulas as well as the second inequality in the strong Wolfe line search (3). Under the usual assumptions, the presented methods are proved to be sufficient descent and globally convergent. Elementary numerical experiments demonstrate that the two proposed methods perform effectively.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

The study was carried out in collaboration with all authors. All authors read and approved the final manuscript.

Acknowledgments

The authors were supported financially by the Natural Science Foundation of Guangxi Province (2018GXNSFAA281099), Middle-aged and Young Teachers' Basic Ability Promotion Project of Guangxi (2017KY0537 and 2018KY0700), and Open Project of Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing (2017CSOBDP0105).

References

[1] B. Balaram, M. D. Narayanan, and P. K. Rajendrakumar, "Optimal design of multi-parametric nonlinear systems using a parametric continuation based genetic algorithm approach," *Nonlinear Dynamics*, vol. 67, no. 4, pp. 2759–2777, 2012.

[2] L. M. Zhang, H. T. Gao, and Z. Q. Chen, "Multi-objective global optimal parafoil homing trajectory optimization via Gauss pseudospectral method," *Nonlinear Dynamics*, vol. 72, no. 1-2, pp. 1–8, 2013.

[3] M. R. Hestenes and E. Stiefel, "Methods of conjugate gradients for solving linear systems," *Journal of Research of the National Bureau of Standards*, vol. 49, no. 6, pp. 409–436, 1952.

[4] R. Fletcher and C. M. Reeves, "Function minimization by conjugate gradients," *The Computer Journal*, vol. 7, no. 6, pp. 163–168, 1964.

[5] B. T. Polyak, "The conjugate gradient method in extremal problems," *USSR Computational Mathematics and Mathematical Physics*, vol. 9, no. 4, pp. 94–112, 1969.

[6] R. Fletcher, "Practical methods of optimization," *I: Unconstrained Optimization*, Wiley, New York, NY, USA, 1987.

[7] Y. Liu and C. Storey, "Efficient generalized conjugate gradient algorithms, part 1: Theory," *Journal of Optimization Theory and Applications*, vol. 69, no. 1, pp. 129–137, 1991.

[8] Y. H. Dai and Y. Yuan, "A nonlinear conjugate gradient method with a strong global convergence property," *SIAM Journal on Optimization*, vol. 10, no. 1, pp. 177–182, 1999.

[9] C. M. Tang, Z. X. Wei, and G. Y. Li, "A new version of the Liu-Storey conjugate gradient method," *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 302–313, 2007.

[10] Y. H. Dai and Y. X. Yuan, "An efficient hybrid conjugate method for unconstrained optimization," *Annals of Operations Research*, vol. 103, no. 1–4, pp. 33–47, 2001.

[11] L. Zhang, "An improved Wei-Yao-Liu nonlinear conjugate gradient method for optimization computation," *Applied Mathematics and Computation*, vol. 215, no. 6, pp. 2269–2274, 2009.

[12] X. Z. Jiang, G. D. Ma, and J. B. Jian, "A new global convergent conjugate gradient method with Wolfe line search," *Chinese Journal of Engineering Mathematics*, vol. 28, no. 6, pp. 779–786, 2011.

[13] X. Z. Jiang and J. B. Jian, "Two modified conjugate gradient methods with disturbance factors for unconstrained optimization," *Nonlinear Dynamics*, vol. 77, no. 1-2, pp. 387–397, 2014.

[14] X. Z. Jiang and J. B. Jian, "Improved Fletcher-Reeves and Dai-Yuan conjugate gradient methods with the strong Wolfe line search," *Journal of Computational and Applied Mathematics*, vol. 348, pp. 525–534, 2019.

[15] N. Andrei, "Accelerated adaptive Perry conjugate gradient algorithms based on the self-scaling memoryless BFGS update," *Journal of Computational and Applied Mathematics*, vol. 325, pp. 149–164, 2017.

[16] G. Zoutendijk, "Nonlinear programming, computational methods," *Integer Nonlinear Programming*, vol. 143, no. 1, pp. 37–86, 1970.

[17] S. Babaie-Kafaki and R. Ghanbari, "Two optimal Dai-Liao conjugate gradient methods," *Optimization*, vol. 64, no. 11, pp. 2277–2287, 2015.

[18] I. Bongartz, A. R. Conn, N. Gould, and P. L. Toint, "CUTE: constrained and unconstrained testing environment," *ACM Transactions on Mathematical Software*, vol. 21, no. 1, pp. 123–160, 1995.

[19] J. J. Moré, B. S. Garbow, and K. E. Hillstrom, "Testing unconstrained optimization software," *ACM Transactions on Mathematical Software (TOMS)*, vol. 7, no. 1, pp. 17–41, 1981.

[20] N. Andrei, "An unconstrained optimization test functions collection," *Advanced Modeling and Optimization*, vol. 10, no. 1, pp. 147–161, 2008.

[21] E. D. Dolan and J. J. Moré, "Benchmarking optimization software with performance profiles," *Mathematical Programming*, vol. 91, no. 2, pp. 201–213, 2002.