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Research Article

Analytical Investigation of Noyes–Field Model for Time-Fractional Belousov–Zhabotinsky Reaction

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In this article, we find the solution of time-fractional Belousov–Zhabotinskii reaction by implementing two well-known analytical techniques. The proposed methods are the modified form of the Adomian decomposition method and homotopy perturbation method with Yang transform. In Caputo manner, the fractional derivative is used. The solution we obtained is in the form of series which helps in investigating the analytical solution of the time-fractional Belousov–Zhabotinskii (B-Z) system. To verify the accuracy of the proposed methods, an illustrative example is taken, and through graphs, the solution is shown. Also, the fractional-order and integer-order solutions are compared with the help of graphs which are easy to understand. It has been verified that the solution obtained by using the given approaches has the desired rate of convergence to the exact solution. The proposed technique's principal benefit is the low amount of calculations required. It can also be used to solve fractional-order physical problems in a variety of domains.

1. Introduction

Fractional calculus is concerned with fractional-order derivatives and integrals [1]. Although fractional calculus has a similar background to classical calculus, it has received little attention for a long time. Fractional calculus and fractional differential equations, on the other hand, have grown in popularity in recent decades as a result of their significant prospective applications in physics and engineering [2–6]. For various physical phenomena, a considerable number of new differential equations (models) including fractional derivatives and integration have been constructed. These models have proven to be effective. Many phenomena in physics, chemistry, engineering, and other sciences can be better understood using fractional calculus methods, such as the theory of fractional noninteger order derivatives and integrals. These mathematical phenomena enable a more accurate description of an actual object than using "integer" methods [7–10].

Fractional partial differential equations (FPDEs) are a contemporary tool in calculus that may be used to simulate a variety of phenomena in applied sciences and engineering. Because it proved difficult to simulate nonlinear real-world processes in ordinary calculus, the subject of fractional calculus became popular among researchers [11–17]. In this context, FPDEs approximation and analytical solutions are critical for accurately describing the dynamics of important physical processes. In view of the above statement, mathematicians have devised and applied a variety of approximate and analytical procedures to identify the solutions to a number of important mathematical models that represent real-world issues. Despite the fact that calculating the analytical and even approximate solutions of certain nonlinear FPDEs and systems of FPDEs is extremely challenging, mathematicians continue to do their best in this respect. Around the years, a great deal of study has been conducted in the fields of science and engineering all over the world, and various methodologies have been developed to provide the best obtainable solutions. This is an unstoppable process, and new techniques are being developed on a daily basis since the world is confronted with new, insoluble, and challenging difficulties and problems [18–22]. Different methods for solving FDEs have been proposed in the literature, including the fractional differential transform method (FDTM) [23], modified homotopy perturbation method (MHPM) [24], Adomian decomposition method (ADM) [25], Sumudu transform method (STM) [26], differential transform method (DTM) [27], homotopy analysis method (HAM) [28], variational iteration method (VIM) [29], and fractional Fourier transform method (FFTM) [30].

The Yang transform decomposition method (YTDM) and the homotopy perturbation Yang transform method (HPYTM) are expanded for time-fractional Belousov–Zhabotinskii (B-Z) system solution in this article. Xiao-Jun Yang proposed the Yang transform, which may be used to solve a variety of differential equations with constant coefficients. On the other hand, the Adomian decomposition method [31, 32] is a well-known technique for solving linear and nonlinear and homogeneous and nonhomogeneous differential and partial differential equations, and integrodifferential and FDEs that provides exact solutions in the form of a convergent series. He introduced HPM in 1998 [33, 34]. The result is considered to be an in series solution with a high number of terms that quickly converge to the actual derived solution in this technique. The method is capable of adequately solving nonlinear PDEs. When the HPTM results were compared with the real solution to the problems, a higher level of accuracy was proven. Nonlinear wave equations [35], nonlinear problem bifurcation [36], and boundary value problems [37] have all been solved using this technique. In this paper, we used a new approximation analytical method called HPYTM. The hybrid form of Yang transform and HPM is the newly designed technique. The current methods are shown to be quite effective in obtaining the analytical solution of the time-fractional Belousov–Zhabotinskii (B-Z) system. The results of the recommended procedures are convincing, providing exact solutions to the targeted problems. The fractional problem findings obtained utilising the given approaches are also used to analyse the problems from a fractional perspective. The existing techniques can be adjusted to solve various fractional PDEs and related systems, which has been confirmed.

Our goal in this paper is to use two analytical techniques for solving a fractional-order nonlinear oscillatory system known as Belousov–Zhabotinskii (B-Z). The B-Z chemical reaction family is fascinating because it can show both temporal and spatial travelling concentration waves, as well as dramatic colour changes [38]. For the B-Z reaction, the simplified fractional Noyes–Field model is as follows [39]:

$$\begin{aligned} D_\tau^\alpha \xi &= \Psi_1 \xi_{\varphi\varphi} + \beta \rangle \zeta + \xi - \xi^2 - \xi\zeta, \quad 0 < \alpha \le 1, \\ D_\tau^\alpha \zeta &= \Psi_2 \zeta_{\varphi\varphi} + \gamma\zeta - \delta\xi\zeta, \end{aligned} \tag{1}$$

where Ψ_1 and Ψ_2 are the diffusing constants for ξ and ζ concentration, respectively, γ and β are given constants, $\delta \neq 1$ and ϱ are positive parameters, and α is the fractional order.

2. Preliminaries

In this section, we presented some basic definitions of fractional calculus along with Yang transform theory properties.

2.1. Definition. The fractional-order derivative in Caputo manner is given as

$$D_\tau^\alpha \xi(\varphi,\tau) = \frac{1}{\Gamma(k-\alpha)} \int_0^\tau (\tau-\rho)^{k-\alpha-1} \xi^{(k)}(\varphi,\rho)\mathrm{d}\rho, \quad k-1 < \alpha \le k,\ k \in N. \tag{2}$$

2.2. Definition. The Yang–Laplace transform was introduced by Xiao-Jun Yang in 2018. The Yang transform for a function $\xi(\tau)$ is determined by $Y\{\xi(\tau)\}$ or M(u) and is given as

$$\begin{aligned} Y\{\xi(\tau)\} &= M(u) \\ &= \int_0^\infty e^{-\tau/u} \xi(\tau)\mathrm{d}\tau, \quad \tau > 0,\ u \in (-\tau_1, \tau_2). \end{aligned} \tag{3}$$

The inverse transform (inverse symbol) is defined as

$$Y^{-1}\{M(u)\} = \xi(\tau), \tag{4}$$

where Y^{-1} is inverse Yang operator.

2.3. Definition. The Yang transform for nth derivatives is defined as

$$Y\{\xi^n(\tau)\} = \frac{M(u)}{u^n} - \sum_{k=0}^{n-1} \frac{\xi^k(0)}{u^{n-k-1}}, \quad \forall n = 1,2,3,\ldots. \tag{5}$$

2.4. Definition. The Yang transform for fractional-order derivatives is defined as

$$Y\{\xi^\alpha(\tau)\} = \frac{M(u)}{u^\gamma} - \sum_{k=0}^{n-1} \frac{\xi^k(0)}{u^{\alpha-(k+1)}}, \quad 0 < \alpha \le n. \tag{6}$$

3. Homotopy Perturbation Yang Transform Method

Consider a general nonlinear homogeneous partial differential equation with initial conditions of the type to demonstrate the basic idea of this method:

$$D_\tau^\alpha \xi(\varphi,\tau) + \mathscr{P}_1 \xi(\varphi,\tau) + \mathscr{Q}_1 \xi(\varphi,\tau) = 0, \quad 0 < \alpha \le 1, \tag{7}$$

with some initial sources such as

$$\xi(\varphi,0) = h(\varphi), \tag{8}$$

where D_τ^α is the fractional differential operator with respect to τ. $\mathscr{P}_1$ and $\mathscr{Q}_1$ are the linear and nonlinear differential operator with respect to φ, respectively.

Using Yang transformation to (7), we have

$$Y\left[D_\tau^\alpha \xi(\varphi,\tau)\right] + Y\left[\mathscr{P}_1 \xi(\varphi,\tau) + \mathscr{Q}_1 \xi(\varphi,\tau)\right] = 0, \tag{9}$$

$$\frac{1}{u^\alpha}\{M(u) - u\xi(0)\} + Y\left[\mathscr{P}_1 \xi(\varphi,\tau) + \mathscr{Q}_1 \xi(\varphi,\tau)\right] = 0. \tag{10}$$

Equation (9) implies that

$$M(\xi) = uh(\varphi) - u^\alpha Y\left[\mathscr{P}_1 \xi(\varphi,\tau) + \mathscr{Q}_1 \xi(\varphi,\tau)\right]. \tag{11}$$

Now, by taking inverse Yang transform, we get

$$\xi(\varphi,\tau) = H(\varphi) - Y^{-1}\left[u^\alpha Y\left[\mathscr{P}_1 \xi(\varphi,\tau) + \mathscr{Q}_1 \xi(\varphi,\tau)\right]\right], \tag{12}$$

where $H(\varphi)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method:

$$\xi(\varphi,\tau) = \sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau), \tag{13}$$

and the nonlinear term can be decomposed as

$$\mathscr{Q}_1 \xi(\varphi,\tau) = \sum_{k=0}^{\infty} p^k H_k(\xi), \tag{14}$$

for a set of He's polynomials H_k defined by

$$H_k(\xi_0,\xi_1,\ldots,\xi_k) = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\left[\mathscr{Q}_1\left(\sum_{i=0}^{\infty} p^i \xi_i\right)\right]_{p=0}, \quad k = 0,1,2,3\ldots \tag{15}$$

Substituting equations (14) and (15) in (12), we get

$$\sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau) = H(\varphi) - p \times \left(Y^{-1}\left[u^\alpha Y\left\{\mathscr{P}_1 \sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau) + \sum_{k=0}^{\infty} p^k H_k(\xi)\right\}\right]\right). \tag{16}$$

The Yang transform and the homotopy perturbation approach using He's polynomials are coupled in this method. The following approximations are obtained by comparing the coefficients of like powers of p.

$$\begin{aligned}
p^0&: \xi_0(\varphi,\tau) = H(\varphi),\\
p^1&: \xi_1(\varphi,\tau) = -Y^{-1}\left[u^\alpha Y\left(\mathscr{P}_1 \xi_0(\varphi,\tau) + H_0(\xi)\right)\right],\\
p^2&: \xi_2(\varphi,\tau) = -Y^{-1}\left[u^\alpha Y\left(\mathscr{P}_1 \xi_1(\varphi,\tau) + H_1(\xi)\right)\right],\\
&\;\;\vdots\\
p^k&: \xi_k(\varphi,\tau) = -Y^{-1}\left[u^\alpha Y\left(\left(\mathscr{P}_1 \xi_{k-1}(\varphi,\tau) + H_{k-1}(\xi)\right)\right)\right], \quad k > 0, k \in N.
\end{aligned} \tag{17}$$

Thus, we can calculate easily component $\xi_k(\varphi,\tau)$, which rapidly leads us to the convergent series. By taking $p \longrightarrow 1$, we get

$$\xi(\varphi,\tau) = \lim_{M\longrightarrow\infty} \sum_{k=1}^{M} \xi_k(\varphi,\tau). \tag{18}$$

The result we get is in form of series and quickly converges to the problem exact solution.

4. Idea of YTDM

The solution by YTDM for partial differential equations having fractional order is described in this section.

$$D_\tau^\alpha \xi(\varphi,\tau) = \mathscr{P}(\varphi,\tau)_1 + \mathscr{Q}_1(\varphi,\tau) + \mathscr{R}_1(\varphi,\tau), \quad 0 < \alpha \le 1, \tag{19}$$

with some initial sources such as

$$\xi(\varphi,0) = \xi(\varphi), \tag{20}$$

where $D_{\tau}^{\alpha} = \partial^{\alpha}/\partial\tau^{\alpha}$ is the fractional derivative in Caputo manner of order α, $\mathscr{P}_1$ and $\mathscr{Q}_1$ are linear and nonlinear functions, and $\mathscr{R}_1$ is the source term, respectively.

By applying Yang transform on both sides of equation (19), we obtain

$$Y\left[D_{\tau}^{\alpha}\xi(\varphi,\tau)\right] = Y\left[\mathscr{P}_1(\varphi,\tau) + \mathscr{Q}_1(\varphi,\tau) + \mathscr{R}_1(\varphi,\tau)\right]. \tag{21}$$

By differentiation property of Yang transform, we obtain

$$\frac{1}{u^{\alpha}}\{M(u) - u\xi(0)\} = Y\left[\mathscr{P}_1(\varphi,\tau) + \mathscr{Q}_1(\varphi,\tau) + \mathscr{R}_1(\varphi,\tau)\right]. \tag{22}$$

Equation (20) implies that

$$M(\xi) = u\xi(0) + u^{\alpha}Y\left[\mathscr{P}_1(\varphi,\tau) + \mathscr{Q}_1(\varphi,\tau) + \mathscr{R}_1(\varphi,\tau)\right], \tag{23}$$

and taking the inverse Yang transform of equation (23), we get

$$\xi(\varphi,\tau) = \xi(0) + Y^{-1}\left[u^{\alpha}Y\left[\mathscr{P}_1(\varphi,\tau) + \mathscr{Q}_1(\varphi,\tau) + \mathscr{R}_1(\varphi,\tau)\right]\right]. \tag{24}$$

YTDM defines the infinite sequence solution of $\xi(\varphi,\tau)$ as

$$\xi(\varphi,\tau) = \sum_{m=0}^{\infty}\xi_m(\varphi,\tau). \tag{25}$$

The Adomian polynomial $\mathscr{Q}_1$ decomposition of nonlinear terms is given as

$$\mathscr{Q}_1(\varphi,\tau) = \sum_{m=0}^{\infty}\mathscr{A}_m. \tag{26}$$

All types of nonlinearity are represented by Adomian polynomials as

$$\mathscr{A}_m = \frac{1}{m!}\left[\frac{\partial^m}{\partial\delta^m}\left\{\mathscr{Q}_1\left(\sum_{k=0}^{\infty}\delta^k\varphi_k, \sum_{k=0}^{\infty}\delta^k\tau_k\right)\right\}\right]_{\delta=0}. \tag{27}$$

Putting equations (23) and (25) into (24) gives

$$\begin{aligned}\sum_{m=0}^{\infty}\xi_m(\varphi,\tau) &= \xi(0) + Y^{-1}\left[u^{\alpha}Y\{\mathscr{R}_1(\varphi,\tau)\}\right]\\ &\quad + Y^{-1}u^{\alpha}\left[Y\left\{\mathscr{P}_1\left(\sum_{m=0}^{\infty}\varphi_m, \sum_{m=0}^{\infty}\tau_m\right)\right.\right.\\ &\quad \left.\left.+\sum_{m=0}^{\infty}\mathscr{A}_m\right\}\right].\end{aligned} \tag{28}$$

The following terms are described:

$$\begin{aligned}\xi_0(\varphi,\tau) &= \xi(0) + Y^{-1}\left[u^{\alpha}Y\{\mathscr{R}_1(\varphi,\tau)\}\right],\\ \xi_1(\varphi,\tau) &= Y^{-1}\left[u^{\alpha}Y\{\mathscr{P}_1(\varphi_0,\tau_0) + \mathscr{A}_0\}\right].\end{aligned} \tag{29}$$

In general, for $m \geq 1$, it is calculated as

$$\xi_{m+1}(\varphi,\tau) = Y^{-1}\left[u^{\alpha}Y\{\mathscr{P}_1(\varphi_m,\tau_m) + \mathscr{A}_m\}\right]. \tag{30}$$

5. Applications

The solutions to the time-fractional Belousov–Zhabotinskii (B-Z) system are derived using YTDM and HPYTM in this section.

5.1. Example

5.1. Example. Consider the time-fractional B-Z system having $\gamma = \beta = 0$, then equation (1) is reduced to

$$\begin{aligned}\frac{\partial^{\alpha}\xi}{\partial\tau^{\alpha}} &= \frac{\partial^2\xi}{\partial\varphi^2} + \xi - \xi^2 - \xi\zeta, \quad 0 < \alpha \leq 1,\\ \frac{\partial^{\alpha}\zeta}{\partial\tau^{\alpha}} &= \frac{\partial^2\zeta}{\partial\varphi^2} - \delta\xi\zeta,\end{aligned} \tag{31}$$

having initial conditions as

$$\begin{cases}(\varphi,0) = \dfrac{1}{\left(e^{\sqrt{\delta/6\varphi}} + 1\right)^2},\\[2ex] \zeta(\varphi,0) = \dfrac{(1-\delta)e^{\sqrt{\delta/6\varphi}}\left(e^{\sqrt{\delta/6\varphi}} + 2\right)}{\left(e^{\sqrt{\delta/6\varphi}} + 1\right)^2}.\end{cases} \tag{32}$$

The exact solution of equation (28) when $\alpha = 1$ is

$$\begin{aligned}\xi(\varphi,\tau) &= \frac{e^{5\delta/3\tau}}{\left(e^{\sqrt{\delta/6\varphi}} + e^{5\delta/6\tau}\right)^2},\\ \zeta(\varphi,\tau) &= \frac{(1-\delta)e^{\sqrt{\delta/6\varphi}}\left(e^{\sqrt{\delta/6\varphi}} + 2e^{5\delta/6\tau}\right)}{\left(e^{\sqrt{\delta/6\varphi}} + e^{5\delta/6\tau}\right)^2\rangle},\end{aligned} \tag{33}$$

where ϱ and $\delta \neq 1$ are positive parameters.

Remark 1. . The exact solution can take the following form as well:

$$\frac{e^{5\delta/3\tau}}{\left(e^{\sqrt{\delta/6}\varphi}+e^{5\delta/6\tau}\right)^2}=\frac{1}{4}\left(\tanh^2\left(\sqrt{\frac{\delta}{24}}\varphi-\frac{5\delta}{12}\tau\right)-1\right)^2,$$
$$\frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\varphi}+2e^{5\delta/6\tau}\right)}{\left(e^{\sqrt{\delta/6}\varphi}+e^{5\delta/6\tau}\right)^2}=\frac{\delta-1}{4\rangle}\left(\tanh^2\left(\sqrt{\frac{\delta}{24}}\varphi-\frac{5\delta}{12}\tau\right)-2\tanh\left(\sqrt{\frac{\delta}{24}}\varphi-\frac{5\delta}{12}\tau\right)-3\right). \tag{34}$$

Taking Yang transformation of equation (28), we get

$$Y\left\{\frac{\partial^\alpha\xi}{\partial\tau^\alpha}\right\}=Y\left[\frac{\partial^2\xi}{\partial\varphi^2}+\xi-\xi^2-\xi\zeta\right],$$
$$Y\left\{\frac{\partial^\alpha\zeta}{\partial\tau^\alpha}\right\}=Y\left[\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right]. \tag{35}$$

Applying the differential property of the Yang transform, we get

$$\frac{1}{u^\alpha}\{M(u)-u\xi(0)\}=Y\left[\frac{\partial^2\xi}{\partial\varphi^2}+\xi-\xi^2-\xi\zeta\right],$$
$$\frac{1}{u^\alpha}\{M(u)-u\zeta(0)\}=Y\left[\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right],$$
$$M(u)=u\xi(0)+u^\alpha Y\left[\frac{\partial^2\xi}{\partial\varphi^2}+\xi-\xi^2-\xi\zeta\right],$$
$$M(u)=u\zeta(0)+u^\alpha Y\left[\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right]. \tag{36}$$

The inverse Yang transform implies that

$$\xi(\varphi,\tau)=\xi(0)+Y^{-1}\left[u^\alpha\left\{Y\left(\frac{\partial^2\xi}{\partial\varphi^2}+\xi-\xi^2-\xi\zeta\right)\right\}\right],$$
$$\zeta(\varphi,\tau)=\zeta(0)+Y^{-1}\left[u^\alpha\left\{Y\left(\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right)\right\}\right],$$
$$(\varphi,\tau)=\frac{1}{\left(e^{\sqrt{\delta/6}\varphi}+1\right)^2}+Y^{-1}\left[u^\alpha\left\{Y\left(\frac{\partial^2\xi}{\partial\varphi^2}+\xi-\xi^2-\xi\zeta\right)\right\}\right],$$
$$\zeta(\varphi,\tau)=\frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\varphi}+2\right)}{\left(e^{\sqrt{\delta/6}\varphi}+1\right)^2\rangle}+Y^{-1}\left[u^\alpha\left\{Y\left(\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right)\right\}\right]. \tag{37}$$

Now, we apply the homotopy perturbation method as follows:

$$\xi(\varphi,\tau)=\xi_0+\xi_1p+\xi_2p^2+\cdots,$$
$$\zeta(\varphi,\tau)=\zeta_0+\zeta_1p+\zeta_2p^2+\cdots,$$
$$\sum_{k=0}^{\infty}p^k\xi_k(\varphi,\tau)=\frac{1}{\left(e^{\sqrt{\delta/6}\varphi}+1\right)^2}+p\left(Y^{-1}\left[u^\alpha Y\left[\left(\sum_{k=0}^{\infty}p^k\xi_k(\varphi,\tau)\right)_{\varphi\varphi}+\sum_{k=0}^{\infty}p^k\xi_k(\varphi,\tau)-\sum_{k=0}^{\infty}p^kH_k^1(\varphi,\tau)\right]\right]\right),$$
$$\sum_{k=0}^{\infty}p^k\zeta_k(\varphi,\tau)=\frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\varphi}+2\right)}{\left(e^{\sqrt{\delta/6}\varphi}+1\right)^2}+p\left(Y^{-1}\left[u^\alpha Y\left[\left(\sum_{k=0}^{\infty}p^k\zeta_k(\varphi,\tau)\right)_{\varphi\varphi}-\sum_{k=0}^{\infty}p^kH_k^2(\varphi,\tau)\right]\right]\right), \tag{38}$$

where the nonlinear terms are represented by He's polynomials $H_k(\varphi)$. For example, the first few components of He's polynomials are given by

$$
\begin{aligned}
H_0^1(\varphi) &= (2\xi_0\xi_1) + (\xi_0\zeta_1 + \xi_1\zeta_0),\\
H_1^1(\varphi) &= \left(2\xi_0\xi_2 + \xi_1^2\right) + (\xi_0\zeta_2 + \xi_1\zeta_1 + \xi_2\zeta_0),\\
H_2^1(\varphi) &= (2\xi_0\xi_3 + 2\xi_1\xi_2) + (\xi_0\zeta_3 + \xi_1\zeta_2 + \xi_2\zeta_1 + \xi_3\zeta_0),\\
&\vdots\\
H_0^2(\varphi) &= \xi_0\zeta_1 + \xi_1\zeta_0,\\
H_1^2(\varphi) &= \xi_0\zeta_2 + \xi_1\zeta_1 + \xi_2\zeta_0,\\
H_2^2(\varphi) &= \xi_0\zeta_3 + \xi_1\zeta_2 + \xi_2\zeta_1 + \xi_3\zeta_0.
\end{aligned}
\tag{39}
$$

When the coefficients of like powers of p are compared, we get

$$
\begin{aligned}
p^0\colon\ \xi_0(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2},\\
\zeta_0(\varphi,\tau) &\frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\,\varphi}+2\right)}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2},\\
p^1\colon\ \xi_1(\varphi,\tau) &= \frac{5\delta e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3},\\
\zeta_1(\varphi,\tau) &= \frac{5\delta(\delta-1)e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3},\\
p^2\colon\ \xi_2(\varphi,\tau) &= \frac{25\delta^2 e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4},\\
{}_2(\varphi,\tau) &= \frac{25\delta^2(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\rangle\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4},\\
p^3\colon\ \xi_3(\varphi,\tau) &= -\frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3 e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi} - 20e^{\sqrt{3\delta/2}\,\varphi} + 6e^{\sqrt{\delta/6}\,\varphi} - 5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6},\\
\zeta_3(\varphi,\tau) &= -\frac{25\delta^3(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi} - 20e^{\sqrt{3\delta/2}\,\varphi} + 6e^{\sqrt{\delta/6}\,\varphi} - 5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6},\\
p^4\colon\ \xi_4(\varphi,\tau) &= -\frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi} - 5e^{\sqrt{\delta/6}\,\varphi} - 1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{25\delta^4 e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{625\delta^4 e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8},\\
\zeta_4(\varphi,\tau)&=-\frac{25\delta^4(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\varrho\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{125\delta^4(\delta-1)\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\varrho\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\
&+\frac{25\delta^4(\delta-1)e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{625\delta^4(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8},\\
&\vdots
\end{aligned}
\tag{40}
$$

Now, taking $p \longrightarrow 1$ gives the approximate solution as follows:

$$
\begin{aligned}
\xi(\varphi,\tau)&=\xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\cdots\\
&=\frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}+\frac{5\delta e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3}+\frac{25\delta^2 e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}\\
&\quad-\frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}-\frac{25\delta^3 e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}\\
&\quad-\frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\
&\quad+\frac{25\delta^4 e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt{[2]}\,2\delta/3\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{625\delta^4 e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt{[2]}\,2\delta/3\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}\cdots.
\end{aligned}
\tag{41}
$$

YTDM Solution
Taking Yang transformation of equation (28), we get

$$
\begin{aligned}
Y\left\{\frac{\partial^\alpha\xi}{\partial\tau^\alpha}\right\}&=Y\left[\frac{\partial^2\xi}{\partial\varphi^2}+\xi-\xi^2-\xi\zeta\right],\\
Y\left\{\frac{\partial^\alpha\zeta}{\partial\tau^\alpha}\right\}&=Y\left[\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right].
\end{aligned}
\tag{42}
$$

Applying the differential property of the Yang transform, we get

$$
\begin{aligned}
\frac{1}{u^\alpha}\{M(u)-u\xi(0)\}&=Y\left[\frac{\partial^2\xi}{\partial\varphi^2}+\xi-\xi^2-\xi\zeta\right],\\
\frac{1}{u^\alpha}\{M(u)-u\zeta(0)\}&=Y\left[\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right].
\end{aligned}
\tag{43}
$$

The inverse Yang transform implies that

$$\begin{aligned}
\zeta(\varphi,\tau) &= \zeta(0) + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\zeta}{\partial\varphi^2} - \delta\xi\zeta\right)\right\}\right],\\
\xi(\varphi,\tau) &= \xi(0) + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\xi}{\partial\varphi^2} + \xi - \xi^2 - \xi\zeta\right)\right\}\right],\\
\xi(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\xi}{\partial\varphi^2} + \xi - \xi^2 - \xi\zeta\right)\right\}\right],\\
\zeta(\varphi,\tau) &= \frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\,\varphi}+2\right)}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\zeta}{\partial\varphi^2} - \delta\xi\zeta\right)\right\}\right].
\end{aligned} \tag{44}$$

Assume that the unknown $\xi(\varphi,\tau)$ and $\zeta(\varphi,\tau)$ functions, in infinite series form, have the following solution:

$$\begin{aligned}
\xi(\varphi,\tau) &= \sum_{m=0}^{\infty}\xi_m(\varphi,\tau),\\
\zeta(\varphi,\tau) &= \sum_{m=0}^{\infty}\xi_m(\varphi,\tau),
\end{aligned} \tag{45}$$

where the Adomian polynomials $\xi^2 = \sum_{m=0}^{\infty}\mathscr{A}_m$ and $\xi\zeta = \sum_{m=0}^{\infty}\mathscr{B}_m$ and the nonlinear terms have been characterised. Using certain terms, equation (44) can be rewritten in the form

$$\begin{aligned}
\sum_{m=0}^{\infty}\xi_m(\varphi,\tau) &= \xi(\varphi,0) + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\xi}{\partial\varphi^2} + \xi - \sum_{m=0}^{\infty}\mathscr{A}_m - \sum_{m=0}^{\infty}\mathscr{B}_m\right]\right],\\
\sum_{m=0}^{\infty}\zeta_m(\varphi,\tau) &= \zeta(\varphi,0) + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\zeta}{\partial\varphi^2} - \delta\sum_{m=0}^{\infty}\mathscr{B}_m\right]\right],\\
\sum_{m=0}^{\infty}\xi_m(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}\\
&\quad + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\xi}{\partial\varphi^2} + \xi - \sum_{m=0}^{\infty}\mathscr{A}_m - \sum_{m=0}^{\infty}\mathscr{B}_m\right]\right],\\
\sum_{m=0}^{\infty}\zeta_m(\varphi,\tau) &= \frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\,\varphi}+2\right)}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}\\
&\quad + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\zeta}{\partial\varphi^2} - \delta\sum_{m=0}^{\infty}\mathscr{B}_m\right]\right].
\end{aligned} \tag{46}$$

All forms of nonlinearity can be represented by the Adomian polynomials as, according to equation (27),

$$\begin{aligned}
\mathscr{A}_0 &= \xi_0^2,\\
\mathscr{A}_1 &= 2\xi_0\xi_1,\\
\mathscr{A}_2 &= 2\xi_0\xi_2 + \xi_1^2,\\
\mathscr{A}_3 &= 2\xi_0\xi_3 + 2\xi_1\xi_2,\\
\mathscr{B}_0 &= \xi_0 w_0,\\
\mathscr{B}_1 &= \xi_1\zeta_0 + \xi_0\zeta_1,\\
\mathscr{B}_2 &= \xi_0\zeta_2 + \xi_1\zeta_1 + \xi_2\zeta_0,\\
\mathscr{B}_3 &= \xi_0\zeta_3 + \xi_1\zeta_2 + \xi_2\zeta_1 + \xi_3\zeta_0.
\end{aligned} \tag{47}$$

Thus, on comparing both sides of equation (46),

$$\begin{aligned}
\xi_0(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2},\\
\zeta_0(\varphi,\tau) &= \frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\,\varphi}+2\right)}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}.
\end{aligned} \tag{48}$$

For $m = 0$,

$$\begin{aligned}
\xi_1(\varphi,\tau) &= \frac{5\delta e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3},\\
\zeta_1(\varphi,\tau) &= \frac{5\delta(\delta-1)e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3}.
\end{aligned} \tag{49}$$

For $m = 1$,

$$\xi_2(\varphi,\tau) = \frac{25\delta^2 e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi} - 1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4},$$
$$\zeta_2(\varphi,\tau) = \frac{25\delta^2(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi} - 1\right)}{18\rangle\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}. \tag{50}$$

For $m = 2$,

$$\xi_3(\varphi,\tau) = -\frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3 e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi} - 20e^{\sqrt{3\delta/2}\,\varphi} + 6e^{\sqrt{\delta/6}\,\varphi} - 5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6},$$
$$\zeta_3(\varphi,\tau) = -\frac{25\delta^3(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\rangle\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi} - 20e^{\sqrt{3\delta/2}\,\varphi} + 6e^{\sqrt{\delta/6}\,\varphi} - 5\right)}{108\rangle\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}. \tag{51}$$

For $m = 3$,

$$\begin{aligned}\xi_4(\varphi,\tau) = &-\frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi} - 5e^{\sqrt{\delta/6}\,\varphi} - 1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi} - 1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\ &+\frac{25\delta^4 e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi} + 100e^{\sqrt{[2]\,2\delta/3\varphi}} + 85e^{\sqrt{\delta/6}\,\varphi} - 4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{625\delta^4 e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt{[2]\,2\delta/3\varphi}} + 1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8},\\ \zeta_4(\varphi,\tau) = &-\frac{25\delta^4(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi} - 5e^{\sqrt{\delta/6}\,\varphi} - 1\right)}{27\varrho\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4(\delta-1)\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi} - 1\right)}{27\varrho\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\ &+\frac{25\delta^4(\delta-1)e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi} + 100e^{\sqrt{[2]\,2\delta/3\varphi}} + 85e^{\sqrt{\delta/6}\,\varphi} - 4\right)}{324\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{625\delta^4(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt{[2]\,2\delta/3\varphi}} + 1\right)}{648\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}.\end{aligned} \tag{52}$$

The remaining YTDM solution elements xi_m and ζ_m for $(m \geq 3)$ are similarly simple to get. As a result, we define the series of possibilities as follows:

$$\xi(\varphi,\tau) = \sum_{m=0}^{\infty} \xi_m(\varphi,\tau) = \xi_0(\varphi,\tau) + \xi_1(\varphi,\tau) + \xi_2(\varphi,\tau) + \xi_3(\varphi,\tau) + \xi_4(\varphi,\tau) + \cdots,$$

$$\zeta(\varphi,\tau) = \sum_{m=0}^{\infty} \zeta_m(\varphi,\tau) = \zeta_0(\varphi,\tau) + \zeta_1(\varphi,\tau) + \zeta(\varphi,\tau) + \zeta_3(\varphi,\tau) + \zeta_4(\varphi,\tau) + \cdots,$$

$$\begin{aligned}
\xi(\varphi,\tau) = {} & \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + \frac{5\delta e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3} + \frac{25\delta^2 e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4} \\
& - \frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3 e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} \\
& - \frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7} \\
& + \frac{25\delta^4 e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[3]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{625\delta^4 e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[3]{2\delta/3}\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}\cdots \\
\zeta(\varphi,\tau) = {} & \frac{(1-\delta)e^{\sqrt{\delta/6}\varphi}\left(e^{\sqrt{\delta/6}\,\varphi}+2\right)}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + \frac{5\delta(\delta-1)e^{\sqrt{\delta/6}\varphi}}{3\rangle\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\varphi}\varphi+1\right)^3} + \frac{25\delta^2(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4} \\
& - \frac{25\delta^3(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\rangle\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} \\
& - \frac{25\delta^4(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4(\delta-1)\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7} \\
& + \frac{25\delta^4(\delta-1)e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[3]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{625\delta^4(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[3]{2\delta/3}\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}\cdots.
\end{aligned} \tag{53}$$

In Figure 1, the exact and analytical solution graph of $\xi(\varphi,\tau)$ is shown. In Figure 2, the different fractional graph of $\alpha = 1, 0.7, 0.5$, and 0.3 of $\xi(\varphi,\tau)$ is shown. Similarly, in Figure 3, the exact and analytical solution graph of $\zeta(\varphi,\tau)$ is shown. In Figure 4, the different fractional graph of $\alpha = 1, 0.7, 0.5$, and 0.3 of $\zeta(\varphi,\tau)$ is shown.

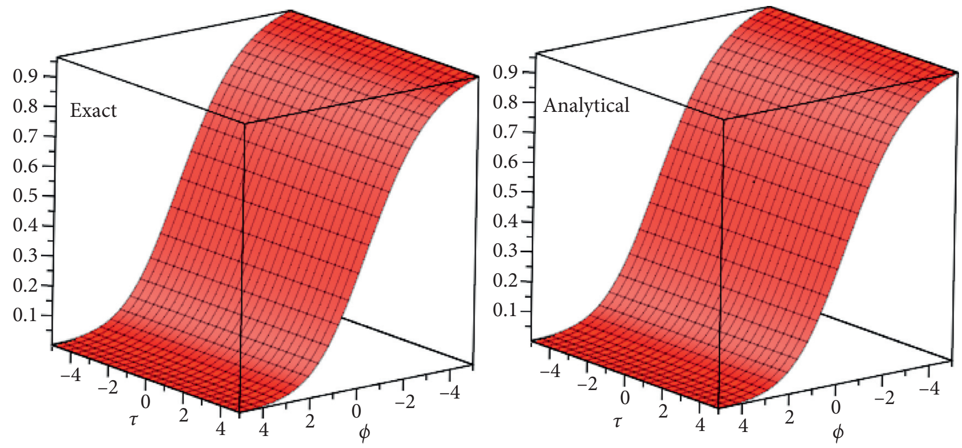

FIGURE 1: The exact solution and analytical solution graph at $\alpha = 1$ for $\xi(\varphi, \tau)$.

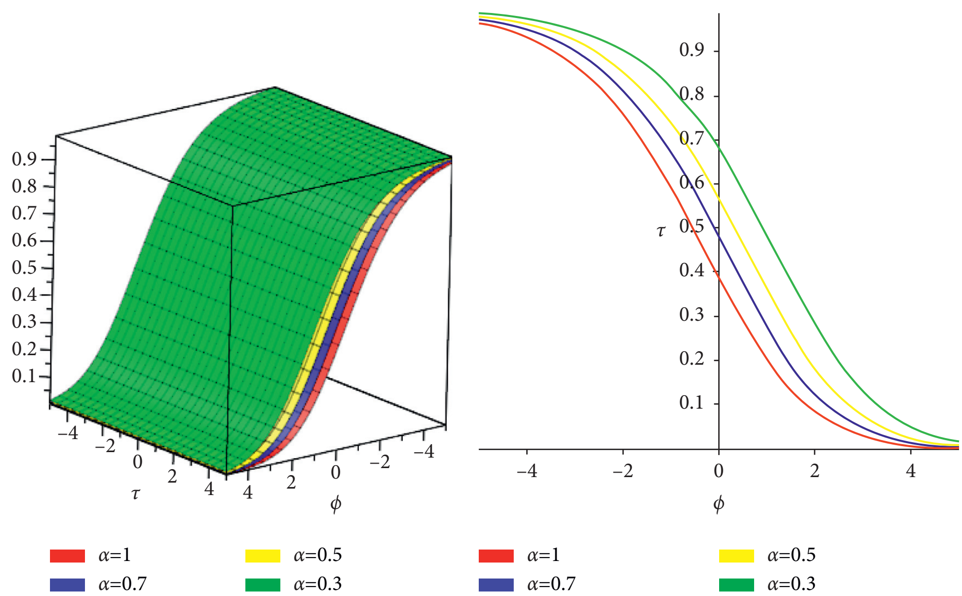

FIGURE 2: The different fractional-order solution graph of α for $\xi(\varphi, \tau)$.

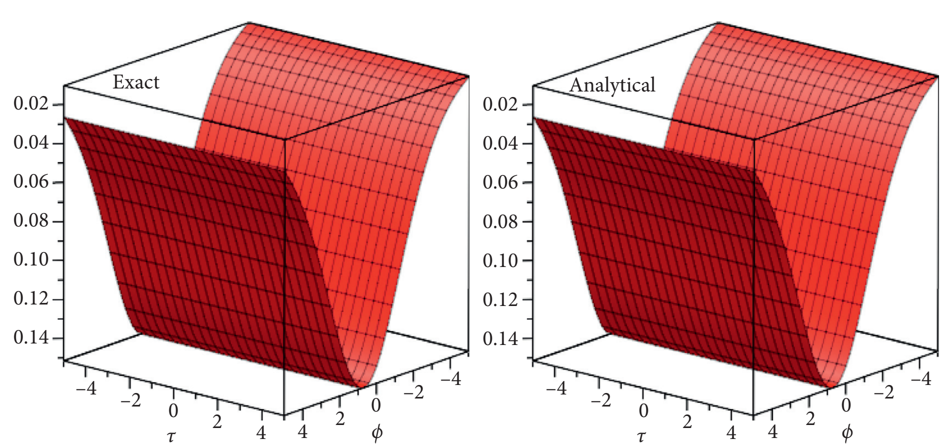

FIGURE 3: The exact solution and analytical solution graph at $\alpha = 1$ for $\zeta(\varphi, \tau)$.

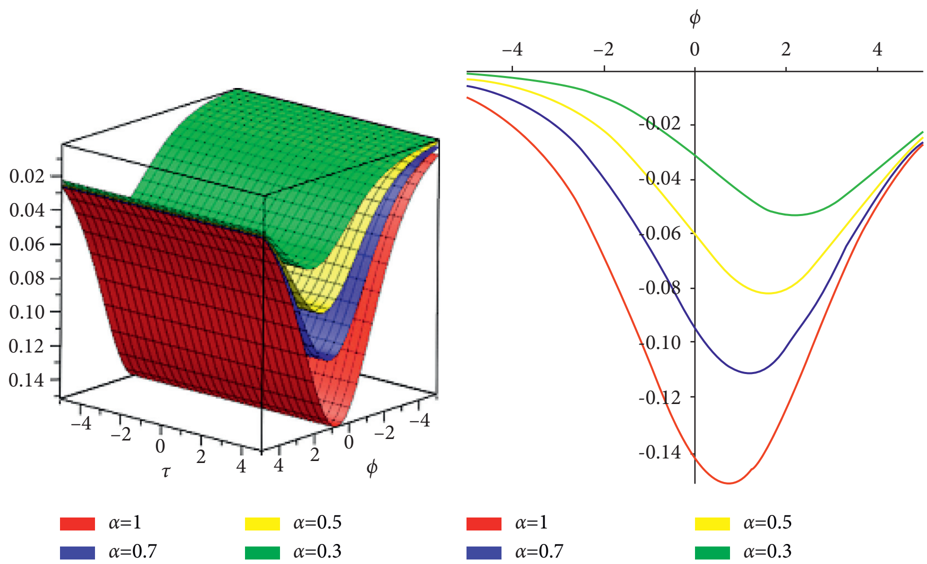

FIGURE 4: The different fractional-order solution graph of α for $\zeta(\varphi, \tau)$.

5.2. Example. Consider the time-fractional B-Z system having $\gamma = \delta$ and $\beta = 1$, then equation (1) is reduced to

$$\begin{aligned} \frac{\partial^\alpha \xi}{\partial \tau^\alpha} &= \frac{\partial^2 \xi}{\partial \varphi^2} + \zeta + \xi - \xi^2 - \xi\zeta, \quad 0 < \alpha \leq 1, \\ \frac{\partial^\alpha \zeta}{\partial \tau^\alpha} &= \frac{\partial^2 \zeta}{\partial \varphi^2} + \delta\zeta - \delta\xi\zeta, \end{aligned} \tag{54}$$

having initial conditions

$$\begin{cases} \xi(\varphi, 0) = \dfrac{1}{\left(e^{\sqrt{\delta/6}\,\varphi} + 1\right)^2}, \\ \zeta(\varphi, 0) = \dfrac{\delta - 1}{\left(e^{\sqrt{\delta/6}\,\varphi} + 1\right)^2}, \end{cases} \tag{55}$$

The exact solution of equation (54) when $\alpha = 1$ is

$$\begin{aligned} \xi(\varphi, \tau) &= \frac{e^{5\delta/3\tau}}{\left(e^{\sqrt{\delta/6}\,\varphi} + e^{5\delta/6\tau}\right)^2}, \\ \zeta(\varphi, \tau) &= \frac{(\delta - 1)e^{5\delta/3\tau\tau}}{\left(e^{\sqrt{\delta/6}\,\varphi} + e^{5\delta/6\tau}\right)^2}, \end{aligned} \tag{56}$$

where $\delta \neq 1$ is a positive parameter.

Remark 2. . The exact solution can take the following form as well:

$$\begin{aligned} \frac{e^{5\delta/3\tau}}{\left(e^{\sqrt{\delta/6}\,\varphi} + e^{5\delta/6\tau}\right)^2} &= \frac{1}{4}\left(\tanh\left(\sqrt{\frac{\delta}{24}}\varphi - \frac{5\delta}{12}\tau\right) - 1\right)^2, \\ \frac{(\delta - 1)e^{5\delta/3\tau}}{\left(e^{\sqrt{\delta/6}\,\varphi} + e^{5\delta/6\tau}\right)^2} &= \frac{\delta - 1}{4}\left(\tanh\left(\sqrt{\frac{\delta}{24}}\varphi - \frac{5\delta}{12}\tau\right) - 1\right)^2. \end{aligned} \tag{57}$$

Taking Yang transformation of equation (54), we get

$$\begin{aligned} Y\left\{\frac{\partial^\alpha \xi}{\partial \tau^\alpha}\right\} &= Y\left[\frac{\partial^2 \xi}{\partial \varphi^2} + \zeta + t\xi n\!-\!\rangle \xi^2 - \xi\zeta\rangle\right], \\ Y\left\{\frac{\partial^\alpha \zeta}{\partial \tau^\alpha}\right\} &= Y\left[\frac{\partial^2 \zeta}{\partial \varphi^2} + \delta\zeta - \delta\xi\zeta\right]. \end{aligned} \tag{58}$$

Applying the differential property of the Yang transform, we get

$$\begin{aligned} \frac{1}{u^\alpha}\{M(u) - u\xi(0)\} &= Y\left[\frac{\partial^2 \xi}{\partial \varphi^2} + \zeta + \xi - \xi^2 - \xi\zeta\right], \\ \frac{1}{u^\alpha}\{M(u) - u\zeta \leq (0)\} &= Y\left[\frac{\partial^2 \zeta}{\partial \varphi^2} + \delta\zeta - \delta\xi\zeta\right], \\ M(u) &= u\xi(0) + u^\alpha Y\left[\frac{\partial^2 \xi}{\partial \varphi^2} + \zeta + \xi - \xi^2 - \xi\zeta\right], \\ M(u) &= u\zeta(0) + u^\alpha Y\left[\frac{\partial^2 \zeta}{\partial \varphi^2} + \delta\zeta - \delta\xi\zeta\right]. \end{aligned} \tag{59}$$

The inverse Yang transform implies that

$$\begin{aligned}
\xi(\varphi,\tau) &= \xi(0) + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2 \xi}{\partial \varphi^2} + \right\rangle \zeta + \xi - \xi^2 - \xi\zeta\right)\right\}\right],\\
\zeta(\varphi,\tau) &= \zeta(0) + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2 \zeta}{\partial \varphi^2} + \delta\zeta - \delta\xi\zeta\right)\right\}\right],\\
\xi(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2 \xi}{\partial \varphi^2} + \right\rangle \zeta + \xi - \xi^2 - \xi\zeta\right)\right\}\right],\\
\zeta(\varphi,\tau) &= \frac{\delta-1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2 \rangle} + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2 \zeta}{\partial \varphi^2} + \delta\zeta - \delta\xi\zeta\right)\right\}\right].
\end{aligned} \tag{60}$$

Now, we apply the homotopy perturbation method as follows:

$$\begin{aligned}
\xi(\varphi,\tau) &= \xi_0 + \xi_1 p + \xi_2 p^2 + \xi_3 p^3 + \xi_4 p^4 + \cdots,\\
\zeta(\varphi,\tau) &= \zeta_0 + \zeta_1 p + \zeta_2 p^2 + \zeta_3 p^3 + \zeta_4 p^4 + \cdots,\\
\sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + p\left(Y^{-1}\left[u^{\alpha}Y\left[\left(\sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau)\right)_{\varphi\varphi} + \left(\sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau)\right) + \sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau) - \sum_{k=0}^{\infty} p^k H_k^1(\varphi,\tau)\right]\right]\right),\\
\sum_{k=0}^{\infty} p^k \zeta_k(\varphi,\tau) &= \frac{\delta-1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + p\left(Y^{-1}\left[u^{\alpha}Y\left[\left(\sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau)\right)_{\varphi\varphi} + \delta\left(\sum_{k=0}^{\infty} p^k \xi_k(\varphi,\tau)\right) - \sum_{k=0}^{\infty} p^k H_k^2(\varphi,\tau)\right]\right]\right),
\end{aligned} \tag{61}$$

where the nonlinear terms are represented by He's polynomials $H_k(\varphi)$. For example, the first few components of He's polynomials are given by

$$\begin{aligned}
H_0^1(\varphi) &= (2\xi_0\xi_1) + (\xi_0\zeta_1 + \xi_1\zeta_0),\\
H_1^1(\varphi) &= \left(2\xi_0\xi_2 + \xi_1^2\right) + (\xi_0\zeta_2 + \xi_1\zeta_1 + \xi_2\zeta_0),\\
H_2^1(\varphi) &= (2\xi_0\xi_3 + 2\xi_1\xi_2) + (\xi_0\zeta_3 + \xi_1\zeta_2 + \xi_2\zeta_1 + \xi_3\zeta_0),\\
&\vdots\\
H_0^2(\varphi) &= \xi_0\zeta_1 + \xi_1\zeta_0,\\
H_1^2(\varphi) &= \xi_0\zeta_2 + \xi_1\zeta_1 + \xi_2\zeta_0,\\
H_2^2(\varphi) &= \xi_0\zeta_3 + \xi_1\zeta_2 + \xi_2\zeta_1 + \xi_3\zeta_0.
\end{aligned} \tag{62}$$

When the coefficients of like powers of p are compared, we get

$$
\begin{aligned}
p^0 &: \xi_0(\varphi,\tau) = \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}, \\
&\zeta_0(\varphi,\tau) = \frac{\delta-1}{\varrho\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}, \\
p^1 &: \xi_1(\varphi,\tau) = \frac{5\delta e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3}, \\
&\zeta_1(\varphi,\tau) = \frac{5\delta(\delta-1)e^{\sqrt{\delta/6}\varphi}}{3\varrho\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3}, \\
p^2 &: \xi_2(\varphi,\tau) = \frac{25\delta^2 e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}, \\
&\zeta_2(\varphi,\tau) = \frac{25\delta^2(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\varrho\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}, \\
p^3 &: \xi_3(\varphi,\tau) = -\frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3 e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}, \\
&\zeta_3(\varphi,\tau) = -\frac{25\delta^3(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\varrho\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6} - \frac{25\delta^3(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\varrho\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}, \\
p^4 &: \xi_4(\varphi,\tau) = -\frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7} \\
&\quad + \frac{25\delta^4 e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{625\delta^4 e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}, \\
&\zeta_4(\varphi,\tau) = -\frac{25\delta^4(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\varrho\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4(\delta-1)\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\varrho\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7} \\
&\quad + \frac{25\delta^4(\delta-1)e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{625\delta^4(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} \\
&\vdots\,.
\end{aligned}
\tag{63}
$$

Now, taking $p \longrightarrow 1$ gives the approximate solution as follows:

$$
\begin{aligned}
\xi(\varphi,\tau) &= \xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\cdots\\
&= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}+\frac{5\delta e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3}+\frac{25\delta^2 e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}\\
&\quad-\frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}-\frac{25\delta^3 e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}\\
&\quad-\frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\
&\quad+\frac{25\delta^4 e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{625\delta^4 e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}\cdots,\\
\zeta(\varphi,\tau) &= \zeta_0+\zeta_1+\zeta_2+\zeta_3+\zeta_4+\cdots\\
&= \frac{\delta-1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2\rangle}+\frac{5\delta(\delta-1)e^{\sqrt{\delta/6}\varphi}}{3\rangle\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}\varphi+1\right)^3}+\frac{25\delta^2(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}\\
&\quad-\frac{25\delta^3(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}-\frac{25\delta^3(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}\\
&\quad-\frac{25\delta^4(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{125\delta^4(\delta-1)\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\rangle\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\
&\quad+\frac{25\delta^4(\delta-1)e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{625\delta^4(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}\cdots.
\end{aligned}
\tag{64}
$$

YTDM Solution

Taking Yang transformation of equation (41), we get

$$
\begin{aligned}
Y\left\{\frac{\partial^\alpha\xi}{\partial\tau^\alpha}\right\} &= Y\left[\frac{\partial^2\xi}{\partial\varphi^2}+\varrho\zeta+\xi-\xi^2-\varrho\xi\zeta\right],\\
Y\left\{\frac{\partial^\alpha\zeta}{\partial\tau^\alpha}\right\} &= Y\left[\frac{\partial^2\zeta}{\partial\varphi^2}-\delta\xi\zeta\right].
\end{aligned}
\tag{65}
$$

Applying the differential property of the Yang transform, we get

$$
\begin{aligned}
\frac{1}{u^\alpha}\{M(u)-u\xi(0)\} &= Y\left[\frac{\partial^2\xi}{\partial\varphi^2}+\varrho\zeta+\xi-\xi^2-\varrho\xi\zeta\right],\\
\frac{1}{u^\alpha}\{M(u)-u\zeta(0)\} &= Y\left[\frac{\partial^2\zeta}{\partial\varphi^2}+\delta\zeta-\delta\xi\zeta\right].
\end{aligned}
\tag{66}
$$

The inverse Yang transform implies that

$$
\begin{aligned}
\xi(\varphi,\tau) &= \xi(0) + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\xi}{\partial\varphi^2} + \zeta + \xi - \xi^2 - \xi\zeta\right)\right\}\right],\\
\zeta(\varphi,\tau) &= \zeta(0) + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\zeta}{\partial\varphi^2} + \delta\zeta - \delta\xi\zeta\right)\right\}\right],\\
\xi(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\xi}{\partial\varphi^2} + \zeta + \xi - \xi^2 - \xi\zeta\right)\right\}\right],\\
\zeta(\varphi,\tau) &= \frac{\delta-1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + Y^{-1}\left[u^{\alpha}\left\{Y\left(\frac{\partial^2\zeta}{\partial\varphi^2} + \delta\zeta - \delta\xi\zeta\right)\right\}\right].
\end{aligned}
\tag{67}
$$

Assume that the unknown $\xi(\varphi,\tau)$ and $\zeta(\varphi,\tau)$ functions, in infinite series form, have the following solution:

$$
\begin{aligned}
\xi(\varphi,\tau) &= \sum_{m=0}^{\infty}\xi_m(\varphi,\tau),\\
\zeta(\varphi,\tau) &= \sum_{m=0}^{\infty}\zeta_m(\varphi,\tau),
\end{aligned}
\tag{68}
$$

where the Adomian polynomials $\xi^2 = \sum_{m=0}^{\infty}\mathscr{A}_m$ and $\xi\zeta = \sum_{m=0}^{\infty}\mathscr{B}_m$ and the nonlinear terms have been characterised. Using certain terms, equation (67) can be rewritten in the form as follows:

$$
\begin{aligned}
\sum_{m=0}^{\infty}\xi_m(\varphi,\tau) &= \xi(\varphi,0) + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\xi}{\partial\varphi^2} + \varrho\zeta + \xi - \sum_{m=0}^{\infty}\mathscr{A}_m - \varrho\sum_{m=0}^{\infty}\mathscr{B}_m\right]\right],\\
\sum_{m=0}^{\infty}\zeta_m(\varphi,\tau) &= \zeta(\varphi,0) + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\zeta}{\partial\varphi^2} + \delta\zeta - \delta\sum_{m=0}^{\infty}\mathscr{B}_m\right]\right],\\
\sum_{m=0}^{\infty}\xi_m(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\xi}{\partial\varphi^2} + \varrho\zeta + \xi - \sum_{m=0}^{\infty}\mathscr{A}_m - \varrho\sum_{m=0}^{\infty}\mathscr{B}_m\right]\right],\\
\sum_{m=0}^{\infty}\zeta_m(\varphi,\tau) &= \frac{\delta-1}{\varrho\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2} + Y^{-1}\left[u^{\alpha}Y\left[\frac{\partial^2\zeta}{\partial\varphi^2} + \delta\zeta - \delta\sum_{m=0}^{\infty}\mathscr{B}_m\right]\right].
\end{aligned}
\tag{69}
$$

All forms of nonlinearity can be represented by the Adomian polynomials, according to equation (25), as

$$
\begin{aligned}
\mathscr{A}_0 &= \xi_0^2,\\
\mathscr{A}_1 &= 2\xi_0\xi_1,\\
\mathscr{A}_2 &= 2\xi_0\xi_2 + \xi_1^2,\\
\mathscr{A}_3 &= 2\xi_0\xi_3 + 2\xi_1\xi_2,\\
\mathscr{B}_0 &= \xi_0 w_0,\\
\mathscr{B}_1 &= \xi_1\zeta_0 + \xi_0\zeta_1,\\
\mathscr{B}_2 &= \xi_0\zeta_2 + \xi_1\zeta_1 + \xi_2\zeta_0,\\
\mathscr{B}_3 &= \xi_0\zeta_3 + \xi_1\zeta_2 + \xi_2\zeta_1 + \xi_3\zeta_0.
\end{aligned}
\tag{70}
$$

Thus, on comparing both side of equation (69),

$$
\begin{aligned}
\xi_0(\varphi,\tau) &= \frac{1}{\left(e^{\sqrt{\delta/6\varphi}}+1\right)^2},\\
\zeta_0(\varphi,\tau) &= \frac{\delta-1}{\left(e^{\sqrt{\delta/6\varphi}}+1\right)^2}.
\end{aligned}
\tag{71}
$$

For $m = 0$,

$$
\begin{aligned}
\xi_1(\varphi,\tau) &= \frac{5\delta e^{\sqrt{\delta/6\varphi}}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^3},\\
\zeta_1(\varphi,\tau) &= \frac{5\delta(\delta-1)e^{\sqrt{\delta/6\varphi}}}{3\Gamma\rangle(\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^3}.
\end{aligned}
\tag{72}
$$

For $m = 1$,

$$
\begin{aligned}
\xi_2(\varphi,\tau) &= \frac{25\delta^2 e^{\sqrt{\delta/6\varphi}}\left(2e^{\sqrt{\delta/6\varphi}}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^4},\\
\zeta_2(\varphi,\tau) &\frac{25\delta^2(\delta-1)e^{\sqrt{\delta/6\varphi}}\left(2e^{\sqrt{\delta/6\varphi}}-1\right)}{18\rangle\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^4}.
\end{aligned}
\tag{73}
$$

For $m = 2$,

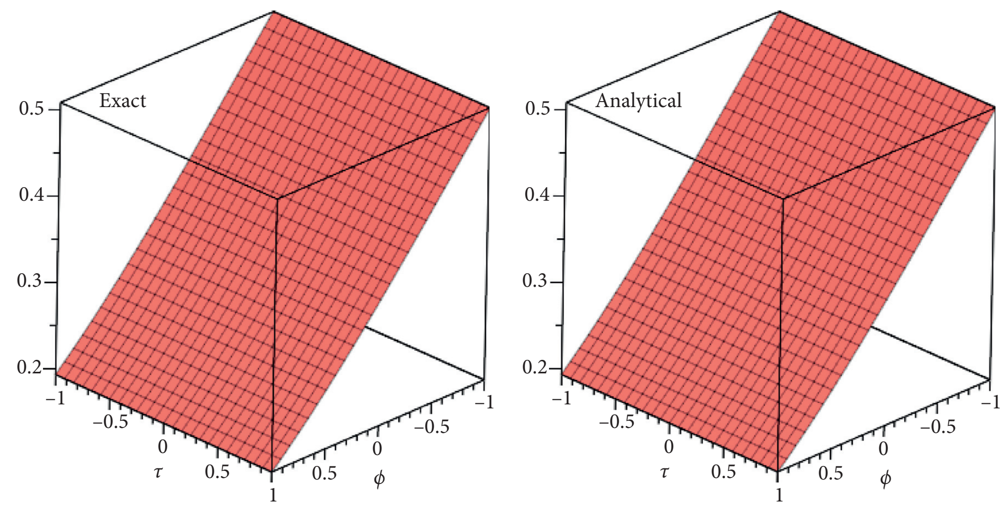

Figure 5: The exact solution and analytical solution graph at $\alpha = 1$ for $\xi(\varphi, \tau)$.

$$\xi_3(\varphi,\tau) = -\frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3\varphi}}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^6} - \frac{25\delta^3 e^{\sqrt{\delta/6\varphi}}\left(15e^{\sqrt{2\delta/3\varphi}} - 20e^{\sqrt{3\delta/2\varphi}} + 6e^{\sqrt{\delta/6\varphi}} - 5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^6},$$
$$\zeta_3(\varphi,\tau) = -\frac{25\delta^3(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3\varphi}}}{9\varrho\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^6} - \frac{25\delta^3(\delta-1) e^{\sqrt{\delta/6\varphi}}\left(15e^{\sqrt{2\delta/3\varphi}} - 20e^{\sqrt{3\delta/2\varphi}} + 6e^{\sqrt{\delta/6\varphi}} - 5\right)}{108\varrho\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^6}. \tag{74}$$

For $m = 3$,

$$\xi_4(\varphi,\tau) = -\frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3\varphi}}\left(11e^{\sqrt{2\delta/3\varphi}} - 5e^{\sqrt{\delta/6\varphi}} - 1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8} - \frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3\varphi}}\left(2e^{\sqrt{\delta/6\varphi}} - 1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^7}$$
$$+\frac{25\delta^4 e^{\sqrt{2\delta/3\varphi}}\left(124e^{\sqrt{2\delta/3\varphi}} + 100e^{\sqrt{[2]\,2\delta/3\varphi}} + 85e^{\sqrt{\delta/6\varphi}} - 4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^8} - \frac{625\delta^4 e^{\sqrt{\delta/6\varphi}}\left(17e^{\sqrt{[2]\,2\delta/3\varphi}} + 1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^8},$$
$$\zeta_4(\varphi,\tau) = -\frac{25\delta^4(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3\varphi}}\left(11e^{\sqrt{2\delta/3\varphi}} - 5e^{\sqrt{\delta/6\varphi}} - 1\right)}{27\varrho\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^8} - \frac{125\delta^4(\delta-1)\Gamma(3\alpha+1)e^{\sqrt{2\delta/3\varphi}}\left(2e^{\sqrt{\delta/6\varphi}} - 1\right)}{27\varrho\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^7}$$
$$+\frac{25\delta^4(\delta-1) e^{\sqrt{2\delta/3\varphi}}\left(124e^{\sqrt{2\delta/3\varphi}} + 100e^{\sqrt{[2]\,2\delta/3\varphi}} + 85e^{\sqrt{\delta/6\varphi}} - 4\right)}{324\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^8} - \frac{625\delta^4(\delta-1) e^{\sqrt{\delta/6\varphi}}\left(17e^{\sqrt{[2]\,2\delta/3\varphi}} + 1\right)}{648\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6\varphi}}+1\right)^8}. \tag{75}$$

The remaining YTDM solution elements xi_m and ζ_m for $(m \geq 3)$ are similarly simple to get. As a result, we define the series of possibilities as follows:

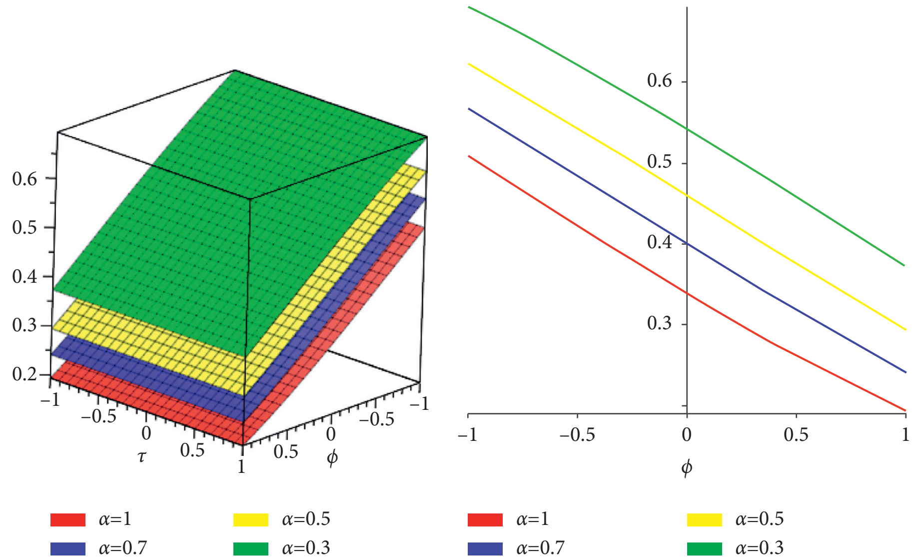

Figure 6: The different fractional-order solution graph of α for $\xi(\varphi,\tau)$.

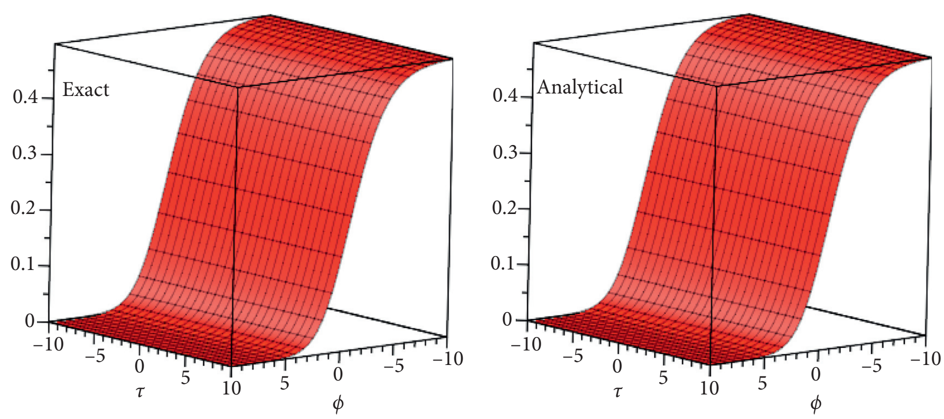

Figure 7: The exact solution and analytical solution graph at $\alpha = 1$ for $\xi(\varphi,\tau)$.

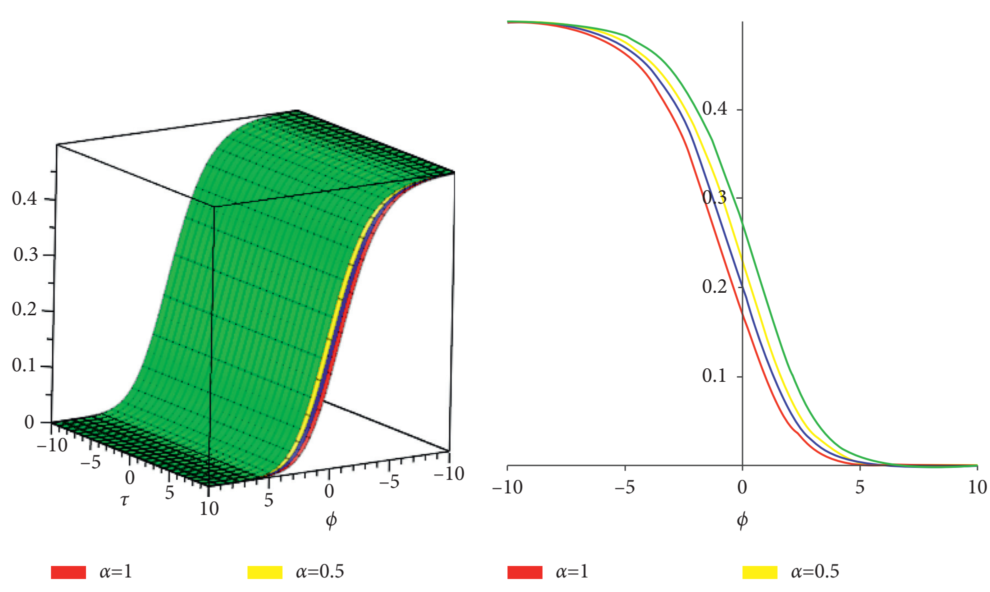

Figure 8: The different fractional-order solution graph of α for $\zeta(\varphi,\tau)$.

$$\xi(\varphi,\tau)=\sum_{m=0}^{\infty}\xi_m(\varphi,\tau)=\xi_0(\varphi,\tau)+\xi_1(\varphi,\tau)+\xi_2(\varphi,\tau)+\xi_3(\varphi,\tau)+\xi_4(\varphi,\tau)+\cdots,$$

$$\zeta(\varphi,\tau)=\sum_{m=0}^{\infty}\zeta_m(\varphi,\tau)=\zeta_0(\varphi,\tau)+\zeta_1(\varphi,\tau)+\zeta(\varphi,\tau)+\zeta_3(\varphi,\tau)+\zeta_4(\varphi,\tau)+\cdots,$$

$$\begin{aligned}
\xi(\varphi,\tau)={}&\frac{1}{\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}+\frac{5\delta e^{\sqrt{\delta/6}\varphi}}{3\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^3}+\frac{25\delta^2 e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}\\
&-\frac{25\delta^3\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}-\frac{25\delta^3 e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}\\
&-\frac{25\delta^4\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{125\delta^4\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\
&+\frac{25\delta^4 e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{625\delta^4 e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}\cdots,\\
\zeta(\varphi,\tau)={}&\frac{\delta-1}{\varrho\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^2}+\frac{5\delta(\delta-1)e^{\sqrt{\delta/6}\varphi}}{3\varrho\Gamma(\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}\varphi+1\right)^3}+\frac{25\delta^2(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{18\varrho\Gamma(2\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^4}\\
&-\frac{25\delta^3(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}}{9\varrho\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}-\frac{25\delta^3(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(15e^{\sqrt{2\delta/3}\,\varphi}-20e^{\sqrt{3\delta/2}\,\varphi}+6e^{\sqrt{\delta/6}\,\varphi}-5\right)}{108\varrho\Gamma(3\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^6}\\
&-\frac{25\delta^4(\delta-1)\Gamma(2\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(11e^{\sqrt{2\delta/3}\,\varphi}-5e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\varrho\Gamma(\alpha+1)^2\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{125\delta^4(\delta-1)\Gamma(3\alpha+1)e^{\sqrt{2\delta/3}\varphi}\left(2e^{\sqrt{\delta/6}\,\varphi}-1\right)}{27\varrho\Gamma(\alpha+1)\Gamma(2\alpha+1)\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^7}\\
&+\frac{25\delta^4(\delta-1)e^{\sqrt{2\delta/3}\varphi}\left(124e^{\sqrt{2\delta/3}\,\varphi}+100e^{\sqrt[2]{2\delta/3}\varphi}+85e^{\sqrt{\delta/6}\,\varphi}-4\right)}{324\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}-\frac{625\delta^4(\delta-1)e^{\sqrt{\delta/6}\varphi}\left(17e^{\sqrt[2]{2\delta/3}\varphi}+1\right)}{648\varrho\Gamma(4\alpha+1)\left(e^{\sqrt{\delta/6}\,\varphi}+1\right)^8}\cdots.
\end{aligned}\tag{76}$$

In Figure 5, the exact and analytical solution graph of $\xi(\varphi,\tau)$ is shown. In Figure 6, the different fractional graph of $\alpha=1, 0.7, 0.5$, and 0.3 of $\xi(\varphi,\tau)$ is shown. Similarly, in Figure 7, the exact and analytical solution graph of $\zeta(\varphi,\tau)$ is shown. In Figure 8, the different fractional graph of $\alpha=1, 0.7, 0.5$, and 0.3 of $\zeta(\varphi,\tau)$ is shown.

6. Conclusion

Finding the analytical solution to fractional partial differential equations is a tough task in most circumstances. This article makes a successful attempt to solve time-fractional Belousov–Zhabotinskii reaction for this purpose. It is confirmed that the proposed methods are the best tool for solving FPDEs. The displayed graphs confirm the strong relationship between the exact and analytical results. The plotted graphs confirmed the accuracy of the suggested techniques. In order to understand the behavior of the given problems, solutions at different fractional order is taken and is shown with the help of graphs. The convergence phenomenon has confirmed the reliability of the suggested method.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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