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Research Article

Extremal Solutions for Caputo Conformable Differential Equations with p-Laplacian Operator and Integral Boundary Condition

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The Caputo conformable derivative is a new Caputo-type fractional differential operator generated by conformable derivatives. In this paper, using Banach fixed point theorem, we obtain the uniqueness of the solution of nonlinear and linear Cauchy problem with the conformable derivatives in the Caputo setting, respectively. We also establish two comparison principles and prove the extremal solutions for nonlinear fractional p-Laplacian differential system with Caputo conformable derivatives by utilizing the monotone iterative technique. An example is given to verify the validity of the results.

1. Introduction

In recent years, fractional calculus has been widely developed in pure mathematics and applied mathematics [1–7]. The characteristic of fractional calculus is that there are many different fractional derivatives or integrals, like Riemann–Liouville (RL), Caputo, Hadamard, Caputo–Hadamard types, and so on [1, 2, 8, 9]. So, the scholars have the opportunity to choose the most appropriate operators to describe complex problems in the real world. We recall some definitions from the traditional fractional calculus [1, 2].

The left RL fractional integral of order $\beta > 0$ is given by

$$ {}_a^R I^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_a^b (t-s)^{\beta-1} f(s) \mathrm{d}s. \tag{1}$$

The left RL fractional derivative of order $\beta > 0$ is defined as

$$ {}_a D^{\beta} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n {}_a I^{n-\beta} f(t) = \frac{(\mathrm{d}/\mathrm{d}t)^n}{\Gamma(n-\beta)} \int_a^t (t-s)^{n-\beta-1} f(s) \mathrm{d}s. \tag{2}$$

The left Caputo fractional derivative of order $\beta > 0$ is given by

$$ {}_a^C D^{\beta} f(t) = {}_a I^{n-\beta} f^{(n)}(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-s)^{n-\beta-1} f^{(n)}(s) \mathrm{d}s. \tag{3}$$

However, some basic properties such as product rule and chain rule are not valid for the RL and Caputo-type fractional derivatives. In 2014, Khalil et al. [10] defined a new fractional differential operator named the conformable derivative which satisfies the product rule and some other properties. In 2015, Abdeljawad [11] defined the left conformable integral ${}_a I^{\alpha}$ and derivative ${}_a T^{\alpha}$ as

$$ \begin{aligned} {}_a I^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^b (s-a)^{\alpha-1} f(s) \mathrm{d}s, \\ {}_a T^{\alpha} f(t) &= \lim_{\varepsilon \longrightarrow 0} \frac{f\left(t+\varepsilon(t-a)^{1-\alpha}\right) - f(t)}{\varepsilon}, \end{aligned} \tag{4}$$

where $\alpha \in (0,1], t > 0, f\colon [a, +\infty) \longrightarrow \mathbb{R}$. If f is differentiable, then ${}_a T^{\alpha} f(t) = (t-a)^{1-\alpha} f'(t)$.

In 2017, Jarad et al. [12] established the conformable calculus in both RL and Caputo setting based on the work of Abdeljawad.

The left RL conformable integral of order $\beta \in \mathbb{C}, \mathrm{Re}(\beta) > 0$, is given by

$$ {}_{a}^{\beta}I^{\alpha} f(t) = \frac{1}{\Gamma(\beta)} \int_{a}^{t} \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha} \right)^{\beta - 1} f(s) \frac{\mathrm{d}s}{(s-a)^{1-\alpha}}. \tag{5} $$

The fractional integral (5) coincides with the traditional RL fractional integral (1) if $\alpha = 1$.

Let $I_{\alpha}([a,b]) = \{f\colon [a,b] \longrightarrow \mathbb{R}; f(t) = {}_{a}I^{\alpha}\psi(t) + f(a)$ for some $\psi \in L_{\alpha}(a)\}$, where $L_{\alpha}(a) = \{\psi\colon [a,b] \longrightarrow \mathbb{R}; {}_{a}I^{\alpha}\psi(t)$ exists for any $t \in [a,b]\}$. If $n \in \mathbb{N}^{+}$, $f \in C_{\alpha,a}^{n}[a,b] = \{f\colon [a,b] \longrightarrow \mathbb{R}; {}_{a}^{n-1}T^{\alpha} f \in I_{\alpha}([a,b])\}$, the left conformable derivative of order $\beta \in \mathbb{C}, \mathrm{Re}(\beta) > 0$, in the Caputo setting is defined by

$$ \begin{aligned} {}_{a}^{C\beta}D^{\alpha} f(t) &= {}_{a}^{n-\beta}I^{\alpha} \left({}_{a}^{n}T^{\alpha} f(t)\right) \\ &= \frac{1}{\Gamma(n-\beta)} \int_{a}^{t} \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha} \right)^{n-\beta-1} \frac{{}_{a}^{n}T^{\alpha} f(s)}{(s-a)^{1-\alpha}} \mathrm{d}s, \end{aligned} \tag{6} $$

where $n = [\beta] + 1$, ${}_{a}^{n}T^{\alpha} = \underbrace{{}_{a}T^{\alpha}\,{}_{a}T^{\alpha} \cdots {}_{a}T^{\alpha}}_{n \text{ times}}$ and ${}_{a}T^{\alpha} f(t) = (t-a)^{1-\alpha} f'(t)$. The fractional derivative (6) coincides with the traditional Caputo fractional derivative (3) if $\alpha = 1$. Readers can see [13, 14] for more details.

It is well known that the monotone iterative technique coupled with the method of upper and lower solutions is an effective mechanism to obtain extremal solutions for nonlinear problems [15]. By using this method, scholars have studied the periodic boundary value problems (BVPs) [16–24], anti-periodic BVPs [25–27], and integral BVPs [28, 29] of integer-order differential equations. Later, this method was widely used to study the initial value problems, periodic BVPs or integral BVPs of RL and Caputo fractional differential equations [30–35].

Mathematical modeling of the real world in physics and mechanical and dynamical systems often involves the p-Laplacian operator. In order to study the turbulent flow in a porous medium, Lejbenson [36] introduced the model of differential equation with the p-Laplacian operator. Many results about the fractional differential equations with the p-Laplacian operator were also studied [37–39]. However, the Caputo conformable fractional differential equations with the p-Laplacian operator have not been considered.

In [37], Liu et al. studied the following problem:

$$ \begin{cases} D_{0^{+}}^{\beta}\left(\phi_{p}\left({}^{C}D_{0^{+}}^{\alpha} x(t)\right)\right) = h\left(t, x(t), {}^{C}D_{0^{+}}^{\alpha} x(t)\right), & t \in (0,1), \\ {}^{C}D_{0^{+}}^{\alpha} x(0) = x'(0) = 0, \\ x(1) = r_{1} x(\eta), {}^{C}D_{0^{+}}^{\alpha} x(1) = r_{2}\, {}^{C}D_{0^{+}}^{\alpha} x(\xi), \end{cases} \tag{7} $$

where $1 < \alpha, \beta \leq 2, r_{1}, r_{2} \geq 0$, $h \in C([0,1] \times t[0,+\infty) n \times q\ (-\infty, 0] h,_{[}0,+\infty))$, and $\phi_{p}(s)$ is the p-Laplacian operator. The extremal solutions were obtained under the assumption that

$$ h(t, w_{1}, z_{1}) \leq h(t, w_{2}, z_{2}), \quad \text{for } 0 \leq w_{1} < w_{2}, z_{1} > z_{2} \geq 0. \tag{8} $$

Inspired by the above work, we study the nonlinear fractional p-Laplacian differential system involving the Caputo conformable derivatives as follows:

$$ \begin{cases} {}_{a}^{C\beta}D^{\alpha}\left(\phi_{p}\left({}_{a}^{C\gamma}D^{\alpha} x(t)\right)\right) = h\left(t, x(t), {}_{a}^{C\gamma}D^{\alpha} x(t)\right), & t \in [a,b], \\ {}_{a}^{k}T^{\alpha} \phi_{p}\left({}_{a}^{C\gamma}D^{\alpha} x(a)\right) = b_{k}, x(a) = \int_{a}^{b} w(s, x(s)) \mathrm{d}s + \rho, \end{cases} \tag{9} $$

where $n-1 < \beta \leq n, n = [\beta] + 1, 0 < \gamma, \alpha \leq 1$, $\rho \geq 0$, $h \in C([a,b] \times \mathbb{R}^{2}, \mathbb{R})$, $w \in C([a,b] \times \mathbb{R}, \mathbb{R})$, $b_{k} (k = 0, 1, \ldots, n-1)$ are real numbers, $\phi_{p}(s) = |s|^{p-2}s\,(p > 1)$ is the p-Laplacian operator, $(\phi_{p})^{-1} = \phi_{q}$, $(1/p) + (1/q) = 1$, and ${}_{a}^{C\delta}D^{\alpha}$ is Caputo conformable derivative with order $\delta (= \beta, \gamma)$.

To obtain the extremal solutions of problem (9), we need consider the nonlinear Cauchy problem

$$ \begin{aligned} &{}_{a}^{C\beta}D^{\alpha} z(t) = g(t, z(t)), \\ &{}_{a}^{k}T^{\alpha} z(a) = b_{k}, \quad t \in (a,b], \end{aligned} \tag{10} $$

and the linear Cauchy problem

$$ \begin{aligned} &{}_{a}^{C\beta}D^{\alpha} z(t) = \sigma(t) - \lambda z(t), \\ &{}_{a}^{k}T^{\alpha} z(a) = b_{k}, \quad t \in [a,b]. \end{aligned} \tag{11} $$

The main contributions of this paper are as follows:

(i) We obtain the unique solution to problem (10) and construct the approximate solutions to problem (11) in terms of Mittag-Leffler function. The corresponding results of problem (10) and problem (11) can be seen as a generalization of Theorem 3.25 and Theorem 4.3 in [1], respectively.

(ii) Based on two comparison principles, we obtain the extremal solutions to problem (9) by using the monotone iterative technique. Different from [37], the restrictive condition of function h is no longer needed in this paper.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries and define some special spaces. In Section 3, we show the uniqueness of the solution for linear and nonlinear Cauchy problems. In Section 4, two comparison principles are established. In Section 5, the extremal solutions for problem (9) are obtained. In Section 6, a numerical example is given.

2. Preliminaries

In this section, we introduce some definitions and lemmas to be used in the sequel.

Let $C^{n}(J, \mathbb{R})$ be a Banach space of all n-order continuously differentiable functions on $J = [a,b]$. For $n-1 < \beta \leq n, 0 < \gamma, \alpha \leq 1$, and $0 \leq \tau < 1$ such that $\tau \leq \beta - k$, we define the spaces $C_{\gamma}(J), C_{\alpha,\tau}(J), C_{T}^{n}(J), C_{T,\tau}^{n}(J)$, and $C_{T,\tau}^{\beta,n}(J)$ as follows:

$$C_{\gamma}(J)=\left\{f: f(t)\in C(J), {}_{a}^{C\gamma}D^{\alpha}f(J)\in C(J)\right\}, \quad (12)$$

under the norm $\|f\|_{\gamma}=\|f\|_{C}+\|{}_{a}^{C\gamma}D^{\alpha}f\|_{C}$, where $\|f\|_{C}=\max_{t\in J}|f(t)|$ and $\|{}_{a}^{C\gamma}D^{\alpha}f\|_{C}=\max_{t\in J}|{}_{a}^{C\gamma}D^{\alpha}f(t)|$.

$$C_{\alpha,\tau}(J)=\left\{f(t)\in C(a,b]:\left(\frac{(t-a)^{\alpha}}{\alpha}\right)^{\tau}f(t)\in C(J)\right\},$$
$$C_{\alpha,0}(J)=C(J), \quad (13)$$

under the norm

$$\|f\|_{C_{\alpha,\tau}}=\left\|\left(\frac{(t-a)^{\alpha}}{\alpha}\right)^{\tau}f\right\|_{C}=\max_{t\in J}\left|\left(\frac{(t-a)^{\alpha}}{\alpha}\right)^{\tau}f(t)\right|,$$
$$C_{T}^{n}(J)=\left\{f(t)\in C(J): {}_{a}^{n}T^{\alpha}f(t)\in C(J)\right\}, \quad (14)$$

under the norm

$$\begin{aligned}
\|f\|_{C_{T}^{n}}&=\sum_{k=0}^{n}\|{}_{a}^{n}T^{\alpha}f\|_{C}=\sum_{k=0}^{n}\max_{t\in J}|{}_{a}^{n}T^{\alpha}f(t)|,\\
C_{T,\tau}^{n}(J)&=\left\{f(t)\in C(J): {}_{a}^{n-1}T^{\alpha}f(t)\in C(J), {}_{a}^{n}T^{\alpha}f(t)\in C_{\alpha,\tau}(J)\right\},\\
C_{T,\tau}^{\beta,n}(J)&=\left\{f(t)\in C_{T}^{n-1}(J): {}_{a}^{C\beta}D^{\alpha}f(t)\in C_{\alpha,\tau}(J)\right\}.
\end{aligned} \quad (15)$$

For convenience, we present the following assumptions:

($\mathbf{H}_1$) For $t\in(a,b]$, $z_1,z_2\in C_{T}^{n-1}(J)$, assume that function g satisfies

$$|g(t,z_1(t))-g(t,z_2(t))|\leq M|z_1(t)-z_2(t)|, \quad M\geq 0. \quad (16)$$

($\mathbf{H}_2$) Assume that $x_0(t)\leq y_0(t), t\in J$, where $x_0(t), y_0(t)\in C_{\gamma}(J)$ are lower and upper solutions of (9), respectively.

($\mathbf{H}_3$) Assume that a constant $\lambda\leq 0$ such that

$$\begin{aligned}
&h\left(t,y(t),{}_{a}^{C\gamma}D^{\alpha}y(t)\right)-h\left(t,x(t),{}_{a}^{C\gamma}D^{\alpha}x(t)\right)\\
&\geq-\lambda\left(\phi_{p}\left({}_{a}^{C\gamma}D^{\alpha}y(t)\right)-\phi_{p}\left({}_{a}^{C\gamma}D^{\alpha}x(t)\right)\right),
\end{aligned} \quad (17)$$

where $x_0(t)\leq x(t)\leq y(t)\leq y_0(t)$, $t\in J$.

($\mathbf{H}_4$) Assume that a constant $0\leq\eta<(1/(b-a))$ such that

$$w(t,y(t))-w(t,x(t))\geq\eta(y(t)-x(t)), \quad (18)$$

where $x_0(t)\leq x(t)\leq y(t)\leq y_0(t)$, $t\in J$.

Definition 1. The function $x_0(t)\in C_{\gamma}(J)$ satisfying $\phi_{p}({}_{a}^{C\gamma}D^{\alpha}x_0(t))\in C_{T}^{n}(J)$ is a lower solution of problem (9) if it satisfies

$$\begin{cases}
{}_{a}^{C\beta}D^{\alpha}\left(\phi_{p}\left({}_{a}^{C\gamma}D^{\alpha}x_0(t)\right)\right)\leq h\left(t,x_0(t),{}_{a}^{C\gamma}D^{\alpha}x_0(t)\right), & t\in J,\\
{}_{a}^{k}T^{\alpha}\phi_{p}\left({}_{a}^{C\gamma}D^{\alpha}x_0(a)\right)\leq b_k,\\
x(a)\leq\int_{a}^{b}w(s,x_0(s))\mathrm{d}s+\rho.
\end{cases} \quad (19)$$

The function $y_0(t)\in C_{\gamma}(J)$ satisfying $\phi_{p}({}_{a}^{C\gamma}D^{\alpha}y_0(t))\in C_{T}^{n}(J)$ is an upper solution of problem (9) if the above inequalities are reversed.

Lemma 1 (see [12]). *For $\alpha>0$, the space $C_{\alpha,a}^{n}(J)$ consists of those and only those functions which are represented in the form*

$$\begin{aligned}
f(t)&=\frac{1}{(n-1)!}\int_{a}^{t}\left(\frac{(t-a)^{\alpha}-(s-a)^{\alpha}}{\alpha}\right)^{n-1}\frac{\psi(s)\mathrm{d}s}{(s-a)^{1-\alpha}}\\
&\quad+\sum_{k=0}^{n-1}\frac{{}_{a}^{k}T^{\alpha}f(a)}{k!}\frac{(x-a)^{\alpha k}}{\alpha^{k}},
\end{aligned} \quad (20)$$

where $\psi(t)={}_{a}^{n}T^{\alpha}f(t)$ and $\psi(t)\in L_{\alpha}(a)$.

Lemma 2 (see [12]). *Let $f\in C_{\alpha,a}^{n}(J)$, $\beta\in\mathbb{C}$. Then,*

$${}_{a}^{\beta}I^{\alpha}\left({}_{a}^{C\beta}D^{\alpha}f(t)\right)=f(t)-\sum_{k=0}^{n-1}\frac{{}_{a}^{k}T^{\alpha}f(a)(t-a)^{\alpha k}}{\alpha^{k}k!}. \quad (21)$$

Remark 1. Lemma 1 still holds if we replace the space $C^n_{\alpha,a}(J)$ with $C^n_{T,\tau}(J)$. In such case, $\psi(t) = {}^n_aT^{\alpha} f(t)$ and $\psi(t) \in C_{\alpha,\tau}(J)$. In particular, $\psi(t) \in C(J)$ when $\tau = 0$. Lemma 2 is also valid for $f \in C^n_T(J)$.

Lemma 3 (see [12]). *Let* $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $0 < m < Re(\beta)$, $m \in \mathbb{N}$. *Then,*

(a) ${}^{\beta}_{a}I^{\alpha}({}^{\gamma}_{a}I^{\alpha}) f(t) = {}^{\beta+\gamma}_{a}I^{\alpha} f(t)$.

(b) ${}^{\beta}_{a}I^{\alpha}(t-a)^{\alpha(\gamma-1)} = (1/\alpha^{\beta})(\Gamma(\gamma)/\Gamma(\beta+\gamma))(t-a)^{\alpha(\beta+\gamma-1)}$.

(c) ${}^{m}_{a}T^{\alpha}({}^{\beta}_{a}I^{\alpha} f(t)) = {}^{\beta-m}_{a}I^{\alpha} f(t)$.

Lemma 4 (see [1]) (Banach fixed point theorem). *Let (U, d) be a nonempty complete metric space, let $0 \le \rho < 1$, and let $A\colon U \longrightarrow U$ be the map such that, for every $x, y \in U$, the relation*

$$d(Ax, Ay) \le \rho d(x, y), \tag{22}$$

holds. Then, the operator A has a unique fixed point $x^* \in U$. Moreover, if A^k ($k \in \mathbb{N}$) is the sequence of operators defined by

$$\begin{aligned} A^1 &= A, \\ A^k &= AA^{k-1} (k \in \mathbb{N}\backslash\{1\}), \end{aligned} \tag{23}$$

then for any $x_0 \in U$, the sequence $\{A^k x_0\}_{k=1}^{\infty}$ converges to the above fixed point x^*.

3. The Unique Solution to the Nonlinear and Linear Cauchy Problems

In this section, we first consider the unique solution of nonlinear Cauchy problem (10) and linear Cauchy problem (11), where the function $g\colon (a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $g \in C_{\alpha,\tau}(J)$.

Let $z(t) \in C^{n-1}_T(J)$, and by Lemma 2 and the initial value condition ${}^k_aT^{\alpha} z(t) = b_k$, problem (10) can be reduced to the Volterra-type integral equation

$$z(t) = \sum_{j=0}^{n-1} \frac{b_j(t-a)^{\alpha j}}{\alpha^j j!} + \frac{1}{\Gamma(\beta)} \int_a^t K^{\beta-1}(t,s) g(s, z(s)) \frac{\mathrm{d}s}{(s-a)^{1-\alpha}}, \tag{24}$$

where $K(t, s) = (((t-a)^{\alpha} - (s-a)^{\alpha})/\alpha), a \le s \le t \le b$. Denoting $z_0 = \sum_{j=0}^{n-1} (b_j(t-a)^{\alpha j}/\alpha^j j!)$, equation (24) can be rewritten as $z(t) = (Az)(t)$, where

$$(Az)(t) = z_0(t) + \frac{1}{\Gamma(\beta)} \int_a^t K^{\beta-1}(t,s) g(s, z(s)) \frac{\mathrm{d}s}{(s-a)^{1-\alpha}}. \tag{25}$$

Theorem 1. *If (H_1) holds, there exists a unique solution $z(t) \in C^{\beta,n-1}_{T,\tau}(J)$ for problem (10).*

Proof. First, we choose t_1 ($a < t_1 < b$) such that

$$\sum_{k=0}^{n-1} \frac{M}{\Gamma(\beta-k+1)} \left(\frac{(t_1-a)^{\alpha}}{\alpha}\right)^{\beta-k} < 1, \tag{26}$$

and prove that (10) has a unique solution $z(t) \in C^{n-1}_T[a, t_1]$.

Applying the operator ${}^k_aT^{\alpha}$ to both sides of (25), by Lemma 3 (c), we can get

$${}^k_aT^{\alpha}(Az(t)) = {}^k_aT^{\alpha} z_0(t) + \frac{1}{\Gamma(\beta-k)} \int_a^t K^{\beta-k-1}(t,s) \frac{g(s, z(s))\mathrm{d}s}{(s-a)^{1-\alpha}}, \tag{27}$$

where ${}^k_aT^{\alpha} z_0(t) = \sum_{j=k}^{n-1} (b_j(t-a)^{(j-k)}/\alpha^{j-k}(j-k)!)$. It is obvious that ${}^k_aT^{\alpha} z_0(t)$ is continuous on $[a, t_1]$. Furthermore, for $0 \le \tau < \beta - k$ and $g \in C_{\alpha,\tau}(J)$, we get by (14) that

$$\begin{aligned} \left|{}^{\beta-k}_{a}I^{\alpha} g\right| &= \left|\frac{1}{\Gamma(\beta-k)} \int_a^t K^{\beta-k-1}(t,s) g(s, z(s)) \frac{\mathrm{d}s}{(s-a)^{1-\alpha}}\right| \\ &\le \frac{1}{\Gamma(\beta-k)} \int_a^t \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha}\right)^{\beta-k-1} |g(s, z(s))| \frac{\mathrm{d}s}{(s-a)^{1-\alpha}} \\ &\le \frac{\|g\|_{C_{\alpha,\tau}[a,t_1]}}{\Gamma(\beta-k)} \int_a^t \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha}\right)^{\beta-k-1} \left(\frac{(s-a)^{\alpha}}{\alpha}\right)^{-\tau} \frac{\mathrm{d}s}{(s-a)^{1-\alpha}} \\ &\le \frac{\Gamma(1-\tau)\|g\|_{C_{\alpha,\tau}[a,t_1]}}{\Gamma(\beta-k+1-\tau)} \left(\frac{(t_1-a)^{\alpha}}{\alpha}\right)^{\beta-k-\tau}, \end{aligned} \tag{28}$$

that is,

$$\left\| {}_a^{\beta-k}I^{\alpha} g \right\|_{C[a,t_1]} \le \frac{\Gamma(1-\tau)\|g\|_{C_{\alpha,\tau}[a,t_1]}}{\Gamma(\beta-k+1-\tau)} \left(\frac{(t_1-a)^{\alpha}}{\alpha} \right)^{\beta-k-\tau}. \tag{29}$$

Inequality (29) implies that the operator ${}_a^{\beta-k}I^{\alpha}$ is bounded from $C_{\alpha,\tau}[a,t_1]$ to $C[a,t_1]$. In particular, if $\tau = 0$, then $g \in C(J)$ and ${}_a^{\beta-k}I^{\alpha}$ is bounded from $C[a,t_1]$ to $C[a,t_1]$ such that

$$\left\| {}_a^{\beta-k}I^{\alpha} g \right\|_{C[a,t_1]} \le \frac{\|g\|_{C[a,t_1]}}{\Gamma(\beta-k+1)} \left(\frac{(t_1-a)^{\alpha}}{\alpha} \right)^{\beta-k}. \tag{30}$$

It follows from (29) and (30) that ${}_a^kT^{\alpha}(Az(t))$ is continuous on $[a,t_1]$, that is, $(Az)(t) \in C_T^{n-1}[a,t_1]$. By (14), (15), and (H_1), we have

$$\begin{aligned}
&\sum_{k=0}^{n-1} \left| {}_a^kT^{\alpha}(Az_1(t)) - {}_a^kT^{\alpha}(Az_2(t)) \right| \\
&\quad = \sum_{k=0}^{n-1} \left| \frac{1}{\Gamma(\beta-k)} \int_a^t K^{\beta-k-1}(t,s) \left[g(s,z_1(s)) - g(s,z_2(s)) \right] \frac{ds}{(s-a)^{1-\alpha}} \right| \\
&\quad \le \sum_{k=0}^{n-1} \frac{1}{\Gamma(\beta-k)} \int_a^t K^{\beta-k-1}(t,s) \left| g(s,z_1(s)) - g(s,z_2(s)) \right| \frac{ds}{(s-a)^{1-\alpha}} \\
&\quad \le \sum_{k=0}^{n-1} \frac{M}{\Gamma(\beta-k)} \int_a^t \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha} \right)^{\beta-k-1} \left| z_1(s) - z_2(s) \right| \frac{ds}{(s-a)^{1-\alpha}} \\
&\quad \le \sum_{k=0}^{n-1} \frac{M\|z_1 - z_2\|_{C[a,t_1]}}{\Gamma(\beta-k+1)} \left(\frac{(t_1-a)^{\alpha}}{\alpha} \right)^{\beta-k} \\
&\quad \le \sum_{k=0}^{n-1} \frac{M\|z_1 - z_2\|_{C_T^{n-1}[a,t_1]}}{\Gamma(\beta-k+1)} \left(\frac{(t_1-a)^{\alpha}}{\alpha} \right)^{\beta-k},
\end{aligned} \tag{31}$$

that is,

$$\begin{aligned}
\|Az_1 - Az_2\|_{C_T^{n-1}[a,t_1]} &= \sum_{k=0}^{n-1} \left\| {}_a^kT^{\alpha}(Az_1(t)) - {}_a^kT^{\alpha}(Az_2(t)) \right\|_{C[a,t_1]} \\
&\le \sum_{k=0}^{n-1} \frac{M\|z_1 - z_2\|_{C_T^{n-1}[a,t_1]}}{\Gamma(\beta-k+1)} \left(\frac{(t_1-a)^{\alpha}}{\alpha} \right)^{\beta-k}.
\end{aligned} \tag{32}$$

From Lemma 4 and (26), we get that there exists a unique solution $z_1^* \in C_T^{n-1}[a,t_1]$ to problem (10). Moreover, $z_1^*(t)$ satisfies

$$\lim_{i\to\infty} \left\| A^i z_g - z_1^* \right\|_{C_T^{n-1}[a,t_1]} = 0, \tag{33}$$

where $z_g(t)$ is any function in $C_T^{n-1}[a,t_1]$ and $A^i z_g(t) = AA^{i-1}z_g(t)$. Let $z_i(t) = A^i z_g(t)$; then,

$$\lim_{i\to\infty} \left\| z_i(t) - z_1^* \right\|_{C_T^{n-1}[a,t_1]} = 0. \tag{34}$$

Next, choose $t_2, t_3, \ldots, t_R$ such that $a = t_0 < t_1 < t_2 < \cdots < t_R = b$. Using the same arguments as above, we get that problem (10) has a unique solution $z_r^* \in C_T^{n-1}[t_{r-1}, t_r]$ $(r = 1, 2, \ldots, R)$. Therefore, (10) has a unique solution $z^* = z_r^* \in C_T^{n-1}(J)$.

Finally, we show that the unique solution $z^*(t)$ belongs to $C_{T,\tau}^{\beta,n-1}(J)$. By (15) and (H_1), we have

$$\begin{aligned}
\left\| {}_a^{C\beta}D^{\alpha} z_i - {}_a^{C\beta}D^{\alpha} z^* \right\|_{C_{\alpha,\tau}(J)} &= \left\| g(t,z_i(t)) - g(t,z^*(t)) \right\|_{C_{\alpha,\tau}(J)} \\
&\le M \left(\frac{(t_1-a)^{\alpha}}{\alpha} \right)^{\tau} \left\| z_i - z^* \right\|_{C(J)} \\
&\le M \left(\frac{(t_1-a)^{\alpha}}{\alpha} \right)^{\tau} \left\| z_i - z^* \right\|_{C_T^{n-1}(J)}.
\end{aligned} \tag{35}$$

Taking the limit as $i \longrightarrow \infty$, we obtain

$$\lim_{i\longrightarrow\infty} \left\| {}_{a}^{C\beta}D^{\alpha} z_i - {}_{a}^{C\beta}D^{\alpha} z^* \right\|_{C_{\alpha,\tau}(J)} = 0, \quad (36)$$

which implies that $z^*(t) \in C_{T,\tau}^{\beta,n-1}(J)$. This ends the proof. □

Remark 2. When $\alpha = 1$, the RL conformable integral and Caputo conformable derivative coincide with the traditional RL fractional integral and Caputo derivative, respectively. Hence, the results of Theorem 3.25 in [1] can be seen as the special case of Theorem 1.

Corollary 1. *If (H_1) holds and the function $g\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $g \in C(J)$, then Cauchy problem (10) has a unique solution $z(t)$ belonging to $C_T^{n-1}(J)$.*

Proof. Corollary 1 can be proven by replacing τ with 0 and using the same argument in Theorem 1. □

Theorem 2. *If $\sigma(t) \in C(J)$ and λ is a constant, then problem (11) has a unique solution $z(t) \in C_T^{n-1}(J)$ which is given by*

$$z(t) = \sum_{j=0}^{n-1} \frac{b_j (t-a)^{j\alpha}}{\alpha^j} E_{\beta,j+1}\left[\frac{-\lambda (t-a)^{\alpha\beta}}{\alpha^{\beta}}\right] + \int_a^t K^{\beta-1}(t,s) E_{\beta,\beta}\left(-\lambda K^{\beta}(t,s)\right) \frac{\sigma(s)\mathrm{d}s}{(s-a)^{1-\alpha}}, \quad (37)$$

where $E_{p,q}(\rho) = \sum_{k=0}^{\infty} (\rho^k / \Gamma(pk+q))$ is the Mittag-Leffler function.

Proof. Clearly $g(t, z(t)) = \sigma(t) - \lambda z(t)$ satisfies (H_1). By Corollary 1, there exists a unique solution $z(t) \in C_T^{n-1}(J)$ to problem (11).

Next, we prove that this unique solution is given by (18). By Lemma 2 and the initial value condition ${}_{a}^{k}T^{\alpha} z(t) = b_k$, problem (11) can be reduced to the equation

$$z(t) = \sum_{j=0}^{n-1} \frac{b_j (t-a)^{\alpha j}}{\alpha^j j!} + (-\lambda) {}_{a}^{\beta}I^{\alpha} z(t) + {}_{a}^{\beta}I^{\alpha} \sigma(t). \quad (38)$$

We apply the successive approximations method to solve equation (38) by taking $z_0(t) = \sum_{j=0}^{n-1} b_j (t-a)^{\alpha j} / \alpha^j j!$ and

$$z_m(t) = z_0(t) + (-\lambda) {}_{a}^{\beta}I^{\alpha} z_{m-1}(t) + {}_{a}^{\beta}I^{\alpha} \sigma(t), \quad m = 1, 2, \ldots. \quad (39)$$

By Lemma 3 (b), for $m = 1$, we have

$$\begin{aligned} z_1(t) &= z_0(t) + (-\lambda) {}_{a}^{\beta}I^{\alpha} z_0(t) + {}_{a}^{\beta}I^{\alpha} \sigma(t) \\ &= z_0(t) + \sum_{j=0}^{n-1} \frac{(-\lambda) b_j}{\alpha^j j!} {}_{a}^{\beta}I^{\alpha} (t-a)^{\alpha j} + {}_{a}^{\beta}I^{\alpha} \sigma(t) \\ &= z_0(t) + \sum_{j=0}^{n-1} \frac{(-\lambda) b_j (t-a)^{\alpha(\beta+j)}}{\alpha^{\beta+j} \Gamma(\beta+j+1)} + {}_{a}^{\beta}I^{\alpha} \sigma(t). \end{aligned} \quad (40)$$

By (a) and (b) of Lemma 3, for $m = 2$,

$$\begin{aligned} z_2(t) &= z_0(t) + (-\lambda) {}_{a}^{\beta}I^{\alpha} z_1(t) + {}_{a}^{\beta}I^{\alpha} \sigma(t) \\ &= z_0(t) + (-\lambda) {}_{a}^{\beta}I^{\alpha} \left[z_0(t) + (-\lambda) {}_{a}^{\beta}I^{\alpha} z_0(t) + {}_{a}^{\beta}I^{\alpha} \sigma(t) \right] + {}_{a}^{\beta}I^{\alpha} \sigma(t) \\ &= z_0(t) + (-\lambda) {}_{a}^{\beta}I^{\alpha} z_0(t) + (-\lambda)^2 {}_{a}^{2\beta}I^{\alpha} z_0(t) + (-\lambda) {}_{a}^{2\beta}I^{\alpha} \sigma(t) + {}_{a}^{\beta}I^{\alpha} \sigma(t) \\ &= z_0(t) + \sum_{j=0}^{n-1} \frac{(-\lambda) b_j (t-a)^{\alpha(\beta+j)}}{\alpha^{\beta+j} \Gamma(\beta+j+1)} + \sum_{j=0}^{n-1} \frac{(-\lambda)^2 b_j (t-a)^{\alpha(2\beta+j)}}{\alpha^{2\beta+j} \Gamma(2\beta+j+1)} \\ &\quad + (-\lambda) {}_{a}^{2\beta}I^{\alpha} \sigma(t) + {}_{a}^{\beta}I^{\alpha} \sigma(t). \end{aligned} \quad (41)$$

Continuing this process, we have

$$\begin{aligned} z_m(t) &= \sum_{j=0}^{n-1} \frac{b_j (t-a)^{j\alpha}}{\alpha^j} \sum_{r=0}^{m} \frac{(-\lambda)^r (t-a)^{r\alpha\beta}}{\alpha^{r\beta} \Gamma(r\beta+j+1)} + \sum_{r=1}^{m} (-\lambda)^{r-1} {}_{a}^{r\beta}I^{\alpha} \sigma(t) \\ &= \sum_{j=0}^{n-1} \frac{b_j (t-a)^{j\alpha}}{\alpha^j} \sum_{r=0}^{m} \frac{(-\lambda)^r (t-a)^{r\alpha\beta}}{\alpha^{r\beta} \Gamma(r\beta+j+1)} \\ &\quad + \int_a^t K^{\beta-1}(t,s) \sum_{r=0}^{m} \frac{(-\lambda)^r K^{r\beta}(t,s)}{\Gamma((r+1)\beta)} \frac{\sigma(s)\mathrm{d}s}{(s-a)^{1-\alpha}}. \end{aligned} \quad (42)$$

Taking the limit as $m \longrightarrow \infty$ and according to the definition of Mittag-Leffler function, we get formula (37). This ends the proof. □

Remark 3. If $\alpha = 1$, we can get the results of Theorem 4.3 in [1].

4. Comparison Principles

In this section, two comparison principles which will be used in the next section are established.

Lemma 5. *Let $\eta \neq (1/b-a)$ and $y(t) \in C(J)$. Then, the following problem:*

$$\begin{cases} {}_a^{C\gamma}D^{\alpha} x(t) = y(t), \quad t \in J, \\ x(a) = \eta \int_a^b x(s)\mathrm{d}s + \rho, \end{cases} \tag{43}$$

is equivalent to

$$x(t) = \int_a^b G(t,s)y(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + \frac{\rho}{1-\eta(b-a)}, \tag{44}$$

where

$$G(t,s) = \frac{1}{\xi}\begin{cases} \dfrac{[1-\eta(b-a)]\gamma(b-a)^{\alpha-1}K^{\gamma-1}(t,s) + \eta K^{\gamma}(b,s)}{1-\eta(b-a)}, & a \le s \le t \le b, \\ \dfrac{\eta K^{\gamma}(b,s)}{1-\eta(b-a)}, & a \le t \le s \le b, \end{cases} \tag{45}$$

$$\xi = (b-a)^{\alpha-1}\Gamma(\gamma+1).$$

Proof. For $0 < \gamma \le 1$, by Lemma 2, equation ${}_a^{C\gamma}D^{\alpha} x(t) = y(t)$ can be reduced to

$$x(t) = {}_a^{\gamma}I^{\alpha} y(t)c_0, \tag{46}$$

where c_0 is a constant. By the boundary condition, we easily get $c_0 = \eta \int_a^b x(s)\mathrm{d}s + \rho$. Hence,

$$x(t) = {}_a^{\gamma}I^{\alpha} y(t) + \eta \int_a^b x(s)\mathrm{d}s + \rho. \tag{47}$$

Let $\Delta = \int_a^b x(t)\mathrm{d}t$, and we can deduce from (47) that

$$\begin{aligned} \Delta &= \int_a^b \frac{1}{\Gamma(\gamma)} \int_a^t \left(\frac{(t-a)^{\alpha} - (s-a)^{\alpha}}{\alpha}\right)^{\gamma-1} y(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}}\mathrm{d}t + \int_a^b (\eta\Delta + \rho)\mathrm{d}t \\ &= \int_a^b \frac{1}{\Gamma(\gamma+1)(b-a)^{\alpha-1}} \left(\frac{(b-a)^{\alpha} - (s-a)^{\alpha}}{\alpha}\right)^{\gamma} \frac{y(s)\mathrm{d}s}{(s-a)^{1-\alpha}} + (\eta\Delta + \rho)(b-a) \\ &= \int_a^b \frac{K^{\gamma}(b,s)y(s)}{\Gamma(\gamma+1)(b-a)^{\alpha-1}} \frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + (\eta\Delta + \rho)(b-a). \end{aligned} \tag{48}$$

Therefore,

$$\Delta = \int_a^b \frac{K^{\gamma}(b,s)y(s)}{[1-\eta(b-a)]\Gamma(\gamma+1)(b-a)^{\alpha-1}} \frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + \frac{\rho(b-a)}{1-\eta(b-a)}. \tag{49}$$

Substituting (49) into (47), we have

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\gamma)}\int_a^t K^{\gamma-1}(t,s)y(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + \frac{\eta\rho(b-a)}{1-\eta(b-a)} + \rho \\ &\quad + \int_a^b \frac{\eta K^{\gamma}(b,s)y(s)}{[1-\eta(b-a)]\Gamma(\gamma+1)(b-a)^{\alpha-1}}\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} \\ &= \frac{1}{\xi}\int_a^t \frac{[1-\eta(b-a)]\gamma(b-a)^{\alpha-1}K^{\gamma-1}(t,s)+\eta K^{\gamma}(b,s)}{1-\eta(b-a)}y(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} \\ &\quad + \frac{1}{\xi}\int_t^b \frac{\eta K^{\gamma}(b,s)}{1-\eta(b-a)}y(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + \frac{\rho}{1-\eta(b-a)} \\ &= \int_a^b G(t,s)y(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + \frac{\rho}{1-\eta(b-a)}. \end{aligned} \tag{50}$$

This ends the proof. □

Remark 4. For $a \le t, s \le b$, $0 \le \eta < (1/(b-a))$, the function $G(t,s)$ is continuous and nonnegative.

Lemma 6. *For $x(t) \in C_\gamma(J)$, the following linear problem:*

$$\begin{cases} {}_a^{C\beta}D^{\alpha}\left(\phi_p\left({}_a^{C\gamma}D^{\alpha}(t)\right)\right) = \sigma(t) - \lambda\phi_p\left({}_a^{C\gamma}D^{\alpha}x(t)\right), & t \in J, \\ {}_a^{k}T^{\alpha}\phi_p\left({}_a^{C\gamma}D^{\alpha}x(a)\right) = b_k, \\ x(a) = \eta\int_a^b x(s)\mathrm{d}s + \rho, \end{cases} \tag{51}$$

has a unique solution.

Proof. Let $z(t) = \phi_p({}_a^{C\gamma}D^{\alpha}x(t))$. By Theorem 2, the problem

$$\begin{aligned} {}_a^{C\beta}D^{\alpha}z(t) &= \sigma(t) - \lambda z(t), \\ {}_a^{k}T^{\alpha}z(t) &= b_k, \quad t \in J \end{aligned} \tag{52}$$

has a unique solution $z(t) \in C_T^{n-1}(J)$, that is, ${}_a^{C\gamma}D^{\alpha}x(t) = \phi_q(z(t)) \in C_T^{n-1}(J)$. Hence, ${}_a^{C\gamma}D^{\alpha}x(t) \in C(J)$.

By Lemma 5, the following problem

$$\begin{aligned} {}_a^{C\gamma}D^{\alpha}x(t) &= \phi_q(z(t)), \\ x(a) &= \eta\int_a^b x(s)\mathrm{d}s + \rho, \quad t \in J, \end{aligned} \tag{53}$$

is equivalent to

$$x(t) = \int_a^b G(t,s)\phi_q(z(s))\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + \frac{\rho}{1-\eta(b-a)}. \tag{54}$$

Considering (52) and (53), we obtain the conclusion that problem (51) has a unique solution which is given by (54). □

Based on the above work, we can get the following comparison principles.

Lemma 7. *If $\lambda \le 0$ and $z(t) \in C_T^{n-1}(J)$ satisfy the following relation:*

$$\left\{ {}_a^{C\beta}D^{\alpha}z(t) \ge -\lambda z(t), {}_a^{k}T^{\alpha}z(a) \ge 0, \right. \tag{55}$$

then for $t \in J, z(t) \ge 0$.

Proof. Let ${}_a^{C\beta}D^{\alpha}z(t) = p(t) - \lambda z(t), {}_a^{k}T^{\alpha}z(a) = a_k$; then, $p(t) \ge 0, a_k \ge 0$. From (37), we can see that $z(t) \ge 0$. This ends the proof. □

Lemma 8. *If $0 \le \eta < (1/(b-a))$ and $x(t) \in C_\gamma(J)$ satisfy the following relation:*

$$\left\{ {}_a^{C\gamma}D^{\alpha}x(t) \ge 0, x(a) \ge \eta\int_a^b x(s)\mathrm{d}s, \right. \tag{56}$$

then for $t \in J, x(t) \ge 0$.

Proof. Let ${}_a^{C\gamma}D^{\alpha}x(t) = q(t), x(a) = \int_a^b \eta x(s)\mathrm{d}s + d$; then, $q(t) \ge 0, d \ge 0$. From (53) and (54), we have that

$$x(t) = \int_a^b G(t,s)q(s)\frac{\mathrm{d}s}{(s-a)^{1-\alpha}} + \frac{d}{1-\eta(b-a)}, \tag{57}$$

which implies that $x(t) \ge 0$ due to $G(t,s) \ge 0$. This ends the proof. □

5. Extremal Solutions for Nonlinear System

The extremal solutions of problem (9) are obtained in this section.

Theorem 3. *If (H_2)–(H_4) hold, then problem (9) has extremal solutions $x^*(t), y^*(t)$ in the sector $[x_0, y_0] = \left\{x(t) \in C_\gamma(J)\colon x_0(t) \le x(t) \le y_0(t), t \in J\right\}$. Moreover,*

$$
\begin{aligned}
&x_0(t) \le x^*(t) \le y^*(t) \le y_0(t), \\
&{}_a^{C\gamma}D^{\alpha} x_0(t) \le {}_a^{C\gamma}D^{\alpha} x^*(t) \le {}_a^{C\gamma}D^{\alpha} y^*(t) \le {}_a^{C\gamma}D^{\alpha} y_0(t).
\end{aligned} \tag{58}
$$

Proof. For $t \in J, n = 1, 2, \ldots,$ define

$$
\begin{cases}
{}_a^{C\beta}D^{\alpha}\left(\phi_p\left({}_a^{C\gamma}D^{\alpha} x_n(t)\right)\right) = h\left(t, x_{n-1}(t), {}_a^{C\gamma}D^{\alpha} x_{n-1}(t)\right) - \lambda\left[\phi_p\left({}_a^{C\gamma}D^{\alpha} x_n(t)\right) - \phi_p\left({}_a^{C\gamma}D^{\alpha} x_{n-1}(t)\right)\right], \\
{}_a^{k}T^{\alpha}\phi_p\left({}_a^{C\gamma}D^{\alpha} x_n(a)\right) = b_k, \\
x_n(a) = \int_a^b \left[w(s, x_{n-1}(s)) + \eta(x_n(s) - x_{n-1}(s))\right] ds + \rho,
\end{cases} \tag{59}
$$

$$
\begin{cases}
{}_a^{C\beta}D^{\alpha}\left(\phi_p\left({}_a^{C\gamma}D^{\alpha} y_n(t)\right)\right) = h\left(t, y_{n-1}(t), {}_a^{C\gamma}D^{\alpha} y_{n-1}(t)\right) - \lambda\left[\phi_p\left({}_a^{C\gamma}D^{\alpha} y_n(t)\right) - \phi_p\left({}_a^{C\gamma}D^{\alpha} y_{n-1}(t)\right)\right], \\
{}_a^{k}T^{\alpha}\phi_p\left({}_a^{C\gamma}D^{\alpha} y_n(a)\right) = b_k, \\
y_n(a) = \int_a^b \left[w(s, y_{n-1}(s)) + \eta(y_n(s) - y_{n-1}(s))\right] ds + \rho.
\end{cases} \tag{60}
$$

By Lemma 6, x_n, y_n are well defined. The proof includes three steps. □

Step 1. We prove the monotone property of $\{x_n\}$ and $\{y_n\}$.

Let $r(t) = \phi_p({}_a^{C\gamma}D^{\alpha} x_1(t)) - \phi_p({}_a^{C\gamma}D^{\alpha} x_0(t))$, and by (H_2) and (59), we get

$$
\left\{ {}_a^{C\beta}D^{\alpha} r(t) \ge -\lambda r(t), \quad t \in J, {}_a^{k}T^{\alpha} r(a) \ge 0. \right. \tag{61}
$$

From Lemma 7, we have $r(t) \ge 0$, i.e., $\phi_p({}_a^{C\gamma}D^{\alpha} x_1(t)) \ge \phi_p({}_a^{C\gamma}D^{\alpha} x_0(t))$. Moreover,

$$
{}_a^{C\gamma}D^{\alpha} x_1(t) \ge {}_a^{C\gamma}D^{\alpha} x_0(t) \tag{62}
$$

holds because of the monotone increasing property of $\phi_p(s)$.

Let $\tilde{r}(t) = x_1(t) - x_0(t)$. From (H_2), (H_4), (59), and (62), we have

$$
\begin{cases}
{}_a^{C\gamma}D^{\alpha}\tilde{r}(t) = {}_a^{C\gamma}D^{\alpha} x_1(t) - {}_a^{C\gamma}D^{\alpha} x_0(t) \ge 0, \\
\tilde{r}(a) \ge \int_a^b \eta\tilde{r}(s) ds.
\end{cases} \tag{63}
$$

From Lemma 8, we have $\tilde{r}(t) \ge 0$, i.e., $x_1(t) \ge x_0(t)$. The same argument holds that ${}_a^{C\gamma}D^{\alpha} y_0(t) \ge {}_a^{C\gamma}D^{\alpha} y_1(t)$, $y_0(t) \ge y_1(t)$. Let $m(t) = \phi_p({}_a^{C\gamma}D^{\alpha} y_1(t)) - \phi_p({}_a^{C\gamma}D^{\alpha} x_1(t))$, and from (H_3), (59), and (60), we get

$$
\begin{aligned}
{}_a^{C\beta}D^{\alpha} m(t) &= h\left(t, y_0(t) {}_a^{C\gamma}D^{\alpha} y_0(t)\right) - h\left(t, x_0(t), {}_a^{C\gamma}D^{\alpha} x_0(t)\right) \\
&\quad - \lambda\left[\phi_p\left({}_a^{C\gamma}D^{\alpha} y_1(t)\right) - \phi_p\left({}_a^{C\gamma}D^{\alpha} y_0(t)\right)\right] \\
&\quad + \lambda\left[\phi_p\left({}_a^{C\gamma}D^{\alpha} x_1(t)\right) - \phi_p\left({}_a^{C\gamma}D^{\alpha} x_0(t)\right)\right] \\
&\ge -\lambda m(t), \\
{}_a^{k}T^{\alpha} m(a) &= 0.
\end{aligned} \tag{64}
$$

From Lemma 7, we have $m(t) \ge 0$, i.e., $\phi_p({}_a^{C\gamma}D^{\alpha} y_1(t)) \ge \phi_p({}_a^{C\gamma}D^{\alpha} x_1(t))$. Hence,

$$
{}_a^{C\gamma}D^{\alpha} y_1(t) \ge {}_a^{C\gamma}D^{\alpha} x_1(t). \tag{65}
$$

Let $\tilde{m}(t) = y_1 - x_1$. We get from (65) and (H_4) that

$$
\begin{cases}
{}_a^{C\gamma}D^{\alpha}\tilde{m}(t) = {}_a^{C\gamma}D^{\alpha} y_1(t) - {}_a^{C\gamma}D^{\alpha} x_1(t) \ge 0, \\
\tilde{m}(a) = \int_a^b \left[w(s, y_0(s)) + \eta(y_1(s) - y_0(s)) - w(s, x_0(s)) - \eta(x_1(s) - x_0(s))\right] ds \ge \int_a^b \eta\tilde{m}(s) ds.
\end{cases} \tag{66}
$$

From Lemma 8, we have $\tilde{m}(t) \ge 0$, i.e. $y_1 \ge x_1$. Therefore, $x_0 \le x_1 \le y_1 \le y_0$ and ${}_a^{C\gamma}D^{\alpha} x_0 \le {}_a^{C\gamma}D^{\alpha} x_1 \le {}_a^{C\gamma}D^{\alpha} y_1 \le {}_a^{C\gamma}D^{\alpha} y_0$.

Next we prove $x_1(t)$, $y_1(t)$ are lower and upper solutions of (9), respectively. From (H_3), (H_4), and (59), we have

$$\begin{aligned}
{}_{a}^{C\beta}D^{\alpha}\left(\phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_1(t)\right)\right) &= h\left(t, x_0(t), {}_{a}^{C\gamma}D^{\alpha}x_0(t)\right) - h\left(t, x_1(t), {}_{a}^{C\gamma}D^{\alpha}x_1(t)\right)\\
&\quad + h\left(t, x_1(t), {}_{a}^{C\gamma}D^{\alpha}x_1(t)\right) - \lambda\left[\phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_1(t)\right) - \phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_0(t)\right)\right]\\
&\leq -\lambda\left[\phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_0(t)\right) - \phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_1(t)\right)\right] + h\left(t, x_1(t), {}_{a}^{C\gamma}D^{\alpha}x_1(t)\right)\\
&\quad - \lambda\left[\phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_1(t)\right) - \phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_0(t)\right)\right] = h\left(t, x_1(t), {}_{a}^{C\gamma}D^{\alpha}x_1(t)\right),\\
{}_{a}^{k}T^{\alpha}\phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_1(a)\right) &= b_k,\\
x_1(a) &= \int_a^b \left[w(s, x_0(s)) - w(s, x_1(s)) + w(s, x_1(s)) + \eta(x_1(s) - x_0(s))\right]\mathrm{d}s + \rho\\
&\leq \int_a^b \left[\eta(x_0(s) - x_1(s)) + \eta(x_1(s) - x_0(s)) + w(s, x_1(s))\right]\mathrm{d}s + \rho\\
&= \int_a^b w(s, x_1(s))\mathrm{d}s + \rho.
\end{aligned} \tag{67}$$

Clearly, $x_1(t)$ is a lower solution of (9). Similarly, $y_1(t)$ is an upper solution of (9). We obtain by applying mathematical induction that

$$\begin{aligned}
&x_0(t) \leq x_1(t) \leq \cdots \leq x_n(t) \leq \cdots \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t),\\
&{}_{a}^{C\gamma}D^{\alpha}x_0(t){}_{a}^{C\gamma}D^{\alpha}x_1(t) \leq \cdots \leq {}_{a}^{C\gamma}D^{\alpha}x_n(t) \leq \cdots {}_{a}^{C\gamma}D^{\alpha}y_n(t)\\
&\leq \cdots \leq {}_{a}^{C\gamma}D^{\alpha}y_1(t) \leq {}_{a}^{C\gamma}D^{\alpha}y_0(t).
\end{aligned} \tag{68}$$

Step 2. We conclude that the sequences $\{x_n\}$ and $\{y_n\}$ satisfy the relations:

$$\begin{aligned}
\lim_{n\longrightarrow\infty} x_n(t) &= x^*(t),\\
\lim_{n\longrightarrow\infty} {}_{a}^{C\gamma}D^{\alpha}x_n(t) &= {}_{a}^{C\gamma}D^{\alpha}x^*(t),
\end{aligned} \tag{69}$$

$$\begin{aligned}
\lim_{n\longrightarrow\infty} y_n(t) &= y^*(t),\\
\lim_{n\longrightarrow\infty} {}_{a}^{C\gamma}D^{\alpha}y_n(t) &= {}_{a}^{C\gamma}D^{\alpha}y^*(t).
\end{aligned} \tag{70}$$

Let $F(x_n)(t) = h(t, x_n(t), {}_{a}^{C\gamma}D^{\alpha}x_n(t)) + \lambda\phi_p({}_{a}^{C\gamma}D^{\alpha}x_n(t))$. We can see that the function F is continuous and nondecreasing from the assumption of h. By (37) and (54), equation (59) can be reduced to the equation

$$\begin{aligned}
x_n(t)) \int_a^b G(t,s)\phi_q &\left[\sum_{j=0}^{n-1} \frac{b_j(s-a)^{j\alpha}}{\alpha^j}E_{\beta,j+1}\left(\frac{-\lambda(s-a)^{\alpha\beta}}{\alpha^\beta}\right)\right.\\
&\left.+ \int_a^s K^{\beta-1}(s,\theta)E_{\beta,\beta}\left(-\lambda K^\beta(s,\theta)\right)\frac{F(x_{n-1}(\theta))\mathrm{d}\theta}{(\theta-a)^{1-\alpha}}\right]\frac{\mathrm{d}s}{(s-a)^{1-\alpha}}\\
&+ \frac{\int_a^b [w(s, x_{n-1}(s)) - \eta x_{n-1}(s)]\mathrm{d}s + \rho}{1-\eta(b-a)}.
\end{aligned} \tag{71}$$

Clearly, $\{x_n\}$ is uniformly bounded in $C_\gamma(J)$. By the continuity of F, G, ϕ_q, and K, we can easily get that $\{x_n\}$ is equicontinuous. By the Arzelà–Ascoli theorem, we have that $\{x_n\}$ satisfies (69). In the same way, we get that $\{y_n\}$ satisfies (70). Moreover, $x^*(t)$ and $y^*(t)$ are solutions of (9).

Step 3. We prove that x^* and y^* are extremal solutions of problem (9).

Assume that any solution $x(t)$ of problem (9) satisfies $x_n(t) \leq x(t) \leq y_n(t)$. Let $u(t) = \phi_p({}_{a}^{C\gamma}D^{\alpha}x(t)) - \phi_p({}_{a}^{C\gamma}D^{\alpha}x_{n+1}(t))$, and by (H_3), we have

$$\begin{aligned} {}_{a}^{C\beta}D^{\alpha}u(t) &= h\left(t, x(t), {}_{a}^{C\gamma}D^{\alpha}x(t)\right) - h\left(t, x_n(t), {}_{a}^{C\gamma}D^{\alpha}x_n(t)\right) \\ &\quad + \lambda\left[\phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_{n+1}(t)\right) - \phi_p\left({}_{a}^{C\gamma}D^{\alpha}x_n(t)\right)\right] \\ &\geq -\lambda u(t), \\ {}_{a}^{k}T^{\alpha}u(a) &= 0. \end{aligned} \tag{72}$$

From Lemma 7, we have $u(t) \geq 0$, i.e., $\phi_p({}_{a}^{C\gamma}D^{\alpha}x(t)) \geq \phi_p({}_{a}^{C\gamma}D^{\alpha}x_{n+1}(t))$. Hence,

$${}_{a}^{C\gamma}D^{\alpha}x(t) \geq {}_{a}^{C\gamma}D^{\alpha}x_{n+1}(t). \tag{73}$$

Let $\tilde{u}(t) = x(t) - x_{n+1}(t)$, and by (H_4), (59), and (73), we have

$$\begin{cases} {}_{a}^{C\gamma}D^{\alpha}\tilde{u}(t) \geq 0, \\ \tilde{u}(a) = \int_a^b \left[w(s, x(s)) - w(s, x_n(s)) - \eta(x_{n+1}(s) - x_n(s))\right]\mathrm{d}s \geq \eta\int_a^b \tilde{u}(s)\mathrm{d}s. \end{cases} \tag{74}$$

We get $\tilde{u}(t) \geq 0$ from Lemma 8, i.e., $x(t) \geq x_{m+1}(t)$. Similarly, ${}_{a}^{C\gamma}D^{\alpha}y_{n+1}(t) \geq {}_{a}^{C\gamma}D^{\alpha}x(t)$, $y_{n+1}(t) \geq x(t)$. Hence, $x_{n+1}(t) \leq x(t) \leq y_{n+1}(t)$ holds. Therefore, $x^*(t) \leq x(t) \leq y^*(t)$ as $n \longrightarrow \infty$, $\forall t \in J$. This ends the proof.

Remark 5. In [37], the authors assume that $h \in C([0,1] \times t[0,+\infty)n \times q(-\infty,0]h,_{[}0,+\infty))$, $h(t, w_1, z_1) \leq h(t, w_2, z_2)$ for $0 \leq w_1 < w_2, z_1 > z_2 \geq 0$, $t \in [0,1]$. The nonlinear term h in this paper satisfies the weaker conditions.

6. Example

We present a numerical example as follows:

$$\begin{cases} {}_{0}^{(1/2)}D^{(1/2)}\phi_2\left({}_{0}^{(1/2)}D^{(1/2)}x(t)\right) = t^{(3/4)} + \dfrac{x(t)}{3\left(t^{(1/4)} + t^{-(3/4)}\right)} + \dfrac{1}{4}{}_{0}^{(1/2)}D^{(1/2)}x(t), \quad t \in [0,1], \\ \phi_2\left({}_{0}^{(1/2)}D^{(1/2)}x(0)\right) = 0, \\ x(0) = \displaystyle\int_0^1 \left[\frac{1}{7}(s+1)x(s) + s\right]\mathrm{d}s + 1, \end{cases} \tag{75}$$

where $k = 0, b_0 = 0, p = 2, a = 0, b = 1, \rho = 1, \beta = \gamma = \alpha = (1/2)$ and

$$\begin{cases} h\left(t, x(t), {}_{a}^{(1/2)}D^{(1/2)}x(t)\right) = t^{(3/4)} + \dfrac{x(t)}{3\left(t^{(1/4)} + t^{-(3/4)}\right)} + \dfrac{1}{4}{}_{0}^{(1/2)}D^{(1/2)}x(t), \\ w(t, x(t)) = \dfrac{1}{7}(t+1)x(t) + t. \end{cases} \tag{76}$$

Taking $x_0(t) = 0, y_0(t) = 3t + 3$, we can get

$$\begin{cases} {}_0^{(1/2)}D^{(1/2)} \phi_2\left({}_0^{(1/2)}D^{(1/2)} x_0(t)\right) = 0 \leq t^{(3/4)} + \dfrac{x(t)}{3\left(t^{(1/4)} + t^{-(3/4)}\right)} + \dfrac{1}{4}{}_0^{(1/2)}D^{(1/2)} x_0(t), & t \in [0,1], \\ \phi_2\left({}_0^{(1/2)}D^{(1/2)} x_0(0)\right) = 0, & x_0(0) < \dfrac{3}{2}, \end{cases}$$
$$\begin{cases} {}_0^{(1/2)}D^{(1/2)} \phi_2\left({}_0^{(1/2)}D^{(1/2)} y_0(t)\right) = {}_0^{(1/2)}D^{(1/2)}\left(\dfrac{4\sqrt{2}}{\sqrt{\pi}} t^{(3/4)}\right) = 3t^{(1/2)} \geq \left(2 + \dfrac{\sqrt{2}}{\sqrt{\pi}}\right) t^{(3/4)}, & t \in [0,1], \\ \phi_2\left({}_0^{(1/2)}D^{(1/2)} y_0(0)\right) = 0, & y_0(0) = 3 > \dfrac{5}{2}. \end{cases} \tag{77}$$

Hence, x_0 and y_0 are lower and upper solutions of (75), respectively. Therefore, (H_2) is satisfied. For $x_0(t) \leq x \leq y \leq y_0(t)$,

$$\begin{aligned} & h\left(t, y(t), {}_0^{(1/2)}D^{(1/2)} y(t)\right) - h\left(t, x(t), {}_0^{(1/2)}D^{(1/2)} x(t)\right) \\ & = \frac{y(t) - x(t)}{3\left(t^{(1/4)} + t^{-(3/4)}\right)} + \frac{1}{4}{}_0^{(1/2)}D^{(1/2)}(y(t) - x(t)) \\ & \geq \frac{1}{4}{}_0^{(1/2)}D^{(1/2)}(y(t) - x(t)) \\ & = \frac{1}{4}\left[\phi_2\left({}_0^{(1/2)}D^{(1/2)} y(t)\right) - \phi_2\left({}_0^{(1/2)}D^{(1/2)} x(t)\right)\right], \\ & w(t, y(t)) - w(t, x(t)) \geq \frac{1}{7}(y(t) - x(t)). \end{aligned} \tag{78}$$

We can see that $\lambda = -(1/4) < 0, \eta = (1/7)$. Therefore, (H_3) and (H_4) hold. In light of Theorem 1, the extremal solutions of (75) can be obtained in $[x_0, y_0]$.

7. Conclusions

In this paper, we mainly use the montone iterative technique to study the Caputo conformable differential equations with p-Laplacian operator and integral boundary condition. A minimal and a maximal solution between the lower and the upper solutions are obtained. This method provides a constructive procedure for the solutions, and it is also useful for the investigation of qualitative properties of solutions. Since the Caputo conformable derivative can be reduced to the traditional Caputo derivative, some results produced from the traditional Caputo differential system can be seen as special cases of this paper. Moreover, the Caputo conformable derivative depends on two parameters naturally and the one coming from conformable operator can better describe long-memory processes. We believe that the Caputo and RL conformable fractional operators will play a key role in studying new types of fractional variational problems, Sturm–Liouville problems, optimal control problems, and modeling of complex systems.

Abbreviations

RL: Riemann–Liouville
BVP: Boundary value condition.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study and read and approved the final manuscript.

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