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Research Article

Positive Solutions for a System of Hadamard-Type Fractional Differential Equations with Semipositone Nonlinearities

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In this paper, we use the fixed-point index and nonnegative matrices to study the existence of positive solutions for a system of Hadamard-type fractional differential equations with semipositone nonlinearities.

1. Introduction

In this paper, we study the existence of positive solutions for the following system of Hadamard-type fractional boundary value problems:

$$\begin{cases} D^{\alpha}D^{\alpha}\xi(t) = f_1(t, \xi(t), \delta\xi(t), -D^{\alpha}\xi(t), \eta(t), \delta\eta(t), -D^{\alpha}\eta(t)), & t \in (1, e), \\ D^{\alpha}D^{\alpha}\eta(t) = f_2(t, \xi(t), \delta\xi(t), -D^{\alpha}\xi(t), \eta(t), \delta\eta(t), -D^{\alpha}\eta(t)), & t \in (1, e), \\ \xi(1) = \delta\xi(1) = \delta\xi(e) = 0, D^{\alpha}\xi(1) = \delta D^{\alpha}\xi(1) = \delta D^{\alpha}\xi(e) = 0, \\ \eta(1) = \delta\eta(1) = \delta\eta(e) = 0, D^{\alpha}\eta(1) = \delta D^{\alpha}\eta(1) = \delta D^{\alpha}\eta(e) = 0, \end{cases} \tag{1}$$

where $\alpha \in (2, 3]$, D^{α} is the Hadamard fractional derivative of order α, $\delta = t(\mathrm{d}/\mathrm{d}t)$ (i.e., if u is ξ or η, then $\delta u(t) = t(\mathrm{d}/\mathrm{d}t)u(t)$; $\delta u(1) = \lim_{t\longrightarrow 1^+} t(\mathrm{d}/\mathrm{d}t)u(t) = (t(\mathrm{d}/\mathrm{d}t)\ u(t)|_{t=1})$ etc.), and the nonlinearities $f_i (i = 1, 2)$ satisfy the semipositone condition:

(H0) $f_i \in C([1, e] \times \mathbb{R}_+^6, \mathbb{R})$, and there exists $M > 0$ such that

$$f_i(t, x_1, x_2, x_3, y_1, y_2, y_3) \geq -M, \quad \text{for } (t, x_1, x_2, x_3, y_1, y_2, y_3) \in [1, e] \times \mathbb{R}_+^6, \tag{2}$$

where $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R} = (-\infty, +\infty)$, $i = 1, 2$.

Fractional problems arise in many applications in aerodynamics, signal and image processing, biophysics, blood flow phenomena, etc. (see, for example, [1–37] and the references therein). In [1], the authors used the Krasnoselskii–Zabreiko fixed-point theorem to study the existence of positive solutions for Riemann–Liouville-type fractional boundary value problems:

$$\begin{cases} D_{0+}^{\alpha}D_{0+}^{\alpha}u = f(t,u,u',-D_{0+}^{\alpha}u), \quad t\in[0,1],\\ u(0)=u'(0)=u'(1)=D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha+1}u(0)=D_{0+}^{\alpha+1}u(1)=0, \end{cases} \tag{3}$$

where f has a semipositone nonlinearity, i.e., it is bounded below and can be sign-changing, and in [2], the authors used the Guo–Krasnoselskii's fixed-point theorem to investigate the existence of positive solutions for fractional multipoint boundary value problems:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\\ D_{0+}^{p}u(1)=\sum_{i=1}^{m}a_iD_{0+}^{q}u(\xi_i), \end{cases} \tag{4}$$

where the nonlinear f may have singularities on the time and the space variables.

Coupled systems of fractional-order differential equations have also been investigated by many authors (see [7–22, 25–36] and the references therein). In [7], the authors used coincidence degree theory to establish an existence result for a coupled system of nonlinear fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t)=f\left(t,v(t),D_{0+}^{\beta-2}v(t),D_{0+}^{\beta-1}v(t)\right), \quad 0<t<1,\\ D_{0+}^{\beta}v(t)=g\left(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t)\right), \quad 0<t<1, \end{cases} \tag{5}$$

with the multipoint boundary conditions:

$$\begin{cases} u(0)=0, \ D_{0+1}^{\alpha-1}u(0)=D_{0+1}^{\alpha-1}u(\eta), \ u(1)=\sum_{i=1}^{m_1}\alpha_iu(\eta_i),\\ v(0)=0, \ D_{0+}^{\beta-1}v(0)=D_{0+}^{\beta-1}u(\xi), \ v(1)=\sum_{i=1}^{m_2}\beta_iv(\xi_i). \end{cases} \tag{6}$$

There are a few results in the literature on Hadamard-type fractional differential equations (see [23–36]). In [23], the authors used fixed-point methods to obtain some existence theorems for Hadamard-type fractional boundary value problems:

$$\begin{cases} -D^{\alpha}u(t)=f(t,u(t)), \quad t\in[1,e],\\ u(1)=\delta u(1)=\delta u(e)=0, \end{cases} \tag{7}$$

where f grows superlinearly and sublinearly at infinity and can be sign-changing. The authors in [24] studied positive solutions for the p-Laplacian Hadamard fractional differential equations with integral boundary value problems:

$$\begin{cases} D^{\beta}\left(\varphi_p(D^{\alpha}u(t))\right)=f(t,u(t)), \quad 1<t<e,\\ u(1)=u'(1)=u'(e)=0,\\ D^{\alpha}u(1)=0,\\ \varphi_p(D^{\alpha}u(e))=\mu\int_1^e\varphi_p(D^{\alpha}u(t))\frac{dt}{t}, \end{cases} \tag{8}$$

where f has a $(p-1)$-superlinear and $(p-1)$-sublinear nonlinearity. In [25], the authors studied fractional impulsive Cauchy problems:

$$\begin{cases} {}_HD_{1^+}^{\alpha}u(t)=f(t,u(t)), & \alpha\in(0,1),\ t\in(1,e]\backslash\{t_1,t_2,\ldots,t_m\},\\ \Delta u(t_i) = {}_HJ_{1^+}^{1-\alpha}u(t_i^+) - {}_HJ_{1^+}^{1-\alpha}u(t_i^-)=p_i, & p_i\in\mathbb{R},\ i=1,2,\ldots,m,\\ {}_HJ_{1^+}^{1-\alpha}u(1^+)=u_0, & u_0\in\mathbb{R}, \end{cases} \tag{9}$$

where ${}_HD_{1^+}^{\alpha}$ denotes the left-sided Hadamard fractional derivative and the nonlinearity f satisfies a Lipschitz condition, and in [26], the authors studied positive solutions for the system of Hadamard fractional differential equations involving coupled integral boundary conditions:

$$\begin{cases} D^{\beta}u(t)+f_1(t,u(t),v(t))=0, \quad 1<t<e,\\ D^{\beta}v(t)+f_2(t,u(t),v(t))=0, \quad 1<t<e,\\ u(1)=v(1)=u'(1)=v'(1)=0,\\ u(e)=\int_1^e h(s)v(s)\frac{ds}{s},\\ v(e)=\int_1^e g(s)u(s)\frac{ds}{s}, \end{cases} \tag{10}$$

where the nonlinearities $f_i(i=1,2)$ grow superlinearly and sublinearly.

Motivated by the above, in this paper, we study the existence of positive solutions for the system of Hadamard-type fractional differential equation (1). The nonlinearities $f_i(i=1,2)$ can be sign-changing and can depend on the unknown functions and their derivatives. We use some appropriate nonnegative matrices to characterize the coupling behavior for the nonlinearities $f_i(i=1,2)$, and the nonlinearities $f_i(i=1,2)$ can grow superlinearly and sublinearly.

2. Preliminaries

For details about Hadamard fractional calculus, see the book [37].

Definition 1. The Hadamard derivative of fractional order q for a function $g : [1,\infty) \longrightarrow \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n \int_1^t (\log t - \log s)^{n-q-1} g(s)\frac{ds}{s}, \quad n-1<q<n, \tag{11}$$

where $n = [q]+1$, $[q]$ denotes the integer part of the real number q, and $\log(\cdot) = \log_e(\cdot)$.

Now, we establish Green's functions associated with the system of (1). In order to do this, we first consider the following auxiliary problem:

$$\begin{cases} D^\alpha D^\alpha x(t) = f(t, x(t), \delta x(t), -D^\alpha x(t)), \\ x(1) = \delta x(1) = \delta x(e) = 0, \\ D^\alpha x(1) = \delta D^\alpha x(1) = \delta D^\alpha x(e) = 0, \end{cases} \tag{12}$$

where α and δ are as in (1), and $f \in C([1,e]\times\mathbb{R}_+^3, \mathbb{R})$ satisfies the following semipositone condition:

(H0)$'$ There exists $M>0$ such that $f(t,x_1,x_2,x_3) \geq -M$, for $(t,x_1,x_2,x_3)\in[1,e]\times\mathbb{R}_+^3$.

Let $-D^\alpha x(t) = y(t)$ for $t\in[1,e]$. Then, from Lemma 2.3 of [36], we have

$$x(t) = c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2} + c_3(\log t)^{\alpha-3} - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t - \log s)^{\alpha-1} y(s)\frac{ds}{s}, \tag{13}$$

where $c_i\in\mathbb{R}, i = 1,2,3$. This, together with the boundary conditions $x(1) = \delta x(1) = \delta x(e) = 0$, implies that

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}(1-\log s)^{\alpha-2} y(s)\frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_1^t (\log t - \log s)^{\alpha-1} y(s)\frac{ds}{s} \\ &= \int_1^e H_1(t,s) y(s)\frac{ds}{s}, \end{aligned} \tag{14}$$

where

$$H_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (\log t)^{\alpha-1}(1-\log s)^{\alpha-2} - (\log t - \log s)^{\alpha-1}, & 1\leq s\leq t\leq e, \\ (\log t)^{\alpha-1}(1-\log s)^{\alpha-2}, & 1\leq t\leq s\leq e. \end{cases} \tag{15}$$

Let

$$H_2(t,s) = \delta H_1(t,s) = \frac{\alpha-1}{\Gamma(\alpha)}\begin{cases} (\log t)^{\alpha-2}(1-\log s)^{\alpha-2} - (\log t - \log s)^{\alpha-2}, & 1\leq s\leq t\leq e, \\ (\log t)^{\alpha-2}(1-\log s)^{\alpha-2}, & 1\leq t\leq s\leq e. \end{cases} \tag{16}$$

Then, we have

$$\delta x(t) = \delta\int_1^e H_1(t,s)y(s)\frac{ds}{s} = \int_1^e H_2(t,s)y(s)\frac{ds}{s}. \tag{17}$$

Therefore, substituting x and δx into problem (12), we have

$$\begin{cases} -D^\alpha y(t) = f\left(t, \int_1^e H_1(t,s)y(s)\frac{ds}{s}, \int_1^e H_2(t,s)y(s)\frac{ds}{s}, y(t)\right), \\ y(1) = \delta y(1) = \delta y(e) = 0. \end{cases} \tag{18}$$

From (13) and (14), we see that problem (18) is equivalent to the following Hammerstein-type integral equation:

$$y(t) = \int_1^e H_1(t,s) f\left(s, \int_1^e H_1(s,\tau)y(\tau)\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)y(\tau)\frac{d\tau}{\tau}, y(s)\right)\frac{ds}{s}. \tag{19}$$

Lemma 1. *Let $\varphi(t) = \log t(1-\log t)^{\alpha-2}$, $t\in[1,e]$. Then, the functions H_i $(i=1,2)$ have the following properties:*

(i) $H_i\in C([1,e]^2, \mathbb{R}_+)$, $i = 1,2$,

(ii) For all $t,s\in[1,e]$, the following inequalities hold:

$$(\log t)^{\alpha-1}\varphi(s) \leq \Gamma(\alpha)H_1(t,s) \leq \varphi(s), \tag{20}$$

$$(\alpha-1)(\alpha-2)(\log t)^{\alpha-2}(1-\log t)\varphi(s) \leq \Gamma(\alpha)H_2(t,s) \leq (\alpha-1)(\log t)^{\alpha-3}\varphi(s). \tag{21}$$

Lemma 1 (ii) is a direct result of [1, Lemma 4], so we omit the proof.

Lemma 2. *Let* $\kappa_1 = (\alpha\Gamma(\alpha-1))/\Gamma(2\alpha)$, $\kappa_2 = 1/(\alpha(\alpha-1)\Gamma(\alpha))$, $\kappa_3 = ((\alpha-1)(\alpha-2)\Gamma(\alpha))/\Gamma(2\alpha)$, *and* $\kappa_4 = (\Gamma(\alpha-1))/(\Gamma(2\alpha-2))$. *Then,*

$$\kappa_1\varphi(s) \le \int_1^e H_1(t,s)\varphi(t)\frac{dt}{t} \le \kappa_2\varphi(s), \quad \text{for } s \in [1,e],$$
$$\kappa_3\varphi(s) \le \int_1^e H_2(t,s)\varphi(t)\frac{dt}{t} \le \kappa_4\varphi(s), \quad \text{for } s \in [1,e]. \tag{22}$$

These inequalities can be deduced from Lemma 1 (ii), so we omit the proof.

Let $w(t) = M\int_1^e H_1(t,s)(ds/s)$, where M is defined in $(H0)'$. Then, w satisfies the following boundary value problem:

$$\begin{cases} -D^\alpha w(t) = M, \\ w(1) = \delta w(1) = \delta w(e) = 0. \end{cases} \tag{23}$$

In what follows, we consider the following boundary value problems:

$$\begin{cases} -D^\alpha y(t) = F\left(t, \int_1^e H_1(t,s)(y(s)-w(s))\frac{ds}{s}, \int_1^e H_2(t,s)(y(s)-w(s))\frac{ds}{s}, (y(t)-w(t))\right), \\ y(1) = \delta y(1) = \delta y(e) = 0, \end{cases} \tag{24}$$

where

$$F(t,x_1,x_2,x_3) = \begin{cases} f(t,x_1,x_2,x_3) + M, & t \in [1,e], \text{ for } x_i \ge 0,\ i = 1,2,3, \\ f(t,0,0,0) + M, & t \in [1,e], \text{for else cases.} \end{cases} \tag{25}$$

Then, $F \in C([1,e] \times \mathbb{R}_+^3, \mathbb{R}_+)$, and we have the following lemma.

Lemma 3

(i) If y^* is a solution of (24) with $y^*(t) \ge w(t)$, $t \in [1,e]$, then $y^* - w$ is a positive solution of (18)

(ii) If y_* is a solution of (18), then $y_* + w$ is a solution of (24)

Proof. Let y^* be a solution of (24) with $y^*(t) \ge w(t)$, $t \in [1,e]$. Then, we substitute $y^* - w$ into problem (18) and obtain

$$\begin{cases} -D^\alpha(y^*-w)(t) = f\left(t, \int_1^e H_1(t,s)(y^*-w)(s)\frac{ds}{s}, \int_1^e H_2(t,s)(y^*-w)(s)\frac{ds}{s}, (y^*-w)(t)\right), \\ (y^*-w)(1) = \delta(y^*-w)(1) = \delta(y^*-w)(e) = 0. \end{cases} \tag{26}$$

Considering (23), we have

$$\begin{aligned} -D^\alpha y^*(t) + D^\alpha w(t) + M &= f\left(t, \int_1^e H_1(t,s)(y^*-w)(s)\frac{ds}{s}, \int_1^e H_2(t,s)(y^*-w)(s)\frac{ds}{s}, (y^*-w)(t)\right) + M, \\ -D^\alpha y^*(t) &= F\left(t, \int_1^e H_1(t,s)(y^*-w)(s)\frac{ds}{s}, \int_1^e H_2(t,s)(y^*-w)(s)\frac{ds}{s}, (y^*-w)(t)\right). \end{aligned} \tag{27}$$

This, together with $y^*(1) = \delta y^*(1) = \delta y^*(e) = 0$, implies that (26) holds, and $y^* - w$ is a positive solution of (18).

Substitute $y_* + w$ into (24), and we get

$$\begin{cases} -D^{\alpha}(y_* + w)(t) = F\left(t, \int_1^e H_1(t,s)y_*(s)\frac{ds}{s}, \int_1^e H_2(t,s)y_*(s)\frac{ds}{s}, y_*(t)\right), \\ (y_* + w)(1) = \delta(y_* + w)(1) = \delta(y_* + w)(e) = 0. \end{cases} \tag{28}$$

Considering (23), we have

$$\begin{aligned} -D^{\alpha}y_*(t) - D^{\alpha}w(t) &= f\left(t, \int_1^e H_1(t,s)y_*(s)\frac{ds}{s}, \int_1^e H_2(t,s)y_*(s)\frac{ds}{s}, y_*(t)\right) + M, \\ -D^{\alpha}y_*(t) &= f\left(t, \int_1^e H_1(t,s)y_*(s)\frac{ds}{s}, \int_1^e H_2(t,s)y_*(s)\frac{ds}{s}, y_*(t)\right). \end{aligned} \tag{29}$$

This, together with $y_*(1) = \delta y_*(1) = \delta y_*(e) = 0$, implies $y_* + w$ is a solution of (24). This completes the proof.

From (13) and (14), we also find that problem (24) is equivalent to the following Hammerstein-type integral equation:

$$y(t) = \int_1^e H_1(t,s)F\left(s, \int_1^e H_1(s,\tau)(y(\tau) - w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(y(\tau) - w(\tau))\frac{d\tau}{\tau}, (y(s) - w(s))\right)\frac{ds}{s}. \tag{30}$$

Let $E := C[1,e], \|y\| := \max_{t\in[1,e]}|y(t)|$, and $P := \{y \in E : y(t) \geq 0, \forall t \in [1,e]\}$. Then, $(E, \|\cdot\|)$ is a real Banach space and P a cone on E. Then, $E \times E$ is also a Banach space, equipped with the norm: $\|(u,v)\| = \max\{\|u\|, \|v\|\}$, where $\|u\|$ and $\|v\|$ are norms in E. From (30), we define an operator $A : P \longrightarrow P$ as follows:

$$(Ay)(t) = \int_1^e H_1(t,s)F\left(s, \int_1^e H_1(s,\tau)(y(\tau) - w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(y(\tau) - w(\tau))\frac{d\tau}{\tau}, (y(s) - w(s))\right)\frac{ds}{s}, \quad y, w \in P. \tag{31}$$

Note that our functions H_1, H_2, F, and w are continuous, so the operator A is a completely continuous operator. Moreover, if there is a $y \in P$, a fixed point of A, and $y(t) \geq w(t), t \in [1,e]$, then we have that $y(t) - w(t)$ is a positive solution of (18). From (12) and (18), we have $x(t) = \int_1^e H_1(t,s)(y - w)(s)(ds/s)$ is a positive solution of (12). Therefore, in what follows, we study the existence of fixed points of the operator A, which are greater than w. □

Lemma 4. *Let* $P_0 = \{y \in P : y(t) \geq (\log t)^{\alpha-1}\|y\|, \forall t \in [1,e]\}$. *Then,* $A(P) \subset P_0$.

This is a direct result from (20) in Lemma 1, so we omit its proof.

Note that our aim is to find a fixed point of A, which is greater than w. If there is a $y \in P$ such that $Ay = y$, and $y(t) \geq w(t), t \in [1,e]$, then if $\|y\| \geq M/((\alpha-1)\Gamma(\alpha))$, Lemma 4 enables us to obtain

$$\begin{aligned} y(t) - w(t) &\geq (\log t)^{\alpha-1}\|y\| \\ &- \frac{M}{\Gamma(\alpha)}\int_1^e (\log t)^{\alpha-1}(1 - \log s)^{\alpha-2}\frac{ds}{s} \geq 0. \end{aligned} \tag{32}$$

Consequently, if A has fixed point $y \in P$ with $\|y\| \geq M/((\alpha-1)\Gamma(\alpha))$, then $y(t) \geq w(t), t \in [1,e]$, and $x(t) = \int_1^e H_1(t,s)(y - w)(s)(ds/s)$ is a positive solution of (12).

Lemma 5 (see [38]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If there exists a $\omega_0 \in P\backslash\{0\}$ such that*

$$\omega - A\omega \neq \lambda\omega_0, \quad \forall\lambda \geq 0, \ \omega \in \partial\Omega \cap P, \tag{33}$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed-point index on P.

Lemma 6 (see [38]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \quad \forall \lambda \in [0,1], \omega \in \partial\Omega \cap P, \tag{34}$$

then $i(A, \Omega \cap P, P) = 1$.

3. Positive Solutions for (1)

From Section 2, let $-D^{\alpha}\xi(t) = u(t), -D^{\alpha}\eta(t) = v(t)$, and then, we should consider the following system of Hadamard-type fractional boundary value problems:

$$\begin{cases} -D^{\alpha}u(t) = F_1\Big(t, \int_1^e H_1(t,s)(u(s)-w(s))\frac{ds}{s}, \int_1^e H_2(t,s)(u(s)-w(s))\frac{ds}{s}, u(t)-w(t), \\ \qquad \int_1^e H_1(t,s)(v(s)-w(s))\frac{ds}{s}, \int_1^e H_2(t,s)(v(s)-w(s))\frac{ds}{s}, v(t)-w(t)\Big) \\ -D^{\alpha}v(t) = F_2\Big(t, \int_1^e H_1(t,s)(u(s)-w(s))\frac{ds}{s}, \int_1^e H_2(t,s)(u(s)-w(s))\frac{ds}{s}, u(t)-w(t), \\ \qquad \int_1^e H_1(t,s)(v(s)-w(s))\frac{ds}{s}, \int_1^e H_2(t,s)(v(s)-w(s))\frac{ds}{s}, v(t)-w(t)\Big) \\ u(1) = \delta u(1) = \delta u(e) = v(1) = \delta v(1) = \delta v(e) = 0, \end{cases} \tag{35}$$

where

$$F_i(t, x_1, x_2, x_3, y_1, y_2, y_3) = \begin{cases} f_i(t, x_1, x_2, x_3, y_1, y_2, y_3) + M, & t \in [1,e], \text{ for } x_j, y_j \geq 0, \ i = 1,2, \ j = 1,2,3, \\ f_i(t,0,0,0,0,0,0) + M, & t \in [1,e], \ i = 1,2, \text{ for else cases.} \end{cases} \tag{36}$$

We define some operators as follows:

$$\begin{aligned} B_1(u,v)(t) &= \int_1^e H_1(t,s)F_1\Big(s, \int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, u(s)-w(s), \\ &\quad \cdot \int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, v(s)-w(s)\Big)\frac{ds}{s}, \\ B_2(u,v)(t) &= \int_1^e H_1(t,s)F_2\Big(s, \int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, u(s)-w(s), \\ &\quad \cdot \int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, v(s)-w(s)\Big)\frac{ds}{s}, \\ B(u,v)(t) &= (B_1, B_2)(u,v)(t), \quad \text{for } u, v \in P, \ t \in [1,e]. \end{aligned} \tag{37}$$

Then, if there is a $(u,v) \in P$ such that $B(u,v) = (u,v)$ with $\|u\|, \|v\| \geq M/((\alpha-1)\Gamma(\alpha))$, i.e., (35) has a solution (u,v) with $u(t), v(t) \geq w(t), t \in [1,e]$, and thus, $(\xi(t), \eta(t)) = (\int_1^e H_1(t,s)(u(s)-w(s))(ds/s), \int_1^e H_1(t,s)(v(s)-w(s))(ds/s))$ is a positive solution for (1) and $t \in [1,e]$.

Now, we list our assumptions for the functions $F_i (i = 1,2)$ as follows:

(H1) There exist $a_{ji}, b_{ji} \geq 0$ and $l_j > 0\,(j = 1, 2, i = 1, 2, 3)$ such that

$$\begin{pmatrix} F_1(t, x_1, x_2, x_3, y_1, y_2, y_3) \\ F_2(t, x_1, x_2, x_3, y_1, y_2, y_3) \end{pmatrix} \geq \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}y_1 + b_{12}y_2 + b_{13}y_3 - l_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}y_1 + b_{22}y_2 + b_{23}y_3 - l_2 \end{pmatrix}, \quad \forall (t, x_1, x_2, x_3, y_1, y_2, y_3) \in [1, e] \times \mathbb{R}_+^6, \tag{38}$$

and the coefficients $a_{ji}, b_{ji}\,(j = 1, 2, i = 1, 2, 3)$ satisfy

$$\begin{gathered} \kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13}) < 1, \\ \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23}) < 1, \\ \det\begin{bmatrix} \kappa_1(b_{11}\kappa_1 + b_{12}\kappa_3 + b_{13}) & \kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13}) - 1 \\ \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23}) - 1 & \kappa_1(a_{21}\kappa_1 + a_{22}\kappa_3 + a_{23}) \end{bmatrix} > 0. \end{gathered} \tag{39}$$

(H2) There exists $Q_i(t)$ in $[1, e]$ such that

$$\begin{gathered} \int_1^e \varphi(t) Q_i(t) \frac{dt}{t} < \frac{M}{(\alpha - 1)}, \\ F_i(t, x_1, x_2, x_3, y_1, y_2, y_3) \leq Q_i(t), \\ \forall (t, x_1, x_2, x_3, y_1, y_2, y_3) \in [1, e] \times \left[0, \frac{M}{(\alpha - 1)\Gamma(\alpha)}\right]^6, \\ i = 1, 2. \end{gathered} \tag{40}$$

(H3) There exist $\widetilde{a}_{ji}, \widetilde{b}_{ji} \geq 0$ and $\widetilde{l}_j > 0\,(j = 1, 2, i = 1, 2, 3)$ such that

$$\begin{pmatrix} F_1(t, x_1, x_2, x_3, y_1, y_2, y_3) \\ F_2(t, x_1, x_2, x_3, y_1, y_2, y_3) \end{pmatrix} \leq \begin{pmatrix} \widetilde{a}_{11}x_1 + \widetilde{a}_{12}x_2 + \widetilde{a}_{13}x_3 + \widetilde{b}_{11}y_1 + \widetilde{b}_{12}y_2 + \widetilde{b}_{13}y_3 + \widetilde{l}_1 \\ \widetilde{a}_{21}x_1 + \widetilde{a}_{22}x_2 + \widetilde{a}_{23}x_3 + \widetilde{b}_{21}y_1 + \widetilde{b}_{22}y_2 + \widetilde{b}_{23}y_3 + \widetilde{l}_2 \end{pmatrix}, \quad \forall (t, x_1, x_2, x_3, y_1, y_2, y_3) \in [1, e] \times \mathbb{R}_+^6, \tag{41}$$

and the coefficients $\widetilde{a}_{ji}, \widetilde{b}_{ji}\,(j = 1, 2, i = 1, 2, 3)$ satisfy

$$\begin{gathered} \kappa_2(\kappa_2\widetilde{a}_{11} + \kappa_4\widetilde{a}_{12} + \widetilde{a}_{13}) < 1, \\ \kappa_2\left(\kappa_2\widetilde{b}_{21} + \kappa_4\widetilde{b}_{22} + \widetilde{b}_{23}\right) < 1, \\ \det\begin{bmatrix} 1 - \kappa_2(\kappa_2\widetilde{a}_{11} + \kappa_4\widetilde{a}_{12} + \widetilde{a}_{13}) & -\kappa_2\left(\kappa_2\widetilde{b}_{11} + \kappa_4\widetilde{b}_{12} + \widetilde{b}_{13}\right) \\ -\kappa_2(\kappa_2\widetilde{a}_{21} + \kappa_4\widetilde{a}_{22} + \widetilde{a}_{23}) & 1 - \kappa_2\left(\kappa_2\widetilde{b}_{21} + \kappa_4\widetilde{b}_{22} + \widetilde{b}_{23}\right) \end{bmatrix} > 0. \end{gathered} \tag{42}$$

(H4) There exists $\widetilde{Q}_i(t)$ in $[1,e]$ and $t_0 \in (1,e]$ such that

$$\int_1^e \varphi(t)\widetilde{Q}_i(t)\frac{dt}{t} > \frac{M}{(\alpha-1)(\log t_0)^{\alpha-1}},$$

$$F_i(t,x_1,x_2,x_3,y_1,y_2,y_3) \geq \widetilde{Q}_i(t),$$

$$\forall (t,x_1,x_2,x_3,y_1,y_2,y_3) \in [1,e]\times\left[0,\frac{M}{(\alpha-1)\Gamma(\alpha)}\right]^6,$$

$$i = 1,2. \tag{43}$$

Let $B_\rho = \{u \in P : \|u\| < \rho\}$ for $\rho > 0$. Then, we have $\overline{B}_\rho = \{u \in P : \|u\| \leq \rho\}$ and $\partial B_\rho = \{u \in P : \|u\| = \rho\}$.

Theorem 1. *Suppose that (H0) to (H2) hold. Then, (1) has at least one positive solution.*

Proof. For convenience, let $\mathscr{M} = M/((\alpha-1)\Gamma(\alpha))$. We next show that there is a $R_1 > \mathscr{M}$ such that

$$(u,v) \neq B(u,v) + \lambda(\phi_0,\phi_0),\quad \text{for } (u,v) \in \partial B_{R_1} \cap (P\times P),\ \forall\lambda \geq 0, \tag{44}$$

where $\phi_0 \in P_0$ is a fixed element. Suppose not, i.e., suppose there exist $(u,v) \in \partial B_{R_1} \cap (P\times P), \lambda_0 \geq 0$ such that

$$(u,v) = B(u,v) + \lambda_0(\phi_0,\phi_0). \tag{45}$$

Combining this and Lemma 4, we have

$$u,v \in P_0. \tag{46}$$

Using (45), we obtain

$$u(t) = B_1(u,v)(t) + \lambda_0\phi(t) \geq B_1(u,v)(t),$$
$$v(t) = B_2(u,v)(t) + \lambda_0\phi(t) \geq B_2(u,v)(t),\quad t \in [1,e]. \tag{47}$$

From (H1), we have

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \geq \begin{pmatrix} \int_1^e H_1(t,s)\Big[a_{11}\int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + a_{12}\int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + a_{13}(u(s)-w(s)) \\ +b_{11}\int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + b_{12}\int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + b_{13}(v(s)-w(s)) - l_1\Big]\frac{ds}{s} \\ \int_1^e H_1(t,s)\Big[a_{21}\int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + a_{22}\int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + a_{23}(u(s)-w(s)) \\ +b_{21}\int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + b_{22}\int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + b_{23}(v(s)-w(s)) - l_2\Big]\frac{ds}{s} \end{pmatrix}. \tag{48}$$

Multiplying by $\varphi(t)$ and integrating from 1 to e, Lemma 2 enables us to obtain

$$\begin{bmatrix} \int_1^e u(t)\varphi(t)\frac{dt}{t} \\ \int_1^e v(t)\varphi(t)\frac{dt}{t} \end{bmatrix} \geq \begin{bmatrix} \kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13})\int_1^e (u(t)-w(t))\varphi(t)\frac{dt}{t} + \kappa_1(b_{11}\kappa_1 + b_{12}\kappa_3 + b_{13})\int_1^e (v(t)-w(t))\varphi(t)\frac{dt}{t} - \frac{l_1\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \\ \kappa_1(a_{21}\kappa_1 + a_{22}\kappa_3 + a_{23})\int_1^e (u(t)-w(t))\varphi(t)\frac{dt}{t} + \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23})\int_1^e (v(t)-w(t))\varphi(t)\frac{dt}{t} - \frac{l_2\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \end{bmatrix}. \tag{49}$$

This implies that

$$\begin{bmatrix} \kappa_1(b_{11}\kappa_1 + b_{12}\kappa_3 + b_{13}) & \kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13}) - 1 \\ \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23}) - 1 & \kappa_1(a_{21}\kappa_1 + a_{22}\kappa_3 + a_{23}) \end{bmatrix} \begin{bmatrix} \int_1^e v(t)\varphi(t)\frac{dt}{t} \\ \int_1^e u(t)\varphi(t)\frac{dt}{t} \end{bmatrix}$$
$$\leq \begin{bmatrix} \kappa_1(\kappa_1(a_{11}+b_{11}) + \kappa_3(a_{12}+b_{12}) + a_{13} + b_{13})\int_1^e w(t)\varphi(t)\frac{dt}{t} + \frac{l_1\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \\ \kappa_1(\kappa_1(a_{21}+b_{21}) + \kappa_3(a_{22}+b_{22}) + a_{23} + b_{23})\int_1^e w(t)\varphi(t)\frac{dt}{t} + \frac{l_2\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \end{bmatrix} := \begin{bmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{bmatrix}. \tag{50}$$

Solving this matrix inequality, we have

$$\begin{bmatrix} \int_1^e v(t)\varphi(t)\frac{dt}{t} \\ \int_1^e u(t)\varphi(t)\frac{dt}{t} \end{bmatrix} \leq \frac{\begin{bmatrix} \kappa_1(a_{21}\kappa_1 + a_{22}\kappa_3 + a_{23}) & 1 - \kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13}) \\ 1 - \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23}) & \kappa_1(b_{11}\kappa_1 + b_{12}\kappa_3 + b_{13}) \end{bmatrix}}{\det\begin{bmatrix} \kappa_1(b_{11}\kappa_1 + b_{12}\kappa_3 + b_{13}) & \kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13}) - 1 \\ \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23}) - 1 & \kappa_1(a_{21}\kappa_1 + a_{22}\kappa_3 + a_{23}) \end{bmatrix}} \begin{bmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{bmatrix}. \tag{51}$$

Consequently, there exist $\mathcal{N}_3, \mathcal{N}_4 > 0$ such that

$$\begin{bmatrix} \int_1^e v(t)\varphi(t)\frac{dt}{t} \\ \int_1^e u(t)\varphi(t)\frac{dt}{t} \end{bmatrix} \leq \begin{bmatrix} \mathcal{N}_3 \\ \mathcal{N}_4 \end{bmatrix}. \tag{52}$$

Note (46) and we have

$$\begin{pmatrix} \|v\| \\ \|u\| \end{pmatrix} \leq \begin{pmatrix} \frac{\mathcal{N}_3\Gamma(\alpha+1)\Gamma(\alpha-1)}{\Gamma(2\alpha)} \\ \frac{\mathcal{N}_4\Gamma(\alpha+1)\Gamma(\alpha-1)}{\Gamma(2\alpha)} \end{pmatrix}. \tag{53}$$

Choose $R_1 > \{\mathcal{M}, ((\mathcal{N}_3\Gamma(\alpha+1)\Gamma(\alpha-1))/\Gamma(2\alpha)), ((\mathcal{N}_4\Gamma(\alpha+1)\Gamma(\alpha-1))/\Gamma(2\alpha))\}$. Then, (44) holds. Lemma 5 gives

$$i\left(B, B_{R_1} \cap (P \times P), P \times P\right) = 0. \tag{54}$$

Next, we prove that

$$(u, v) \neq \lambda B(u, v), \quad \text{for } (u, v) \in \partial B_{\mathcal{M}} \cap (P \times P), \forall \lambda \in [0, 1]. \tag{55}$$

Suppose not. Then, there exist $(u, v) \in \partial B_{\mathcal{M}} \cap (P \times P)$, $\lambda_1 \in [0, 1]$ such that

$$(u, v) = \lambda_1 B(u, v). \tag{56}$$

This implies that

$$\begin{aligned} \|u\| &\leq \|B_1(u, v)\|, \\ \|v\| &\leq \|B_2(u, v)\|. \end{aligned} \tag{57}$$

However, from (H2), we have

$$\begin{aligned} B_1(u,v)(t) &= \int_1^e H_1(t,s)F_1\bigg(s, \int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, u(s)-w(s), \\ &\quad \cdot \int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, v(s)-w(s)\bigg)\frac{ds}{s} \\ &\leq \frac{1}{\Gamma(\alpha)}\int_1^e \varphi(s)Q_1(s)\frac{ds}{s} \\ &< \frac{M}{(\alpha-1)\Gamma(\alpha)}. \end{aligned} \tag{58}$$

Note that from (H2), we have $\|u\| = M/((\alpha-1)\Gamma(\alpha))$. Hence, we obtain $\|B_1(u,v)\| < \|u\|$. Similarly, $\|B_2(u,v)\| < \|v\|$. This is a contradiction. As a result, (55) holds. From Lemma 6, we have

$$i\left(B, B_{\mathscr{M}} \cap (P\times P), P\times P\right) = 1. \tag{59}$$

From (54) and (59), we have

$$\begin{aligned} i\left(B, \left(B_{R_1}\backslash \overline{B}_{\mathscr{M}}\right) \cap (P\times P), P\times P\right) &= i\left(B, B_{R_1} \cap (P\times P), P\times P\right) \\ &\quad - i\left(B, B_{\mathscr{M}} \cap (P\times P), P\times P\right) \\ &= 0 - 1 = -1. \end{aligned} \tag{60}$$

Therefore, the operator B has at least one fixed point on $(B_{R_1}\backslash \overline{B}_{\mathscr{M}}) \cap (P\times P)$. Thus, (1) has at least one positive solution. This completes the proof. □

Theorem 2. *Suppose that (H0), (H3) and (H4) hold. Then, (1) has at least one positive solution.*

Proof. We first show that there is a $R_2 > M/((\alpha-1)\Gamma(\alpha))$ such that

$$(u,v) \neq \lambda B(u,v), \quad \text{for } (u,v) \in \partial B_{R_2} \cap (P\times P),\ \forall \lambda \in [0,1]. \tag{61}$$

If this claim is false, there exist $(u,v) \in \partial B_{R_2} \cap (P\times P)$, $\lambda_2 \in [0,1]$ such that

$$(u,v) = \lambda_2 B(u,v). \tag{62}$$

Lemma 4 implies that

$$u, v \in P_0. \tag{63}$$

Moreover, from (H3), we have

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \leq \begin{pmatrix} \displaystyle\int_1^e H_1(t,s)\left[\widetilde{a}_{11}\int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{a}_{12}\int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{a}_{13}(u(s)-w(s)) + \widetilde{b}_{11}\right. \\ \displaystyle\left.\cdot\int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{b}_{12}\int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{b}_{13}(v(s)-w(s)) + \widetilde{l}_1\right]\frac{ds}{s} \\ \displaystyle\int_1^e H_1(t,s)\left[\widetilde{a}_{21}\int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{a}_{22}\int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{a}_{23}(u(s)-w(s)) + \widetilde{b}_{21}\right. \\ \displaystyle\left.\cdot\int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{b}_{22}\int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau} + \widetilde{b}_{23}(v(s)-w(s)) + \widetilde{l}_2\right]\frac{ds}{s} \end{pmatrix}. \tag{64}$$

Multiply by $\varphi(t)$ on both sides of the above and integrate over $[1,e]$ and use Lemma 2 to obtain

$$\begin{bmatrix} \displaystyle\int_1^e u(t)\varphi(t)\frac{dt}{t} \\ \displaystyle\int_1^e v(t)\varphi(t)\frac{dt}{t} \end{bmatrix} \leq \begin{bmatrix} \displaystyle\kappa_2(\kappa_2\widetilde{a}_{11} + \kappa_4\widetilde{a}_{12} + \widetilde{a}_{13})\int_1^e (u(t)-w(t))\varphi(t)\frac{dt}{t} + \kappa_2\left(\kappa_2\widetilde{b}_{11} + \kappa_4\widetilde{b}_{12} + \widetilde{b}_{13}\right)\int_1^e (v(t)-w(t))\varphi(t)\frac{dt}{t} + \frac{\widetilde{l}_1\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \\ \displaystyle\kappa_2(\kappa_2\widetilde{a}_{21} + \kappa_4\widetilde{a}_{22} + \widetilde{a}_{23})\int_1^e (u(t)-w(t))\varphi(t)\frac{dt}{t} + \kappa_2\left(\kappa_2\widetilde{b}_{21} + \kappa_4\widetilde{b}_{22} + \widetilde{b}_{23}\right)\int_1^e (v(t)-w(t))\varphi(t)\frac{dt}{t} + \frac{\widetilde{l}_2\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \end{bmatrix}. \tag{65}$$

This implies that

$$\begin{bmatrix} 1-\kappa_2(\kappa_2\widetilde{a}_{11} + \kappa_4\widetilde{a}_{12} + \widetilde{a}_{13}) & -\kappa_2\left(\kappa_2\widetilde{b}_{11} + \kappa_4\widetilde{b}_{12} + \widetilde{b}_{13}\right) \\ -\kappa_2(\kappa_2\widetilde{a}_{21} + \kappa_4\widetilde{a}_{22} + \widetilde{a}_{23}) & 1-\kappa_2\left(\kappa_2\widetilde{b}_{21} + \kappa_4\widetilde{b}_{22} + \widetilde{b}_{23}\right) \end{bmatrix} \begin{bmatrix} \displaystyle\int_1^e u(t)\varphi(t)\frac{dt}{t} \\ \displaystyle\int_1^e v(t)\varphi(t)\frac{dt}{t} \end{bmatrix} \leq \begin{bmatrix} \dfrac{\widetilde{l}_1\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \\ \dfrac{\widetilde{l}_2\Gamma(\alpha-1)}{\alpha^2(\alpha-1)\Gamma^2(\alpha)} \end{bmatrix}. \tag{66}$$

Solving this matrix inequality, we obtain

$$\begin{bmatrix} \int_1^e u(t)\varphi(t)\frac{dt}{t} \\ \int_1^e v(t)\varphi(t)\frac{dt}{t} \end{bmatrix} \leq \frac{\begin{bmatrix} 1-\kappa_2\left(\kappa_2\widetilde{b}_{21}+\kappa_4\widetilde{b}_{22}+\widetilde{b}_{23}\right) & \kappa_2\left(\kappa_2\widetilde{b}_{11}+\kappa_4\widetilde{b}_{12}+\widetilde{b}_{13}\right) \\ \kappa_2\left(\kappa_2\widetilde{a}_{21}+\kappa_4\widetilde{a}_{22}+\widetilde{a}_{23}\right) & 1-\kappa_2\left(\kappa_2\widetilde{a}_{11}+\kappa_4\widetilde{a}_{12}+\widetilde{a}_{13}\right) \end{bmatrix} \begin{bmatrix} \left(\widetilde{l}_1\Gamma(\alpha-1)\right)/\left(\alpha^2(\alpha-1)\Gamma^2(\alpha)\right) \\ \left(\widetilde{l}_2\Gamma(\alpha-1)\right)/\left(\alpha^2(\alpha-1)\Gamma^2(\alpha)\right) \end{bmatrix}}{\det\begin{bmatrix} 1-\kappa_2\left(\kappa_2\widetilde{a}_{11}+\kappa_4\widetilde{a}_{12}+\widetilde{a}_{13}\right) & -\kappa_2\left(\kappa_2\widetilde{b}_{11}+\kappa_4\widetilde{b}_{12}+\widetilde{b}_{13}\right) \\ -\kappa_2\left(\kappa_2\widetilde{a}_{21}+\kappa_4\widetilde{a}_{22}+\widetilde{a}_{23}\right) & 1-\kappa_2\left(\kappa_2\widetilde{b}_{21}+\kappa_4\widetilde{b}_{22}+\widetilde{b}_{23}\right) \end{bmatrix}}. \tag{67}$$

Note that $u, v \in P_0$, there exist $\mathcal{N}_5, \mathcal{N}_6 > 0$ such that

$$\begin{pmatrix} \|u\| \\ \|v\| \end{pmatrix} \leq \begin{pmatrix} \frac{\mathcal{N}_5\Gamma(\alpha+1)\Gamma(\alpha-1)}{\Gamma(2\alpha)} \\ \frac{\mathcal{N}_6\Gamma(\alpha+1)\Gamma(\alpha-1)}{\Gamma(2\alpha)} \end{pmatrix}. \tag{68}$$

Choose $R_2 > \max\{\mathcal{M}, ((\mathcal{N}_5\Gamma(\alpha+1)\Gamma(\alpha-1))/\Gamma(2\alpha)), ((\mathcal{N}_6\Gamma(\alpha+1)\Gamma(\alpha-1))/\Gamma(2\alpha))\}$. Then, (61) holds. From Lemma 6, we have

$$i\left(B, B_{R_2} \cap (P\times P), P\times P\right) = 1. \tag{69}$$

Next, we prove that

$$\begin{gathered} (u,v) \neq B(u,v) + \lambda(\phi_1, \phi_1), \\ \text{for } (u,v) \in \partial B_{\mathcal{M}} \cap (P\times P),\ \forall \lambda \geq 0, \end{gathered} \tag{70}$$

where $\phi_1 \in P$ is a fixed element. Suppose not. Then, there exist $(u,v) \in \partial B_{\mathcal{M}} \cap (P\times P)$, $\lambda_3 \geq 0$ such that

$$(u,v) = B(u,v) + \lambda_3(\phi_1, \phi_1). \tag{71}$$

This implies that

$$\begin{aligned} \|u\| &\geq \left\|B_1(u,v)\right\|, \\ \|v\| &\geq \left\|B_2(u,v)\right\|. \end{aligned} \tag{72}$$

However, from (H4), we have

$$\begin{aligned} B_1(u,v)(t_0) &= \int_1^e H_1(t_0,s)F_1 s, \int_1^e H_1(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(u(\tau)-w(\tau))\frac{d\tau}{\tau}, u(s) \\ &\quad - w(s), \int_1^e H_1(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, \int_1^e H_2(s,\tau)(v(\tau)-w(\tau))\frac{d\tau}{\tau}, v(s)-w(s)\frac{ds}{s} \\ &\geq \frac{(\log t_0)^{\alpha-1}}{\Gamma(\alpha)} \int_1^e \varphi(s)\widetilde{Q}_1(s)\frac{ds}{s} \\ &> \frac{(\log t_0)^{\alpha-1}}{\Gamma(\alpha)} \frac{M}{(\alpha-1)(\log t_0)^{\alpha-1}}. \end{aligned} \tag{73}$$

Note that from (H2), we have $\|u\| = M/((\alpha-1)\Gamma(\alpha))$. Hence, we obtain $\|B_1(u,v)\| \geq B_1(u,v)(t_0) > \|u\|$. Similarly, $\|B_2(u,v)\| > \|v\|$. This has a contradiction, and thus, (70) holds. From Lemma 5, we obtain

$$i(B, B_{\mathcal{M}} \cap (P\times P), P\times P) = 0. \tag{74}$$

From (69) and (74), we have

$$\begin{aligned} i\left(B, \left(B_{R_2}\backslash\overline{B}_{\mathcal{M}}\right) \cap (P\times P), P\times P\right) &= i\left(B, B_{R_2} \cap (P\times P), P\times P\right) \\ &\quad - i(B, B_{\mathcal{M}} \cap (P\times P), P\times P) \\ &= 1 - 0 = 1. \end{aligned} \tag{75}$$

Therefore, the operator B has at least one fixed point on $(B_{R_2}\backslash\overline{B}_{\mathcal{M}}) \cap (P\times P)$. Thus, (1) has at least one positive solution. This completes the proof. □

Let $\alpha = 2.5$. Then, we have $\kappa_1 = 0.09, \kappa_2 = 0.2, \kappa_3 = 0.004, \kappa_4 = 0.44$, $M/((\alpha-1)\Gamma(\alpha)) = 0.502M$, and $\int_1^e \varphi(t)(dt/t) = 0.27$.

Example 1. Let $a_{11} = 50, a_{12} = 1000, a_{13} = 2$, $b_{21} = 52, b_{22} = 1002, b_{23} = 2.2$, $a_{21} = 140, a_{22} = 2900, a_{23} = 15$, $b_{11} = 130$, $b_{12} = 2800, b_{13} = 12$, and

$$F_1(t,x_1,x_2,x_3,y_1,y_2,y_3) = (2005)^{-\gamma_1} M^{1-\gamma_1} (50x_1 + 1000x_2 + 2x_3 + 130y_1 + 2800y_2 + 12y_3)^{\gamma_1},$$
$$F_2(t,x_1,x_2,x_3,y_1,y_2,y_3) = (2064)^{-\gamma_1} M^{1-\gamma_1} (140x_1 + 2900x_2 + 15x_3 + 52y_1 + 1002y_2 + 2.2y_3)^{\gamma_2}, \tag{76}$$

for $(t,x_1,x_2,x_3,y_1,y_2,y_3) \in [1,e] \times \mathbb{R}_+^6$, where $\gamma_1,\gamma_2 > 1$. Then, we have

$$\kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13}) = 0.945 < 1, \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23}) = 0.98 < 1,$$
$$\begin{vmatrix} \kappa_1(b_{11}\kappa_1 + b_{12}\kappa_3 + b_{13}) & \kappa_1(a_{11}\kappa_1 + a_{12}\kappa_3 + a_{13}) - 1 \\ \kappa_1(b_{21}\kappa_1 + b_{22}\kappa_3 + b_{23}) - 1 & \kappa_1(a_{21}\kappa_1 + a_{22}\kappa_3 + a_{23}) \end{vmatrix} = \begin{vmatrix} 3.141 & -0.055 \\ -0.02 & 3.528 \end{vmatrix} > 0. \tag{77}$$

Moreover, note that when $(t,x_1,x_2,x_3,y_1,y_2,y_3) \in [1,e] \times [0,(M/((\alpha-1)\Gamma(\alpha)))]^6$, we have $F_i \le M$, $i=1,2$. Consequently, if we choose $Q_i(t) \equiv M$, $t \in [1,e]$, $i=1,2$, then $\int_1^e \varphi(t)Q_i(t)(\mathrm{d}t/t) < M/(\alpha-1)$.

Note that we have

$$\liminf_{a_{11}x_1+a_{12}x_2+a_{13}x_3+b_{11}y_1+b_{12}y_2+b_{13}y_3 \longrightarrow +\infty} \frac{F_1(t,x_1,x_2,x_3,y_1,y_2,y_3)}{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}y_1 + b_{12}y_2 + b_{13}y_3}$$
$$= \liminf_{a_{11}x_1+a_{12}x_2+a_{13}x_3+b_{11}y_1+b_{12}y_2+b_{13}y_3 \longrightarrow +\infty} \frac{(2005)^{-\gamma_1} M^{1-\gamma_1} (50x_1 + 1000x_2 + 2x_3 + 130y_1 + 2800y_2 + 12y_3)^{\gamma_1}}{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}y_1 + b_{12}y_2 + b_{13}y_3} = +\infty,$$
$$\liminf_{a_{21}x_1+a_{22}x_2+a_{23}x_3+b_{21}y_1+b_{22}y_2+b_{23}y_3 \longrightarrow +\infty} \frac{F_2(t,x_1,x_2,x_3,y_1,y_2,y_3)}{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}y_1 + b_{22}y_2 + b_{23}y_3}$$
$$= \liminf_{a_{21}x_1+a_{22}x_2+a_{23}x_3+b_{21}y_1+b_{22}y_2+b_{23}y_3 \longrightarrow +\infty} \frac{(2064)^{-\gamma_1} M^{1-\gamma_1} (140x_1 + 2900x_2 + 15x_3 + 52y_1 + 1002y_2 + 2.2y_3)^{\gamma_2}}{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}y_1 + b_{22}y_2 + b_{23}y_3} = +\infty. \tag{78}$$

As a result, (H1) and (H2) hold.

Example 2. Let $t_0 = e, \tilde{a}_{11} = 0.075, \tilde{a}_{12} = 0.034, \tilde{a}_{13} = 0.015, \tilde{b}_{21} = 0.08, \tilde{b}_{22} = 0.035, \tilde{b}_{23} = 0.016, \tilde{b}_{11} = 0.09, \tilde{b}_{12} = 0.04, \tilde{b}_{13} = 0.02, \tilde{a}_{21} = 0.08, \tilde{a}_{22} = 0.05, \tilde{a}_{23} = 0.018$, and

$$F_1(t,x_1,x_2,x_3,y_1,y_2,y_3) = 2.7Me^{0.14M} \exp(-(0.075x_1 + 0.034x_2 + 0.015x_3 + 0.09y_1 + 0.04y_2 + 0.02y_3)),$$
$$F_2(t,x_1,x_2,x_3,y_1,y_2,y_3) = 3Me^{0.14M} \exp(-(0.08x_1 + 0.05x_2 + 0.018x_3 + 0.08y_1 + 0.035y_2 + 0.016y_3)), \tag{79}$$

for $(t,x_1,x_2,x_3,y_1,y_2,y_3) \in [1,e] \times \mathbb{R}_+^6$. Then, we have $(\log t_0)^{\alpha-1} = 1$, and

$$\kappa_2(\kappa_2\tilde{a}_{11} + \kappa_4\tilde{a}_{12} + \tilde{a}_{13}) = 0.009 < 1,$$
$$\kappa_2(\kappa_2\tilde{b}_{21} + \kappa_4\tilde{b}_{22} + \tilde{b}_{23}) = 0.009 < 1,$$
$$\begin{vmatrix} 1-\kappa_2(\kappa_2\tilde{a}_{11} + \kappa_4\tilde{a}_{12} + \tilde{a}_{13}) & -\kappa_2(\kappa_2\tilde{b}_{11} + \kappa_4\tilde{b}_{12} + \tilde{b}_{13}) \\ -\kappa_2(\kappa_2\tilde{a}_{21} + \kappa_4\tilde{a}_{22} + \tilde{a}_{23}) & 1-\kappa_2(\kappa_2\tilde{b}_{21} + \kappa_4\tilde{b}_{22} + \tilde{b}_{23}) \end{vmatrix} = \begin{vmatrix} 0.991 & -0.011 \\ -0.0112 & 0.991 \end{vmatrix} > 0. \tag{80}$$

Moreover, when $(t, x_1, x_2, x_3, y_1, y_2, y_3) \in [1, e] \times [0, (M/((\alpha - 1)\Gamma(\alpha)))]^6$, we obtain $F_1 \geq 2.7M$ and $F_2 \geq 3M$, and if we choose $\widetilde{Q}_1(t) \equiv 2.7M$ and $\widetilde{Q}_2(t) \equiv 3M$, $t \in [1, e]$, we also have $\int_1^e \varphi(t)\widetilde{Q}_i(t)(dt/t) > M/((\alpha - 1)(\log t_0)^{\alpha-1})$.

Note that we have

$$\begin{aligned}
&\limsup_{\widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3+\widetilde{b}_{11}y_1+\widetilde{b}_{12}y_2+\widetilde{b}_{13}y_3 \longrightarrow +\infty} \frac{F_1(t, x_1, x_2, x_3, y_1, y_2, y_3)}{\widetilde{a}_{11}x_1 + \widetilde{a}_{12}x_2 + \widetilde{a}_{13}x_3 + \widetilde{b}_{11}y_1 + \widetilde{b}_{12}y_2 + \widetilde{b}_{13}y_3} \\
&= \limsup_{\widetilde{a}_{11}x_1+\widetilde{a}_{12}x_2+\widetilde{a}_{13}x_3+\widetilde{a}_{11}y_1+\widetilde{b}_{12}y_2+\widetilde{b}_{13}y_3 \longrightarrow +\infty} \frac{2.7Me^{0.14M}\exp(-(0.075x_1 + 0.034x_2 + 0.015x_3 + 0.09y_1 + 0.04y_2 + 0.02y_3))}{\widetilde{a}_{11}x_1 + \widetilde{a}_{12}x_2 + \widetilde{a}_{13}x_3 + \widetilde{b}_{11}y_1 + \widetilde{b}_{12}y_2 + \widetilde{b}_{13}y_3} = 0, \\
&\limsup_{\widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3+\widetilde{b}_{21}y_1+\widetilde{b}_{22}y_2+\widetilde{b}_{23}y_3 \longrightarrow +\infty} \frac{F_2(t, x_1, x_2, x_3, y_1, y_2, y_3)}{\widetilde{a}_{21}x_1 + \widetilde{a}_{22}x_2 + \widetilde{a}_{23}x_3 + \widetilde{b}_{21}y_1 + \widetilde{b}_{22}y_2 + \widetilde{b}_{23}y_3} \\
&= \limsup_{\widetilde{a}_{21}x_1+\widetilde{a}_{22}x_2+\widetilde{a}_{23}x_3+\widetilde{b}_{21}y_1+\widetilde{b}_{22}y_2+\widetilde{b}_{23}y_3 \longrightarrow +\infty} \frac{3Me^{0.14M}\exp(-(0.08x_1 + 0.05x_2 + 0.018x_3 + 0.08y_1 + 0.035y_2 + 0.016y_3))}{\widetilde{a}_{21}x_1 + \widetilde{a}_{22}x_2 + \widetilde{a}_{23}x_3 + \widetilde{b}_{21}y_1 + \widetilde{b}_{22}y_2 + \widetilde{b}_{23}y_3} = 0.
\end{aligned} \tag{81}$$

As a result, (H3) and (H4) hold.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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References

[1] J. Xu, Z. Wei, and Y. Ding, "Positive solutions for a boundary-value problem with Riemann–Liouville fractional derivative," *Lithuanian Mathematical Journal*, vol. 52, no. 4, pp. 462–476, 2012.

[2] R. Pu, X. Zhang, Y. Cui, P. Li, and W. Wang, "Positive solutions for singular semipositone fractional differential equation subject to multipoint boundary conditions," *Journal of Function Spaces*, vol. 2017, Article ID 5892616, 7 pages, 2017.

[3] Y. Zou and G. He, "On the uniqueness of solutions for a class of fractional differential equations," *Applied Mathematics Letters*, vol. 74, pp. 68–73, 2017.

[4] Z. Yue and Y. Zou, "New uniqueness results for fractional differential equation with dependence on the first order derivative," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 38, 9 pages, 2019.

[5] Z. Fu, S. Bai, D. O'Regan, and J. Xu, "Nontrivial solutions for an integral boundary value problem involving Riemann–Liouville fractional derivatives," *Journal of Inequalities and Applications*, vol. 2019, no. 1, Article ID 104, 16 pages, 2019.

[6] K. Zhang, D. O'Regan, J. Xu, and Z. Fu, "Nontrivial solutions for a higher order nonlinear fractional boundary value problem involving Riemann–Liouville fractional derivatives," *Journal of Function Spaces*, vol. 2019, Article ID 2381530, 11 pages, 2019.

[7] Y. Zhang, "Existence results for a coupled system of nonlinear fractional multi-point boundary value problems at resonance," *Journal of Inequalities and Applications*, vol. 2018, no. 1, Article ID 198, 17 pages, 2018.

[8] Y. Zhang, Z. Bai, and T. Feng, "Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance," *Computers & Mathematics with Applications*, vol. 61, no. 4, pp. 1032–1047, 2011.

[9] T. Qi, Y. Liu, and Y. Zou, "Existence result for a class of coupled fractional differential systems with integral boundary value conditions," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 7, pp. 4034–4045, 2017.

[10] T. Qi, Y. Liu, and Y. Cui, "Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions," *Journal of Function Spaces*, vol. 2017, Article ID 6703860, 9 pages, 2017.

[11] J. Xu, C. S. Goodrich, and Y. Cui, "Positive solutions for a system of first-order discrete fractional boundary value problems with semipositone nonlinearities," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 2, pp. 1343–1358, 2019.

[12] W. Cheng, J. Xu, Y. Cui, and Q. Ge, "Positive solutions for a class of fractional difference systems with coupled boundary conditions," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 249, 22 pages, 2019.

[13] W. Cheng, J. Xu, and Y. Cui, "Positive solutions for a system of nonlinear semipositone fractional q-difference equations with q-integral boundary conditions," *The Journal of*

Nonlinear Sciences and Applications, vol. 10, no. 8, pp. 4430–4440, 2017.

[14] F. Wang and Y. Cui, "Positive solutions for an infinite system of fractional order boundary value problems," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 38, 11 pages, 2019.

[15] X. Qiu, J. Xu, D. O'Regan, and Y. Cui, "Positive solutions for a system of nonlinear semipositone boundary value problems with Riemann–Liouville fractional derivatives," *Journal of Function Spaces*, vol. 2018, Article ID 7351653, 10 pages, 2018.

[16] X. Hao, H. Wang, L. Liu, and Y. Cui, "Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator," *Boundary Value Problems*, vol. 2017, no. 1, Article ID 182, 18 pages, 2017.

[17] X. Zhang, L. Liu, and Y. Zou, "Fixed-point theorems for systems of operator equations and their applications to the fractional differential equations," *Journal of Function Spaces*, vol. 2018, Article ID 7469868, 9 pages, 2018.

[18] X. Zhang, L. Liu, Y. Wu, and Y. Zou, "Existence and uniqueness of solutions for systems of fractional differential equations with Riemann–Stieltjes integral boundary condition," *Advances in Difference Equations*, vol. 2018, no. 1, Article ID 204, 15 pages, 2018.

[19] H. Li and J. Zhang, "Positive solutions for a system of fractional differential equations with two parameters," *Journal of Function Spaces*, vol. 2018, Article ID 1462505, 9 pages, 2018.

[20] Y. Zhao, X. Hou, Y. Sun, and Z. Bai, "Solvability for some class of multi-order nonlinear fractional systems," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 23, 10 pages, 2019.

[21] Y. Zhao, Y. Sun, Y. Wang, and Z. Bai, "Asymptotical stabilization of the nonlinear upper triangular fractional-order systems," *Advances in Difference Equations*, vol. 2019, no. 1, 11 pages, 2019.

[22] Y. Cui, "Multiplicity results for positive solutions to differential systems of singular coupled integral boundary value problems," *Mathematical Problems in Engineering*, vol. 2017, Article ID 3608352, 7 pages, 2017.

[23] K. Zhang and Z. Fu, "Solutions for a class of Hadamard fractional boundary value problems with sign-changing nonlinearity," *Journal of Function Spaces*, vol. 2019, Article ID 9046472, 7 pages, 2019.

[24] K. Zhang, J. Wang, and W. Ma, "Solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations," *Journal of Function Spaces*, vol. 2018, Article ID 2193234, 10 pages, 2018.

[25] J. Wang and Y. Zhang, "On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives," *Applied Mathematics Letters*, vol. 39, pp. 85–90, 2015.

[26] J. Jiang, D. O'Regan, J. Xu, and Z. Fu, "Positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, Article ID 204, 18 pages, 2019.

[27] W. Yukunthorn, B. Ahmad, S. K. Ntouyas, and J. Tariboon, "On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions," *Nonlinear Analysis: Hybrid Systems*, vol. 19, pp. 77–92, 2016.

[28] S. Aljoudi, B. Ahmad, J. J. Nieto, and A. Alsaedi, "A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions," *Chaos, Solitons & Fractals*, vol. 91, pp. 39–46, 2016.

[29] J. Tariboon, S. K. Ntouyas, and W. Sudsutad, "Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 1, pp. 295–308, 2016.

[30] Q. Ma, R. Wang, J. Wang, and Y. Ma, "Qualitative analysis for solutions of a certain more generalized two-dimensional fractional differential system with Hadamard derivative," *Applied Mathematics and Computation*, vol. 257, pp. 436–445, 2015.

[31] Q. Ma, J. Wang, R. Wang, and X. Ke, "Study on some qualitative properties for solutions of a certain two-dimensional fractional differential system with Hadamard derivative," *Applied Mathematics Letters*, vol. 36, pp. 7–13, 2014.

[32] Z. Gong, D. Qian, C. Li, and P. Guo, "On the Hadamard type fractional differential system," in *Fractional Dynamics and Control*, Springer, New York, NY, USA, 2012.

[33] U. Riaz, A. Zada, Z. Ali, Y. Cui, and J. Xu, "Analysis of coupled systems of implicit impulsive fractional differential equations involving Hadamard derivatives," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 226, 27 pages, 2019.

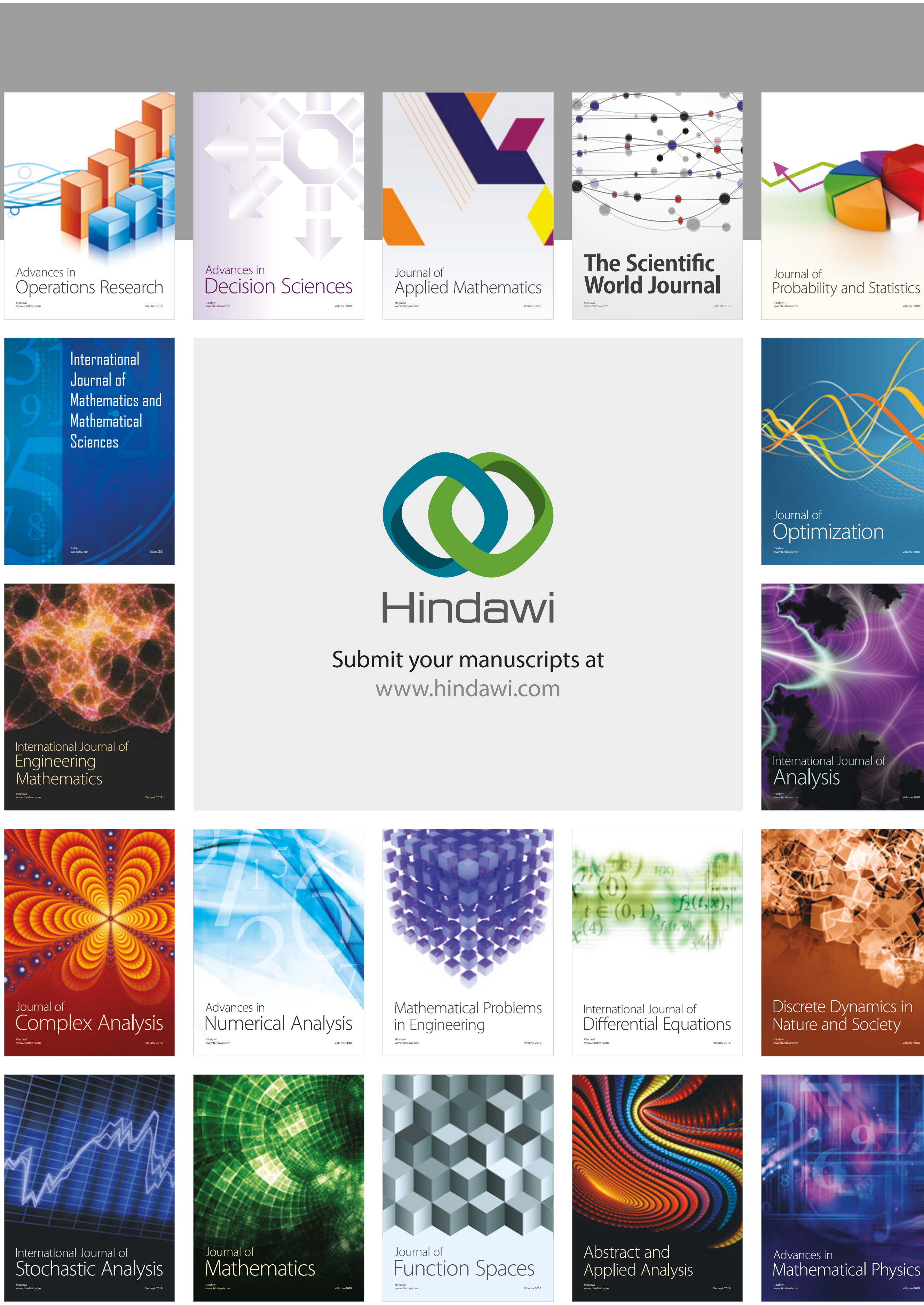
[34] U. Riaz, A. Zada, Z. Ali, M. Ahmad, J. Xu, and Z. Fu, "Analysis of nonlinear coupled systems of impulsive fractional differential equations with Hadamard derivatives," *Mathematical Problems in Engineering*, vol. 2019, Article ID 5093572, 20 pages, 2019.

[35] C. Zhai, W. Wang, and H. Li, "A uniqueness method to a new Hadamard fractional differential system with four-point boundary conditions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, 16 pages, 2018.

[36] W. Yang, "Positive solutions for singular coupled integral boundary value problems of nonlinear Hadamard fractional differential equations," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 2, pp. 110–129, 2015.

[37] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Boston, MA, USA, 2006.

[38] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL, USA, 1988.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