Research Article

An Approach to the Geometric-Arithmetic Index for Graphs under Transformations' Fact over Pendent Paths

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Graph theory is a dynamic tool for designing and modeling of an interconnection system by a graph. The vertices of such graph are processor nodes and edges are the connections between these processors nodes. The topology of a system decides its best use. Geometric-arithmetic index is one of the most studied graph invariant to characterize the topological aspects of underlying interconnection networks or graphs. Transformation over graph is also an important tool to define new network of their own choice in computer science. In this work, we discuss transformed family of graphs. Let $\Gamma_n^{k,l}$ be the connected graph comprises by k number of pendent path attached with fully connected vertices of the n -vertex connected graph Γ . Let $A_\alpha(\Gamma_n^{k,l})$ and $A_\alpha^\beta(\Gamma_n^{k,l})$ be the transformed graphs under the fact of transformations A_α and A_α^β , $0 \leq \alpha \leq l$, $0 \leq \beta \leq k-1$, respectively. In this work, we obtained new inequalities for the graph family $\Gamma_n^{k,l}$ and transformed graphs $A_\alpha(\Gamma_n^{k,l})$ and $A_\alpha^\beta(\Gamma_n^{k,l})$ which involve $GA(\Gamma)$. The presence of $GA(\Gamma)$ makes the inequalities more general than all those which were previously defined for the GA index. Furthermore, we characterize extremal graphs which make the inequalities tight.

1. Introduction

The advancement in technology mainly networking, computer, biological, and electrical networks made practicable the accurate data transfer within very small duration. The Internet, social media, biological, ecological, and neural networks are few examples of such networks. Telecommunication is based on interconnection networks which used to share data files. Similarly, data exchange using computing devices is also based on computer network through data linkage, optical fiber cable (OFC), and wireless media such as Wi-Fi. Different algorithms are used for directing, arranging/determining numerical calculations, and image

processing. Multiprocessor interconnection networks (MINs) are used to design powerful microprocessors and memory chips [1, 2].

Graph theory provides a fundamental tool for designing and analyzing such networks. Naturally, the interconnection system is modeled by the graph with processor nodes as vertices and links between these nodes as edges of such graph. Graph theory and interconnection networks provide a thorough understanding of these interrelated topics through their topology. The topology of a graph provides information about the manner in which vertices joined in a graph. The topological indices are graph invariants used to study the topology of graphs. Other than computer

networks, graph theory is considered as a powerful tool in different areas of research, such as in coding theory, database management system, circuit design, secret sharing schemes, and theoretical chemistry [3]. The topological descriptors of several interconnection networks are already been computed in [4–6]. Along with interconnection networks, these invariants are equally important in chemical graph theory which deals with problems in chemistry using associated graph of chemical compounds [7].

The study of underlying substance using their graph with the help of graph invariants plays an important role in cheminformatics, pharmaceutical sciences, materials science, engineering, and so forth [8, 9]. Among theoretical molecular descriptors, topological indices have an impact in chemistry due to the prediction of physio-chemical properties of the underlying substance. Its role in the QSPR/QSAR analysis to model physical and chemical properties of molecules is also remarkable [10–12]. Actually, topological indices are designed on the ground of transformation which associates a numeric value with the graph which characterizes its topology [13]. The first topological index, named Winner index, was proposed in 1947 by Winner [14]. It provides best correlation with the boiling points of alkanes. The discovery of the Winner index provides emerging research platform to the research community. The interest in accurate prediction of physio-chemical properties encouraged the researchers to propose a large class of topological indices. For the first time, an index is defined on the base of end vertices' degrees of the edges by Milan Randić named Randić connectivity index [15]:

$$RA(\Gamma) = \sum_{uv \in E(\Gamma)} \frac{1}{\sqrt{\deg_u \deg_v}} \quad (1)$$

Due to this reason, it has attained a great attraction of the researchers till now. In 2009, Vukičević and Furtula [16] introduced the geometric-arithmetic index:

$$GA(\Gamma) = \sum_{st \in E(\Gamma)} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} \quad (2)$$

GA has correlation coefficient of 0.972 with heat of formation of benzene hydrocarbons. Also, in case of “standard enthalpy of vaporization,” its accuracy is 9% more than the Randić index. Due to this reason, GA was studied more than all other indices in the last decade. The bonds and extremal characterization of graphs regarding the GA index were studied at some extent in [17–24]. It encouraged us to study the GA index for $\Gamma_n^{k,l}$ and transformed graphs $A_\alpha(\Gamma_n^{k,l})$ and $A_\alpha^\beta(\Gamma_n^{k,l})$ under the fact of transformations A_α and A_α^β , $0 \leq \alpha \leq l$ $0 \leq \beta \leq k-1$, respectively. We characterize extremal graphs for all of these families of graphs.

2. Results and Discussion

Throughout this work, let graph $\Gamma_n^{k,l}$ comprise with n -vertex simple connected graph Γ along with k pendent paths of length $l \geq 2$ attached with $v \in \Gamma$, having degree $d_v \geq 2$. The order of $\Gamma_n^{k,l}$ is $n + kl$, size is $m + kl$, and $\deg_1 = \delta_\Gamma \leq \deg_2 \leq \deg_3 \leq \dots \leq \Delta_\Gamma + 1$ is its degree sequence.

Let graph $\Gamma = \Gamma(V, E)$ be with the degree of vertex $u \in \Gamma$ and $\delta_\Gamma \leq \deg_u \leq \Delta_\Gamma$ and $\delta_\Gamma \leq \deg_v \leq \Delta_\Gamma + 1$ be the degrees of $v \in \Gamma_n^{k,l}$. For validity of our proved results, we defined the following list of useful graphs.

Type I: let $\delta_\Gamma \leq \deg_u \leq \Delta_\Gamma$, where $u \in V(\Gamma)$. $\Gamma_n^{k,l}$ of type I is obtained by attaching pendent paths of length l with vertices of degree $\deg_u \geq 2$ in such a way that the vertices with pendent path are adjacent to the vertices without pendent paths.

The graph of type I is shown in Figure 1(a).

Type II: $\Gamma_n^{k,l}$ of type II is the graph of type I with $\deg_u = \Delta_\Gamma$, $\forall u \in V(\Gamma)$.

The graph of type II is shown in Figure 1(b).

Before attempting the major problem, we prove the following preposition.

Proposition 1. Let $x \geq 2$; then,

$$\frac{2\sqrt{x(x-1)}}{2x-1} \leq \frac{2\sqrt{x(x+1)}}{2x+1}. \quad (3)$$

Proof. Let $x \geq 2$:

$$\begin{aligned} & \frac{2\sqrt{x(x+1)}}{2x+1} - \frac{2\sqrt{x(x-1)}}{2x-1} \\ &= \frac{2\sqrt{x(x+1)}(2x-1) - 2\sqrt{x(x-1)}(2x+1)}{(2x+1)(2x-1)} \\ &= \frac{2\sqrt{x}[x\sqrt{(x+1)} + (x-1)\sqrt{(x+1)} - x\sqrt{(x-1)} - (x+1)\sqrt{(x-1)}]}{(2x+1)(2x-1)} \\ &= \frac{2\sqrt{x}\left[\sqrt{x+1}\left(x - \sqrt{x^2-1}\right) - \sqrt{x-1}\left(x - \sqrt{x^2-1}\right)\right]}{(2x+1)(2x-1)} \\ &= \frac{2\sqrt{x}\left(x - \sqrt{x^2-1}\right)(\sqrt{x+1} - \sqrt{x-1})}{(2x+1)(2x-1)} \geq 0. \end{aligned} \quad (4)$$

FIGURE 1: Graph $\Gamma_n^{k,l}$. (a) Graph $\Gamma_n^{k,l}$ of type I. (b) Graph $\Gamma_n^{k,l}$ of type II.

The above calculations implies

$$\frac{2\sqrt{x(x-1)}}{2x-1} \leq \frac{2\sqrt{x(x+1)}}{2x+1}. \quad (5)$$

□

Theorem 1. Let graph $\Gamma_n^{k,l}$ comprise of n -vertex simple connected graph Γ along with k pendent paths of length $l \geq 2$ attached with $v \in \Gamma$ of degree $d_v \geq 2$, maximum degree $\Delta_\Gamma + 1$, and minimum δ_Γ . Then,

$$GA(\Gamma_n^{k,l}) \geq k \left[\frac{3(l - \Delta_\Gamma - 2) + 2\sqrt{2}}{3} + 2\sqrt{\Delta_\Gamma + 1} \left(\frac{\sqrt{2}}{(\Delta_\Gamma + 3)} + \frac{\Delta_\Gamma \sqrt{\delta_\Gamma}}{(\Delta_\Gamma + \delta_\Gamma + 1)} \right) \right] + GA(\Gamma). \quad (6)$$

Equality holds for graphs of type II. And,

$$GA(\Gamma_n^{k,l}) \leq k \left[\frac{2\sqrt{2} + 3l - 6}{3} + \frac{2\sqrt{2(\delta_\Gamma + 1)}}{\delta_\Gamma + 3} + 2k\Delta_\Gamma^{3/2} \left(\frac{\sqrt{\Delta_\Gamma + 1}}{2\Delta_\Gamma + 1} - \frac{\sqrt{\delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma} \right) \right] + GA(\Gamma). \quad (7)$$

Equality holds for graph of type II.

Proof. Let a simple graph Γ be of order n , size m , maximum degree Δ_Γ , and minimum δ_Γ . $\Gamma_n^{k,l}$ be the graph formed by k number of paths having length l pendent at distinct vertices $u \in \Gamma$ such that $2 \leq \deg_u \leq \delta_\Gamma$. The geometric-arithmetic index of any graph Γ is

$$GA(\Gamma) = \sum_{st \in E(\Gamma)} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t}. \quad (8)$$

The construction of $\Gamma_n^{k,l}$, $l \geq 2$ implies $|E(\Gamma_n^{k,l})| = m + kl$ and for $st \in E(\Gamma_n^{k,l})$ $(\deg_s + \deg_t) \in \{3, 4, \deg_u + 2, \deg_u + \deg_v, \deg_u + \deg_v + 1\}$.

The edge set of $\Gamma_n^{k,l}$ partitioned as

$$A_3 = \{st \in \Gamma_n^{k,l} : \deg_s = 1, \deg_t = 2\},$$

$$A_4 = \{st \in \Gamma_n^{k,l} : \deg_s = \deg_t = 2\},$$

$$A_{\deg_u + 2} = \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u \leq \Delta_\Gamma, \deg_t = 2\},$$

$$A_{\deg_u + \deg_v} = \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u, \deg_t = \deg_v \leq \Delta_\Gamma\}, \quad (9)$$

$$\text{and } A_{\deg_u + \deg_v + 1} = \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u, \deg_t = \deg_v + 1 \leq \Delta_\Gamma\},$$

$$\begin{aligned}
\text{GA}(\Gamma_n^{k,l}) &= \sum_{\substack{st \text{ are edges of} \\ \text{pendent paths}}} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} + \sum_{st \text{ are edges of } \Gamma} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t}, \\
\text{GA}(\Gamma_n^{k,l}) &= \sum_{st \in A_3} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} + \sum_{st \in A_4} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} + \sum_{st \in A_{\deg_u+3}} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} \\
&\quad + \sum_{st \in A_{\deg_u+\deg_v+1}} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} + \sum_{st \in A_{\deg_u+\deg_v}} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t}.
\end{aligned} \tag{10}$$

The construction of $\text{GA}(\Gamma_n^{k,l})$ implies that the cardinality of A_3 is k , i.e., $|A_3| = k$, $|A_4| = k(l-2)$, $|A_{\deg_u+3}| = k$, $|A_{\deg_u+\deg_v+1}| \leq k\Delta_\Gamma$, and $|A_{\deg_u+\deg_v}| \leq k\Delta_\Gamma$. The function $f(x) = 2\sqrt{ax}/a + x$ is decreasing, where $a \leq x$ is a constant. So, for δ_Γ minimum degree of vertices of Γ and maximum degree Δ_Γ , we have

$= 2\sqrt{ax}/a + x$ is decreasing, where $a \leq x$ is a constant. So, for δ_Γ minimum degree of vertices of Γ and maximum degree Δ_Γ , we have

$$\frac{2\sqrt{2 * (\deg_s + 1)}}{2 + (\deg_s + 1)} \geq \frac{2\sqrt{2 * (\Delta_\Gamma + 1)}}{2 + (\Delta_\Gamma + 1)}, \frac{2\sqrt{(\Delta_\Gamma + 1)\delta_\Gamma}}{(\Delta_\Gamma + 1) + \delta_\Gamma} \leq \frac{2\sqrt{(\deg_s + 1)\deg_t}}{(\deg_s + 1) + \deg_t}, \frac{2\sqrt{\deg_s + \deg_t}}{\deg_s + \deg_t} \geq \text{GA}(\Gamma) - \Delta_\Gamma k. \tag{11}$$

From equation (10), we have

$$\text{GA}(\Gamma_n^{k,l}) \geq \frac{2k\sqrt{2}}{3} + k(l-2) + \frac{2k\sqrt{2 * (\Delta_\Gamma + 1)}}{2 + (\Delta_\Gamma + 1)} + \frac{2\Delta_\Gamma k \sqrt{\delta_\Gamma (\Delta_\Gamma + 1)}}{\delta_\Gamma + \Delta_\Gamma + 1} - k\Delta_\Gamma + \text{GA}(\Gamma). \tag{12}$$

After simplification, we obtain

$$\text{GA}(\Gamma_n^{k,l}) \geq k \left[\frac{3(l - \Delta_\Gamma - 2) + 2\sqrt{2}}{3} + 2\sqrt{\Delta_\Gamma + 1} \left(\frac{\sqrt{2}}{(\Delta_\Gamma + 3)} + \frac{\Delta_\Gamma \sqrt{\delta_\Gamma}}{(\Delta_\Gamma + \delta_\Gamma + 1)} \right) \right] + \text{GA}(\Gamma). \tag{13}$$

Now, again set

$$\begin{aligned}
\frac{2\sqrt{2 * (\deg_s + 1)}}{2 + (\deg_s + 1)} &\leq \frac{2\sqrt{2 * (\delta_\Gamma + 1)}}{2 + (\delta_\Gamma + 1)}, \sum_{st \in A_{\deg_u+\deg_v+1}} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} + \sum_{st \in A_{\deg_u+\deg_v}} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t} \\
&\leq \frac{2\sqrt{(\Delta_\Gamma + 1)\Delta_\Gamma}}{(\Delta_\Gamma + 1) + \Delta_\Gamma} + \text{GA}(\Gamma) - \frac{2k\Delta_\Gamma \sqrt{\delta_\Gamma \Delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma},
\end{aligned} \tag{14}$$

which implies from Proposition 1 and the characteristics of $f(x) = 2\sqrt{xy}/x + y$ in equation (10). We get the following inequality:

$$\text{GA}(\Gamma_n^{k,l}) \leq \frac{2k\sqrt{2}}{3} + k(l-2) + \frac{2k\sqrt{2 * (\delta_\Gamma + 1)}}{2 + (\delta_\Gamma + 1)} + \frac{2\Delta_\Gamma k \sqrt{\delta_\Gamma (\Delta_\Gamma + 1)}}{\delta_\Gamma + \Delta_\Gamma + 1} + \text{GA}(\Gamma) - \frac{2\Delta_\Gamma k \sqrt{\delta_\Gamma \Delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma}. \tag{15}$$

After simplification, we obtain

$$GA(\Gamma_n^{k,l}) \leq k \left[\frac{2\sqrt{2} + 3l - 6}{3} + \frac{2\sqrt{2(\delta_\Gamma + 1)}}{\delta_\Gamma + 3} + 2k\Delta_\Gamma^{3/2} \left(\frac{\sqrt{\Delta_\Gamma + 1}}{2\Delta_\Gamma + 1} - \frac{\sqrt{\delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma} \right) \right] + GA(\Gamma). \quad (16)$$

Inequalities (13) and (16) complete the proof.

Corollary 1 shows generalization of the above defined inequalities. One can get more inequalities of their desire by replacing $GA(\Gamma)$ with already defined bonds of the GA index. $\square$

Corollary 1. Let graph $\Gamma_n^{k,l}$ comprise of n -vertex simple connected graph Γ along with k pendent paths of length $l \geq 2$ attached with $v \in \Gamma$ of degree $d_v \geq 2$, maximum degree $\Delta_\Gamma + 1$, and minimum δ_Γ . Then,

$$\begin{aligned} GA(\Gamma_n^{k,l}) &\geq k \left[\frac{3(l - \Delta_\Gamma - 2) + 2\sqrt{2}}{3} + 2\sqrt{\Delta_\Gamma + 1} \left(\frac{\sqrt{2}}{(\Delta_\Gamma + 3)} + \frac{\Delta_\Gamma \sqrt{\delta_\Gamma}}{(\Delta_\Gamma + \delta_\Gamma + 1)} \right) \right] + \frac{2m\sqrt{\Delta_\Gamma \delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma}, \\ GA(\Gamma_n^{k,l}) &\leq k \left[\frac{2\sqrt{2} + 3l - 6}{3} + \frac{2\sqrt{2(\delta_\Gamma + 1)}}{\delta_\Gamma + 3} + 2k\Delta_\Gamma^{3/2} \left(\frac{\sqrt{\Delta_\Gamma + 1}}{2\Delta_\Gamma + 1} - \frac{\sqrt{\delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma} \right) \right] + m. \end{aligned} \quad (17)$$

Equality holds for regular graph of the type II.

Proof. Using results of Theorem 1 and inequality regarding the geometric index proved in [25, 26], as

$$\frac{2m\sqrt{\Delta_\Gamma \delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma} \leq GA(\Gamma) \leq m, \quad (18)$$

we get desired results. $\square$

2.1. Graph Transformations. Let $H(\Gamma) \subset E(\Gamma)$, $\Gamma' = \Gamma - H$ be the new graph generated by removing set edges of $H(\Gamma)$, and $\Gamma'' = \Gamma - V_1(\Gamma)$ be the new graph generated by deleting set of vertices $V_1(\Gamma) \subset V(\Gamma)$. We use the following transformations as used in [27]. These transformations have solid effect over GA of $\Gamma_n^{k,l}$.

Transformation A: let $w_j \in V(\Gamma)$, $\deg_{w_j} \geq 2$, for $1 \leq j \leq k \leq n$, and paths pendent at w_j of the form $\{w_j u_j^1, u_j^1 u_j^2, u_j^2 u_j^3, \dots, u_j^{l-1} u_j^l\}$ comprise $\Gamma_n^{k,l}$. Then,

$$\begin{aligned} A(\Gamma_n^{k,l}) &= \Gamma_n^{k,l} - \sum_{j=1}^k \{u_j^2 u_j^3, u_j^3 u_j^4, \dots, u_j^{l-1} u_j^l\} \\ &\quad + \sum_{j=1}^k \{w_j u_j^2, u_j^2 u_j^3, \dots, u_j^{l-1} u_j^l\}. \end{aligned} \quad (19)$$

The transformation A is shown in Figure 2.

In Theorem 2, we discuss the effect of transformation A over the GA index.

Theorem 2. Let graph $\Gamma_n^{k,l}$ comprise of n -vertex simple connected graph Γ along with k pendent paths of length $l \geq 2$ attached with $v \in \Gamma$ of degree $d_v \geq 2$, maximum degree of $v \in \Gamma_n^{k,l}$ is $\Delta_\Gamma + 1$, and minimum δ_Γ . Then,

$$\begin{aligned} GA(A_\alpha(\Gamma_n^{k,l})) &\geq 2k\sqrt{\Delta_\Gamma + \alpha + 1} \left[\frac{\alpha}{\Delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\Delta_\Gamma + \alpha + 3} + \frac{\Delta_\Gamma \sqrt{\delta_\Gamma}}{(\Delta_\Gamma + 1 + \alpha) + \delta_\Gamma} \right] \\ &\quad + \frac{2k\sqrt{2}}{3} + GA(\Gamma) + kl - k(2 + \alpha + \Delta_\Gamma). \end{aligned} \quad (20)$$

Equality holds for all graphs of type II:

$$\begin{aligned} GA(A_\alpha(\Gamma_n^{k,l})) &\leq 2k\sqrt{\delta_\Gamma + \alpha + 1} \left[\frac{\alpha}{\delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\delta_\Gamma + \alpha + 3} \right] + \frac{2k\sqrt{2}}{3} \\ &\quad + k(l - 2 - \alpha) + GA(\Gamma) + 2k\Delta_\Gamma^{3/2} \left[\frac{\sqrt{\Delta_\Gamma + 1}}{2\Delta_\Gamma + 1} - \frac{\sqrt{\delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma} \right]. \end{aligned} \quad (21)$$

FIGURE 2: Transformation A.

Equality holds for all graphs of the type II and $\alpha = 0$.

Proof. Let a simple graph Γ be of order n , size m , minimum degree δ_Γ , and maximum Δ_Γ . Let $\Gamma_n^{k,l}$ be the graph formed by k number of paths of length l pendent at distinct fully connected vertices of Γ . The geometric-arithmetic index of any graph Γ is

$$GA(\Gamma) = \sum_{st \in E(G)} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t}. \quad (22)$$

The construction of $\Gamma_n^{k,l}$, $l \geq 2$, implies $|E(\Gamma_n^{k,l})| = m + kl$. After successive applications of transformation A as A_α , $\alpha \leq l-1$, the edge set of $A_\alpha(\Gamma_n^{k,l})$ is partitioned as $E_{(\deg_s + \deg_t)}(A_\alpha(\Gamma_n^{k,l}))$, $(\deg_s + \deg_t) \in \{3, 4, \deg_u + \alpha + 2, \deg_u + \alpha + 3, \deg_u + \deg_v, \deg_u + \alpha + 1 + \deg_v\}$:

$$\begin{aligned} E_3(A_\alpha(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \deg_s = 1, \deg_t = 2\}, \\ E_4(A_\alpha(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \deg_s = \deg_t = 2\}, \\ E_{\deg_u + \alpha + 2}(A_\alpha(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u + \alpha + 1 \leq \Delta_\Gamma + \alpha + 1, \deg_t = 1\}, \\ E_{\deg_u + \alpha + 3}(A_\alpha(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u + \alpha + 1 \leq \Delta_\Gamma + \alpha + 1, \deg_t = 2\}, \\ E_{\deg_u + \alpha + 1}(A_\alpha(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u + \alpha \leq \Delta_\Gamma, \deg_t = 1\}, \\ E_{\deg_u + \deg_v}(A_\alpha(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u, \deg_t = \deg_v \leq \Delta_\Gamma\}, \\ E_{\deg_u + \alpha + 1 + \deg_v}(A_\alpha(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_v, \deg_u \leq \Delta_\Gamma, \deg_s = \deg_u + \alpha + 1, \deg_t = \deg_v\}, \end{aligned} \quad (23)$$

$$GA(A_\alpha(\Gamma_n^{k,l})) = \sum_{\substack{E_{(\deg_s + \deg_t)}(A_\alpha(\Gamma_n^{k,l})) \\ \subseteq E(A_\alpha(\Gamma_n^{k,l}))}} \sum_{E_{(\deg_s + \deg_t)}(A_\alpha(\Gamma_n^{k,l}))} \sum \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t}. \quad (24)$$

The cardinality of A_3 is k , i.e., $|E_3(A_\alpha(\Gamma_n^{k,l}))| = k$, $|E_4(A_\alpha(\Gamma_n^{k,l}))| = k(l - \alpha - 2)$, $|E_{\deg_v + \alpha + 2}(A_\alpha(\Gamma_n^{k,l}))| = k\alpha$, and $|E_{\deg_u + \alpha + 3}(A_\alpha(\Gamma_n^{k,l}))| = k$. The function $f(x) = 2\sqrt{ax}/a + x$ is decreasing, where $a \leq x$ is a constant. So, for δ_Γ minimum degree of Γ and Δ_Γ maximum, for any graph,

$$\frac{2\sqrt{2 * (\deg_s + 1 + \alpha)}}{2 + (\deg_s + 1 + \alpha)} \geq \frac{2\sqrt{2 * (\Delta_\Gamma + 1 + \alpha)}}{2 + (\Delta_\Gamma + 1 + \alpha)},$$

$$\frac{2\sqrt{(\Delta_\Gamma + \alpha + 1)\delta_\Gamma}}{(\Delta_\Gamma + \alpha + 1) + \delta_\Gamma} \leq \frac{2\sqrt{(\deg_s + \alpha + 1)\deg_t}}{(\deg_s + \alpha + 1) + \deg_t} \quad (25)$$

$$\sum_{st \in A_{\deg_u + \deg_v}} \frac{2\sqrt{\deg_s + \deg_t}}{\deg_s + \deg_t} \geq GA(\Gamma) - \Delta_\Gamma k.$$

Substituting these changes in equation (24), we have the following inequality:

$$GA(A_\alpha(\Gamma_n^{k,l})) \geq \frac{2k\sqrt{2}}{3} + k(l - 2 - \alpha) + \frac{2\alpha k \sqrt{1 * (\Delta_\Gamma + \alpha + 1)}}{1 + (\Delta_\Gamma + \alpha + 1)} + \frac{2k\sqrt{2(\Delta_\Gamma + \alpha + 1)}}{2 + (\Delta_\Gamma + \alpha + 1)} + \frac{2k\Delta_\Gamma \sqrt{(\Delta_\Gamma + \alpha + 1)\delta_\Gamma}}{(\Delta_\Gamma + \alpha + 1) + \delta_\Gamma} + GA(\Gamma) - \Delta_\Gamma K. \quad (26)$$

After simplification, we get the required result

$$GA(A_\alpha(\Gamma_n^{k,l})) \geq 2k\sqrt{\Delta_\Gamma + \alpha + 1} \left[\frac{\alpha}{\Delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\Delta_\Gamma + \alpha + 3} + \frac{\Delta_\Gamma \sqrt{\delta_\Gamma}}{(\Delta_\Gamma + 1 + \alpha) + \delta_\Gamma} \right] + \frac{2k\sqrt{2}}{3} + GA(\Gamma) + kl - k(2 + \alpha + \Delta_\Gamma). \quad (27)$$

Now, again, from equation (24) and inequalities,

$$\sum_{st \in A_{\deg_u + \deg_v}} \frac{2\sqrt{\deg_s + \deg_t}}{\deg_s + \deg_t} \geq GA(\Gamma) - \frac{2\Delta_\Gamma \sqrt{\Delta_\Gamma \delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma},$$

$$\frac{2\sqrt{\deg_s + \alpha + 1}}{1 + (\deg_s + \alpha + 1)} \leq \frac{2\sqrt{\delta_\Gamma + \alpha + 1}}{1 + (\delta_\Gamma + \alpha + 1)},$$

$$\frac{2\Delta_\Gamma k \sqrt{(\Delta_\Gamma + 1)\Delta_\Gamma}}{(\Delta_\Gamma + 1) + \Delta_\Gamma} \geq \sum_{st \in A_{\deg_u + \deg_v + \alpha + 1}} \frac{2\sqrt{(\deg_s + \alpha + 1)\deg_t}}{(\deg_s + \alpha + 1) + \deg_t},$$

$$\frac{2\sqrt{2 * (\deg_s + \alpha + 1)}}{2 + (\deg_s + \alpha + 1)} \leq \frac{2\sqrt{2 * (\delta_\Gamma + \alpha + 1)}}{2 + (\delta_\Gamma + \alpha + 1)}, \quad (28)$$

$$GA(A_\alpha(\Gamma_n^{k,l})) \leq \frac{2k\sqrt{2}}{3} + \frac{2\alpha k \sqrt{1 * (\delta_\Gamma + \alpha + 1)}}{1 + (\delta_\Gamma + \alpha + 1)} + \frac{2k\sqrt{2 * (\delta_\Gamma + \alpha + 1)}}{2 + (\delta_\Gamma + \alpha + 1)} + \frac{2\Delta_\Gamma k \sqrt{(\Delta_\Gamma + 1)\Delta_\Gamma}}{(\Delta_\Gamma + 1) + \Delta_\Gamma} + k(l - 2 - \alpha) + GA(\Gamma) - \frac{2k\Delta_\Gamma \sqrt{\Delta_\Gamma \delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma}. \quad (29)$$

After simplification, we obtain

$$GA(A_\alpha(\Gamma_n^{k,l})) \leq 2k\sqrt{\delta_\Gamma + \alpha + 1} \left[\frac{\alpha}{\delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\delta_\Gamma + \alpha + 3} \right] + \frac{2k\sqrt{2}}{3} + k(l - 2 - \alpha) + GA(\Gamma) + 2k\Delta_\Gamma^{3/3} \cdot \left[\frac{\sqrt{\Delta_\Gamma + 1}}{2\Delta_\Gamma + 1} - \frac{\sqrt{\delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma} \right]. \quad (30)$$

Inequalities (27) and (30) complete the proof.

Transformation B : let $w_j \in V(\Gamma)$, $\deg_{w_j} \geq 2$, for $1 \leq j \leq k \leq n$, and paths pendent at w_j of the form $\{w_j u_j^1, u_j^1 u_j^2, u_j^2 u_j^3, \dots, u_j^{l-1} u_j^l\}$ which comprises $\Gamma_n^{k,l}$. Then, for fixed vertex w_1 ,

$$B(\Gamma_n^{k,l}) = \Gamma_n^{k,l} - \{u_j^1 u_j^2, u_j^2 u_j^3, \dots, u_j^{l-1} u_j^l\} + \{w_1 u_j^1, u_j^1 u_j^2, u_j^2 u_j^3, \dots, u_j^{l-1} u_j^l\}. \quad (31)$$

The transformation B is shown in Figure 3 and A_α^β shown in Figure 4.

Transformation A_α^β : let $0 \leq \alpha \leq l - 1$ and $0 \leq \beta \leq k - 1$. The transformation A_α^β is the composition of successive applications of transformation A and B as A_α and B_β , respectively [27].

In Theorem 3, we discuss the effect of transformation A_α^β over the GA index. $\square$

Theorem 3. Let graph $\Gamma_n^{k,l}$ comprise of n -vertex simple connected graph Γ along with k pendent paths of length $l \geq 2$ attached with $v \in \Gamma$ of degree $d_v \geq 2$, maximum degree of $v \in \Gamma_n^{k,l}$ is $\Delta_\Gamma + 1$, and minimum δ_Γ . Then,

FIGURE 3: Transformation B for fixed vertex w_1 . FIGURE 4: Transformation A_α^β .

$$\begin{aligned}
 \text{GA}(A_\alpha^\beta(\Gamma_n^{k,l})) &\geq 2(k - \beta - 1)\sqrt{\Delta_\Gamma + \alpha + 1} \left(\frac{\alpha}{\Delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\Delta_\Gamma + \alpha + 3} + \frac{\Delta_\Gamma \sqrt{\delta_\Gamma}}{\Delta_\Gamma + \alpha + 1 + \delta_\Gamma} \right) \\
 &\quad + 2\sqrt{\Delta_\Gamma + (\beta + 1)(1 + \alpha)} \left(\frac{\alpha(\beta + 1)}{1 + \Delta_\Gamma + (\beta + 1)(1 + \alpha)} + \frac{\sqrt{2}(\beta + 1)}{2 + \Delta_\Gamma + (\beta + 1)(1 + \alpha)} \right) \\
 &\quad + 2 \frac{\Delta_\Gamma \sqrt{(\Delta_\Gamma + (\beta + 1)(1 + \alpha))\delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma + (\beta + 1)(1 + \alpha)} + k(l - 2 - \alpha) - \Delta_\Gamma(k - \beta) + \frac{2k\sqrt{2}}{3} + \text{GA}(\Gamma),
 \end{aligned}$$

$$\begin{aligned}
\text{GA}(A_\alpha^\beta(\Gamma_n^{k,l})) &\leq 2(k - \beta - 1)\sqrt{\delta_\Gamma + \alpha + 1} \left(\frac{\alpha}{\delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\delta_\Gamma + \alpha + 3} \right) + k(l - 2 - \alpha) + \frac{2k\sqrt{2}}{3} \\
&\quad + 2\sqrt{\delta_\Gamma + (\beta + 1)(1 + \alpha)} \left(\frac{\alpha(\beta + 1)}{1 + \delta_\Gamma + (\beta + 1)(1 + \alpha)} + \frac{\sqrt{2}(\beta + 1)}{2 + \delta_\Gamma + (\beta + 1)(1 + \alpha)} \right) \\
&\quad + 2\Delta_\Gamma\sqrt{\Delta_\Gamma}(k - \beta) \left(\frac{\sqrt{\Delta_\Gamma + 1}}{2\Delta_\Gamma + 1} - \frac{\sqrt{\delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma} \right) + \text{GA}(\Gamma).
\end{aligned} \tag{32}$$

Equality holds for graph of the type II with $\alpha = 0$ and $\beta = 0$.

$$\text{GA}(\Gamma) = \sum_{st \in E(G)} \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t}. \tag{33}$$

Proof. Let a simple graph Γ of order n , size m , minimum degree δ_Γ , and maximum Δ_Γ . Let $\Gamma_n^{k,l}$ be the graph formed by k number of paths of length l pendent at distinct fully connected vertices of Γ . The geometric-arithmetic index of any graph Γ is

The construction of $\Gamma_n^{k,l}$, $l \geq 2$, implies $|E(\Gamma_n^{k,l})| = m + kl$. Let u^* be the fixed vertex. Applications of transformation A_α^β has an effect over the edge set partition as $E_{(\deg_s + \deg_t)}(A_\alpha^\beta(\Gamma_n^{k,l}))$, $(\deg_s + \deg_t) \in \{3, 4, \deg_u + \alpha + 2, \deg_u + \alpha + 3, \deg_u + \deg_v, \deg_u + \alpha + 1 + \deg_v, \deg_{u^*} + (\beta + 1)(\alpha + 1) + \deg_v, \deg_{u^*} + (\beta + 1)(\alpha + 1) + 1, \deg_{u^*} + (\beta + 1)(\alpha + 1) + 2\}$:

$$\begin{aligned}
E_3(A_\alpha^\beta(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \deg_s = 1, \deg_t = 2\}, \\
E_4(A_\alpha^\beta(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \deg_s = \deg_t = 2\}, \\
E_{\deg_u + \alpha + 2}(A_\alpha^\beta(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u + \alpha + 1 \leq \Delta_\Gamma + \alpha + 1, \deg_t = 1, E_{\deg_u + \alpha + 3}(A_\alpha^\beta(\Gamma_n^{k,l}))\} \\
&= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u + \alpha + 1 \leq \Delta_\Gamma + \alpha + 1, \deg_t = 2, E_{\deg_u + \deg_v}(A_\alpha^\beta(\Gamma_n^{k,l}))\} \\
&= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_s = \deg_u, \deg_t = \deg_v \leq \Delta_\Gamma\}, \\
E_{\deg_u + \alpha + 1 + \deg_v}(A_\alpha^\beta(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_v = \deg_t, \deg_u \leq \Delta_\Gamma, \deg_s = \deg_u + \alpha + 1\}, \\
E_{\deg_{u^*} + (\beta + 1)(\alpha + 1) + \deg_v}(A_\alpha^\beta(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_v = \deg_t, \deg_u \leq \Delta_\Gamma, \deg_s = \deg_u + (\beta + 1)(\alpha + 1)\}, \\
E_{\deg_{u^*} + (\beta + 1)(\alpha + 1) + 1}(A_\alpha^\beta(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_v, \deg_u \leq \Delta_\Gamma, \deg_s = \deg_u + (\beta + 1)(\alpha + 1), \deg_t = 1\}, \\
E_{\deg_{u^*} + (\beta + 1)(\alpha + 1) + 2}(A_\alpha^\beta(\Gamma_n^{k,l})) &= \{st \in \Gamma_n^{k,l} : \delta_\Gamma \leq \deg_v, \deg_u \leq \Delta_\Gamma, \deg_s = \deg_u + (\beta + 1)(\alpha + 1), \deg_t = 2\}, \\
\text{GA}(A_\alpha^\beta(\Gamma_n^{k,l})) &= \sum_{\substack{E_{(\deg_s + \deg_t)}(A_\alpha^\beta(\Gamma_n^{k,l})) \\ \subseteq E(A_\alpha^\beta(\Gamma_n^{k,l}))}} \sum_{\substack{st \in \\ E_{(\deg_s + \deg_t)}(A_\alpha^\beta(\Gamma_n^{k,l}))}} \sum \frac{2\sqrt{\deg_s \deg_t}}{\deg_s + \deg_t}.
\end{aligned} \tag{35}$$

The cardinality of E_3 is k , i.e., $|E_3(A_\alpha^\beta(\Gamma_n^{k,l}))| = k$, $|E_4(A_\alpha^\beta(\Gamma_n^{k,l}))| = k(l - \alpha - 2)$, $|E_{\deg_u + \alpha + 2}(A_\alpha^\beta(\Gamma_n^{k,l}))| = \alpha(k - \beta - 1)$, $|E_{\deg_u + \alpha + 3}(A_\alpha^\beta(\Gamma_n^{k,l}))| = k - \beta - 1$, $|E_{\deg_{u^*} + (\beta + 1)(\alpha + 1) + 1}(A_\alpha^\beta(\Gamma_n^{k,l}))| = \alpha(\beta + 1)$, and

$$|E_{\deg_{u^*} + (\beta + 1)(\alpha + 1) + 2}(A_\alpha^\beta(\Gamma_n^{k,l}))| = \beta + 1. \tag{36}$$

The function $f(x) = 2\sqrt{ax}/a + x$ is decreasing, where $a \leq x$ is a constant. So, for δ_Γ minimum degree of Γ and Δ_Γ maximum, we have

$$\begin{aligned}
\frac{2\sqrt{2(\deg_s + 1 + \alpha)}}{2 + (\deg_s + 1 + \alpha)} &\geq \frac{2\sqrt{2(\Delta_\Gamma + 1 + \alpha)}}{2 + (\Delta_\Gamma + 1 + \alpha)}, \frac{2\sqrt{(\Delta_\Gamma + \alpha + 1) * 1}}{(\Delta_\Gamma + \alpha + 1) + 1} \leq \frac{2\sqrt{(\deg_s + \alpha + 1)\deg_t}}{(\deg_s + \alpha + 1) + \deg_t}, \\
\frac{2\sqrt{2 * (\deg_s + (\beta + 1)(1 + \alpha))}}{2 + (\deg_s + (\beta + 1)(1 + \alpha))} &\geq \frac{2\sqrt{2 * (\Delta_\Gamma + (\beta + 1)(1 + \alpha))}}{2 + (\Delta_\Gamma + (\beta + 1)(1 + \alpha))}, \\
\frac{2\sqrt{(\Delta_\Gamma + (\beta + 1)(1 + \alpha))\delta_\Gamma}}{(\Delta_\Gamma + (\beta + 1)(1 + \alpha)) + \delta_\Gamma} &\leq \frac{2\sqrt{(\deg_s + (\beta + 1)(1 + \alpha))\deg_t}}{(\deg_s + (\beta + 1)(1 + \alpha)) + \deg_t},
\end{aligned} \tag{37}$$

$$\begin{aligned}
\sum_{st \in A_{\deg_u + \deg_v}} \frac{2\sqrt{\deg_s + \deg_t}}{\deg_s + \deg_t} &\geq \text{GA}(\Gamma) + \frac{2\Delta_\Gamma(k - \beta - 1)\sqrt{(\Delta_\Gamma + \alpha + 1)\delta_\Gamma}}{(\Delta_\Gamma + \alpha + 1) + \delta_\Gamma} \\
&+ \frac{2\Delta_\Gamma\sqrt{(\Delta_\Gamma + (\beta + 1)(1 + \alpha))\delta_\Gamma}}{(\Delta_\Gamma + (\beta + 1)(1 + \alpha)) + \delta_\Gamma} - \Delta_\Gamma(k - \beta).
\end{aligned} \tag{38}$$

Substituting these changes in equation (35), we obtained the following inequality:

$$\begin{aligned}
\text{GA}(A_\alpha^\beta(\Gamma_n^{k,l})) &\geq \frac{2k\sqrt{2}}{3} + \frac{2\alpha(k - \beta - 1)\sqrt{1 * (\Delta_\Gamma + \alpha + 1)}}{1 + (\Delta_\Gamma + \alpha + 1)} + \frac{2(k - \beta - 1)\sqrt{2(\Delta_\Gamma + \alpha + 1)}}{2 + (\Delta_\Gamma + \alpha + 1)} \\
&+ \frac{2\alpha(\beta + 1)\sqrt{1 * (\Delta_\Gamma + (\beta + 1)(1 + \alpha))}}{1 + (\Delta_\Gamma + (\beta + 1)(1 + \alpha))} \\
&+ \frac{2(\beta + 1)\sqrt{2(\Delta_\Gamma + (\beta + 1)(1 + \alpha))}}{2 + (\Delta_\Gamma + (\beta + 1)(1 + \alpha))} \\
&+ \frac{2\Delta_\Gamma(k - \beta - 1)\sqrt{(\Delta_\Gamma + \alpha + 1)\delta_\Gamma}}{(\Delta_\Gamma + \alpha + 1) + \delta_\Gamma} + \frac{2\Delta_\Gamma\sqrt{(\Delta_\Gamma + (\beta + 1)(1 + \alpha))\delta_\Gamma}}{(\Delta_\Gamma + (\beta + 1)(1 + \alpha)) + \delta_\Gamma} \\
&+ \text{GA}(\Gamma) + k(l - 2 - \alpha) - \Delta_\Gamma(k - \beta).
\end{aligned} \tag{39}$$

After simplification, we get the required result:

$$\begin{aligned}
 \text{GA}(A_\alpha^\beta(\Gamma_n^{k,l})) &\geq 2(k - \beta - 1)\sqrt{\delta_\Gamma + \alpha + 1} \left(\frac{\alpha}{\Delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\Delta_\Gamma + \alpha + 3} + \frac{\Delta_\Gamma \sqrt{\delta_\Gamma}}{\Delta_\Gamma + \alpha + 1 + \delta_\Gamma} \right) + 2\sqrt{\Delta_\Gamma + (\beta + 1)(1 + \alpha)} \\
 &\times \left(\frac{\alpha(\beta + 1)}{1 + \Delta_\Gamma + (\beta + 1)(1 + \alpha)} + \frac{\sqrt{2}(\beta + 1)}{2 + \Delta_\Gamma + (\beta + 1)(1 + \alpha)} \right) + 2 \frac{\Delta_\Gamma \sqrt{(\Delta_\Gamma + (\beta + 1)(1 + \alpha))\delta_\Gamma}}{\delta_\Gamma + \Delta_\Gamma + (\beta + 1)(1 + \alpha)} \\
 &+ k(l - 2 - \alpha) - \Delta_\Gamma(k - \beta) + \frac{2k\sqrt{2}}{3} + \text{GA}(\Gamma).
 \end{aligned} \tag{40}$$

Now, again substituting the following inequalities in equation (35),

$$\begin{aligned}
 \frac{2\sqrt{2(\deg_s + 1 + \alpha)}}{2 + (\deg_s + 1 + \alpha)} &\leq \frac{2\sqrt{2(\delta_\Gamma + 1 + \alpha)}}{2 + (\delta_\Gamma + 1 + \alpha)}, \\
 \frac{2\sqrt{(\deg_s + \alpha + 1)\deg_t}}{(\deg_s + \alpha + 1) + \deg_t} &\leq \frac{2\sqrt{(\delta_\Gamma + \alpha + 1)}}{(\delta_\Gamma + \alpha + 1) + 1}, \\
 \frac{2\sqrt{2(\deg_s + (\beta + 1)(1 + \alpha))}}{2 + (\deg_s + (\beta + 1)(1 + \alpha))} &\leq \frac{2\sqrt{2(\delta_\Gamma + (\beta + 1)(1 + \alpha))}}{2 + (\delta_\Gamma + (\beta + 1)(1 + \alpha))}, \\
 \frac{2\sqrt{(\deg_s + (\beta + 1)(1 + \alpha))}}{1 + (\deg_s + (\beta + 1)(1 + \alpha))} &\leq \frac{2\sqrt{(\delta_\Gamma + (\beta + 1)(1 + \alpha))}}{1 + (\delta_\Gamma + (\beta + 1)(1 + \alpha))}, \\
 \sum_{st \in A_{\deg_{st} + \deg_v}} \frac{2\sqrt{\deg_s + \deg_t}}{\deg_s + \deg_t} &\leq \text{GA}(\Gamma) + \frac{2\Delta_\Gamma(k - \beta - 1)\sqrt{(\Delta_\Gamma + 1)\Delta_\Gamma}}{(\Delta_\Gamma + 1) + \Delta_\Gamma} \\
 &+ \frac{2\Delta_\Gamma\sqrt{(\Delta_\Gamma + 1)\Delta_\Gamma}}{(\Delta_\Gamma + 1) + \Delta_\Gamma} - \frac{2\Delta_\Gamma(k - \beta)\sqrt{\Delta_\Gamma\delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma}.
 \end{aligned} \tag{41}$$

We obtain

$$\begin{aligned}
 \text{GA}(A_\alpha^\beta(\Gamma_n^{k,l})) &\leq \frac{2k\sqrt{2}}{3} + \frac{2\alpha(k-\beta-1)\sqrt{1 * (\delta_\Gamma + \alpha + 1)}}{1 + (\delta_\Gamma + \alpha + 1)} + \frac{2(k-\beta-1)\sqrt{2(\delta_\Gamma + \alpha + 1)}}{2 + (\delta_\Gamma + \alpha + 1)} \\
 &\quad + \frac{2\alpha(\beta+1)\sqrt{1 * (\delta_\Gamma + (\beta+1)(1+\alpha))}}{1 + (\delta_\Gamma + (\beta+1)(1+\alpha))} \\
 &\quad + \frac{2(\beta+1)\sqrt{2(\delta_\Gamma + (\beta+1)(1+\alpha))}}{2 + (\delta_\Gamma + (\beta+1)(1+\alpha))} + \text{GA}(\Gamma) + \frac{2\Delta_\Gamma(k-\beta-1)\sqrt{(\Delta_\Gamma+1)\Delta_\Gamma}}{(\Delta_\Gamma+1) + \Delta_\Gamma} \\
 &\quad + \frac{2\Delta_\Gamma\sqrt{(\Delta_\Gamma+1)\Delta_\Gamma}}{(\Delta_\Gamma+1) + \Delta_\Gamma} - \frac{2\Delta_\Gamma(k-\beta)\sqrt{\Delta_\Gamma\delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma}.
 \end{aligned} \tag{42}$$

After simplification, we get the required result:

$$\begin{aligned}
 \text{GA}(A_\alpha^\beta(\Gamma_n^{k,l})) &\leq 2(k-\beta-1)\sqrt{\delta_\Gamma + \alpha + 1} \left(\frac{\alpha}{\delta_\Gamma + \alpha + 2} + \frac{\sqrt{2}}{\delta_\Gamma + \alpha + 3} \right) + k(l-2-\alpha) + \frac{2k\sqrt{2}}{3} \\
 &\quad + 2\sqrt{\delta_\Gamma + (\beta+1)(1+\alpha)} \left(\frac{\alpha(\beta+1)}{1 + \delta_\Gamma + (\beta+1)(1+\alpha)} + \frac{\sqrt{2}(\beta+1)}{2 + \delta_\Gamma + (\beta+1)(1+\alpha)} \right) \\
 &\quad + 2\Delta_\Gamma\sqrt{\Delta_\Gamma}(k-\beta) \left(\frac{\sqrt{\Delta_\Gamma+1}}{2\Delta_\Gamma+1} - \frac{\sqrt{\delta_\Gamma}}{\Delta_\Gamma + \delta_\Gamma} \right) + \text{GA}(\Gamma).
 \end{aligned} \tag{43}$$

Inequalities (40) and (43) complete the proof. $\square$

3. Conclusion

The study of mathematical aspect regarding topological indices is a partially open problem [20, 28, 29]. For which member family of graphs the certain index has a minimal or maximal value? In this work, we discussed, for this fundamental question, general graphs with pendent paths for the most studied index named geometric-arithmetic index GA and developed tight bounds by characterizing graphs. In Theorems 2 and 3, for the first time, we defined tight bonds for the transformed graphs under the effect of transformations defined in [27].

Data Availability

No data were used to support this study.

Disclosure

This paper has not been published elsewhere and it will not be submitted anywhere else for publication.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have equally contributed to the study.

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