Research Article

Global Exponential Stability and Periodicity of Nonautonomous Impulsive Neural Networks with Time-Varying Delays and Reaction-Diffusion Terms

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In this paper, we investigate the global exponential stability and periodicity of nonautonomous cellular neural networks with reaction-diffusion, impulses, and time-varying delays. By establishing a new differential inequality for nonautonomous systems, using the properties of M-matrix and inequality techniques, some new sufficient conditions for the global exponential stability of the system are obtained. Moreover, sufficient conditions for the periodic solutions of the system are obtained by using the Poincaré mapping and the fixed point theory. The validity and superiority of the main results are verified by numerical examples and simulations.

1. Introduction

Since Chua proposed the cellular neural networks (CNNs) in the 1980s [1], the neural network models have been widely studied and applied in the fields of signal recognition, image processing, pattern classification, and so on. Indeed, all these applications rely on the dynamic behaviors of neural networks. The key to using neural networks to solve these problems is that the neural network must be globally dynamic stable; that is to say, each of its loci must converge to a unique balance. In the practical application of neural networks, time delay is inevitable because of the limited signal transmission and switching speed. As we all know, the time delay may make the system lose stability or produce nonlinear oscillation, which makes the dynamic behaviors of the system more complicated. Under the circumstances, the neural network model with time delay is introduced to better reflect the actual state. Hence, how to ensure the stability and good performance of the system is particularly important in the case of time delay. The dynamic analysis of time-delay neural networks has received considerable attention; for example, see [2–7].

In practical applications, neurons change over time and space. In order to better describe the state of neurons, a partial differential equation that considers diffusion is introduced in space. So, in 1995, Chua et al. [8] established the structure of reaction-diffusion CNNs. In [9,10], the authors have investigated the global exponential stability and the global exponential robust stability of reaction-diffusion neural networks with time-varying delays, by using partial differential equations to express the diffusion term of the CNNs, some interesting results were obtained. In [11], the authors used the properties and inequality analysis of the nonsingular M-matrix to study the exponential stability of CNNs with diffusion. In [12], considering the global exponential stability under the Neumann boundary conditions, using the properties of the M-matrix and algebraic inequality techniques, some sufficient conditions for the global exponential stability of the system are obtained. In [13], under Neumann boundary conditions, several sufficient conditions based on linear matrix inequalities for the global asymptotic stability of reaction-diffusion CNNs with distributed delay are given. In [14, 15], the Dirichlet boundary conditions of the rectangular region

are introduced into the neural network with reaction-diffusion. In [16], under the Dirichlet boundary conditions, a diffusive-dependent Lyapunov functional related to the diffusion term is proposed to analyze the effect of the diffusion coefficient of neurons on the dynamics of the model, and the stabilization effect of the reaction-diffusion term is studied by using the extended Wirtinger inequality. In [17, 18], the authors considered the exponential stability of reaction-diffusion neural networks with time delays in general spatial regions and obtained some new sufficient conditions for system stability by using Poincaré's inequality technique.

As we all know, the phenomenon of nonautonomy arises in a lot of real systems frequently. In particular, when a system is in a dynamic behavior for a long time, the parameters of the system often change with time. Compared with autonomous CNNs, the study of nonautonomous CNNs is equally important. In fact, many authors have explored the dynamic behavior of different types of nonautonomous CNNs with time delays and produced some interesting results by constructing some Lyapunov–Krasovskii functionals [19–26]. In addition, some functions in these methods must be differentiable, while

other features require additional constraints in terms of delay. In [27], the author improved some of the earlier Halanay-type inequalities and obtained the exponential stability of CNNs and the existence of periodic solutions through these new inequalities. In [28], sufficient conditions for the global exponential stability of a special class of nonautonomous differential systems with infinite time delay are given, without proving the existence of equilibrium point, periodic solution, or almost periodic solution. In addition, during the physical simulation of the neural network, the neural network will sometimes be interrupted due to the transient disturbance, which is called the impulse effect. Impulse may cause system instability or other more complex dynamic behavior. The influence of pulses on neural networks has been considered in many literatures [29–33]. Therefore, it is of great significance to study the stability and periodicity of neural network models with dynamic impulses, time delays, and reaction-diffusion terms.

Here, we introduce a class of new nonautonomous impulsive neural networks with time-varying delays and reaction-diffusion terms as follows:

$$\frac{\partial z_p(t, x)}{\partial t} = \sum_{k=1}^m \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial z_p(t, x)}{\partial x_k} \right) - \beta_p(t) z_p(t, x) + \sum_{q=1}^n a_{pq}(t) f_q(z_q(t, x)) \quad (1a)$$

$$+ \sum_{q=1}^n b_{pq}(t) g_q(z_q(t - \tau_{pq}(t), x)) + J_p(t), \quad t \neq t_l, p = 1, 2, \dots, n, x \in \Xi,$$

$$z_p(t_l^+, x) = I_{pl}(z_p(t_l^-, x)), \quad p = 1, 2, \dots, n, x \in \Xi, l \in N \triangleq \{0, 1, 2, \dots\}, \quad (1b)$$

$$z_p(t, x) = 0, \quad t \geq 0, x \in \partial \Xi, \quad (1c)$$

$$z_p(s, x) = \mu_p(s, x), \quad s \in [-\tau, 0], p = 1, 2, \dots, n, \quad (1d)$$

where $z_p(t, x)$ is the state of the p th neuron in space x and time t . $x = (x_1, x_2, \dots, x_m)^T \in \Xi \subset R^m$. $\Xi = \{x = (x_1, x_2, \dots, x_m)^T \mid |x_k| < L_k, k = 1, 2, \dots, m\}$ is a bounded compact set with smooth boundary $\partial \Xi$ and $\text{mes } \Xi > 0$ in space R^m (L_k is a positive constant). $z(t, x) = (z_1(t, x), z_2(t, x), \dots, z_n(t, x))^T \in R^n$. $\beta_p \in C(R_+, R_+)$ denotes the rate at which the p th neuron resets its potential to the isolated resting state when disconnected from the network and external input; $a_{pq} \in C(R_+, R)$ denotes the intensity of the p th element on the q th element at time t ; $b_{pq} \in C(R_+, R)$ represents the strength of the q th element due to delay at the p th element. $\tau_{pq}(t)$ represents the transmission delay from the p th element to

the q th element and satisfies $0 \leq \tau_{pq}(t) \leq \tau$ ($\tau = \max_{1 \leq p, q \leq n} \{\tau_{pq}\}$); $f_q \in C(R, R)$ and $g_q \in C(R, R)$ denote the activation function and activation function caused by delay, respectively; $J_p(t)$ represents the external input of the p th neuron at time t . In (1b), $t_l > 0$ satisfies $t_l < t_{l+1}$, $\lim_{l \rightarrow +\infty} t_l = +\infty$; $z_p(t_l^-, x)$ and $z_p(t_l^+, x)$ represent the left and right limits at t_l , respectively; I_{pl} shows impulsive perturbation of the p th neuron at time t_l . Let $z_p(t_l^+, x) = z_p(t_l, x)$, $l \in N$. Equation (1c) denotes the Dirichlet boundary conditions and (1d) denotes the initial conditions.

If impulsive operator $I_{pl}(z_p) = 0$, $p = 1, 2, \dots, n, l \in N$, systems (1a)–(1d) will become a model of the following form:

$$\frac{\partial z_p(t, x)}{\partial t} = \sum_{k=1}^m \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial z_p(t, x)}{\partial x_k} \right) - \beta_p(t) z_p(t, x) + \sum_{q=1}^n a_{pq}(t) f_q(z_q(t, x))$$

$$+ \sum_{q=1}^n b_{pq}(t) g_q(z_q(t - \tau_{pq}(t), x)) + J_p(t), \quad p = 1, 2, \dots, n, x \in \Xi,$$

(2a)

$$z_p(t, x) = 0, \quad t \geq 0, x \in \partial \Xi,$$

(2b)

$$z_p(s, x) = \mu_p(s, x), \quad s \in [-\tau, 0], p = 1, 2, \dots, n.$$

(2c)

Systems (2a)–(2c) are called the continuous form of impulsive systems (1a)–(1d).

The main contributions of this work are as follows:

- (I) We have formulated a class of new neural network models which assembles nonautonomous neural networks, reaction-diffusion cellular neural networks with time-varying delays, impulses, and the Dirichlet boundary conditions.
- (II) We have obtained several new criteria which guarantee the global exponential stability and periodicity of systems (1a)–(1d). These criteria are shown in simple algebraic inequality forms, and they rely only on the system parameters.
- (III) An optimization method is proposed in order to estimate the exponential convergence rate of systems (1a)–(1d).

The remainder of this paper is organized as follows. In Section 2, some assumptions and the necessary knowledge are provided. We give our main results in Section 3. In Section 4, two examples are given to show the effectiveness of the results obtained here. Finally, we give the conclusion in Section 5.

2. Preliminaries

In this section, we make some assumptions and introduce some notions.

(A1) For the activation functions $f_p(z)$ and $g_p(z)$, there exist positive diagonal matrices $F = \text{diag}(F_1, F_2, \dots, F_n)$ and $G = \text{diag}(G_1, G_2, \dots, G_n)$ such that

$$F_p = \sup_{z \neq w} \left| \frac{f_p(z) - f_p(w)}{z - w} \right|,$$

(3)

$$G_p = \sup_{z \neq w} \left| \frac{g_p(z) - g_p(w)}{z - w} \right|,$$

for all $z, w \in R(z \neq w)$, $p = 1, 2, \dots, n$.

(A2) There exists a positive diagonal matrix $\mathfrak{F}_l = \text{diag}(I_{1l}, \dots, I_{nl})$ such that

$$|I_{pl}(z) - I_{pl}(v)| \leq I_{pl}|z - v|,$$

(4)

for all $z, v \in R(z \neq v)$, $p \in \{1, 2, \dots, n\}$, $l \in N$.

(A3) $a_{pq}(t)$, $b_{pq}(t)$, $\beta_p(t)$, $J_p(t)$, and $\tau_{pq}(t)$ are continuous bounded functions defined on $t \in [0, +\infty)$ for all $p, q \in \{1, 2, \dots, n\}$.

We define $\|z_p(t, x)\|_2 = [\int_{\Xi} |z_p(t, x)|^2 dx]^{(1/2)}$, $p = 1, 2, \dots, n$. $R_+ = [0, +\infty)$ and $h^+(t) = \max\{0, h(t)\}$, where $h(t)$ is a continuous function. For any $\omega(s, x) = (\omega_1(s, x), \omega_2(s, x), \dots, \omega_n(s, x))^T \in PC(\Xi)$, the norm on $PC(\Xi)$ is defined by

$$\|\omega\| = \sup_{-\tau \leq s \leq 0} \sum_{p=1}^n \|\omega_p(s, x)\|_2. \quad (5)$$

$PC(\Xi) \triangleq \{\omega: [-\tau, 0] \times \Xi \rightarrow R^n | \omega(s, x) \text{ is bounded on } [-\tau, 0] \times \Xi \text{ and } \omega(s^+, x) = \omega(s, x) \text{ for } s \in [-\tau, 0]. \omega(s^-, x) \text{ exists for } s \in [-\tau, 0] \text{ and } \omega(s^-, x) = \omega(s, x) \text{ for all but a finite number of points } s \in [-\tau, 0]\}$.

$PC \triangleq \{\omega: [-\tau, 0] \rightarrow R^n | \omega(s) \text{ is bounded on } [-\tau, 0] \text{ and } \omega(s^+) = \omega(s) \text{ for } s \in [-\tau, 0]. \omega(s^-) \text{ exists for } s \in [-\tau, 0] \text{ and } \omega(s^-) = \omega(s) \text{ for all but a finite number of points } s \in [-\tau, 0]\}$.

For $C, D \in R^{m \times n}$, we define the Schur product by $C \otimes D = (c_{pq} d_{pq})_{m \times n}$ and $C \geq D$ ($C > D$) means that the inequality $c_{pq} \geq d_{pq}$ ($c_{pq} > d_{pq}$) holds. $\mathbf{e} = (1, 1, \dots, 1)^T \in R^n$ and E is a n -dimensional identity matrix.

Definition 1. A function $z(t, x)$ ($z: [-\tau, +\infty) \times \Xi \rightarrow R^n$) is said to be the solution of system (1a)–(1d) if $z(t, x)$ satisfies the following conditions:

- (1) For t , $z(t, x)$ is piecewise continuous with the first kind of discontinuity at the points t_l , $l \in N$, and $z(t, x)$ is right-continuous at every discontinuity point
- (2) $z(t, x)$ satisfies (1a)–(1c) for all $t \geq 0$, and $z(t, x) = \mu(t, x)$ for $t \in [-\tau, 0]$

Therefore, $z(t, x)$ is a particular solution that satisfies the initial conditions for systems (1a)–(1d), and $z(t, \mu, x)$ represents the special solution of systems (1a)–(1d) under initial condition $\mu \in PC(\Xi)$.

Definition 2. $z(t, \omega, x)$ is a particular solution of systems (1a)–(1d) and satisfies the initial conditions $\omega \in PC(\Xi)$. In the initial condition of $\bar{\omega} \in PC(\Xi)$, $z(t, \bar{\omega}, x)$ is any solution

to systems (1a)–(1d). If there is a positive number $\lambda > 0$ and $M \geq 1$ such that

$$\|z(t, \omega, x) - z(t, \bar{\omega}, x)\| \leq M\|\omega - \bar{\omega}\|e^{-\lambda t}, \quad \text{for all } t \geq 0, \quad (6)$$

then systems (1a)–(1d) are said to be globally exponentially stable.

Definition 3 (see [34]). A real matrix $D = (d_{pq})_{n \times n}$ is said to be a nonsingular M -matrix if $d_{pq} \leq 0$, $p, q = 1, 2, \dots, n$, $p \neq q$, and all successive principal minors of D are positive.

Lemma 1 (see [34]). Let $D = (d_{pq})_{n \times n}$ with $d_{pq} \leq 0$ ($p \neq q$), then D is a nonsingular M -matrix if and only if the diagonal elements of D are all positive, and there exists a positive vector d such that $Dd > 0$ or $D^T d > 0$.

Lemma 2 (see [35]). Let Ξ be a cube $|x_k| < L_k$ ($k = 1, 2, \dots, m$), and let $h(x)$ be a real-valued function belonging to $C^1(\Xi)$ which vanishes on the boundary of Ξ , i.e., $h(x)|_{\partial\Xi} = 0$. Then,

$$\int_{\Xi} h^2(x) dx \leq L_k^2 \int_{\Xi} \left| \frac{\partial h}{\partial x_k} \right|^2 dx. \quad (7)$$

Lemma 3. If $a(x)$ and $b(x)$ are continuous on $[x_0, +\infty)$ with $a(x) - b(x) \geq \rho > 0$ for all $x \in [x_0, +\infty)$ and there is a positive number L such that $0 \leq b(x) \leq L$ for all $x \geq x_0$, then $\lambda^* \equiv \inf_{x \geq x_0} \{\lambda > 0: \lambda - a(x) + b(x)e^{\lambda\tau(x)} = 0\} > 0$, where $\tau(x)$ is nonnegative function (i.e., $\tau(x) \geq 0$ for all $x \geq 0$).

Proof. Set $F(\lambda) = \lambda - a(x) + b(x)e^{\lambda\tau(x)}$. Note that

$$F(0) = -a(x) + b(x) < 0, \quad (8)$$

$$\frac{dF}{d\lambda} = 1 + \tau(x)b(x)e^{\lambda\tau(x)} > 0 \text{ and } F(+\infty) > 0,$$

then we know that $F(\lambda)$ is a strictly monotone increasing function. Hence, for any $x \geq x_0$, there exists a unique positive $\lambda(x)$ such that

$$\lambda(x) - a(x) + b(x)e^{\lambda(x)\tau(x)} = 0. \quad (9)$$

Therefore, λ^* exists and $\lambda^* \geq 0$. We will prove $\lambda^* > 0$ in the following.

When $\tau(x) = 0$, we can obtain that $\lambda^* = a(x) - b(x) \geq \rho > 0$.

When $\tau(x) > 0$, $\lambda^* < 0$. Here, for any $\sigma < \min\{(\rho/2), (1/\tau(x))n(1 + (\rho/2L))\}$, there exists $x^* > 0$ such that $\lambda^*(x^*) < \sigma$ and satisfies

$$\lambda^*(x^*) - a(x^*) + b(x^*)e^{\lambda^*(x^*)\tau(x^*)} = 0. \quad (10)$$

It follows that

$$\begin{aligned} 0 &= \lambda^*(x^*) - a(x^*) + b(x^*)e^{\lambda^*(x^*)\tau(x^*)} \\ &< \sigma - a(x^*) + b(x^*)\left(1 + \frac{\rho}{2L}\right) \\ &< \frac{\rho}{2} - (a(x^*) - b(x^*)) + b(x^*)\frac{\rho}{2L} \\ &< \frac{\rho}{2} - \rho + \frac{\rho}{2} = 0, \end{aligned} \quad (11)$$

and this is a contradiction. So, $\lambda^* > 0$. The proof is completed. $\square$

Remark 1. Lemma 3 is a generalization of Lemma 4.1 in [36].

Lemma 4. Let $\tau > 0$, $a < b \leq +\infty$, suppose that $Z(t) = (Z_1(t), Z_2(t), \dots, Z_n(t))^T \in C[[a, b], R^n]$ satisfies the following differential inequality:

$$\begin{cases} D^+ Z(t) \leq P(t)Z(t) + (Q(t) \otimes \bar{Z}(t))\mathbf{e}, & a \leq t < b, \\ Z(a+s) \in PC, & -\tau \leq s < 0, \end{cases} \quad (12)$$

where $P(t) = (p_{pq}(t))_{n \times n}$ with $p_{pq}(t) \geq 0$ ($p \neq q$), $Q(t) = (q_{pq}(t))_{n \times n}$ with $q_{pq}(t) \geq 0$, $\bar{Z}(t) = (Z(t - \tau_{pq}(t)))_{n \times n}$. If the initial condition satisfies

$$Z(t) \leq \kappa \xi e^{-\lambda(t)}, \quad \kappa \geq 0, t \in [a - \tau, a], \quad (13)$$

in which the scalar $\lambda > 0$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$ are determined by

$$[\lambda E + P(t) + Q(t) \otimes \zeta(\lambda)]\xi < 0, \quad \zeta(\lambda) = (e^{\lambda\tau_{pq}(t)})_{n \times n}, \quad (14)$$

then $Z(t) \leq \kappa \xi e^{-\lambda(t-a)}$, for $t \in [a, b]$.

Proof. For $p \in \{1, 2, \dots, n\}$ and arbitrary $\varepsilon > 0$, set $w_p(t) \triangleq (\kappa + \varepsilon)\xi_p e^{-\lambda(t-a)}$. We prove that

$$Z_p(t) \leq w_p(t) = (\kappa + \varepsilon)\xi_p e^{-\lambda(t-a)}, \quad t \in [a, b], p = 1, 2, \dots, n. \quad (15)$$

If this is not true, then there exist a number $t^* \in [a, b]$ and some integer r such that

$$\begin{aligned} Z_r(t^*) &= w_r(t^*), \\ D^+ Z_r(t^*) &\geq \dot{w}_r(t^*), \\ Z_p(t) &\leq w_p(t), \quad t \in [a, t^*], p = 1, 2, \dots, n. \end{aligned} \quad (16)$$

From Lemma 4 and (15), we have

$$\begin{aligned}
D^+ Z_r(t^*) &\leq \sum_{q=1}^n [p_{rq}(t^*) Z_q(t^*) + q_{rq}(t^*) Z_q(t^* - \tau_{rq}(t^*))] \\
&\leq \sum_{q=1}^n [p_{rq}(t^*) (\kappa + \varepsilon) \xi_q e^{-\lambda(t^*-a)} + q_{rq}(t^*) (\kappa + \varepsilon) \xi_q e^{-\lambda(t^*-\tau_{rq}(t^*)-a)}] \\
&= \sum_{q=1}^n [p_{rq}(t^*) (\kappa + \varepsilon) \xi_q e^{-\lambda(t^*-a)} + q_{rq}(t^*) e^{\lambda \tau_{rq}(t^*)} (\kappa + \varepsilon) \xi_q e^{-\lambda(t^*-a)}] \\
&\leq \sum_{q=1}^n [p_{rq}(t^*) + q_{rq}(t^*) e^{\lambda \tau_{rq}(t^*)}] (\kappa + \varepsilon) \xi_q e^{-\lambda(t^*-a)}.
\end{aligned} \tag{17}$$

According to $[\lambda E + P(t) + Q(t) \otimes \zeta(\lambda)] \xi < 0$, $p_{pq}(t^*) \geq 0$ ($p \neq q$), $q_{pq}(t^*) \geq 0$, and $e^{\lambda \tau_{pq}(t^*)} > 0$, it follows that

$$\sum_{q=1}^n [p_{rq}(t^*) + q_{rq}(t^*) e^{\lambda \tau_{rq}(t^*)}] \xi_q < -\lambda \xi_r < 0. \tag{18}$$

From (17), we have

$$D^+ Z_r(t^*) < -\lambda \xi_r (\kappa + \varepsilon) e^{-\lambda(t^*-l)} = \dot{w}_r(t^*), \tag{19}$$

that is,

$$D^+ Z_r(t^*) < \dot{w}_r(t^*). \tag{20}$$

It is clear that this inequality and (16) contradict each other, and this implies that inequality (15) holds for all $t \in [a, b]$.

Letting $\varepsilon \rightarrow 0$, we obtain that

$$Z_p(t) \leq \kappa \xi_p e^{-\lambda(t-a)}, \quad p = 1, 2, \dots, n, t \in [a, b]. \tag{21}$$

The proof is completed. $\square$

3. Main Results

In this part, we establish a series of sufficient conditions to ensure the global exponential stability of systems (1a)–(1d).

Theorem 1. Under assumptions (A1)–(A3), suppose that

(C1) There exist a vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$ and a constant $\lambda > 0$ such that

$$[\lambda E - D(t) + A(t)F + B(t)G \otimes \zeta(\lambda)] \xi < 0, \quad t \geq 0, \tag{22}$$

where $D(t) = \text{diag}(d_1(t), d_2(t), \dots, d_n(t))$ with $d_p(t) = \beta_p(t) + \sum_{k=1}^m (D_{pk}/L_k^2)$, $\zeta(\lambda) = (e^{\lambda \tau_{pq}(t)})_{n \times n}$, $A(t) = (a_{pq}^+(t))_{n \times n}$, $B(t) = (b_{pq}^+(t))_{n \times n}$, $F = \text{diag}(F_1, F_2, \dots, F_n)$, and $G = \text{diag}(G_1, G_2, \dots, G_n)$;

(C2) There is a positive number $\gamma > 0$ such that

$$\sup_{l \in \mathbb{N}} \left\{ \frac{\ln \gamma_l}{t_l - t_{l-1}} \right\} \leq \gamma < \lambda, \tag{23}$$

where $\gamma_l = \max_{1 \leq p \leq n} \{1, I_{pl}\}$, $l \in \mathbb{N}$; then, systems (1a)–(1d) are globally exponentially stable.

Proof. For $\omega, \bar{\omega} \in PC(\Xi)$, we denote, respectively, the solutions of systems (1a)–(1d) through $(0, \omega)$ and $(0, \bar{\omega})$:

$$\begin{aligned}
z(t, \omega, x) &= (z_1(t, \omega, x), z_2(t, \omega, x), \dots, z_n(t, \omega, x))^T, \\
z(t, \bar{\omega}, x) &= (z_1(t, \bar{\omega}, x), z_2(t, \bar{\omega}, x), \dots, z_n(t, \bar{\omega}, x))^T.
\end{aligned} \tag{24}$$

Define $z_t(t, \omega, x) = z(t+s, \omega, x)$, $z_t(\bar{\omega}, x) = z(t+s, \bar{\omega}, x)$, $-\tau < s \leq 0$, $t \geq 0$, then $z_t(\omega, x)$, $z_t(\bar{\omega}, x) \in PC(\Xi)$ for $t \geq 0$.

Let $Z_p(t, \psi, x) = z_p(t, \omega, x) - z_p(t, \bar{\omega}, x)$, $p = 1, 2, \dots, n$, $\psi = \omega - \bar{\omega}$, we have

$$\begin{aligned}
\frac{\partial Z_p(t, \psi, x)}{\partial t} &= \sum_{k=1}^m \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial Z_p(t, \psi, x)}{\partial x_k} \right) - \beta_p(t) Z_p(t, \psi, x) \\
&\quad - \sum_{q=1}^n a_{pq}(t) [f_q(z_q(t, \omega, x)) - f_q(z_q(t, \bar{\omega}, x))] \\
&\quad - \sum_{q=1}^n b_{pq}(t) [g_q(z_q(t - \tau_{pq}(t), \omega, x)) - g_q(z_q(t - \tau_{pq}(t), \bar{\omega}, x))],
\end{aligned} \tag{25}$$

for $t \neq t_l, x \in \Xi, p = 1, 2, \dots, n$.

When we multiply both sides of (11) by $Z_p(t, \psi, x)$ and integrate them, we can get

$$\begin{aligned}
 & \frac{d}{dt} \int_{\Xi} (Z_p(t, \psi, x))^2 dx \\
 &= 2 \int_{\Xi} Z_p(t, \psi, x) \sum_{k=1}^m \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial Z_p(t, \psi, x)}{\partial x_k} \right) dx - 2 \int_{\Xi} \beta_p(t) Z_p^2(t, \psi, x) dx \\
 &+ 2 \sum_{q=1}^n a_{pq}(t) \int_{\Xi} Z_p(t, \psi, x) [f_q(z_q(t, \omega, x)) - f_q(z_q(t, \bar{\omega}, x))] dx \\
 &+ 2 \sum_{q=1}^n b_{pq}(t) \int_{\Xi} Z_p(t, \psi, x) [g_q(z_q(t - \tau_{pq}(t), \omega, x)) - g_q(z_q(t - \tau_{pq}(t), \bar{\omega}, x))] dx.
 \end{aligned} \tag{26}$$

From the Dirichlet boundary conditions and Greens formula, we can obtain

By Lemma 2, we have

$$\begin{aligned}
 & \int_{\Xi} Z_p(t, \psi, x) \sum_{k=1}^m \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial Z_p(t, \psi, x)}{\partial x_k} \right) dx \\
 &= - \sum_{k=1}^m \int_{\Xi} D_{pk} \left(\frac{\partial (Z_p(t, \psi, x))}{\partial x_k} \right)^2 dx.
 \end{aligned} \tag{27}$$

$$\int_{\Xi} Z_p(t, \psi, x) \sum_{k=1}^m \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial Z_p(t, \psi, x)}{\partial x_k} \right) dx \leq - \sum_{k=1}^m \frac{D_{pk}}{L_k^2} \int_{\Xi} (Z_p(t, \psi, x))^2 dx = - \sum_{k=1}^m \frac{D_{pk}}{L_k^2} \|Z_p(t, \psi, x)\|_2^2. \tag{28}$$

From (A1)–(A3) and Hoder inequality, we can get

In a similar way, we have

$$\begin{aligned}
 & \sum_{q=1}^n a_{pq}(t) \int_{\Xi} Z_p(t, \psi, x) [f_q(z_q(t, \omega, x)) - f_q(z_q(t, \bar{\omega}, x))] dx \\
 &\leq \sum_{q=1}^n a_{pq}^+(t) \int_{\Xi} |Z_p(t, \psi, x)| |f_q(z_q(t, \omega, x)) - f_q(z_q(t, \bar{\omega}, x))| dx \\
 &\leq \sum_{q=1}^n a_{pq}^+(t) \int_{\Xi} |Z_p(t, \psi, x)| |Z_q(t, \psi, x)| F_q dx \\
 &\leq \sum_{q=1}^n a_{pq}^+(t) \|Z_p(t, \psi, x)\|_2 F_q \|Z_q(t, \psi, x)\|_2.
 \end{aligned} \tag{29}$$

$$\begin{aligned}
 & \sum_{q=1}^n b_{pq}(t) \int_{\Xi} Z_p(t, \psi, x) [g_q(z_q(t - \tau_{pq}(t), \omega, x)) - g_q(z_q(t - \tau_{pq}(t), \bar{\omega}, x))] dx \\
 &\leq \sum_{q=1}^n b_{pq}^+(t) \|Z_q(t - \tau_{pq}(t), \psi, x)\|_2 G_q \|Z_p(t, \psi, x)\|_2.
 \end{aligned} \tag{30}$$

Applying (26)–(30) to (25), we can obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|Z_p(t, \psi, x)\|_2^2 \leq & - \left(\beta_p(t) + \sum_{k=1}^m \frac{D_{pk}}{L_k^2} \right) \|Z_p(t, \psi, x)\|_2^2 + \sum_{q=1}^n a_{pq}^+(t) \|Z_p(t, \psi, x)\|_2 F_q \|Z_p(t, \psi, x)\|_2 \\ & + \sum_{q=1}^n b_{pq}^+(t) \|Z_q(t - \tau_{pq}(t), \psi, x)\|_2 G_q \|Z_p(t, \psi, x)\|_2, \end{aligned} \quad (31)$$

i.e.,

$$D^+ \|Z_p(t, \psi, x)\|_2 \leq - \left(\beta_p(t) + \sum_{k=1}^m \frac{D_{pk}}{L_k^2} \right) \|Z_p(t, \psi, x)\|_2 + \sum_{q=1}^n a_{pq}^+(t) F_q \|Z_q(t, \psi, x)\|_2 + \sum_{q=1}^n b_{pq}^+(t) \|Z_q(t - \tau_{pq}(t), \psi, x)\|_2 G_q. \quad (32)$$

Let $Z_p(t) = \|Z_p(t, \psi, x)\|_2$, $Z(t) = (Z_1(t), Z_2(t), \dots, Z_n(t))^T$, $\bar{Z}(t) = (Z_p(t - \tau_{pq}(t)))_{n \times n}$, $d_p(t) = \beta_p(t) + \sum_{k=1}^m (D_{pk}/L_k^2)$, $p = 1, 2, \dots, n$, $D(t) = \text{diag}(d_1(t), d_2(t), \dots, d_n(t))$, and $P(t) = -D(t) + A(t)F$, $Q(t) = B(t)G$. Then, (31) can be reduced to the following form:

$$D^+ Z(t) \leq P(t)Z(t) + (Q(t) \otimes \bar{Z}(t))e. \quad (33)$$

From condition (C1), there exist a vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$ and a constant $\lambda > 0$ such that

$$[\lambda E - D(t) + A(t)F + B(t)G \otimes \xi(\lambda)]\xi < 0. \quad (34)$$

Taking $\kappa = \|\psi\|/\min_{1 \leq p \leq n} \{\xi_p\}$, obviously

$$Z(t) \leq \kappa \xi e^{-\lambda t}, \quad t \in [-\tau, t_0], t_0 = 0. \quad (35)$$

By Lemma 4, we have

$$Z(t) \leq \kappa \xi e^{-\lambda t}, \quad t \in [t_0, t_1]. \quad (36)$$

Assume that the following inequality holds for $m \leq l$:

$$Z(t) \leq \kappa \gamma_0, \dots, \gamma_{m-1} \xi e^{-\lambda t}, \quad t \in [t_{m-1}, t_m], \gamma_0 = 1. \quad (37)$$

When $m = l + 1$, we can obtain

$$\begin{aligned} Z(t_l) &= \|z(t_l, \omega, x) - z(t_l, \bar{\omega}, x)\|_2 \\ &= \|I_l(z(t_l^-, \omega, x)) - I_l(z(t_l^-, \bar{\omega}, x))\|_2 \\ &\leq \mathfrak{F}_l \|Z(t_l^-, \psi, x)\|_2 \\ &= \mathfrak{F}_l Z(t_l^-) \\ &\leq \kappa \gamma_0, \dots, \gamma_{l-1} \mathfrak{F}_l \xi \lim_{t \rightarrow t_l^-} e^{-\lambda t} \\ &\leq \kappa \gamma_0, \dots, \gamma_{l-1} \gamma_l \xi e^{-\lambda t_l}. \end{aligned} \quad (38)$$

By (37) and (38) and $\gamma_l \geq 1$, we obtain

$$Z(t) \leq \kappa \gamma_0, \dots, \gamma_{l-1} \gamma_l \xi e^{-\lambda t}, \quad t_l - \tau \leq t \leq t_l. \quad (39)$$

Combining (34), (35), and (39) and Lemma 4, we get

$$Z(t) \leq \kappa \gamma_0, \dots, \gamma_{l-1} \gamma_l \xi e^{-\lambda t}, \quad t_l \leq t < t_{l+1}. \quad (40)$$

Applying mathematical induction, we have

$$Z(t) \leq \kappa \gamma_0, \dots, \gamma_{l-1} \xi e^{-\lambda t}, \quad t_{l-1} \leq t < t_l, l \in N. \quad (41)$$

From (C2) and (41), we obtain

$$Z(t) \leq \kappa e^{\gamma t_1} e^{\gamma(t_2 - t_1)}, \dots, e^{\gamma(t_{l-1} - t_{l-2})} \xi e^{-\lambda t} \leq \kappa \xi e^{\gamma t} e^{-\lambda t} = \kappa \xi e^{-(\lambda - \gamma)t}, \quad (42)$$

for all $l \in N$, $t_{l-1} \leq t < t_l$.

This implies that

$$\begin{aligned} \|z(t, \omega, x) - z(t, \bar{\omega}, x)\| &= \sum_{p=1}^n \|z_p(t, \omega, x) - z_p(t, \bar{\omega}, x)\|_2 \\ &= \sum_{p=1}^n Z_p(t) \\ &\leq \sum_{p=1}^n \kappa \xi_p e^{-(\lambda - \gamma)t} \\ &= \frac{\sum_{p=1}^n \xi_p}{\min_{1 \leq p \leq n} \{\xi_p\}} \|\omega - \bar{\omega}\| e^{-(\lambda - \gamma)t}, \end{aligned} \quad (43)$$

for all $t \geq 0$.

That is,

$$\|z_t(\omega, x) - z_t(\bar{\omega}, x)\| \leq M \|\omega - \bar{\omega}\| e^{-(\lambda - \gamma)t}, \quad t \geq 0, \quad (44)$$

where $M = (\sum_{p=1}^n \xi_p / \min_{1 \leq p \leq n} \{\xi_p\})$. The proof is completed. $\square$

Remark 2. In Theorem 1, the condition (C2) ($\gamma = \sup_{l \in N} \{\ln \gamma_l / (t_l - t_{l-1})\}$) describes the influence of the impulsive intensity and the impulsive interval on the global exponential stability of systems (1a)–(1d). In the absence of impulses, the following optimization problem can be solved in order to estimate the exponential convergence rate of systems (1a)–(1d):

$$(\text{OP}) \begin{cases} \max & \lambda \\ \text{s.t.} & (C1) \text{ holds.} \end{cases} \quad (45)$$

$$\lambda^* = \inf_{t \geq 0} \{ \lambda(t) > 0 : [\lambda(t)E - D(t) + A(t)F + B(t)G \otimes \zeta(t)] \xi = 0 \} > 0, \quad \zeta(t) = (e^{\lambda(t)\tau_{pq}(t)})_{n \times n}. \quad (46)$$

Proof. By Lemma 3, λ^* exists and $\lambda^* > 0$. Let $\lambda = \lambda^* - \varepsilon$ for all $\varepsilon \in (0, \lambda^*)$, then

$$[(\lambda^* - \varepsilon)E - D(t) + A(t)F + B(t)G \otimes \zeta(\lambda^* - \varepsilon)] \xi < 0, \quad (47)$$

for all $t \geq 0$. That is,

$$[\lambda E - D(t) + A(t)F + B(t)G \otimes \zeta(t)] \xi < 0, \quad (48)$$

for all $t \geq 0$. Therefore, $\max \lambda = \lambda^*$. The proof is completed. $\square$

Remark 3. Some existing neural network models become special cases of systems (2a)–(2c) formulated in this paper; for example, see [15, 16]. It is not hard to find out that all criteria in [15] are improved. Comparing with LMIs method in [16], our results are more concise, and these results do require the differentiability on time-varying delays, so our method has different advantages and disadvantages with the method in [16]. In addition, although those models in [37, 38] are different from systems (2a)–(2c) because of the difference in reaction-diffusion terms and boundary conditions, all criteria in [37] are independent of diffusion effect. In [38], the authors had, respectively, obtained global exponential stability criteria of nonautonomous neural networks with time-varying delays and reaction-diffusion terms under the Neumann boundary conditions and the Dirichlet

Obviously, λ is associated with all time delays, the diffusion coefficients, the Dirichlet boundary conditions, and system parameters. When $\gamma \in [0, \lambda)$, Theorem 1 shows that systems (1a)–(1d) are globally exponentially stable and its exponential convergence rate equals $\lambda - \gamma$. About the exponential convergence rate λ , we have the following result.

Theorem 2. If condition (C1) holds, then $\max \lambda = \lambda^*$, where

boundary conditions. However, our approach is similar to the one in [38] when the Dirichlet boundary conditions are considered.

Remark 4. Let $\Delta(t) = D(t) - A(t)F - B(t)G$, then condition (C1) is equivalent to that $\Delta(t)$ is a nonsingular M-matrix for any $t \geq 0$. In fact, if $\Delta(t)$ is a nonsingular M-matrix for any $t \geq 0$, from Lemma 1, there exists a vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$ such that

$$[D(t) - A(t)F - B(t)G] \xi > 0. \quad (49)$$

Because of the uniform continuity, there exists a real number $\lambda > 0$ such that

$$[\lambda E - D(t) + A(t)F + B(t)G \otimes \theta(\lambda)] \xi < 0, \quad (50)$$

and this shows that condition (C1) holds. Reversely, let $\lambda = 0$ in (C1); it is easy to obtain that $\Delta(t)$ is a nonsingular M-matrix for $t \geq 0$.

Based on the above discussion, we have the following conclusions.

Corollary 1. Under assumptions (A1) – (A3), suppose that

(C1') There exist some numbers $\xi_p > 0$ ($p = 1, 2, \dots, n$) and a number $\lambda > 0$ such that

$$\left(\beta_p(t) + \sum_{k=1}^m \frac{D_{pk}}{L_k^2} \right) - \sum_{q=1}^n \frac{\xi_q}{\xi_p} a_{pq}^+(t) F_q - \sum_{q=1}^n \frac{\xi_q}{\xi_p} b_{pq}^+(t) e^{\lambda \tau_{pq}(t)} G_q > \lambda > 0, \quad \text{for all } t \geq 0, p = 1, 2, \dots, n. \quad (51)$$

(C2) There is a positive number $\gamma > 0$ such that

$$\sup_{l \in N} \left\{ \frac{\ln \gamma_l}{t_l - t_{l-1}} \right\} \leq \gamma < \lambda, \quad (52)$$

where $\gamma_l = \max_{1 \leq p \leq n} \{1, I_{pl}\}$, $l \in N$; then, systems (1a)–(1d) are globally exponentially stable.

Proof. Note that (C1) and (C1') are equivalent; it follows from Theorem 1 that Corollary 1 holds. $\square$

Corollary 2. Under assumptions (A1) and (A3), if condition (C1) holds, then systems (2a)–(2c) are globally exponentially stable.

Corollary 3. Under assumptions (A1) and (A3), if there exist some numbers $\xi_p > 0$ ($p = 1, 2, \dots, n$) and a number $\lambda > 0$ such that

$$\xi_p \left(\beta_p(t) + \sum_{k=1}^m \frac{D_{pk}}{L_k^2} \right) - \sum_{q=1}^n \xi_q a_{pq}^+(t) F_q - \sum_{q=1}^n \xi_q b_{pq}^+(t) G_q > \lambda > 0, \quad \text{for all } t \geq 0, p = 1, 2, \dots, n, \quad (53)$$

then systems (2a)–(2c) are globally exponentially stable.

Corollary 4. Under assumptions (A1) – (A3), suppose that

(C1'') There exist a vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$ and a positive number $\lambda > 0$ such that

$$[\lambda E - D(t) + A(t)F + B(t)G \otimes \bar{\zeta}(\lambda)] \xi < 0, \quad \bar{\zeta}(\lambda) = (e^{\lambda \tau})_{n \times n}, t \geq 0, \quad (54)$$

where $D(t) = \text{diag}(d_1(t), d_2(t), \dots, d_n(t))$ with $d_p(t) = \beta_p(t) + \sum_{k=1}^m (D_{pk}/L_k^2)$, $A(t) = (a_{pq}^+(t))_{n \times n}$, $B(t) = (b_{pq}^+(t))_{n \times n}$, $F = \text{diag}(F_1, F_2, \dots, F_n)$, and $G = \text{diag}(G_1, G_2, \dots, G_n)$;

(C2) There is a positive number $\gamma > 0$ such that

$$\sup_{l \in \mathbb{N}} \left\{ \frac{\ln \gamma_l}{t_l - t_{l-1}} \right\} \leq \gamma < \lambda, \quad (55)$$

where $\gamma_l = \max_{1 \leq p \leq n} \{1, I_{pl}\}$, $l \in \mathbb{N}$; then, systems (1a)–(1d) are globally exponentially stable.

Proof. Note that $\bar{\zeta}(\lambda) \geq \zeta(\lambda)$, it follows that

$$B(t)G \otimes \bar{\zeta}(\lambda) \geq B(t)G \otimes \zeta(\lambda), \quad t \geq 0. \quad (56)$$

This shows that (C1'') ensures (C1).

In this part, we consider the case of the periodic solution of systems (1a)–(1d). Let us make the following assumptions:

(A4) For $p, q \in \{1, 2, \dots, n\}$, $\beta_p(t)$, $a_{pq}(t)$, $b_{pq}(t)$, $\tau_{pq}(t)$, and $J_p(t)$ are periodic continuous functions and has a common period $\Omega > 0$ for all $t \geq 0$.

(A5) For $\mathfrak{I}_l = \text{diag}(I_{1l}, \dots, I_{nl})$ and the impulsive moments $\{t_l\}_{l \in \mathbb{N}}$, there exists an integer $r > 0$ such that

$$\begin{aligned} I_{p(l+r)} &= I_{pl}, \\ t_{l+r} &= t_l + \Omega. \end{aligned} \quad (57)$$

For the existence and the global exponential stability of periodic solutions for periodic systems (1a)–(1d), we have the following results. $\square$

Theorem 3. Under assumptions (A1) – (A5), suppose that

(C1) There exist a vector $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T > 0$ and a constant $\lambda > 0$ such that

$$[\lambda E - D(t) + A(t)F + B(t)G \otimes \zeta(\lambda)] \xi < 0, \quad t \geq 0, \quad (58)$$

where $D(t) = \text{diag}(d_1(t), d_2(t), \dots, d_n(t))$ with $d_p(t) = \beta_p(t) + \sum_{k=1}^m (D_{pk}/L_k^2)$, $\zeta(\lambda) = (e^{\lambda \tau_{pq}(t)})_{n \times n}$, $A(t) = (a_{pq}^+(t))_{n \times n}$, $B(t) = (b_{pq}^+(t))_{n \times n}$, $F = \text{diag}(F_1, F_2, \dots, F_n)$, and $G = \text{diag}(G_1, G_2, \dots, G_n)$;

(C2) There is a positive number $\gamma > 0$ such that

$$\sup_{l \in \mathbb{N}} \left\{ \frac{\ln \gamma_l}{t_l - t_{l-1}} \right\} \leq \gamma < \lambda, \quad (59)$$

where $\gamma_l = \max_{1 \leq p \leq n} \{1, I_{pl}\}$, $l \in \mathbb{N}$; then, systems (1a)–(1d) have exactly one globally exponentially stable Ω -periodic solution.

Proof. On the basis of Theorem 1, there exists a integer $K > 0$ such that $Me^{-(\lambda - \gamma)K\Omega} \leq (1/2)$, and we define a Poincare mapping as follows:

$$\Gamma: PC(\Xi) \longrightarrow PC(\Xi), \quad \text{by } \Gamma(\omega) = z_\Omega(\omega, x), \quad (60)$$

and it follows that $\Gamma^K(\omega) = z_{K\Omega}(\omega, x)$. Setting $t = K\Omega$, we get

$$\|\Gamma^K(\omega) - \Gamma^K(\bar{\omega})\| \leq \frac{1}{2} \|\omega - \bar{\omega}\|. \quad (61)$$

Obviously, Γ^K is a contraction mapping, and then there exists a unique fixed point $\omega^* \in PC(\Xi)$ such that

$$\Gamma^K(\omega^*) = \omega^*. \quad (62)$$

And then, we have

$$\Gamma^K(\Gamma(\omega^*)) = \Gamma(\Gamma^K(\omega^*)) = \Gamma(\omega^*), \quad (63)$$

and this implies that $\Gamma(\omega^*) \in PC(\Xi)$ is also a fixed point of Γ^K . So, we have

$$\Gamma(\omega^*) = \omega^*, \quad \text{i.e. } z_\Omega(\omega^*, x) = \omega^*. \quad (64)$$

Hence, if $z(t, \omega^*, x)$ is a solution of systems (1a)–(1d) through $(0, \omega^*)$, then $z(t + \Omega, \omega^*, x)$ is also a solution to systems (1a)–(1d). It follows that

$$z_{t+\Omega}(\omega^*, x) = z_t(z_\Omega(\omega^*, x)) = z_t(\omega^*, x), \quad t \geq 0, \quad (65)$$

that is,

$$z(t + \Omega, \omega^*, x) = z(t, \omega^*, x). \quad (66)$$

This demonstrates that $z(t, \omega^*, x)$ has a solution for systems (1a)–(1d) with Ω -period, and all other solutions of systems (1a)–(1d) converge exponentially to it as $t \rightarrow +\infty$. The proof is completed. $\square$

Corollary 5. Under assumptions (A1) – (A5),

(C1') There exist constants $\xi_p > 0$ ($p = 1, 2, \dots, n$) and a number $\lambda > 0$ such that

$$\left(\beta_p(t) + \sum_{k=1}^m \frac{D_{pk}}{L_k^2} \right) - \sum_{q=1}^n \frac{\xi_q}{\xi_p} a_{pq}^+(t) F_q - \sum_{q=1}^n \frac{\xi_q}{\xi_p} b_{pq}^+(t) e^{\lambda \tau_{pq}(t)} G_q > \lambda > 0, \quad \text{for all } t \geq 0, p = 1, 2, \dots, n. \quad (67)$$

(C2) There exists a positive number $\gamma > 0$ such that

$$\sup_{l \in N} \left\{ \frac{\ln \gamma_l}{t_l - t_{l-1}} \right\} \leq \gamma < \lambda, \quad (68)$$

where $\gamma_l = \max_{1 \leq p \leq n} \{1, I_{pl}\}$, $l \in N$; then, systems (1a)–(1d) have exactly one globally exponentially stable Ω -periodic solution.

Corollary 6. Under assumptions (A1) and (A4), if condition (C1) holds, then systems (2a)–(2c) have exactly one globally exponentially stable Ω -periodic solution.

4. Illustrative Examples

In this section, we give two examples to illustrate the conditions required for our theorems.

Example 1. Consider a two-neuron impulsive system with reaction-diffusion terms:

$$\frac{\partial z_p(t, x)}{\partial t} = \sum_{k=1}^1 \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial z_p(t, x)}{\partial x_k} \right) - \beta_p(t) z_p(t, x) + \sum_{q=1}^n a_{pq}(t) f_q(z_q(t, x)) \quad (69a)$$

$$+ \sum_{q=1}^n b_{pq}(t) g_q(z_q(t - \tau_{pq}(t)), x), \quad t \geq 0, t \neq t_l, x \in \Xi,$$

$$z_p(t_l^+, x) = 1.4 z_p(t_l^-, x), \quad x \in \Xi, l \in N, \quad (69b)$$

$$z_p(t, x) = 0, \quad t \geq 0, x \in \partial \Xi, \quad (69c)$$

$$z_p(s, x) = \mu_p(s, x), \quad s \in [-\tau, 0], \quad (69d)$$

for $p = 1, 2$, where $\Xi = [-1, 1]$, $t_l = 1.5\pi l$, and

$$(D_{p1}) = \begin{pmatrix} 0.15 \\ 0.15 \end{pmatrix},$$

$$\beta(t) = \begin{pmatrix} |\sin \pi t| & 0 \\ 0 & |\cos 3 t| \end{pmatrix},$$

$$A(t) = \begin{pmatrix} |\sin \pi t| & \cos 4 t \\ \sin t & |\cos 3 t| \end{pmatrix},$$

$$B(t) = \begin{pmatrix} \frac{1}{25}(1 - e^{-t}) & \cos 2 t \\ \sin t & \frac{1 - e^{-t}}{30(1 + e^{-t})} \end{pmatrix}, \quad (70)$$

$$(\tau(t))_{2 \times 2} = \begin{pmatrix} |\sin t| & |\sin t| \\ |\cos t| & |\cos t| \end{pmatrix},$$

$$f_q(z) = g_q(z) = \frac{|z + 1| - |z - 1|}{2}.$$

It is easy to verify that assumptions (A1)–(A3) are satisfied, and we can obtain the following results:

$$\begin{aligned}\zeta(\lambda) &= \begin{pmatrix} e^{\lambda|\sin t|} & e^{\lambda|\sin t|} \\ e^{\lambda|\cos t|} & e^{\lambda|\cos t|} \end{pmatrix}, \\ D(t) &= \begin{pmatrix} 0.15 + |\sin \pi t| & 0 \\ 0 & 0.15 + |\cos 3t| \end{pmatrix}, \\ F = G &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}\quad (71)$$

$\lambda \in [0, 1)$, $\gamma_l = \max\{1, 1.4\} = 1.4 \geq 1$, $\gamma = \sup_{l \in \mathbb{N}} (\ln \gamma_l / (t_l - t_{l-1})) = (\ln 1.4 / 1.5\pi) \approx 0.0714$.

By solving the following optimization problem:

$$(\text{OP}) \begin{cases} \max & \lambda \\ \text{s.t.} & (C1) \text{ holds,} \end{cases} \quad (72)$$

we can obtain that $\lambda \approx 0.1054 > 0.0714 = \gamma$ and $\xi = (233240509, 227642892) > 0$. Thus, the conditions in Theorem 1 hold, then the solution of systems (69a)–(69d) is globally exponentially stable, and the exponential convergence rate is estimated as $\lambda - \gamma \approx 0.1054 - 0.0714 = 0.0340$. The numerical simulation of systems (69a)–(69d) is shown in Figure 1.

Example 2. Consider a two-neuron periodic impulsive system:

$$\frac{\partial z_p(t, x)}{\partial t} = \sum_{k=1}^1 \frac{\partial}{\partial x_k} \left(D_{pk} \frac{\partial z_p(t, x)}{\partial x_k} \right) - \beta_p(t) z_p(t, x) + \sum_{q=1}^n a_{pq}(t) f_q(z_q(t, x)) \quad (73a)$$

$$+ \sum_{q=1}^n b_{pq}(t) g_q(z_q(t - \tau_{pq}(t)), x), \quad t \geq 0, t \neq t_l, x \in \Xi,$$

$$z_p(t_l^+, x) = 1.5 z_p(t_l^-, x), \quad x \in \Xi, l \in \mathbb{N}, \quad (73b)$$

$$z_p(t, x) = 0, \quad t \geq 0, x \in \partial \Xi, \quad (73c)$$

$$z_p(s, x) = \mu_p(s, x), \quad s \in [-\tau, 0], \quad (73d)$$

for $p = 1, 2$, where $\Xi = [-1, 1]$, $t_l = 1.5\pi l$, and

$$\begin{aligned}(D_{p1}) &= \begin{pmatrix} 0.15 \\ 0.15 \end{pmatrix}, \\ \beta(t) &= \begin{pmatrix} |\sin t| & 0 \\ 0 & |\cos t| \end{pmatrix}, \\ A(t) &= \begin{pmatrix} |\sin t| & \cos t \\ \sin t & |\cos t| \end{pmatrix}, \\ B(t) &= \begin{pmatrix} \frac{1}{20} |\sin t| & \cos t \\ \sin t & \frac{1}{30} |\cos t| \end{pmatrix}, \\ (\tau(t))_{2 \times 2} &= \begin{pmatrix} |\sin t| & |\sin t| \\ |\cos t| & |\cos t| \end{pmatrix}, \\ f_q(z) = g_q(z) &= \frac{|z+1| - |z-1|}{2}.\end{aligned}\quad (74)$$

It is easy to verify that assumptions (A1)–(A5) are satisfied, and we can obtain the following results:

$$\begin{aligned}\zeta(\lambda) &= \begin{pmatrix} e^{\lambda|\sin t|} & e^{\lambda|\sin t|} \\ e^{\lambda|\cos t|} & e^{\lambda|\cos t|} \end{pmatrix}, \\ D(t) &= \begin{pmatrix} 0.15 + |\sin t| & 0 \\ 0 & 0.15 + |\cos t| \end{pmatrix}, \\ F = G &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}\quad (75)$$

$\lambda \in [0, 1)$, $\gamma_l = \max\{1, 1.5\} = 1.5 \geq 1$, $\gamma = \sup_{l \in \mathbb{N}} (\ln \gamma_l / (t_l - t_{l-1})) = (\ln 1.5 / 1.5\pi) \approx 0.0860$.

By solving the following optimization problem:

$$(\text{OP}) \begin{cases} \max & \lambda \\ \text{s.t.} & (C1) \text{ holds,} \end{cases} \quad (76)$$

we can obtain that $\lambda \approx 0.0950 > 0.0860 = \gamma$ and $\xi = (106069006, 130884350) > 0$. Thus, the conditions in Theorem 3 hold, then systems (73a)–(73d) have exactly one globally exponentially stable 2π -periodic solution, and the exponential convergence rate is estimated as $\lambda - \gamma \approx 0.0950 - 0.0860 = 0.0090$. The numerical simulation of systems (73a)–(73d) is shown in Figure 2.

FIGURE 1: Computer simulation of the neuronal states $z_1(t, x)$ and $z_2(t, x)$.

FIGURE 2: Computer simulation of the neuronal states $z_1(t, x)$ and $z_2(t, x)$.

5. Conclusion

We have formulated and investigated a class of new neural network models which assembles nonautonomous neural networks, reaction-diffusion cellular neural networks with time-varying delays, impulses, and the Dirichlet boundary conditions. Several new sufficient conditions have been obtained to ensure the global exponential stability and periodicity of systems (1a)–(1d), and these criteria are shown in

simple algebraic inequality forms. In particular, an optimization method is proposed in order to estimate the exponential convergence rate of systems (1a)–(1d), and this method depends on the diffusion coefficient, the Dirichlet boundary conditions, the delays, system parameters, and impulses. Comparing with the method of Lyapunov functional in many previous publications, our method is simpler and more effective for stability and periodicity analysis of nonautonomous impulsive neural networks with time-varying delays and reaction-diffusion terms. Two examples have shown that our results improve and generalize previously known criteria. In near future, we will continue to study global exponential stability and periodicity of nonautonomous impulsive neural networks with distributed delays or leakage delays under the Dirichlet boundary conditions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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