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Research Article

Quantum Approach to Damped Three Coupled Nano-Optomechanical Oscillators

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We investigate quantum features of three coupled dissipative nano-optomechanical oscillators. The Hamiltonian of the system is somewhat complicated due not only to the coupling of the optomechanical oscillators but to the dissipation in the system as well. In order to simplify the problem, a spatial unitary transformation approach and a matrix-diagonalization method are used. From such procedures, the Hamiltonian is eventually diagonalized. In other words, the complicated original Hamiltonian is transformed to a simple one which is associated to three independent simple harmonic oscillators. By utilizing such a simplification of the Hamiltonian, complete solutions (wave functions) of the Schrödinger equation for the optomechanical system are obtained. We confirm that the probability density converges to the origin of the coordinate in a symmetric manner as the optomechanical energy dissipates. The wave functions that we have derived can be used as a basic tool for evaluating diverse quantum consequences of the system, such as quadrature fluctuations, entanglement entropy, energy evolution, transition probability, and the Wigner function.

1. Introduction

Physical systems in nature do not behave independently in most cases because they are not isolated in usual. The coupling of a system to another one often results in various mutual phenomena, such as energy exchange, dissipation, entanglement, amplitude fluctuations, and decoherence [1, 2]. The model of a chain of oscillatory motions can be utilized in analyzing the dynamical characteristics of coupled optomechanical [3–6], nanoelectromechanical [7, 8], chemical [9, 10], and biological systems [11]. The mechanical analyses of coupled oscillators have been extensively explored so far through different approaches. Such systems can be usually investigated using invariant operator methods [12, 13], Bogoliubov transformation methods [14, 15], Jaynes–Cummings approaches [16], path integral methods [17], and adiabatic approaches [2]. Based on these approaches, we can elucidate diverse mechanical properties of the coupled oscillatory systems even when they are described by a time-dependent Hamiltonian.

Coupled harmonic oscillators are important topics especially in optomechanical systems which utilize interaction and entanglement between optical and mechanical modes. Coupling of nano- and micromechanical oscillators with photons [3] and/or other systems such as electrons [18] and atoms [19] provides a substantial tool for realizing next-generation quantum technologies. For instance, photonic couplings provide a basic technology platform for manipulation of phononic structures [20], slow/fast-light technology [21], zero-point cooling [22], and mechanical-frequency shifting [23]. Besides such applicability, coupled optomechanical oscillatory systems are required for highly sensitive measuring quantum devices which are crucial in quantum state tomographies and information processing with quantum states. Moreover, abundant physical

phenomena along this line can be utilized in nonclassical state preparation in hybrid quantum-information systems including quantum networks with appropriate quantum protocols.

In order to know underlying quantum properties related to entanglement [24, 25], quantum coherence and decoherence [26], nonclassical correlations [27], and concurrence [28], exact quantum theory which explains the evolution of the coupled system is necessary. It is known that such a system can be quantized with the assistance of a dynamical invariant [29] or imposing an adiabatic condition [2]. In this paper, we will show that we can also unfold the quantum dynamics of optomechanical physical systems described by a Hamiltonian of dissipative three coupled oscillators through neither introducing a dynamical invariant nor imposing an adiabatic condition in the dissipation. The complete wave functions of the system will be derived by diagonalizing the Hamiltonian directly, instead of the diagonalization of the dynamical invariant.

Before we carry out the main diagonalization, we first transform the Hamiltonian by a unitary operator into an appropriate form that can be diagonalizable without any approximation and condition. Indeed, unitary transformation procedure is a useful tool that enables us to treat a complicated Hamiltonian in a simplified transformed space. This greatly helps us to solve the analytical quantum solutions of the mechanical systems.

The present paper is organized as follows. Our problem for dissipative nano-optomechanical three coupled oscillators is formulated by basic equations of mechanics in Section 2. In Section 3, we deal with the quantization of the system and its physical states. At first, we simplify the Hamiltonian of the system by using the unitary transformation method. Such a simplified Hamiltonian will be represented as a rigorous matrix form for further investigation. The matrix Hamiltonian will be diagonalized so that we can easily manage it. By taking advantage of such a diagonalization, the quantum solutions of the system will be investigated. The concluding remarks are given in the last section.

2. Preliminaries

Before the actual investigation of optomechanical oscillatory systems, let us briefly show the method for describing mechanical oscillators subjected to a dissipational nonconservative force. In classical mechanics, the motion of a damped mechanical system can be described by the Newtonian equation

$$\ddot{x} + \delta\dot{x} = -\frac{\overrightarrow{\nabla} V(x)}{m}. \tag{1}$$

The second term on the left-hand side is a dissipative frictional force proportional to velocity, where δ is the damping coefficient. From the theoretical point of view, there is no limitation on the magnitude of δ; i.e., it can be any real number. For a mechanical oscillatory system that we are interested, the term in the right-hand side is of the form

$$\overrightarrow{\nabla} V(x) = kx. \tag{2}$$

Whereas the system is underdamped provided that $\delta < \omega/2$ where $\omega = (k/m)^{1/2}$, it is well known that the system becomes an overdamped oscillator when $\delta > \omega/2$. For the case of the overdamped oscillator, its mathematical treatment is somewhat difficult from quantum mechanical point of view. The Hamiltonian that gives the equation of motion, equation (1) with equation (2), is given by

$$H(t) = \frac{1}{2}\left[\frac{P^2}{e^{\delta t}m} + e^{\delta t}kx^2\right]. \tag{3}$$

The role of this Hamiltonian is limited to a generator of the classical equation of motion. Regarding this, it is well known that the energy of the system in this case is another problem and it is given by [30–32]

$$E_n(t) = \frac{1}{2}\left[\frac{\langle P^2\rangle}{e^{2\delta t}m} + k\langle x^2\rangle\right], \tag{4}$$

where $\langle\cdots\rangle$ is an expectation value in the Fock state. While equation (4) is represented in terms of the canonical momentum, we can also represent it in terms of the physical momentum P_k using the relation that $P_k = m\dot{x} = Pe^{-\delta t}$:

$$E_n(t) = \frac{1}{2}\left[\frac{\langle P_k^2\rangle}{m} + k\langle x^2\rangle\right]. \tag{5}$$

This implies that the quantum energy of the system dissipates according to the decrease of both $\langle P_k^2\rangle$ and $\langle x^2\rangle$ over time. This basic description of 1D system can be easily extended to coupled oscillatory systems.

In this work, we treat an optomechanical system shown in Figure 1 as a generalization of 1D system described by equation (3). As can be seen from Figure 1, optical fields are coupled with a mechanical mode through a nonlinear interaction caused by radiation pressure. The system can be linearized to some extent [3] and the resultant Hamiltonian can be parameterized by the optical quadratures (x_1, x_2) and a position of the mechanical membrane (x_3) [29, 33–35]:

$$\begin{aligned} H(t) = {} & \frac{1}{2}\sum_{i=1}^{3}\left[\frac{P_i^2}{e^{\delta t}m_i} + e^{\delta t}k_i x_i^2\right] \\ & + \frac{1}{2}e^{\delta t}\left[k_{12}(x_1 - x_2)^2 + k_{13}(x_1 - x_3)^2 + k_{23}(x_2 - x_3)^2\right], \end{aligned} \tag{6}$$

where m_1 and m_2 are the effective electric permittivities in the cavity 1 and cavity 2, respectively, in Figure 1 (although an electric permittivity is written as ε in many cases, we denote it as m for the convenience of mathematical expressions), m_3 is the effective mass of the nanomembrane, $k_i\,(i = 1, 2, 3)$ are the stiffness constants, and k_{12}, k_{13}, and k_{23} are the coupling constants; the convention of the subscript i (including j) which is given here will also be used subsequently. We manage only the case of underdamped oscillation of the system for simplicity. For a simple case where the system is a coupling of two oscillators (instead of three)

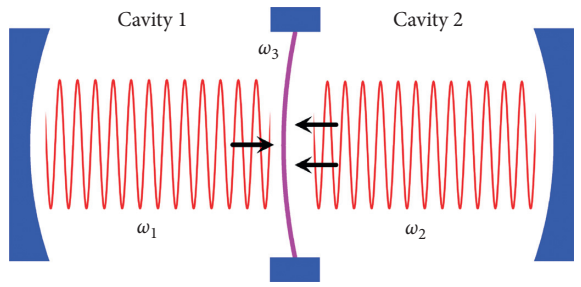

FIGURE 1: Schematic of three coupled nano-optomechanical oscillators that we consider. ω_1 and ω_2 are the optical modes, whereas ω_3 is a mechanical mode. It represents the interaction of the cavity fields with a flexible nanomembrane via radiation-pressure forces.

with the condition $\delta \longrightarrow 0$, the system corresponds to that of [3] and the relevant Hamiltonian, equation (6), reduces to equation (1) in the same reference.

In phase space, the coordinates and the conjugate momenta satisfy the commutation relations

$$\begin{aligned}\left[x_i, P_j\right] &= i\hbar\delta_{ij},\\ \left[x_i, x_j\right] &= \left[P_i, P_j\right] = 0.\end{aligned} \tag{7}$$

We are interested in quantum mechanical treatment of the optomechanical system. To study the quantum behavior of our system, we consider the Schrödinger equation of the form

$$i\hbar\partial_t|\Psi(t)\rangle = H(t)|\Psi(t)\rangle. \tag{8}$$

The evolution of the state vector associated with this equation and its physical interpretation is usually made within the framework of the Copenhagen interpretation of quantum mechanics. However, because the Hamiltonian in equation (6) is a somewhat complicated form, it may be not an easy task to solve the solutions of equation (8). In order to overcome such a difficulty, we will manage the system based on the unitary transformation method and a diagonalization procedure in the next section.

3. Results and Discussion

3.1. Unitary Transformations. Unitary transformation methods have often been used to solve the time-dependent Schrödinger equation (TDSE) [36–42] for a complicated dynamical system. If we use this method, it is possible to reduce the complicated form of the Schrödinger equation to a simpler one. This may facilitate the derivation of associated quantum solutions.

In order to achieve an appropriate transformation, a suitable choice of a unitary operator depending on the Hamiltonian is necessary. In our case, we choose the unitary operator as [29, 36, 40, 41]

$$U(t) = \left\{\prod_{i=1}^{3}\exp\left[\frac{i}{2\hbar}\left(P_i x_i + x_i P_i\right)\left(\ln\sqrt{m_i} + \frac{\delta}{2}t\right)\right]\right\} \times \exp\left(-\frac{i\delta}{4\hbar}\sum_{i=1}^{3}x_i^2\right). \tag{9}$$

This is a product of two unitary operators. The first operator is represented by considering the symmetry between x_i and P_i. Later on, we will see that the choice of this operator simplifies the Hamiltonian through a transformation. In terms of $U(t)$, the transformation of the wave function can be carried out in a way that

$$\psi(t) = U^{-1}(t)\Psi(t), \tag{10}$$

where $\psi(t)$ is the transformed wave function.

Under this unitary transformation, the Schrödinger equation of the original Hamiltonian system, equation (8), is mapped into

$$i\hbar\partial_t\psi(t) = \mathscr{H}\psi(t), \tag{11}$$

where the new Hamiltonian $\mathscr{H}$ has the form

$$\mathscr{H} = U^{-1}(t)H(t)U(t) - i\hbar U^{-1}(t)\partial_t U(t). \tag{12}$$

The transformation of canonical variables using equation (9) results in

$$\begin{aligned}U^{-1}(t)x_i U(t) &= \frac{e^{-\delta t/2}}{\sqrt{m_i}}x_i,\\ U^{-1}(t)P_i U(t) &= \sqrt{m_i}e^{\delta t/2}\left(P_i - \frac{\delta}{2}x_i\right).\end{aligned} \tag{13}$$

From the use of equation (6) with the above relations, we see that equation (12) becomes

$$\mathscr{H} = \frac{1}{2}\sum_{i=1}^{3}\left(P_i^2 + \Omega_i^2 x_i^2\right) + \frac{1}{2}\left(J_{12}x_1x_2 + J_{13}x_1x_3 + J_{23}x_2x_3\right), \tag{14}$$

where

$$\begin{aligned}
\Omega_1 &= \left[\frac{k_1 + k_{12} + k_{13}}{m_1} - \frac{\delta^2}{4}\right]^{1/2},\\
\Omega_2 &= \left[\frac{k_2 + k_{12} + k_{23}}{m_2} - \frac{\delta^2}{4}\right]^{1/2},\\
\Omega_3 &= \left[\frac{k_3 + k_{13} + k_{23}}{m_3} - \frac{\delta^2}{4}\right]^{1/2},\\
J_{12} &= \frac{-2k_{12}}{\sqrt{m_1m_2}},\\
J_{13} &= \frac{-2k_{13}}{\sqrt{m_1m_3}},\\
J_{23} &= \frac{-2k_{23}}{\sqrt{m_2m_3}}.
\end{aligned} \tag{15}$$

We see from equation (14) that we have taken the Hamiltonian which is simplified to some extent, thanks to the transformation with the use of $U(t)$ given in equation (9). However, the coupling terms x_ix_j still remained in the Hamiltonian. We will further simplify the Hamiltonian in the subsequent subsection by removing the terms x_ix_j using a diagonalization method.

3.2. Diagonalization of the Hamiltonian. Let us now consider diagonalization of the Hamiltonian. To this end, we represent the Hamiltonian $\mathscr{H}$ in a matrix form such that

$$\mathscr{H} = \frac{1}{2}\sum_{i,j=1}^{3} P_i\delta_{ij}P_j + \frac{1}{2}\sum_{i,j=1}^{3} x_i\Gamma_{ij}x_j, \tag{16}$$

where Γ_{ij} are elements of the matrix

$$\Gamma = \begin{pmatrix} \Omega_1^2 & \frac{1}{2}J_{12} & \frac{1}{2}J_{13} \\ \frac{1}{2}J_{12} & \Omega_2^2 & \frac{1}{2}J_{23} \\ \frac{1}{2}J_{13} & \frac{1}{2}J_{23} & \Omega_3^2 \end{pmatrix}, \tag{17}$$

which corresponds to ith row and jth column for each.

If we think that the dimension of the matrix Γ is (3×3), Γ is a diagonalizable square matrix [43, 44]. The diagonalizability of a matrix is in principle equivalent to the existence of a basis of eigenvectors, which makes it possible to define a diagonalizable endomorphism of a vector space. To diagonalize the Hamiltonian matrix, equation (16), it is necessary to seek its eigenvalues ϖ_i^2 and the corresponding eigenvectors $\vec{V}_i$. In our case, by solving the secular equation for Γ given in equation (17), the eigenvalues of the matrix Γ are given as follows:

$$\begin{aligned}
\varpi_1^2 &= \Omega_1^2 + \frac{J_{12} + J_{13}}{2},\\
\varpi_2^2 &= \Omega_2^2 - \frac{1}{2}J_{23} + \omega^2,\\
\varpi_3^2 &= \Omega_3^2 - \frac{1}{2}J_{23} - \omega^2,
\end{aligned} \tag{18}$$

where

$$\omega^2 = \frac{1}{2}\left[J_{12}^2 + J_{13}^2 + J_{23}^2 - \left(J_{12}J_{13} + J_{12}J_{23} + J_{13}J_{23}\right)\right]^{1/2}. \tag{19}$$

From a standard procedure associated with the eigenvalue problem of a matrix, we can easily see that the corresponding normalized eigenvectors are given by

$$\begin{aligned}
\vec{V}_1 &= \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\\
\vec{V}_\pm &= \lambda_\pm \begin{pmatrix} \frac{1}{2}J_{12} - \frac{1}{2}J_{23} \mp \omega^2 \\ \frac{1}{2}J_{23} - \frac{1}{2}J_{12} \pm \omega^2 \\ \frac{1}{2}J_{23} - \frac{1}{2}J_{13} \end{pmatrix},
\end{aligned} \tag{20}$$

where

$$\lambda_\pm = \frac{1}{J_{23} - J_{13}}\left[\frac{2}{3} \pm \frac{J_{12} - (J_{13} + J_{23})/2}{3\omega^2}\right]^{1/2}. \tag{21}$$

Note that the vectors $\vec{V}_i\,(i = 1;\ \pm)$ are orthonormal to each other because Γ is symmetric. Thus, Γ can be diagonalized by $\Gamma = \mathbb{R}D\mathbb{R}^{-1}$, where

$$\mathbb{R} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \lambda_+\left[\frac{1}{2}J_{12} - \frac{1}{2}J_{23} - \omega^2\right] & \lambda_-\left[\frac{1}{2}J_{12} - \frac{1}{2}J_{23} + \omega^2\right] \\ \frac{1}{\sqrt{3}} & \lambda_+\left[-\frac{1}{2}J_{12} + \frac{1}{2}J_{23} + \omega^2\right] & \lambda_-\left[-\frac{1}{2}J_{12} + \frac{1}{2}J_{23} - \omega^2\right] \\ \frac{1}{\sqrt{3}} & \lambda_+\left[-\frac{1}{2}J_{13} + \frac{1}{2}J_{23}\right] & \lambda_-\left[-\frac{1}{2}J_{13} + \frac{1}{2}J_{23}\right] \end{pmatrix}. \tag{22}$$

$$\begin{aligned} D &= \mathbb{R}^{-1}\Gamma\mathbb{R} \\ &= \mathrm{diag}\left[\varpi_1^2, \varpi_2^2, \varpi_3^2\right]. \end{aligned} \tag{23}$$

Now, we introduce new coordinates q_i and p_i as

$$\begin{aligned} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} &= \mathbb{R}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\ \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} &= \mathbb{R}\begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}. \end{aligned} \tag{24}$$

By using equation (22), we can easily obtain the full expressions of q_i and p_i; we have represented them in Appendix for completeness. Now, from a straightforward algebraic procedure using those expressions (Equation (A.1)–(A.6)), we see that $\mathscr{H}$ takes the form

$$\mathscr{H} = \frac{1}{2}\sum_{i=1}^{3}\left(p_i^2 + \varpi_i^2 q_i^2\right). \tag{25}$$

Thus, the transformed system is eventually decoupled (diagonalized); i.e., it is reduced to a collection of three decoupled harmonic oscillators. During the diagonalization procedure, we have imposed no conditions or restrictions. Hence, the consequence (equation (25)) justifies the appropriateness of the chosen unitary operator together with the subsequently performed diagonalization. Based on this, the solutions of our problem can be easily obtained. Indeed, from a mathematical point of view, the diagonalization method plays a major role in the simplification of the problem without modification of the behavior of the physical system.

We note that the coupled optomechanical oscillatory systems can also be diagonalized by using the Bogoliubov transformation or its advanced technics, provided that $\delta = 0$. However, the resulting expression of the Hamiltonian which is diagonalized based on such a method is somewhat different from that obtained using the method used here (see, for example, equations (54–57) in Ref. [15]).

3.3. Quantum Solutions. We now inspect solutions of the TDSE for the Hamiltonian given in equation (25) at first. Then, based on the inverse unitary transformation of such solutions, we find the solutions of the original optomechanical physical system of which Hamiltonian is equation (6). As we have indicated previously, we can easily derive the solutions of energy spectrum of the new Hamiltonian $\mathscr{H}$ which is the sum of three individual simple-harmonic-oscillator Hamiltonians with frequencies ϖ_1, ϖ_2, and ϖ_3. Consequently, the solutions $\psi(q_1, q_2, q_3, t)$ of the Schrödinger equation, equation (11), for the transformed Hamiltonian are given by

$$\begin{aligned} \psi(q_1, q_2, q_3, t) = \psi(q_1, t)\psi(q_2, t)\psi(q_3, t) &= \left[\frac{\sqrt{\varpi_1\varpi_2\varpi_3}}{(\pi\hbar)^{3/2} n_1! n_2! n_3! 2^{n_1+n_2+n_3}}\right]^{1/2} \exp\left[i\alpha_{n_1,n_2,n_3}(t)\right] \\ &\times H_{n_1}\left(\sqrt{\frac{\varpi_1}{\hbar}}q_1\right) H_{n_2}\left(\sqrt{\frac{\varpi_2}{\hbar}}q_2\right) H_{n_3}\left(\sqrt{\frac{\varpi_3}{\hbar}}q_3\right) \\ &\times \exp\left[\frac{-1}{2\hbar}\left(\varpi_1 q_1^2 + \varpi_2 q_2^2 + \varpi_3 q_3^2\right)\right], \end{aligned} \tag{26}$$

where H_{n_i} are n_ith-order Hermite polynomials, and the phases take the form

$$\alpha_{n_1,n_2,n_3}(t) = -\sum_{i=1}^{3} \varpi_i \left(n_i + \frac{1}{2}\right) t. \tag{27}$$

Based on this, it is possible to determine the solutions associated with the original optomechanical system by using equations (10), (26), and (27). Thence, we finally have the full wave functions in the form

$$\begin{aligned}\Psi_{n_1,n_2,n_3}(x_1,x_2,x_3,t) = N H_{n_1}(\xi)_1 H_{n_2}(\xi_2) H_{n_3}(\xi_3) \times \exp\left[-i\left(\sum_{i=1}^{3} \varpi_i \left(n_i + \frac{1}{2}\right) t\right)\right] \\ \times \exp\left[\left(\mu_1 - \frac{i\delta}{4\hbar} m_1 e^{\delta t}\right) x_1^2 + \left(\mu_2 - \frac{i\delta}{4\hbar} m_2 e^{\delta t}\right) x_2^2\right. \\ \left. + \left(\mu_3 - \frac{i\delta}{4\hbar} m_3 e^{\delta t}\right) x_3^2 + \mu_{12} x_1 x_2 + \mu_{13} x_1 x_3 + \mu_{23} x_2 x_3\right],\end{aligned} \tag{28}$$

where N is the normalization factor:

$$N = \left(\frac{\sqrt{\varpi_1 \varpi_2 \varpi_3 m_1 m_2 m_3}\; e^{(3/2)\delta t}}{(\pi\hbar)^{3/2} n_1! n_2! n_3! 2^{n_1+n_2+n_3}}\right)^{1/2}. \tag{29}$$

Also, ξ_1, ξ_2, and ξ_3 are given as

$$\begin{aligned}\xi_1 &= \left(\frac{\varpi_1}{\hbar} e^{\delta t}\right)^{1/2} \left[\frac{1}{\sqrt{3}} \sqrt{m_1}\, x_1 + \lambda_+ \left(\frac{1}{2} J_{12} - \frac{1}{2} J_{23} - \omega^2\right) \sqrt{m_2}\, x_2 \right. \\ &\quad \left. + \lambda_- \left(\frac{1}{2} J_{12} - \frac{1}{2} J_{23} + \omega^2\right) \sqrt{m_3}\, x_3\right], \\ \xi_2 &= \left(\frac{\varpi_2}{\hbar} e^{\delta t}\right)^{1/2} \left[\frac{1}{\sqrt{3}} \sqrt{m_1}\, x_1 + \lambda_+ \left(-\frac{1}{2} J_{12} + \frac{1}{2} J_{23} + \omega^2\right) \sqrt{m_2}\, x_2 \right. \\ &\quad \left. + \lambda_- \left(-\frac{1}{2} J_{12} + \frac{1}{2} J_{23} - \omega^2\right) \sqrt{m_3}\, x_3\right], \\ \xi_3 &= \left(\frac{\varpi_3}{\hbar} e^{\delta t}\right)^{1/2} \left[\frac{1}{\sqrt{3}} \sqrt{m_1}\, x_1 + \lambda_+ \left(-\frac{1}{2} J_{13} + \frac{1}{2} J_{23}\right) \sqrt{m_2}\, x_2 \right. \\ &\quad \left. + \lambda_- \left(-\frac{1}{2} J_{13} + \frac{1}{2} J_{23}\right) \sqrt{m_3}\, x_3\right],\end{aligned} \tag{30}$$

while the time-dependent coefficients $\mu_1, \mu_2, \mu_3, \mu_{12}, \mu_{13}$, and μ_{23} are expressed to be

$$\mu_1 = -\frac{m_1 e^{\delta t}}{6\hbar} (\varpi_1 + \varpi_2 + \varpi_3), \tag{31}$$

$$\begin{aligned}\mu_2 = -\frac{m_2 e^{\delta t}}{2\hbar} \lambda_+^2 \left[\varpi_1 \left(\frac{1}{2} J_{12} - \frac{1}{2} J_{23} - \omega^2\right)^2 + \varpi_2 \left(-\frac{1}{2} J_{12} + \frac{1}{2} J_{23} + \omega^2\right)^2 \right. \\ \left. + \varpi_3 \left(-\frac{1}{2} J_{13} + \frac{1}{2} J_{23}\right)^2\right],\end{aligned} \tag{32}$$

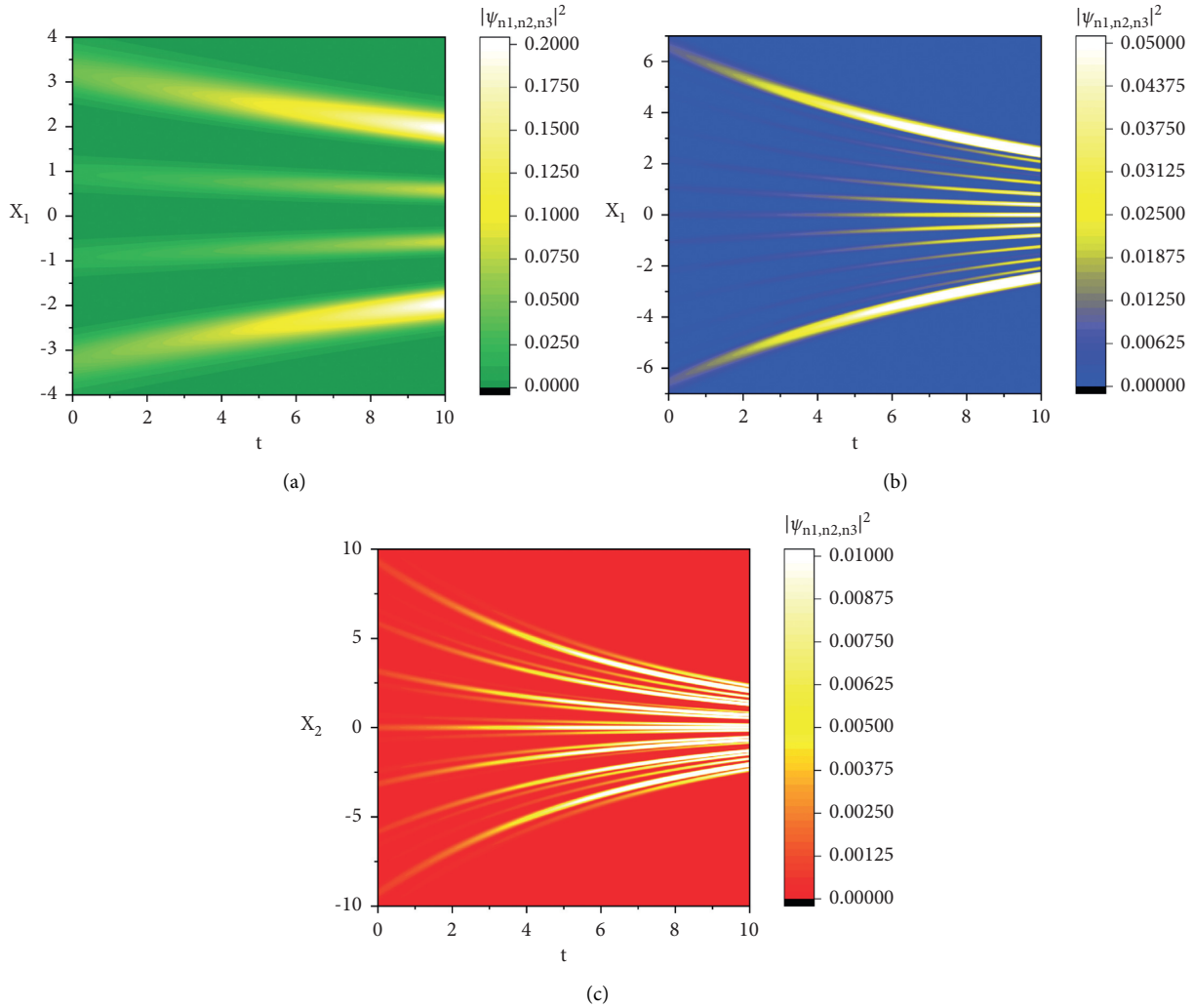

Figure 2: Contour plot of the time evolution of the probability density $|\Psi_{n_1,n_2,n_3}(x_1,x_2,x_3,t)|^2$ as a function of x_1 (a, b) and x_2 (c), where $x_2 = x_3 = 0$ for (a) and (b), while $x_1 = x_3 = 0$ for (c). The chosen quantum numbers (n_1, n_2, n_3) are (3, 3, 3) (a), (10, 10, 10) (b), and (20, 20, 20) (c), whereas the damping factor δ is 0.10 (a), 0.20 (b), and 0.30 (c). We used $\hbar = 1$, $m_1 = m_2 = m_3 = 1$, $k_1 = k_2 = k_3 = 1$, $k_{12} = 0.1$, $k_{13} = 0.2$, and $k_{23} = 0.3$.

$$\mu_3 = -\frac{m_3 e^{\delta t}}{2\hbar}\lambda_-^2\left[\varpi_1\left(\frac{1}{2}J_{12} - \frac{1}{2}J_{23} + \omega^2\right)^2 + \varpi_2\left(-\frac{1}{2}J_{12} + \frac{1}{2}J_{23} - \omega^2\right)^2 + \varpi_3\left(-\frac{1}{2}J_{13} + \frac{1}{2}J_{23}\right)^2\right], \tag{33}$$

$$\mu_{12} = -\frac{\sqrt{m_1 m_2}e^{\delta t}}{\sqrt{3}\hbar}\lambda_+\left[\varpi_1\left(\frac{1}{2}J_{12} - \frac{1}{2}J_{23} - \omega^2\right) + \varpi_2\left(-\frac{1}{2}J_{12} + \frac{1}{2}J_{23} + \omega^2\right) + \varpi_3\left(-\frac{1}{2}J_{13} + \frac{1}{2}J_{23}\right)\right], \tag{34}$$

$$\mu_{13} = -\frac{\sqrt{m_1 m_3}e^{\delta t}}{\sqrt{3}\hbar}\lambda_-\left[\varpi_1\left(\frac{1}{2}J_{12} - \frac{1}{2}J_{23} + \omega^2\right) + \varpi_2\left(-\frac{1}{2}J_{12} + \frac{1}{2}J_{23} - \omega^2\right) + \varpi_3\left(-\frac{1}{2}J_{13} + \frac{1}{2}J_{23}\right)\right], \tag{35}$$

$$\mu_{23} = -\frac{\sqrt{m_2 m_3}e^{\delta t}\lambda_+\lambda_-}{\sqrt{3}\hbar}\left\{\varpi_1\left[\left(\frac{1}{2}J_{12}-\frac{1}{2}J_{23}\right)^2-\omega^4\right]\right.$$
$$\left.+\varpi_2\left[\left(-\frac{1}{2}J_{12}+\frac{1}{2}J_{23}\right)^2-\omega^4\right]+\varpi_3\left(-\frac{1}{2}J_{13}+\frac{1}{2}J_{23}\right)^2\right\}. \quad (36)$$

Notice that we have used no approximation and perturbation methods during the derivation of equation (28) with equations (29)–(36). Hence, the wave functions in equation (28) are exact even if they are somewhat complicated. The normalization factor, equation (29), is chosen in a way that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi^*_{n_1,n_2,n_3}\Psi_{n_1,n_2,n_3}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = 1. \quad (37)$$

Equation (29) involves $e^{3\delta t/4}$, and this reduces to $e^{\delta t/4}$ for a 1D case, which is a previously known consequence [30]. The obtained wave functions can be used as a basic tool for further investigation of the quantum dynamics of the nano-optomechanical system. As mentioned earlier, we considered only the case of underdamped oscillation. For the cases of critically and overdamped systems, the quantum mechanism of the oscillation is somewhat different. Then, the wave functions are not described by quantum number n_i but are similar to those of an unbound system [30].

We have plotted the time behavior of the probability density $|\Psi_{n_1,n_2,n_3}(x_1,x_2,x_3,t)|^2$ for several different choices of quantum numbers n_i and the damping factor δ in Figure 2. The numerical data associated to the graphics in this figure were obtained from the use of Mathematica program (Wolfram Research) with equations (28)–(36). The probability density converges to the origin of the coordinate in a symmetric manner as time goes by according to the dissipation of energy, regardless of the chosen values of n_i. The convergence is fast when δ is large as expected.

4. Conclusions

We have investigated quantum mechanical features of dissipative three coupled nano-optomechanical oscillators. Exact solutions of the TDSE of the system have been derived using the unitary transformation method and a diagonalization procedure. From the unitary transformation at first, the complicated time-dependent original quantal Hamiltonian was transformed to a simple one which has unit masses; however, the coupling terms $x_i x_j$ in the Hamiltonian still remained even after this transformation. In order to remove the coupling terms $x_i x_j$, we used a diagonalization method. Through these procedures, the Hamiltonian eventually diagonalized. The finally transformed Hamiltonian was given in the form associated to three independent harmonic oscillators. As a consequence, the transformed Hamiltonian was easily treated and we have identified the corresponding quantum solutions without difficulty. By inverse transformation of the solutions of the TDSE for the transformed Hamiltonian, we finally had the complete quantum solutions of the original optomechanical system. The probability density, which is the absolute square of the wave function, converges to the origin of the coordinate as the oscillatory energy dissipates. The wave functions, which we obtained here, can be used to evaluate not only the quantum mechanical expectation values of various observables, such as physical momentum and quantum energy, but also probability densities, optomechanical dissipation, and fluctuations of the canonical variables.

One of the important research tasks for coupled optomechanical systems is characterizing entanglement between oscillators based on, for example, entropy. Usually, the derivation of entanglement entropy for coupled nonstationary oscillators along this line was carried out up until now with an implicit assumption that all masses (and electric permittivities) of the system are unity or equal to each other [45–50]. Notice that such a limited treatment was entirely due to the difficulty in the associated diagonalization procedure of the Hamiltonian. The significance of this research is that we neither adopted such an unnecessary assumption nor used a mathematical approximation when we unfold our theory starting from equation (6). Such clear treatment was possible thanks to the managing of the system in a hybrid way, i.e., by combining the unitary transformation approach and the matrix-diagonalization method together.

An obvious trend in current electronic and optical science is that the related devices become smaller and smaller towards atomic scale as the technology advances. Notice that quantum effects are prominent in devices miniaturized, especially below the scale of the Fermi wavelength [51]. Consequently, quantum treatment of devices including optomechanical ones is important in such cases, while classical mechanics is inadequate in describing their characteristics. The quantum results of this research may provide a theoretical foundation which enables the investigation of the entanglement problem for coupled optomechanical oscillators without a necessity of certain assumptions.

Appendix

The Full Expressions of q_i and p_i

The formulae of q_i and p_i appeared in equation (25) are given by

$$q_1 = \frac{1}{\sqrt{3}}x_1 + \frac{\lambda_+}{2}\left(J_{12} - J_{23} - 2\omega^2\right)x_2 + \frac{\lambda_-}{2}\left(J_{12} - J_{23} + 2\omega^2\right)x_3, \tag{A.1}$$

$$q_2 = \frac{1}{\sqrt{3}}x_1 + \frac{\lambda_+}{2}\left(J_{23} - J_{12} + 2\omega^2\right)x_2 + \frac{\lambda_-}{2}\left(J_{23} - J_{12} - 2\omega^2\right)x_3, \tag{A.2}$$

$$q_3 = \frac{1}{\sqrt{3}}x_1 + \frac{\lambda_+}{2}\left(J_{23} - J_{13}\right)x_2 + \frac{\lambda_-}{2}\left(J_{23} - J_{13}\right)x_3, \tag{A.3}$$

$$p_1 = \frac{1}{\sqrt{3}}P_1 + \frac{\lambda_+}{2}\left(J_{12} - J_{23} - 2\omega^2\right)P_2 + \frac{\lambda_-}{2}\left(J_{12} - J_{23} + 2\omega^2\right)P_3, \tag{A.4}$$

$$p_2 = \frac{1}{\sqrt{3}}P_1 + \frac{\lambda_+}{2}\left(J_{23} - J_{12} + 2\omega^2\right)P_2 + \frac{\lambda_-}{2}\left(J_{23} - J_{12} - 2\omega^2\right)P_3, \tag{A.5}$$

$$p_3 = \frac{1}{\sqrt{3}}P_1 + \frac{\lambda_+}{2}\left(J_{23} - J_{13}\right)P_2 + \frac{\lambda_-}{2}\left(J_{23} - J_{13}\right)P_3. \tag{A.6}$$

Data Availability

The data used to support the findings of this study can be reproducible from analytical representation of the results in the text.

Conflicts of Interest

The authors declare no conflicts of interest.

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