Research Article

Spectral Sufficient Conditions on Pancyclic Graphs

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Received 6 May 2021; Revised 22 June 2021; Accepted 30 June 2021; Published 15 July 2021

Academic Editor: Shaohui Wang

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A pancyclic graph of order n is a graph with cycles of all possible lengths from 3 to n . In fact, it is NP-complete that deciding whether a graph is pancyclic. Because the spectrum of graphs is convenient to be calculated, in this study, we try to use the spectral theory of graphs to study this problem and give some sufficient conditions for a graph to be pancyclic in terms of the spectral radius and the signless Laplacian spectral radius of the graph.

1. Introduction

Let $G = (V(G), E(G))$ be a finite simple undirected graph. Write by $n = |V(G)|$ the number of vertex and $m = |E(G)|$ the number of edges of the graph G , respectively. Write by $\delta(G)$ the minimum degree of G , $N_G(v_i)$ the set of neighbours of a vertex v_i in G , and $d_i = d_G(v_i) = |N_G(v_i)|$. If $(d_1, d_2, \dots, d_n)$ can be the degree sequence of a graph, $d_1 \leq d_2 \leq \dots \leq d_n$, and then, we call it a graphic sequence. Let G and H be two disjoint graphs. Denote by $G + H$ the disjoint union of G and H , which is the graph with the vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G_1 = G_2 = \dots = G_k$, we denote $G_1 + G_2 + \dots + G_k$ by kG_1 . Denote by $G \vee H$ the joint of G and H , which is the graph obtained from disjoint union of G and H by adding edges joining every vertex of G to every vertex of H . Denote by $K_{n-1} + v$ the complete graph on $n-1$ vertices together with an isolated vertex v and $K_{n-1} - e$ the complete graph on $n-1$ vertices deleting the edge. Graph theory is widely used in many fields [1, 2]. Here, we mainly study using the spectral theory of graphs to determine the cycles' properties of the graph.

The adjacency matrix of G of order n is the matrix $A(G) = [a_{ij}]$ of order n , where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. Denote by $\mu(G)$ the spectral radius of G , which is the largest eigenvalue of $A(G)$. Write by $D(G)$ the degree diagonal matrix of G . The signless Laplacian matrix of G is defined to be the matrix

$Q(G) = D(G) + A(G)$. Denote by $q(G)$ the signless Laplacian spectral radius of G , which is the largest eigenvalue of $Q(G)$.

A graph G is Hamiltonian (respectively, traceable) if it contains a cycle (respectively, path) containing all vertices of the graph. A cycle of length l is called an l -cycle. A graph G is pancyclic if it contains all l -cycle, $3 \leq l \leq n$. Indeed, it is NP-complete that deciding whether a given graph is Hamiltonian.

Because the spectrum of graphs is convenient to be calculated, some researchers try to use the spectrum of the graph to character the Hamiltonian of graph. In 2010, Fiedler and Nikiforov [3] first gave sufficient conditions for the given graph to be Hamiltonian (or traceable) in terms of the spectral radius of the graph. In fact, it provides an easy and efficient determination method. And then, many good conclusions have been given [2, 4–14]. But there are no spectral sufficient conditions on pancyclic graphs. In this study, we first give a new sufficient condition for a graph to be pancyclic with the edge number of the graph; then basing on the edge number sufficient condition, we give spectral radius or signless Laplacian spectral radius conditions for a graph to be pancyclic.

In this study, we can use some graphs, $G_1 \sim G_8$, in the proof of the theorem, as shown in Figure 1.

G_1 is the graph obtained from $K_2 \vee (2K_1 + K_{n-3})$ by adding two pending edges on one vertex of degree 2. G_2 is

FIGURE 1: Some special graphs used in this study. (a) G_1 . (b) G_2 . (c) G_3 . (d) G_4 . (e) G_5 . (f) G_6 . (g) G_7 . (h) G_8 .

the graph obtained from S_5 by adding one pedending edge on one vertex of degree 1. G_3 is the graph obtained from $P_3 \vee 4K_1$ by adding all edges from two vertexes with degree 5 to K_1 . G_4 is the graph obtained from $4K_1 \vee 2K_1$ by adding one pedending edge on one vertex of degree 4. G_5 is the graph obtained from $K_2 \vee 3K_1$ by adding one pedending edge on two vertexes of degree 4, respectively. G_6 is the graph obtained from $P_3 \vee 3K_1$ by adding all edges from two vertexes of degree 4 to K_1 . G_7 is the graph obtained from $3K_1 \vee 2K_1$ by adding one pedending edge on one vertex of degree 3. G_8 is the graph obtained from $K_2 \vee 2K_1$ by adding one pedending edge on two vertexes of degree 3, respectively.

2. Preliminary

Given a graph G with n vertex, a vector $X \in R^n$ is called to be defined on G , if there is a 1-1 map φ from $V(G)$ to the entries of X , simply written $X_u = \varphi(u)$.

If X is an eigenvector of $A(G)$ (or $Q(G)$), then we call that X is defined on G , and X_u is the entry of X which is corresponded to the vertex u . In fact, when λ is an eigenvalue of G which is corresponded to the eigenvector X ,

$$\lambda X_v = \sum_{u \in N_G(v)} X_u, \quad \text{for } v \in V(G). \quad (1)$$

We call equation (1) is eigen equation of G . When q is a signless Laplacian eigenvalue of G which is corresponded to the eigenvector X , we have

$$[q - d_G(v)]X_v = \sum_{u \in N_G(v)} X_u, \quad \text{for } v \in V(G). \quad (2)$$

We call equation (2) is signless Laplacian eigen equation of G .

Lemma 1 (see [15]). *Let G be a graph with $|V(G)| = n$, which has degree sequence $d_1 \leq \dots \leq d_n$. If for all positive integers k , such that $d_k \leq k < n/2$ and $d_{n-k} \geq n - k$, then G is a pancyclic graph or bipartite graph.*

Lemma 2 (see [16]). *Let G be a connected graph with $|V(G)| = n$. We have*

$$\mu(G) \leq \sqrt{2|E(G)| - |V(G)| + 1}, \quad (3)$$

and the equality holds if and only if $G = K_n$ or $G = K_{1,n-1}$.

Lemma 3 (see [2]). *Let G be a graph with $|V(G)| = n$. Then,*

$$q(G) \leq \frac{2|E(G)|}{|V(G)| - 1} + |V(G)| - 2. \quad (4)$$

If G is connected, the equality holds if and only if $G = K_{1,n-1}$ or $G = K_n$. Otherwise, the equality holds if and only if $G = K_{n-1} + v$.

3. Main Results

Theorem 1. *Let G be a simple connected graph, where $|V(G)| = n (\geq 5)$, $|E(G)| = m$, $\delta(G) \geq 2$. If*

$$m \geq \binom{n-2}{2} + 3, \quad (5)$$

then G is a pancyclic graph, unless G is a bipartite graph or

$$\begin{aligned}
G \in \mathbb{NP}_1 = \{ & \{K_3 \vee (K_3 + 3K_1), K_4 \vee (K_2 + 4K_1), K_2 \vee (K_{n-4} + 2K_1), 2K_1 \vee (K_{n-4} + 2K_1), K_2 \vee ((K_{n-4} - e) + 2K_1), \\
& K_1 \vee g_1, K_1 \vee (K_2 + K_{n-3}), K_5 \vee 6K_1, K_3 \vee K_{2,6}, K_4 \vee (K_1 + K_{1,5}), K_3 \vee (K_2 + 3K_1), K_1 \vee (2K_1 \vee (K_2 + 3K_1)), \\
& K_2 \vee (K_1 + K_1 \vee (K_2 + 2K_1)), K_2 \vee G_2, K_3 \vee 5K_1, K_4 \vee 5K_1, K_3 \vee (K_1 + K_{1,4}), (K_2 \vee 2K_1) \vee 5K_1, \\
& K_3 \vee (2K_1 + K_{1,3}), K_2 \vee G_5, K_2 \vee (K_1 + K_2 \vee 4K_1), K_2 \vee G_4, K_1 \vee (5K_1 \vee (K_2 + K_1)), K_2 \vee G_3, \\
& 5K_1 \vee C_4, K_3 \vee 4K_1, K_{1,2} \vee 4K_1, K_2 \vee (K_1 + K_{1,3}), G_6, 4K_1 \vee (K_1 + K_2), K_2 \vee (2K_2 + K_1), \\
& K_2 \vee (2K_1 + K_{1,2}), K_1 \vee G_7, K_1 \vee G_8 \}.
\end{aligned} \tag{6}$$

Proof. Let G satisfy the condition of theorem, but it is neither a pancyclic graph nor a bipartite graph. By Lemma 1 and the condition of theorem, there exists a positive integer k , such that $d_k \leq k < n/2$ and $d_{n-k} \leq n - k - 1$. Then,

$$\begin{aligned}
2m &= \sum_{i=1}^k d_i + \sum_{i=k+1}^{n-k} d_i + \sum_{i=n-k+1}^n d_i \\
&\leq k^2 + (n-2k)(n-k-1) + k(n-1) \\
&= n^2 - n + 3k^2 + (1-2n)k \\
&= 2 \binom{n-2}{2} + 8 - (k-2)(2n-3k-7).
\end{aligned} \tag{7}$$

Thus,

$$m \leq \binom{n-2}{2} + 3 - \frac{1}{2}((k-2)(2n-3k-7) - 2). \tag{8}$$

Because $\binom{n-2}{2} + 3 \leq m \leq \binom{n-2}{2} + 3 - 1/2((k-2)(2n-3k-7) - 2)$, $(k-2)(2n-3k-7) - 2 \leq 0$, i.e., $(k-2)(2n-3k-7) \leq 2$. Next, we discuss in the following four cases.

Case 1: $(k-2)(2n-3k-7) = 2$, i.e., $k-2 = 2$ and $2n-3k-7 = 1$ or $k-2 = 1$ and $2n-3k-7 = 2$. We can get $k = 3, n = 9$ or $k = 4, n = 10$.

In this time, $m = \binom{n-2}{2} + 3$, and all inequalities in the above should take mark of equality.

Case 1.1: $k = 3$ and $n = 9$. Then, G is a graph with $d_1 = d_2 = d_3 = 3$, $d_4 = d_5 = d_6 = 5$, and $d_7 = d_8 = d_9 = 8$. The three vertices with degree 8 must be adjacent to every vertex, and it induces K_3 . The three vertices with degree 3 are not adjacent to other vertices, and they induce $3K_1$. For the rest of 3 vertices with degree 5, in order to make sure the requirement of degree 5, they must be adjacent to each other and induce K_3 . By the above analysis, G must be isomorphic to $K_3 \vee (K_3 + 3K_1)$, as given in Table 1.

Case 1.2: $k = 4$ and $n = 10$. Then, G is a graph with $d_1 = d_2 = d_3 = d_4 = 4$, $d_5 = d_6 = 5$, and $d_7 = d_8 = d_9 = d_{10} = 9$. The four vertices with degree 9 must be adjacent to every vertex, and they induce K_4 . The four vertices with degree 4 are not adjacent to other vertices, and they induce $4K_1$. For the remaining 2

vertices of degree 5, in order to make sure the requirement of the degree 5, they must be adjacent to each other, and they induce K_2 . By the above analysis, G must be isomorphic to $K_4 \vee (K_2 + 4K_1)$, as given in Table 1.

Case 2: $(k-2)(2n-3k-7) = 1$, i.e., $k-2 = 1$ and $2n-3k-7 = 1$. We cannot find k and n fill the bill.

Case 3: $(k-2)(2n-3k-7) = 0$, i.e., $k-2 = 0$ or $2n-3k-7 = 0$.

Case 3.1: $k-2 = 0$, i.e., $k = 2$. Thus, $d_2 \leq 2$, $d_{n-2} \leq n-3$ and $(n-2)(n-3) + 6 \leq \sum_{i=1}^n d_i \leq (n-2)(n-3) + 8$. Because $\sum_{i=1}^n d_i$ is even number, thus $\sum_{i=1}^n d_i = (n-2)(n-3) + 8$ or $\sum_{i=1}^n d_i = (n-2)(n-3) + 6$. We can get their graphic sequence and corresponding graphs as given in Table 2.

Case 3.2: $2n-3k-7 = 0$. Then, $n \leq 13$ because $k < n/2$, and $n = 11$ and $k = 5$ or $n = 8$ and $k = 3$.

Case 4: $(k-2)(2n-3k-7) < 0$, i.e., $k \geq 3$ and $2n-3k-7 < 0$. In this case, we have $6 \leq 2k < n \leq 13$.

Case 4.1: $n = 13$. We have $k \leq 6$, $2n-3k-7 > 0$.

Case 4.2: $n = 12$. We have $k \leq 5$, $2n-3k-7 > 0$.

Case 4.3: $n = 11$. We have $k \leq 5$, $2n-3k-7 \geq 0$.

Case 4.4: $n = 10$. We have $k \leq 4$, $2n-3k-7 > 0$.

Case 4.5: $n = 8$. We have $k \leq 3$, $2n-3k-7 = 0$. The above five cases all contradict to $2n-3k-7 < 0$.

Case 4.6: $n = 9$. Then, $k \leq 4$. When $k = 3$, then $2n-3k-7 = 2 > 0$, in contradiction with $2n-3k-7 < 0$.

When $k = 4$, then $2n-3k-7 = -1 < 0$; thus, $d_4 \leq 4$, $d_5 \leq 4$, and we have $48 \leq \sum_{i=1}^9 d_i \leq 52$ by (5) and (8). Because $\sum_{i=1}^9 d_i$ is an even number, so $\sum_{i=1}^9 d_i = 52$ or $\sum_{i=1}^9 d_i = 50$ or $\sum_{i=1}^9 d_i = 48$. We can get their graphic sequence and corresponding graphs as given in Table 5.

Case 4.7: $n = 7$. Then, $k \leq 3$, and because $k \geq 3$, i.e., $k = 3$. Thus, $d_3 \leq 3$, $d_4 \leq 3$, and $26 \leq \sum_{i=1}^7 d_i \leq 30$ by (5) and (8). Because $\sum_{i=1}^7 d_i$ is an even number, so $\sum_{i=1}^7 d_i = 30$ or $\sum_{i=1}^7 d_i = 28$ or $\sum_{i=1}^7 d_i = 26$. We can get their graphic sequence and corresponding graphs as given in Table 6.

In Tables 1–6, we can find the following:

- (i) The corresponding graphs of the graphic sequence $(5, 5, 5, 5, 5, 5, 8, 10, 10, 10, 10)$ contain cycles of every length l , $3 \leq l \leq 11$

TABLE 1: The graphic sequence of G and its corresponding graph.

	Graphic sequence	Nonpancyclic
$n = 9$ $k = 3$ $2m = 48$	$(3, 3, 3, 5, 5, 5, 8, 8, 8)$	$G \cong K_3 \vee (K_3 + 3K_1)$
$n = 10$ $k = 4$ $2m = 62$	$(4, 4, 4, 4, 5, 5, 9, 9, 9, 9)$	$G \cong K_4 \vee (K_2 + 4K_1)$

TABLE 2: The graphic sequence of G and its corresponding graph.

	Graphic sequence	Nonpancyclic
$ V(G) = n$ $k = 2$ $2m = (n-2)(n-3) + 8$	$(2, 2, n-3, n-3, n-3, \dots, n-3, n-1, n-1)$	$G \cong K_2 \vee (K_{n-4} + 2K_1)$
$ V(G) = n$ $k = 2$ $2m = (n-2)(n-3) + 8$	$(2, 2, n-3, n-3, n-3, \dots, n-3, n-2, n-2)$ $(2, 2, n-4, n-4, n-3, \dots, n-3, n-1, n-1)$ $(2, 2, n-4, n-3, n-3, \dots, n-3, n-2, n-1)$ $(2, 2, n-3, n-3, n-3, \dots, n-3, n-3, n-1)$	$G \cong 2K_1 \vee (K_{n-4} + 2K_1)$ $G \cong K_2 \vee ((K_{n-4} - e) + 2K_1)$ $G \cong K_1 \vee G_1$ $G \cong K_1 \vee (K_2 + K_{n-3})$

TABLE 3: The graphic sequence of G and its corresponding graph.

	Graphic sequence	Nonpancyclic
$n = 11$ $k = 5$ $2m = 80$	$(5, 5, 5, 5, 5, 5, 10, 10, 10, 10, 10)$	$G \cong K_5 \vee 6K_1$
$n = 11$ $k = 5$ $2m = 78$	$(5, 5, 5, 5, 5, 5, 9, 9, 10, 10, 10)$ $(5, 5, 5, 5, 5, 5, 8, 10, 10, 10, 10)$ $(4, 5, 5, 5, 5, 5, 9, 10, 10, 10, 10)$	$G \cong K_3 \vee K_{2,6}$ Every graph is pancyclic $G \cong K_4 \vee (K_1 + K_{1,5})$

TABLE 4: The graphic sequence of G and its corresponding graph.

	Graphic sequence	Nonpancyclic
$n = 8$ $k = 3$ $2m = 38$	$(3, 3, 3, 4, 4, 7, 7, 7)$	$G \cong K_3 \vee (K_2 + 3K_1)$
$n = 8$ $k = 3$ $2m = 36$	$(3, 3, 3, 4, 4, 6, 6, 7)$ $(2, 3, 3, 4, 4, 6, 7, 7)$ $(3, 3, 3, 3, 4, 6, 7, 7)$ $(3, 3, 3, 4, 4, 5, 7, 7)$ $(3, 3, 3, 3, 3, 7, 7, 7)$	$G \cong K_1 \vee (2K_1 \vee (K_2 + 3K_1))$ $G \cong K_2 \vee (K_1 + K_1 \vee (K_2 + 2K_1))$ $G \cong K_2 \vee G_2$ Every graph is pancyclic $G \cong K_3 \vee 5K_1$

TABLE 5: The graphic sequence of G and its corresponding graph.

	Graphic sequence	Nonpancyclic
$n = 9$ $k = 4$ $2m = 52$	$(4, 4, 4, 4, 4, 8, 8, 8, 8)$	$G \cong K_4 \vee 5K_1$
$n = 9$ $k = 4$ $2m = 50$	$(3, 4, 4, 4, 4, 7, 8, 8, 8)$ $(4, 4, 4, 4, 4, 6, 8, 8, 8)$ $(4, 4, 4, 4, 4, 7, 7, 8, 8)$ $(4, 4, 4, 4, 4, 4, 8, 8, 8)$ $(3, 4, 4, 4, 4, 5, 8, 8, 8)$ $(3, 3, 4, 4, 4, 6, 8, 8, 8)$ $(4, 4, 4, 4, 4, 6, 6, 8, 8)$ $(3, 3, 4, 4, 4, 7, 7, 8, 8)$ $(2, 4, 4, 4, 4, 7, 7, 8, 8)$ $(3, 4, 4, 4, 4, 6, 7, 8, 8)$ $(4, 4, 4, 4, 4, 5, 7, 8, 8)$ $(4, 4, 4, 4, 4, 6, 7, 7, 8)$ $(3, 4, 4, 4, 4, 7, 7, 7, 8)$ $(4, 4, 4, 4, 4, 7, 7, 7, 7)$	$G \cong K_3 \vee (K_1 + K_{1,4})$ Every graph is pancyclic $G \cong (K_2 \vee 2K_1) \vee 5K_1$ Every graph is pancyclic Every graph is pancyclic $G \cong K_3 \vee (2K_1 + K_{1,3})$ Every graph is pancyclic $G \cong K_2 \vee G_5$ $G \cong K_2 \vee (K_1 + K_2 \vee 4K_1)$ $G \cong K_2 \vee G_4$ Every graph is pancyclic $G \cong K_1 \vee (5K_1 \vee (K_2 + K_1))$ $G \cong K_1 \vee G_3$ $G \cong 5K_1 \vee C_4$

$3 \leq l \leq 7$, respectively. Namely, they are the pancyclic graph, a contradiction. The other graphs in Tables 1 ~ 6 are neither pancyclic nor bipartite, and $K_3 \vee (K_3 + 3K_1)$ and $K_4 \vee (K_2 + 4K_1)$ are neither pancyclic nor bipartite.

The proof is complete. $\square$

Theorem 2. Let G be a simple connected graph, where $|V(G)| = n (\geq 5)$, $|E(G)| = m$, $\delta(G) \geq 2$. If

$$\mu(G) \geq \sqrt{n^2 - 6n + 13}, \quad (9)$$

then G is a pancyclic graph, unless G is a bipartite graph or $G \in \{K_3 \vee 4K_1, G_6, K_2 \vee 3K_1\}$.

- (ii) The corresponding graphs of the graphic sequence $(4, 4, 4, 4, 4, 8, 8, 8)$, $(3, 4, 4, 4, 4, 5, 8, 8, 8)$, $(4, 4, 4, 4, 4, 6, 6, 8, 8)$, $(4, 4, 4, 4, 4, 5, 7, 8, 8)$, and $(4, 4, 4, 4, 4, 6, 8, 8, 8)$ contain cycles of every length l , $3 \leq l \leq 9$, respectively
- (iii) The corresponding graphs of the graphic sequence $(3, 3, 3, 4, 4, 5, 7, 7)$ contain cycles of every length l , $3 \leq l \leq 8$
- (iv) The corresponding graphs of the graphic sequence $(3, 3, 3, 3, 4, 4, 6)$, $(3, 3, 3, 3, 3, 5, 6)$, and $(3, 3, 3, 3, 4, 6, 6)$ contain cycles of every length l ,

TABLE 6: The graphic sequence of G and its corresponding graph.

	Graphic sequence	Nonpancyclic
$n = 7$ $k = 3$ $2m = 30$	$(3, 3, 3, 3, 6, 6, 6)$	$G \cong K_3 \vee 4K_1$
$n = 7$ $k = 3$ $2m = 28$	$(3, 3, 3, 3, 5, 5, 6)$ $(3, 3, 3, 3, 4, 6, 6)$ $(2, 3, 3, 3, 5, 6, 6)$	$G \cong K_{1,2} \vee 4K_1$ Every graph is pancyclic $G \cong K_2 \vee (K_1 + K_{1,3})$
$n = 7$	$(2, 3, 3, 3, 5, 5, 5)$ $(3, 3, 3, 3, 4, 5, 5)$ $(2, 3, 3, 3, 3, 6, 6)$ $(2, 2, 3, 3, 4, 6, 6)$	$G \cong G_6$ $G \cong 4K_1 \vee (K_1 + K_2)$ $G \cong K_2 \vee (2K_2 + K_1)$ $G \cong K_2 \vee (2K_1 + K_{1,2})$
$k = 3$	$(3, 3, 3, 3, 4, 4, 6)$ $(3, 3, 3, 3, 3, 5, 6)$ $(2, 3, 3, 3, 4, 5, 6)$ $(2, 2, 3, 3, 5, 5, 6)$	Every graph is pancyclic Every graph is pancyclic $G \cong K_1 \vee G_7$ $G \cong K_1 \vee G_8$

Proof. Let G satisfy the condition of theorem, but it is neither a pancyclic graph nor a bipartite graph. Because K_n is pancyclic and $\delta(K_{1,n-1}) = 1$, by Lemma 2,

$$\sqrt{n^2 - 6n + 13} \leq \mu(G) < \sqrt{2m - n + 1}, \quad (10)$$

then,

$$m > \binom{n-2}{2} + 3. \quad (11)$$

By Theorem 1, we get $G \in \mathbb{NP}_1$. Because when $G \in \{K_3 \vee (K_3 + 3K_1), K_4 \vee (K_2 + 4K_1)\}$,

$$m = \binom{n-2}{2} + 3, \quad (12)$$

is a contradiction. So,

$$\begin{aligned} G \in \mathbb{NP}_2 = \{ & K_2 \vee (K_{n-4} + 2K_1), 2K_1 \vee (K_{n-4} + 2K_1), K_2 \vee ((K_{n-4} - e) + 2K_1), K_1 \vee G_1, K_1 \vee (K_2 + K_{n-3}), K_5 \vee 6K_1, \\ & K_3 \vee K_{2,6}, K_4 \vee (K_1 + K_{1,5}), K_3 \vee (K_2 + 3K_1), K_1 \vee (2K_1 \vee (K_2 + 3K_1)), K_2 \vee (K_1 + K_1 \vee (K_2 + 2K_1)), K_2 \vee G_2, K_3 \vee 5K_1, \\ & K_4 \vee 5K_1, K_3 \vee (K_1 + K_{1,4}), (K_2 \vee 2K_1) \vee 5K_1, K_3 \vee (2K_1 + K_{1,3}), K_2 \vee G_5, K_2 \vee (K_1 + K_2 \vee 4K_1), K_2 \vee G_4, \\ & K_1 \vee (5K_1 \vee (K_2 + K_1)), K_2 \vee G_3, 5K_1 \vee C_4, K_3 \vee 4K_1, K_{1,2} \vee 4K_1, K_2 \vee (K_1 + K_{1,3}), \\ & G_6, 4K_1 \vee (K_1 + K_2), K_2 \vee (2K_2 + K_1), K_2 \vee (2K_1 + K_{1,2}), K_1 \vee G_7, K_1 \vee G_8 \}. \end{aligned} \quad (13)$$

- (i) For $K_2 \vee (K_{n-4} + 2K_1)$, by theorem condition, we have $n \geq 5$. It is easy to check by WolframAlpha (<http://www.wolframalpha.com>) that $\mu(K_2 \vee (K_{n-4} + 2K_1))$ is the largest root of $f(x) = x^3 + (4-n)x^2 - (n+1)x + 4n - 20$, and $f'(x) = 3x^2 + (8-2n)x - n - 1$. The roots of $f'(x) = 0$ are $x_1 = 1/3(-\sqrt{n^2 - 5n + 19} + n - 4)$ and $x_2 = 1/3(\sqrt{n^2 - 5n + 19} + n - 4)$. Then by analyzing, when $x > x_2$, we have $f'(x) > 0$, so $f(x)$ is monotonically increasing. We can get when $n = 5$, $f(\sqrt{n^2 - 6n + 13}) < 0$, and when $n \geq 6$, $f(\sqrt{n^2 - 6n + 13}) > 0$. Because $\mu(G) \geq \sqrt{n^2 - 6n + 13}$, then $n = 5$, $G \cong K_2 \vee 3K_1$.

- (ii) For $2K_1 \vee (K_{n-4} + 2K_1)$, by theorem condition, we have $n \geq 5$. Using the same method,

$\mu(2K_1 \vee (K_{n-4} + 2K_1))$ is the largest root of $f(x) = x^3 + (5-n)x^2 + (4-2n)x + 4n - 20$, and $f'(x) = 3x^2 + (10-2n)x + 4 - 2n$. The roots of $f'(x) = 0$ are $x_1 = 1/3(-\sqrt{n^2 - 4n + 13} + n - 5)$ and $x_2 = 1/3(\sqrt{n^2 - 4n + 13} + n - 5)$. Then by analyzing, when $x > x_2$, we have $f'(x) > 0$, so $f(x)$ is monotonically increasing. Because when $n \geq 5$, we have $f(\sqrt{n^2 - 6n + 13}) > 0$; this implies $\mu(2K_1 \vee (K_{n-4} + 2K_1)) < \sqrt{n^2 - 6n + 13}$, a contradiction.

- (iii) For $K_2 \vee ((K_{n-4} - e) + 2K_1)$, we can get $n \geq 6$ from the graph structure. Using the same method, we get $\mu(K_2 \vee ((K_{n-4} - e) + 2K_1))$ is the largest root of $f(x) = x^4 + (-n+6)x^3 + (9-3n)x^2 + (2n-20)x + 8n - 48$. Then by analyzing, $f(\sqrt{n^2 - 6n + 13}) > 0$,

and this implies $\mu(K_2 \vee ((K_{n-4} - e) + 2K_1)) < \sqrt{n^2 - 6n + 13}$, a contradiction.

- (iv) For $K_1 \vee G_1$, we can get $n \geq 6$ from the graph structure. Using the same method, we get $\mu(K_1 \vee G_1)$ is the largest root of $f(x) = x^5 + (6 - n)x^4 + (9 - 3n)x^3 + (2n - 20)x^2 + (8n - 42)x + 2n - 8$. By analyzing, when $n \geq 6$, then $f(\sqrt{n^2 - 6n + 13}) > 0$, so $\mu(K_1 \vee G_1) < \sqrt{n^2 - 6n + 13}$, a contradiction.

- (v) For $K_1 \vee (K_2 + K_{n-3})$, by theorem condition, we have $n \geq 5$. Using the same method, we have $\mu(K_1 \vee (K_2 + K_{n-3}))$ is the largest root of $f(x) = x^3 + (3 - n)x^2 - 3x + 3n - 11$, and $f'(x) = 3x^2 + (6 - 2n)x - 3$. The roots of $f'(x) = 0$ are $x_1 = 1/3(-\sqrt{n^2 - 6n + 18} + n - 3)$, and $x_2 = 1/3(\sqrt{n^2 - 6n + 18} + n - 3)$. Then by analyzing, when $x > x_2$, we have $f'(x) > 0$, so $f(x)$ is monotonically increasing. Because when $n \geq 5$, we have $f(\sqrt{n^2 - 6n + 13}) > 0$; this implies $\mu(K_1 \vee (K_2 + K_{n-3})) < \sqrt{n^2 - 6n + 13}$, a contradiction.

- (vi) For $G = (K_2 \vee 2K_1) \vee 5K_1$, let $X = (X_1, X_2, \dots, X_9)^T$ be the eigenvector corresponding to $\mu(G)$, where $X_i (1 \leq i \leq 5)$ correspond to the vertex of degree 4, $X_i (6 \leq i \leq 7)$ correspond to the vertex of degree 7, and $X_i (8 \leq i \leq 9)$ correspond to the vertex of degree 8. Then, by eigen equation (1), we have

$$\begin{cases} X_1 = X_2 = \dots = X_5, X_6 = X_7, X_8 = X_9, \\ \mu(G)X_1 = 2X_6 + 2X_8, \\ \mu(G)X_6 = 5X_1 + 2X_8, \\ \mu(G)X_8 = 5X_1 + 2X_6 + X_8. \end{cases} \quad (14)$$

Transform above into a matrix equation $(A'(G) - \mu(G)I)X' = 0$, where $X' = (X_1, X_6, X_8)^T$ and

$$A'(G) = \begin{bmatrix} 0 & 2 & 2 \\ 5 & 0 & 2 \\ 5 & 2 & 1 \end{bmatrix}. \quad (15)$$

Let $f(x) := \det(xI - A'(G))$; then, $f(x) = x^3 - x^2 - 24x - 30$, and $\mu(G)$ is the largest root of $f(x) = 0$. Through calculation, $\mu(G) = 5.9150 < \sqrt{9^2 - 6 \times 9 + 13} = 6.3246$, a contradiction.

Using the same method, we get the spectral radius of the other graphs in $\mathbb{NP}_2$, as given in Table 7.

From Table 7, all graphs in $\mathbb{NP}_2$ satisfy $\mu(G) < \sqrt{n^2 - 6n + 13}$, except $G \in \{K_3 \vee 4K_1, G_6, K_2 \vee 3K_1\}$, a contradiction.

The proof is complete. $\square$

Theorem 3. Let G be a simple connected graph, where $|V(G)| = n (\geq 5)$, $|E(G)| = m$, and $\delta(G) \geq 2$. If

$$q(G) \geq \frac{6}{n-1} + 2n - 6, \quad (16)$$

then G is a pancyclic graph, unless G is a bipartite graph or $G \in \{K_3 \vee (K_{1,4} + K_1), K_4 \vee 5K_1, K_3 \vee 5K_1, K_3 \vee 4K_1, K_3 \vee (K_2 + 3K_1), K_2 \vee (2K_1 + K_{1,2}), K_2 \vee (K_1 + K_{1,3}), K_2 \vee (K_{n-4} + 2K_1) (5 \leq n \leq 11), K_1 \vee 2K_2\}$.

Proof. Let G satisfy the condition of theorem, but it is neither a pancyclic graph nor a bipartite graph. Because K_n is pancyclic and $\delta(K_{1,n-1}) = 1$, $\delta(K_{n-1} + v) = 0$. By Lemma 3,

$$\frac{6}{n-1} + 2n - 6 \leq q(G) < \frac{2m}{n-1} + n - 2; \quad (17)$$

then,

$$m > \binom{n-2}{2} + 3. \quad (18)$$

By Theorem 1, we get $G \in \mathbb{NP}_1$. Because when $G \in \{K_3 \vee (K_3 + 3K_1), K_4 \vee (K_2 + 4K_1)\}$,

$$m = \binom{n-2}{2} + 3, \quad (19)$$

a contradiction. So,

$$\begin{aligned} G \in \mathbb{NP}_2 = \{ & K_2 \vee (K_{n-4} + 2K_1), 2K_1 \vee (K_{n-4} + 2K_1), K_2 \vee ((K_{n-4} - e) + 2K_1), K_1 \vee G_1, K_1 \vee (K_2 + K_{n-3}), K_5 \vee 6K_1, \\ & K_3 \vee K_{2,6}, K_4 \vee (K_1 + K_{1,5}), K_3 \vee (K_2 + 3K_1), K_1 \vee (2K_1 \vee (K_2 + 3K_1)), K_2 \vee (K_1 + K_1 \vee (K_2 + 2K_1)), \\ & K_2 \vee G_2, K_3 \vee 5K_1, K_4 \vee 5K_1, K_3 \vee (K_1 + K_{1,4}), (K_2 \vee 2K_1) \vee 5K_1, K_3 \vee (2K_1 + K_{1,3}), \\ & K_2 \vee G_5, K_2 \vee (K_1 + K_2 \vee 4K_1), K_2 \vee G_4, K_1 \vee (5K_1 \vee (K_2 + K_1)), K_2 \vee G_3, 5K_1 \vee C_4, \\ & K_3 \vee 4K_1, K_{1,2} \vee 4K_1, K_2 \vee (K_1 + K_{1,3}), G_6, 4K_1 \vee (K_1 + K_2), K_2 \vee (2K_2 + K_1), \\ & K_2 \vee (2K_1 + K_{1,2}), K_1 \vee G_7, K_1 \vee G_8 \}. \end{aligned} \quad (20)$$

- (i) For $K_2 \vee (K_{n-4} + 2K_1)$, by the theorem condition, we have $n \geq 5$. It is easy to check by Wolfram Alpha (<http://www.wolframalpha.com/>)

that $q(K_2 \vee (K_{n-4} + 2K_1))$ is the largest root of $f(x) = x^3 + (-3n + 6)x^2 + (2n^2 - 4n - 12)x - 4n^2 + 28n - 48$. Then, $f(6/n - 1 + 2n - 6) =$

TABLE 7: The spectral radius of G .

G	$\mu(G)$	$\sqrt{n^2 - 6n + 13}$	G	$\mu(G)$	$\sqrt{n^2 - 6n + 13}$
$K_{1,2} \vee 4K_1$	4.2182	4.4721	$K_4 \vee 5K_1$	6.2170	6.3246
$K_2 \vee (K_1 + K_{1,3})$	4.3723	4.4721	$K_3 \vee 4K_1$	4.6056	4.4721
$K_5 \vee 6K_1$	7.8310	8.2462	$K_3 \vee (K_2 + 3K_1)$	5.1758	5.3852
$K_3 \vee (K_1 + K_{1,4})$	6.0322	6.3246	$K_3 \vee K_{2,6}$	7.5826	8.2462
$K_4 \vee (K_1 + K_{1,5})$	7.6779	8.2462	$K_1 \vee (2K_1 \vee (K_2 + 3K_1))$	4.8130	5.3852
$K_2 \vee (K_1 + K_1 \vee (K_2 + 2K_1))$	4.9837	5.3852	$K_2 \vee G_2$	4.9273	5.3852
$K_3 \vee 5K_1$	5.0000	5.3852	$K_2 \vee G_5$	5.8367	6.3246
$K_2 \vee (K_1 + K_2 \vee 4K_1)$	5.8904	6.3246	$K_2 \vee G_4$	5.7407	6.3246
$K_1 \vee (5K_1 \vee (K_2 + K_1))$	5.6293	6.3246	$K_1 \vee G_3$	5.7165	6.3246
$5K_1 \vee C_4$	5.5826	6.3246	$K_3 \vee (2K_1 + K_{1,3})$	5.8503	6.3246
G_6	8.2749	4.4721	$4K_1 \vee (K_1 + K_2)$	3.8482	4.4721
$K_2 \vee (2K_2 + K_1)$	4.0839	4.4721	$K_2 \vee (2K_1 + K_{1,2})$	4.1377	4.4721
$K_1 \vee G_7$	3.9964	4.4721	$K_1 \vee G_8$	4.1160	4.4721

$4(n^4 - 18n^3 + 90n^2 - 211n + 192)/(n-1)^3$. When $5 \leq n \leq 11$, $f(6/n - 1 + 2n - 6) < 0$, and when $n \geq 12$, $f(6/n - 1 + 2n - 6) > 0$. Because $q(G) \geq 6/n - 1 + 2n - 6$, then $5 \leq n \leq 11$, and $G = K_2 \vee (K_{n-4} + 2K_1)$.

(ii) For $2K_1 \vee (K_{n-4} + 2K_1)$ because $n \geq 5$. Using the same method, we have $q(2K_1 \vee (K_{n-4} + 2K_1))$ is the largest root of $f(x) = x^3 + (-3n + 8)x^2 + (2n^2 - 8n)x - 4n^2 + 36n - 80$. Then, $f(6/n - 1 + 2n - 6) = 4(n-4)(3n^3 - 14n^2 + 34n - 41)/(n-1)^3 > 0$. So, $q(2K_1 \vee (K_{n-4} + 2K_1)) < 6/n - 1 + 2n - 6$, a contradiction.

(iii) For $K_2 \vee ((K_{n-4} - e) + 2K_1)$ because $n \geq 6$. Using the same method, we get $q(K_2 \vee ((K_{n-4} - e) + 2K_1))$ is the largest root of $f(x) = x^4 + (-4n + 12)x^3 + (5n^2 - 28n + 28)x^2 + (-2n^3 + 12n^2 + 12n - 120)x + 4n^3 - 52n^2 + 224n - 320$. Then, $f(6/n - 1 + 2n - 6) > 0$. So, $q(K_2 \vee ((K_{n-4} - e) + 2K_1)) < 6/n - 1 + 2n - 6$, a contradiction.

(iv) For $K_1 \vee G_1$, we can get $n \geq 6$. Using the same method, we get $q(K_1 \vee G_1)$ is the largest root of $f(x) = x^5 + (-5n + 14)x^4 + (9n^2 - 48n + 54)x^3 + (-7n^3 + 50n^2 - 78n - 42)x^2 + (2n^4 - 12n^3 - 36n^2 + 346n - 544)x - 4n^4 + 60n^3 - 332n^2 + 796n - 688$. Then, $f(6/n - 1 + 2n - 6) = 4(3n^8 - 54n^7 + 437n^6 - 2150n^5 + 7066n^4 - 16131n^3 + 25286n^2 - 25525n + 13012)/(n-1)^5$. We can find that when $n \geq 6$, $f(6/n - 1 + 2n - 6)$ is monotone increasing because $f(6/n - 1 + 2n - 6)|_{n=6} > 0$; thus, $q(K_1 \vee G_1) < 6/n - 1 + 2n - 6$, a contradiction.

(v) For $K_1 \vee (K_2 + K_{n-3})$, by theorem condition, $n \geq 5$. Using the same method, $q(K_1 \vee (K_2 + K_{n-3}))$ is the largest root of $f(x) = x^3 + (5 - 3n)x^2 + (2n^2 - n - 16)x - 6n^2 + 34n - 44$. The roots of $f(x) = 0$ are $x_1 = 1/2(-\sqrt{4n^2 - 36n + 97} + 2n - 3)$, $x_2 = n - 2$, and $x_3 = 1/2(\sqrt{4n^2 - 36n + 97} +$

$2n - 3) < 6/n - 1 + 2n - 6$, when $n \geq 6$. So, $q(K_1 \vee (K_2 + K_{n-3})) < 6/n - 1 + 2n - 6$; when $n = 5$, $q(K_1 \vee (K_2 + K_{n-3})) > 6/n - 1 + 2n - 6$; and when $n \geq 6$, $q(K_1 \vee (K_2 + K_{n-3})) < 6/n - 1 + 2n - 6$ ($n \geq 6$). Because $q(G) \geq 6/n - 1 + 2n - 6$, then $n = 5$, and $G = K_1 \vee (K_2 + K_{n-3}) = K_1 \vee 2K_2$.

(vi) For $(K_2 \vee 2K_1) \vee 5K_1$, let $X = (X_1, X_2, \dots, X_9)^T$ be the eigenvector corresponding to $q(G)$, where X_i ($1 \leq i \leq 5$) correspond to the vertex of degree 4, X_i ($6 \leq i \leq 7$) correspond to the vertex of degree 7, and X_i ($8 \leq i \leq 9$) correspond to the vertex of degree 8. Then, by signless Laplacian eigen equation (2), we have

$$\begin{cases} X_1 = X_2 = \dots = X_5, X_6 = X_7, X_8 = X_9, \\ (q(G) - 4)X_1 = 2X_6 + 2X_8, \\ (q(G) - 7)X_6 = 5X_1 + 2X_8, \\ (q(G) - 8)X_8 = 5X_1 + 2X_6 + X_8. \end{cases} \quad (21)$$

Transform above into a matrix equation $(Q'(G) - q(G)I)\tilde{X} = 0$, where $\tilde{X} = (X_1, X_6, X_8)^T$ and

$$Q'(G) = \begin{bmatrix} 4 & 2 & 2 \\ 5 & 7 & 2 \\ 5 & 2 & 9 \end{bmatrix}. \quad (22)$$

Let $g(x) = \det(xI - Q'(G))$, then $g(x) = x^3 - 20x^2 + 103x - 116$, and $q(G)$ is the largest root of $g(x) = 0$. Through calculation, $q(G) = 12.5052 < 6/9 - 1 + 2 \times 9 - 6 = 12.7500$. Using the same method, we get the signless Laplacian spectral radius of the other graphs in $\mathbb{NP}_2$, as given in Table 8.

From Table 8, all graphs in $\mathbb{NP}_2$ except $G \in \{K_3 \vee (K_{1,4} + K_1), K_4 \vee 5K_1, K_3 \vee 5K_1, K_3 \vee 4K_1, K_3 \vee (K_2 + 3K_1), K_2 \vee (2K_1 + K_{1,2}), K_2 \vee (K_1 + K_{1,3})\}$, other graphs satisfy $q(G) < 10/n - 1 + 2n - 6$, a contradiction.

The proof is complete. $\square$

TABLE 8: The signless Laplacian spectral radius of G .

G	$q(G)$	$6/n - 1 + 2n - 6$	G	$q(G)$	$6/n - 1 + 2n - 6$
$K_{1,2} \vee 4K_1$	8.8965	9.0000	$K_4 \vee 5K_1$	13.1789	12.7500
$K_2 \vee (K_1 + K_{1,3})$	9.3408	9.0000	$K_3 \vee 4K_1$	9.7720	9.0000
$K_5 \vee 6K_1$	16.5887	16.6000	$K_3 \vee (K_2 + 3K_1)$	11.0959	10.8571
$K_3 \vee (K_1 + K_{1,4})$	12.8381	12.7500	$K_3 \vee K_{2,6}$	16.0352	16.6000
$K_4 \vee (K_1 + K_{1,5})$	16.3062	16.6000	$K_1 \vee (2K_1 \vee (K_2 + 3K_1))$	10.2372	10.8571
$K_2 \vee (K_1 + K_1 \vee (K_2 + 2K_1))$	10.7116	10.8571	$K_2 \vee G_2$	10.6393	10.8571
$K_3 \vee 5K_1$	10.8990	10.8571	$K_2 \vee G_5$	12.4661	12.7500
$K_2 \vee (K_1 + K_2 \vee 4K_1)$	12.5303	12.7500	$K_2 \vee G_4$	12.2386	12.7500
$K_1 \vee (5K_1 \vee (K_2 + K_1))$	11.8787	12.7500	$K_1 \vee G_3$	12.0914	12.7500
$5K_1 \vee C_4$	11.6235	12.7500	$K_3 \vee (2K_1 + K_{1,3})$	12.5601	12.7500
G_6	8.2749	9.0000	$4K_1 \vee (K_1 + K_2)$	8.0000	9.0000
$K_2 \vee (2K_2 + K_1)$	8.9164	9.0000	$K_2 \vee (2K_1 + K_{1,2})$	9.0000	9.0000
$K_1 \vee G_7$	8.5650	9.0000	$K_1 \vee G_8$	8.8405	9.0000

4. Conclusions

There are many researchers who studied the spectral theory of graphs in the Hamiltonian graph, but few considered the spectral sufficient conditions on pancyclic graphs. In this study, in terms of the edge number, the spectral radius, and the signless Laplacian spectral radius of the graph, three theorems are established for a graph to be pancyclic. The proofs are also given accordingly.

Data Availability

No data were used to support this study.

Disclosure

In order to prove the originality of the manuscript, this manuscript was prepublished on Avix.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Huan Xu contributed to the calculation and derivation of the formula in Section 3, especially for the proof of Theorem 3.

Acknowledgments

This study was supported by the Natural Science Foundation of China (11871077), NSF of Anhui Province (1808085MA04), NSF of Department of Education of Anhui Province (KJ2020A0894), and Research and Innovation Team of Hefei Preschool Education College (KCTD202001).

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