Research Article

Linear Diophantine Uncertain Linguistic Power Einstein Aggregation Operators and Their Applications to Multiattribute Decision Making

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Linear Diophantine uncertain linguistic set (LDULS) is a modified variety of the fuzzy set (FS) to manage problematic and inconsistent information in actual life troubles. LDULS covers the grade of truth, grade of falsity, and their reference parameters with the uncertain linguistic term (ULT) with a rule $0 \leq \alpha_{AMG} u_{AMG}(x) + \beta_{ANG} v_{AMG}(x) \leq 1$, where $0 \leq \alpha_{AMG} + \beta_{ANG} \leq 1$. In this study, the principle of LDULS and their useful laws are elaborated. Additionally, the power Einstein (PE) aggregation operator (AO) is a conventional sort of AO utilized in innovative decision-making troubles, which is effective to aggregate the family of numerical elements. To determine the interrelationship between any numbers of arguments, we elaborate the linear Diophantine uncertain linguistic PE averaging (LDULPEA), linear Diophantine uncertain linguistic PE weighted averaging (LDULPEWA), linear Diophantine uncertain linguistic PE geometric (LDULPEG), and linear Diophantine uncertain linguistic PE weighted geometric (LDULPEWG) operators; then, we discuss their useful results. Conclusively, a decision-making methodology is utilized for the multiattribute decision-making (MADM) dilemma with elaborated information. A sensible illustration is specified to demonstrate the accessibility and rewards of the intended technique by comparison with certain prevailing techniques. The intended AOs are additional comprehensive than the prevailing ones to exploit the ambiguous and inaccurate knowledge. Numerous remaining operators are chosen as individual incidents of the suggested one. Ultimately, the supremacy and advantages of the elaborated operators are also discussed with the help of the geometrical form to show the validity and consistency of explored operators.

1. Introduction

MADM is a technique to discover the ideal option between a family of possibilities and a family of certain opinions. The perception of MADM has extensively been utilized in numerous disciplines. Ambiguity usually happened in genuine

life troubles owing to the contribution of some convoluted restrictions, unavailability of evidence/information, and unpredictability of the challenge. To survive with this worry, Zadeh [1] expounded the theory of the fuzzy set (FS), which covers the grade of truth $u_{AMG}(x)$ restricted to $[0, 1]$. When the theory of FS was elaborated, then numerous intellectuals

have exploited it in the natural environment of several areas. For instance, Molodtsov [2] initiated the soft sets, Mahmood [2] elaborated the new bipolar soft sets, Somjanta et al. [3] utilized the fuzzy sets in UP-algebra, Noppharat Dokkhamdang et al. [4] utilized the generalized fuzzy sets in UP-algebra, Tanamoon et al. [5] explored the Q-fuzzy sets and utilized in UP-algebra, Kawila et al. [6] explored the bipolar fuzzy UP-algebra, Mahmood and Hayat [7] initiated the characterizations of hemirings by their bipolar-valued fuzzy h-ideals, and Mahmood et al. [8] initiated the lattice ordered soft near ring. However, in certain situations, the principle of FS cannot be working effectively; for instance, when an individual face such sorts of information, which contains the truth and falsity grades, then FS failed. To survive with such circumstances, Atanassov [9] elaborated the theory of intuitionistic FS (IFS), which covers the grade of truth and falsity grades with the rule $0 \leq u_{\text{AMG}}(x) + v_{\text{ANG}}(x) \leq 1$. When the theory of IFS was elaborated, then numerous scholars have utilized it in the environment of several areas. For instance, Beg and Rashid [10] initiated the intuitionistic hesitant fuzzy sets, Atanassov [11] explored the interval-valued intuitionistic fuzzy sets, Kumari and Mishra [12] investigated the parametric measures for IFSs, Jana and Pal [13] explored bipolar intuitionistic fuzzy soft sets, Joshi and Kumar [14] developed the fuzzy time series model based on intuitionistic fuzzy sets, Fu et al. [15] proposed the correlation measures by using the interval-valued intuitionistic fuzzy sets, and Meng and He [16] elaborated the geometric interaction aggregation operators by using the intuitionistic fuzzy sets.

Again, in certain situations, the principle of IFS cannot be working effectively; for instance, when an individual faces such sorts of information which contains the truth and falsity grades with the rule $0 \leq u_{\text{AMG}}^2(x) + v_{\text{ANG}}^2(x) \leq 1$, then IFS failed. To survive with such circumstances, Yager [17] elaborated the theory of Pythagorean FS (PFS), which covers the grade of truth and falsity grades with the rule $0 \leq u_{\text{AMG}}^2(x) + v_{\text{ANG}}^2(x) \leq 1$. When the theory of PFS was elaborated, then numerous scholars have utilized it in the environment of several areas. For instance, Garg [18] explored the linguistic PFS, Wei and Wei [19] developed the similarity measures for PFS, Xiao and Ding [20] investigated the diverges measures by using the PFS and their application in decision making, Ullah et al. [21] explored some similarity measures based on a novel complex PFS and their application in pattern recognition, Li and Lu [22] proposed some similarity and distance measures for PFSs, Garg [23] proposed the novel improved accuracy function for interval-valued PFS and their application in decision making, and Yang and Hussain [24] investigated some entropy measures for PFS and their application.

Again, in certain situations, the principle of PFS cannot be working effectively; for instance, when an individual face such sorts of information which contains the truth and falsity grades with the rule $0 \leq u_{\text{AMG}}^q(x) + v_{\text{ANG}}^q(x) \leq 1$, then PFS failed. To survive with such circumstances, Yager [25] elaborated the theory of q-rung orthopair FS (q-ROFS), which covers the grade of truth and falsity grades with the rule $0 \leq u_{\text{AMG}}^q(x) + v_{\text{ANG}}^q(x) \leq 1$. When the theory of

q-ROFS was elaborated, then numerous scholars have utilized it in the environment of several areas; for instance, Ali [26] created another view on q-ROFSs. Liu and Wang [27] elaborated the aggregation operators for q-ROFSs. Peng and Liu [28] investigated the information measures for q-ROFSs. Wang et al. [29] explored the similarity measures for q-ROFSs. Ali and Mahmood [30] investigated the complex q-ROFSs and their Maclaurin symmetric mean operators. Liu et al. [31] proposed cosine similarity and distance measures for q-ROFSs. Liu and Wang [32] investigated the Archimedean Bonferroni mean operators for q-ROFSs. Liu et al. [33] explored the linguistic q-ROFSs. Garg [34] investigated the possibility degree for interval-valued q-ROFSs. The power aggregation operators for complex q-ROFSs were developed by Garg et al. [35].

To handle such sorts of concerns, Zadeh [36] investigated the theory of linguistic variable (LV) to describe the preferences of decision-makers. Moreover, the theory of a 2-tuple linguistic set was developed by Herrera and Martinez [37]. Liu and Jin [38] investigated the uncertain LV (ULV). In approximate actual existence troubles, the sum of truth and falsity grades to which an option filling an ascribe offered by the decision-maker could not hold the rule of IFS, PFS, and q-ROFSs; then, the theory of IFS and PFS fails in such situations. To survive with such circumstances, Riaz and Hashmi [39] elaborated the theory of linear Diophantine FS (LDFS), which covers the grade of truth and falsity grades and their reference parameters with the rule $0 \leq \alpha_{\text{AMG}} u_{\text{AMG}}(\varrho) + \beta_{\text{ANG}} v_{\text{AMG}}(\varrho) \leq 1$. When the theory of LDFS was elaborated, then numerous scholars have utilized it in the environment of several areas; for instance, Riaz et al. [40] discovered the theory of linear Diophantine fuzz soft rough sets and their applications. Some algebraic structures based on LDFs were developed by Kamaci [41]. But, up to date, no study explored the theory of LDULVs and their operational laws.

The concepts of intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PFSs), q-rung orthopair fuzzy sets (q-ROFSs), and linear Diophantine fuzzy sets have numerous applications in various fields of real life, but these theories have their own limitations related to the membership and nonmembership grades. To eradicate these restrictions, we introduce the novel concept of the linear Diophantine uncertain linguistic set (LDULS) with the addition of reference parameters and uncertain linguistic terms. The proposed model of LDULS is more efficient and flexible rather than other approaches due to the use of reference parameters and ULVs. LDULS also categorizes the data in MADM problems by changing the physical sense of reference parameters and ULVs. This set covers the spaces of existing structures and enlarge the space for membership and nonmembership grades with the help of reference parameters and ULVs. The motivation of the proposed model is given step by step in the whole manuscript. Now, we discuss some important objectives of this study.

- (1) The theory of LDULS is more generalized than IFSs, PFSs, q-ROFSs, LDFSs, and ULVs
- (2) If we choose the information in the form of (0.5, 0.6), then by using the condition of IFSs, that is, the sum

of both terms is limited to unit interval, but $0.5 + 0.6 = 1.1 > 1$, the theory of IFS has been failed for coping with such sorts of issues, and the theory of LDULS is very comfortable to resolve the above issues. For this, we choose the reference parameters such as $(0.1, 0.2)$; then, by using the condition of LDULS, $0.1 * 0.5 + 0.2 * 0.6 = 0.05 + 0.12 = 0.17 < 1$. We clear that the IFS is the special case of the proposed LDULS.

- (3) If we choose the information in the form of $(0.8, 0.9)$, then by using the condition of PFSs, that is, the sum of the square of both terms is limited to unit interval, but $0.8^2 + 0.9^2 = 0.64 + 0.81 = 1.45 > 1$, the theory of PFS has been failed for coping with such sorts of issues, and the theory of LDULS is very comfortable to resolve the above issues. For this, we choose the reference parameters such as $(0.2, 0.2)$; then, by using the condition of LDULS, $0.2 * 0.8 + 0.2 * 0.9 = 0.16 + 0.18 = 0.34 < 1$. We clear that the theory of PFS is the special case of the proposed LDULS.
- (4) If we choose the information in the form of $(0.1, 0.1)$, then by using the condition of q-ROFSs, that is, the sum of the q -powers of both terms is limited to unit interval, but $1 + 1 = 2 > 1$, the theory of q-ROFS has been failed for coping with such sorts of issues, and the theory of LDULS is very comfortable to resolve the above issues. For this, we choose the reference parameters such as $(0, 0.1)$; then, by using the condition of LDULS, $0.0 * 1 + 0.1 * 1 = 0 + 0.1 = 0.1 < 1$. We clear that the theory of q-ROFS is the special case of the proposed LDULS.
- (5) If we choose the information in the form of $([s_1, s_2], (0.5, 0.3), (0.5, 0.4))$, then using the condition of IFSs, PFSs, q-ROFSs, and LDFS has been failed; for coping with such sorts of issues, the theory of LDULS is very proficient and reliable technique to resolve with it. From the above analysis, the theories of IFSs, PFSs, q-ROFSs, and LDFSs are the special case of the proposed LDULS.

The opinions of IFSs, PFSs, q-ROFSs, and LDFSs have repeated applications in uncountable fields of genuine existence, but these philosophies have their shortcomings associated with the truth and falsity grades. To eliminate these constraints, we announce the narrative hypothesis of LDULS with the supplement of situation parameters. The suggested version of LDULS is additionally inexpensive and accommodating more accurately than other methodologies expected to the usage of suggestion parameters. LDULS also compartmentalizes the information in MADM troubles by modifying the physical meaning of orientation parameters. This set encompasses the areas of accessible assemblies and expands the space for truth and falsity grades with the help of reference parameters. The inspiration of the suggested pattern is offered step by step in the entire script.

The rest of this manuscript as follows. In Section 2, we briefly recall some definitions such as LDFSs, ULSs, and their operational laws. The theory of power aggregation (PA) operator is also reviewed. In Section 3, we notified the novel idea of LDULS and elaborated their fundamental laws. In Section 4, we elaborated the LDULPEA, LDULPEWA, LDULPEG, and LDULPEWG operators and their properties are also discovered. In Section 5, by using these operators, we acquire a MADM procedure based on elaborated operators. To determine the consistency and validity of the elaborated operators, we illustrate some examples by using explored operators. Finally, the superiority and comparative analysis of the elaborated operators with some existing operators are also determined and justify with the help of a graphical point of view. In Section 6, we discussed the conclusion of this study.

As shown above, the advantages of the operators and keeping the superiority of the elaborated approaches, the main points of the elaborated approaches are discussed as shown in Figure 1.

2. Preliminaries

Certain scholars have proposed numerous theories such as fuzzy sets, intuitionistic fuzzy sets, and their extensions to cope with awkward and complicated information in real-life issues. But these theories have their own limitations related to the membership and nonmembership grades. To eradicate these restrictions, the theory of the linear Diophantine fuzzy set (LDFS) with the addition of reference parameters was explored by Riaz and Hashmi [39]. The proposed model of LDFS is more efficient and flexible rather than other approaches due to the use of reference parameters. LDFS also categorize the data in MADM problems by changing the physical sense of reference parameters. This set covers the spaces of existing structures and enlarge the space for membership and nonmembership grades with the help of reference parameters. Keeping the advantages of the existing LDFS, in this section, we briefly recall some definitions such as LDFSs, ULSs, and their operational laws. The theory of the PA operator is also reviewed. In an overall study, the universal set is denoted by X .

Definition 1 (see [39]). A LDFS A_{LD} is elaborated by

$$A_{LD} = \left\{ \frac{(x, (u_{AMG}(x), v_{ANG}(x)), (\alpha_{AMG}, \beta_{ANG}))}{x \in X} \right\}, \quad (1)$$

with a rule $0 \leq \alpha_{AMG}u_{AMG}(x) + \beta_{ANG}v_{ANG}(x) \leq 1$, with $0 \leq \alpha_{AMG} + \beta_{ANG} \leq 1$. The symbol $\xi_{A_{LD}}\pi(x)_{A_{LD}} = 1 - (\alpha_{AMG}u_{AMG}(x) + \beta_{ANG}v_{ANG}(x))$ expressed the refusal grade. Simply, $A_{LD} = ((u_{AMG}(x), t_{v_{ANG}}(x)), t(\alpha_{AMG}, \beta_{ANG}))$ is called linear Diophantine fuzzy number (LDFN). To determine the interrelationships between any numbers of attributes, we revised some algebraic operation laws which are very important for proposed work in the next study.

FIGURE 1: Geometrical expressions of the explored approaches.

Definition 2 (see [39]). For any two LDFNs $A_{LD} = ((u_{AMG}(x), tv_{ANG}n(x)), t(\alpha_{AMG}, \beta_{ANG}))$ and $B_{LD} = ((u_{BMG}(x), tv_{BNG}n(x)), t(\alpha_{BMG}, \beta_{BNG}))$,

$$A_{LD} \oplus B_{LD} = ((u_{AMG} + u_{BMG} - u_{AMG}u_{BMG}, v_{AMG}v_{BNG}), (\alpha_{AMG} + \alpha_{BMG} - \alpha_{AMG}\alpha_{BMG}, \beta_{ANG}\beta_{BNG})), \quad (2)$$

$$A_{LD} \otimes B_{LD} = ((u_{AMG}u_{BMG}, v_{AMG} + v_{BNG} - v_{AMG}v_{BNG}), (\alpha_{AMG}\alpha_{BMG}, \beta_{ANG} + \beta_{BNG} - \beta_{ANG}\beta_{BNG})), \quad (3)$$

$$\lambda A_{LD} = \left(\left((1 - (1 - u_{AMG}(x))^\lambda, (v_{ANG}(x))^\lambda), (1 - (1 - \alpha_{AMG})^\lambda, (\beta_{ANG}))^\lambda \right) \right), \quad (4)$$

$$A_{LD}^\lambda = \left(\left((u_{AMG}(x))^\lambda, 1 - (1 - (v_{ANG}(x))^\lambda) \right), ((\alpha_{AMG})^\lambda, 1 - (1 - \beta_{ANG})^\lambda) \right). \quad (5)$$

Furthermore, by using any number of LDFNs, we revised the score values for determining the ordering between any two LDFNs.

Definition 3 (see [39]). For any LDFN $A_{LD_1} = ((u_{AMG_1}(x), tv_{ANG_1}n(x)), t(\alpha_{AMG_1}, \beta_{ANG_1}))$, the score value (SV) is elaborated by

$$s_{A_{LD}} = s(A_{LD}) = \frac{1}{2} (u_{AMG}(x) - v_{ANG}(x) + (\alpha + \beta)). \quad (6)$$

Furthermore, if equation (6) has been failed by comparing any number of LDFNs, then we used the technique of accuracy function, so, for this, we revised the accuracy values for determining the ordering between any two LDFNs.

Definition 4 (see [39]). For any LDFN $A_{LD_1} = ((u_{AMG_1}(x), tv_{ANG_1}n(x)), t(\alpha_{AMG_1}, \beta_{ANG_1}))$, the accuracy value (AV) is elaborated by

$$a_{A_{LD}} = a(A_{LD}) = \frac{1}{2} \left(\frac{(u_{AMG}(x) + v_{ANG}(x))}{2} + (\alpha + \beta) \right). \quad (7)$$

Definition 5 (see [39]). For any two LDFNs $A_{LD} = ((u_{AMG}(x), tv_{ANG}n(x)), t(\alpha_{AMG}, \beta_{ANG}))$ and $B_{LD} = ((u_{BMG}(x), tv_{BNG}n(x)), t(\alpha_{BMG}, \beta_{BNG}))$,

(1) If $s_{A_{LD}} < s_{B_{LD}}$, then $A_{LD} < B_{LD}$

(2) If $s_{A_{LD}} > s_{B_{LD}}$, then $A_{LD} > B_{LD}$

(3) If $s_{A_{LD}} = s_{B_{LD}}$, then

(1) If $a_{A_{LD}} < a_{B_{LD}}$, then $A_{LD} < B_{LD}$

(2) If $a_{A_{LD}} > a_{B_{LD}}$, then $A_{LD} > B_{LD}$

(3) If $a_{A_{LD}} = a_{B_{LD}}$, then $A_{LD} \approx B_{LD}$

When faced with problems that are too complex or ill-defined to be solved by quantitative expressions, linguistic variables can be an effective tool because the use of linguistic information enhances the reliability and flexibility of classical decision models. To handle such sorts of concerns, Zadeh [36] investigated the theory of linguistic variable (LV) to describe the preferences of decision-makers. Moreover, the theory of a 2-tuple linguistic set was developed by Herrera and Martinez [37]. Liu and Jin [38] investigated the uncertain LV (ULV).

Definition 6 (see [36]). A LTS is elaborated by

$$S = \{s_1, s_2, \dots, s_{l-1}\}, \quad (8)$$

where l should be odd and hold the following conditions:

- (1) If $l > l'$, then $s_l > s_{l'}$
- (2) The negative operator $\text{neg}(s_l) = s_{l'}$ with a condition $l + l' = l + 1$
- (3) If $l \geq l'$, $\max(s_l, s_{l'}) = s_l$, and if $l \leq l'$, $\max(s_l, s_{l'}) = s_{l'}$.

Let $\tilde{s} = [s_a, s_b]$ where $s_a, s_b \in S$, s_a and s_b are the lower and the upper limits, respectively. We call $\tilde{s}$ the uncertain linguistic variable. To determine the interrelationships between any numbers of attributes, we revised some operation laws for ULVs which are very important for the proposed work in next study.

Definition 7 (see [38]). For any two ULVs $\tilde{s}_1 = [s_{a1}, s_{b1}]$ and $\tilde{s}_2 = [s_{a2}, s_{b2}]$,

$$\begin{aligned} \tilde{s}_1 \oplus \tilde{s}_2 &= [s_{a1}, s_{b1}] \oplus [s_{a2}, s_{b2}] = [s_{a1+a2}, s_{b1+b2}], \\ \tilde{s}_1 \otimes \tilde{s}_2 &= [s_{a1}, s_{b1}] \otimes [s_{a2}, s_{b2}] = [s_{a1 \times a2}, s_{b1 \times b2}], \\ \lambda \tilde{s}_1 &= \lambda [s_{a1}, s_{b1}] = [s_{\lambda \times a1}, s_{\lambda \times b1}], \\ (\tilde{s}_1)^\lambda &= [s_{(a1)^\lambda}, s_{(b1)^\lambda}]. \end{aligned} \quad (9)$$

Definition 8 (see [33]). The PA operator is elaborated by

$$\text{PA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) = \frac{\sum_{i=1}^n (1 + T(A_{\text{LD}-i})) * A_{\text{LD}-i}}{\sum_{i=1}^n (1 + T(A_{\text{LD}-i}))}, \quad (10)$$

where $T(A_{\text{LD}-i}) = \sum_{j=1}^n \sup(A_{\text{LD}-i}, A_{\text{LD}-j})$, and $\sup(A_{\text{LD}-i}, A_{\text{LD}-j})$ means the support for $A_{\text{LD}-i}$ from $A_{\text{LD}-j}$, which holds the following rules:

- (1) $\sup(A_{\text{LD}-i}, A_{\text{LD}-j}) = \sup(A_{\text{LD}-j}, A_{\text{LD}-i})$
- (2) $\sup(A_{\text{LD}-i}, A_{\text{LD}-j}) \in [0, 1]$
- (3) $\sup(A_{\text{LD}-i}, A_{\text{LD}-j}) \geq \sup(A_{\text{LD}-k}, A_{\text{LD}-l})$, if $|A_{\text{LD}-i}, A_{\text{LD}-j}| \leq |A_{\text{LD}-k}, A_{\text{LD}-l}|$

3. Linear Diophantine Uncertain Linguistic Set

As shown above, the existing theories are very useful in the circumstances of investigation of any theories. In this study, by using the existing theories, we elaborated on the novel concept of LDULSs and their operational laws. These investigated theories are also justifying with the help of examples.

Definition 9. A LDULS A_{LDULS} is elaborated by

$$A_{\text{LDULS}} = \left\{ \frac{x \left[[s_{\theta(X)}, s_{\tau(X)}], (u_{\text{AMG}}(x), v_{\text{ANG}}(x)), (\alpha_{\text{AMG}}, \beta_{\text{ANG}}) \right]}{x \in X} \right\}, \quad (11)$$

with the rule $0 \leq \alpha_{\text{AMG}} u_{\text{AMG}}(x) + \beta_{\text{ANG}} v_{\text{ANG}}(x) \leq 1$, with $0 \leq \alpha_{\text{AMG}} + \beta_{\text{ANG}} \leq 1$. The symbol $\xi_{A_{\text{LD}}} \pi(x)_{A_{\text{LD}}} = 1 - (\alpha_{\text{AMG}} u_{\text{AMG}}(x) + \beta_{\text{ANG}} v_{\text{ANG}}(x))$ expressed the refusal grade with $s_{\theta(X)}, s_{\tau(X)} \in S$. Simply, $A_{\text{LD}} = ([s_{\theta(X)}, s_{\tau(X)}], (u_{\text{AMG}}(x), v_{\text{ANG}}(x)), (\alpha_{\text{AMG}}, \beta_{\text{ANG}}))$ expresses the linear Diophantine uncertain linguistic number (LDULN).

Definition 10. For any two LDULNs $A_{\text{LD}} = ([s_{A-\theta(X)}, s_{A-\tau(X)}], (u_{\text{AMG}}(x), v_{\text{ANG}}(x)), (\alpha_{\text{AMG}}, \beta_{\text{ANG}}))$ and $B_{\text{LD}} = ([s_{B-\theta(X)}, s_{B-\tau(X)}], (u_{\text{BMG}}(x), v_{\text{BNG}}(x)), (\alpha_{\text{BMG}}, \beta_{\text{BNG}}))$, then

$$\begin{aligned} A_{\text{LD}} \oplus B_{\text{LD}} &= ([s_{A-\theta(X)+B-\theta(X)}, s_{A-\tau(X)+B-\tau(X)}], (u_{\text{AMG}} + u_{\text{BMG}} - u_{\text{AMG}}u_{\text{BMG}}, v_{\text{AMG}}v_{\text{BMG}}), (\alpha_{\text{AMG}} + \alpha_{\text{BMG}} - \alpha_{\text{AMG}}\alpha_{\text{BMG}}, \beta_{\text{ANG}}\beta_{\text{BNG}})), \\ A_{\text{LD}} \otimes B_{\text{LD}} &= ([s_{A-\theta(X)*B-\theta(X)}, s_{A-\tau(X)*B-\tau(X)}], (u_{\text{AMG}}u_{\text{BMG}}, v_{\text{AMG}} + v_{\text{BMG}} - v_{\text{AMG}}v_{\text{BMG}}), (\alpha_{\text{AMG}}\alpha_{\text{BMG}}, \beta_{\text{ANG}} + \beta_{\text{BNG}} - \beta_{\text{ANG}}\beta_{\text{BNG}})), \\ \lambda A_{\text{LD}} &= ([s_{\lambda \times A-\theta(X)}, s_{\lambda \times A-\tau(X)}], ((1 - (1 - u_{\text{AMG}}(x))^\lambda), (v_{\text{ANG}}(x))^\lambda), (1 - (1 - \alpha_{\text{AMG}})^\lambda, \beta_{\text{ANG}})^\lambda)), \\ A_{\text{LD}}^\lambda &= ([s_{(A-\theta(X))^\lambda}, s_{(A-\tau(X))^\lambda}], ((u_{\text{AMG}}(x))^\lambda, 1 - (1 - (v_{\text{ANG}}(x))^\lambda)), ((\alpha_{\text{AMG}})^\lambda, 1 - (1 - \beta_{\text{ANG}})^\lambda)). \end{aligned} \quad (12)$$

Example 1. For any two LDULNs $A_{\text{LD}} = ([s_1, s_2], (0.5, 0.3), (0.5, 0.4))$ and $B_{\text{LD}} = ([s_2, s_3], (0.7, 0.5), (0.1, 0.3))$, with $\lambda = 2$, then by using Definition 10, we get

$$\begin{aligned}
A_{LD} \oplus B_{LD} &= ([s_3, s_5], (0.42, 0.15), (0.55, 0.12)), \\
A_{LD} \otimes B_{LD} &= ([s_2, s_6], (0.15, 0.42), (0.12, 0.55)), \\
2A_{LD} &= ([s_2, s_4], (0.75, 0.09), (0.75, 0.16)), \\
A_{LD}^2 &= ([s_1, s_4], (0.09, 0.75), (0.16, 0.75)).
\end{aligned} \tag{13}$$

Theorem 1. For any two LDULNs $A_{LD} = ([s_{A-\theta(X)}, s_{A-\tau(X)}], (u_{AMG}(x), v_{ANG}(x)), (\alpha_{AMG}, \beta_{ANG}))$ and $B_{LD} = ([s_{B-\theta(X)}, s_{B-\tau(X)}], (u_{BMG}(x), v_{BNG}(x)), (\alpha_{BMG}, \beta_{BNG}))$,

- (1) $A_{LD} \oplus B_{LD} = B_{LD} \oplus A_{LD}$
- (2) $A_{LD} \otimes B_{LD} = B_{LD} \otimes A_{LD}$
- (3) $\lambda(A_{LD} \oplus B_{LD}) = \lambda A_{LD} \oplus \lambda B_{LD}, \lambda > 0$
- (4) $\lambda_1 A_{LD} \oplus \lambda_2 A_{LD} = (\lambda_1 + \lambda_2) A_{LD}, \lambda_1, \lambda_2 > 0$
- (5) $A_{LD}^{\lambda_1} \otimes A_{LD}^{\lambda_2} = A_{LD}^{\lambda_1 + \lambda_2}, \lambda_1, \lambda_2 > 0$
- (6) $A_{LD}^{\lambda_1} \otimes A_{LD}^{\lambda_2} = (A_{LD} \otimes B_{LD})^{\lambda_1}, \lambda_1 > 0$

Definition 11. For any LDULN $A_{LD} = ([s_{A-\theta(X)}, s_{A-\tau(X)}], (u_{AMG}(x), v_{ANG}(x)), (\alpha_{AMG}, \beta_{ANG}))$, the expected value (EV) is elaborated by

$$E(A_{LD}) = s((A-\theta(X)+A-\tau(X)) \times t(1/2)n(u_{AMG}(x)+1-v_{ANG}(x)+(\alpha+\beta)))/4. \tag{14}$$

Example 2. For any LDULN $A_{LD} = ([s_1, s_2], (0.5, 0.3), (0.5, 0.4))$, the expected value (EV) is elaborated as

$$E(A_{LD}) = s((1+2) \times (1/2)(0.5+1-0.3+0.5+0.4))/4 = s((3) \times 1.05)/4 = s_{0.7875}. \tag{15}$$

Definition 12. For any LDULN $A_{LD} = ([s_{A-\theta(X)}, s_{A-\tau(X)}], (u_{AMG}(x), v_{ANG}(x)), (\alpha_{AMG}, \beta_{ANG}))$, the accuracy value (AV) is elaborated by

$$\varsigma_{A_{LD}} = \varsigma(A_{LD}) = s((u(a_1)+v(a_1)) \times (1/2)(u_{AMG}(x)+t v_{ANG}(x)q+h(\alpha+\beta)))/4. \tag{16}$$

Definition 13. For any two LDULNs $A_{LD} = ([s_{A-\theta(X)}, s_{A-\tau(X)}], (u_{AMG}(x), v_{ANG}(x)), (\alpha_{AMG}, \beta_{ANG}))$ and $B_{LD} = ([s_{B-\theta(X)}, s_{B-\tau(X)}], (u_{BMG}(x), v_{BNG}(x)), (\alpha_{BMG}, \beta_{BNG}))$, then

- (1) If $\varsigma_{A_{LD1}} < \varsigma_{A_{LD2}}$, then $A_{LD1} < A_{LD2}$
- (2) If $\varsigma_{A_{LD1}} > \varsigma_{A_{LD2}}$, then $A_{LD1} > A_{LD2}$
- (3) If $\varsigma_{A_{LD1}} = \varsigma_{A_{LD2}}$, then
 - (1) If $\varrho_{A_{LD1}} < \varrho_{A_{LD2}}$, then $A_{LD1} < A_{LD2}$
 - (2) If $\varrho_{A_{LD1}} > \varrho_{A_{LD2}}$, then $A_{LD1} > A_{LD2}$
 - (3) If $\varrho_{A_{LD1}} = \varrho_{A_{LD2}}$, then $A_{LD1} \approx A_{LD2}$

4. The Power Aggregation (PA) Operator

Here, we review the basic Einstein t-norm and t-conorm, which are useful for the elaborated approaches.

- (1) $a \otimes_\epsilon b = (a + b) / (1 + a \cdot b), a, b \in [0, 1]$
- (2) $a \oplus_\epsilon b = (a \cdot b) / (1 + (1 - a) \cdot (1 - b)), a, b \in [0, 1]$

By using any two LDULNs $A_{LD} = ([s_{A-\theta(X)}, s_{A-\tau(X)}], (u_{AMG}(x), v_{ANG}(x)), (\alpha_{AMG}, \beta_{ANG}))$ and $B_{LD} = ([s_{B-\theta(X)}, s_{B-\tau(X)}], (u_{BMG}(x), v_{BNG}(x)), (\alpha_{BMG}, \beta_{BNG}))$,

$$\begin{aligned}
A_{LD} \otimes_\epsilon B_{LD} &= \left([s_{A-\theta(X)+B-\theta(X)}, s_{A-\tau(X)+B-\tau(X)}], \left(\frac{u_{AMG}(x) \cdot u_{BMG}(x)}{1 + (1 - u_{AMG}(x)) \cdot (1 - u_{BMG}(x))}, \frac{v_{ANG}(x) + v_{BNG}(x)}{1 + v_{ANG}(x) \cdot v_{BNG}(x)} \right), \right. \\
&\quad \cdot \left(\frac{\alpha_{AMG} \cdot \alpha_{BMG}}{1 + (1 - \alpha_{AMG}) \cdot (1 - \alpha_{BMG})}, \frac{\beta_{ANG} + \beta_{BNG}}{1 + \beta_{ANG} \cdot \beta_{BNG}} \right) \Big), \\
A_{LD} \oplus_\epsilon B_{LD} &= \left([s_{A-\theta(X)+B-\theta(X)}, s_{A-\tau(X)+B-\tau(X)}], \left(\frac{u_{AMG}(x) + u_{BMG}(x)}{1 + u_{AMG}(x) \cdot u_{BMG}(x)}, \frac{v_{ANG}(x) \cdot v_{BNG}(x)}{1 + (1 - v_{ANG}(x)) \cdot (1 - v_{BNG}(x))} \right), \right. \\
&\quad \cdot \left(\frac{\alpha_{AMG} + \alpha_{BMG}}{1 + \alpha_{AMG} \cdot \alpha_{BMG}}, \frac{\beta_{ANG} \cdot \beta_{BNG}}{1 + (1 - \beta_{ANG}) \cdot (1 - \beta_{BNG})} \right) \Big), \\
\lambda A_{LD} &= \left([s_{\lambda \cdot A-\theta(X)}, s_{\lambda \cdot A-\tau(X)}], \left(\frac{(1 + u_{AMG}(x))^\lambda - (1 - u_{AMG}(x))^\lambda}{(1 + u_{AMG}(x))^\lambda + (1 - u_{AMG}(x))^\lambda}, \frac{2(v_{ANG}(x))^\lambda}{(2 - v_{ANG}(x))^\lambda + (v_{ANG}(x))^\lambda} \right), \right. \\
&\quad \cdot \left(\frac{(1 + \alpha_{AMG})^\lambda - (1 - \alpha_{AMG})^\lambda}{(1 + \alpha_{AMG})^\lambda + (1 - \alpha_{AMG})^\lambda}, \frac{2(\beta_{ANG})^\lambda}{(2 - \beta_{ANG})^\lambda + (\beta_{ANG})^\lambda} \right) \Big),
\end{aligned}$$

$$A_{LD}^\lambda = \left(\left[s_{A-\theta(X)}^\lambda, s_{A-\tau(X)}^\lambda \right], \left(\frac{2(u_{AMG}(x))^\lambda}{(2 - u_{AMG}(x))^\lambda + (u_{AMG}(x))^\lambda}, \frac{(1 + v_{ANG}(x))^\lambda - (1 - v_{ANG}(x))^\lambda}{(1 + v_{ANG}(x))^\lambda + (1 - v_{ANG}(x))^\lambda} \right), \right. \\ \left. \cdot \left(\frac{2(\alpha_{AMG})^\lambda}{(2 - \alpha_{AMG})^\lambda + (\alpha_{AMG})^\lambda}, \frac{(1 + \beta_{ANG})^\lambda - (1 - \beta_{ANG})^\lambda}{(1 + \beta_{ANG})^\lambda + (1 - \beta_{ANG})^\lambda} \right) \right). \quad (17)$$

Theorem 2. For any two LDULNs $A_{LD} = ([s_{A-\theta(X)}, s_{A-\tau(X)}], (u_{AMG}(x), v_{ANG}(x)), (\alpha_{AMG}, \beta_{ANG}))$ and $B_{LD} = ([s_{B-\theta(X)}, s_{B-\tau(X)}], (u_{BMG}(x), v_{BNG}(x)), (\alpha_{BMG}, \beta_{BNG}))$, then

- (1) $A_{LD} \otimes_\varepsilon B_{LD} = B_{LD} \otimes_\varepsilon A_{LD}$
- (2) $A_{LD} \oplus_\varepsilon B_{LD} = B_{LD} \oplus_\varepsilon A_{LD}$
- (3) $\lambda(A_{LD} \otimes_\varepsilon B_{LD}) = \lambda A_{LD} \otimes_\varepsilon \lambda B_{LD}, \lambda > 0$

$$(4) \lambda_1 A_{LD} \otimes_\varepsilon \lambda_2 A_{LD} = (\lambda_1 + \lambda_2) \otimes_\varepsilon A_{LD}, \lambda_1, \lambda_2 > 0$$

$$(5) A_{LD}^{\lambda_1} \otimes_\varepsilon A_{LD}^{\lambda_2} = A_{LD}^{\lambda_1 + \lambda_2}, \lambda_1, \lambda_2 > 0$$

$$(6) A_{LD}^{\lambda_1} \otimes_\varepsilon A_{LD}^{\lambda_2} = (A_{LD} \otimes_\varepsilon B_{LD})^{\lambda_1}, \lambda_1 > 0$$

Proof. The proof of the first two parts is trivial. Additionally, we prove that part 3 is

$$A_{LD} \otimes_\varepsilon B_{LD} = \left(\left[s_{A-\theta(X)+B-\theta(X)}, s_{A-\tau(X)+B-\tau(X)} \right], \left(\frac{u_{AMG}(x) + u_{BMG}(x)}{1 + u_{AMG}(x) \cdot u_{BMG}(x)}, \frac{v_{ANG}(x) \cdot v_{BNG}(x)}{1 + (1 - v_{ANG}(x)) \cdot (1 - v_{BNG}(x))} \right), \right. \\ \left. \cdot \left(\frac{\alpha_{AMG} + \alpha_{BMG}}{1 + \alpha_{AMG} \cdot \alpha_{BMG}}, \frac{\beta_{ANG} \cdot \beta_{BNG}}{1 + (1 - \beta_{ANG}) \cdot (1 - \beta_{BNG})} \right) \right); \quad (18)$$

then,

$$\lambda(A_{LD} \otimes_\varepsilon B_{LD}) = \left(\left[s_{\lambda(A-\theta(X)+B-\theta(X))}, s_{\lambda(A-\tau(X)+B-\tau(X))} \right], \right. \\ \left(\frac{((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda - ((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda}{((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda + ((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda}, \right. \\ \left. \cdot \frac{2(v_{ANG}(x) \cdot v_{BNG}(x))^\lambda}{(4 - 2v_{ANG}(x) - 2v_{BNG}(x) + v_{ANG}(x) \cdot v_{BNG}(x))^\lambda + (v_{ANG}(x) \cdot v_{BNG}(x))^\lambda}, \right. \\ \left. \left(\frac{((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda - ((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda}{((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda + ((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda}, \frac{2(\beta_{ANG} \cdot \beta_{BNG})^\lambda}{(4 - 2\beta_{ANG} - 2\beta_{BNG} + \beta_{ANG} \cdot \beta_{BNG})^\lambda + (\beta_{ANG} \cdot \beta_{BNG})^\lambda} \right) \right). \quad (19)$$

For the right hand of the part (3),

$$\lambda A_{LD} = \left(\left[s_{\lambda \cdot A-\theta(X)}, s_{\lambda \cdot A-\tau(X)} \right], \left(\frac{(1 + u_{AMG}(x))^\lambda - (1 - u_{AMG}(x))^\lambda}{(1 + u_{AMG}(x))^\lambda + (1 - u_{AMG}(x))^\lambda}, \frac{2(v_{ANG}(x))^\lambda}{(2 - v_{ANG}(x))^\lambda + (v_{ANG}(x))^\lambda} \right), \right. \\ \left. \cdot \left(\frac{(1 + \alpha_{AMG})^\lambda - (1 - \alpha_{AMG})^\lambda}{(1 + \alpha_{AMG})^\lambda + (1 - \alpha_{AMG})^\lambda}, \frac{2(\beta_{ANG})^\lambda}{(2 - \beta_{ANG})^\lambda + (\beta_{ANG})^\lambda} \right) \right),$$

$$\lambda B_{LD} = \left(\left[s_{\lambda \cdot B - \theta(X)}, s_{\lambda \cdot B - \tau(X)} \right], \left(\frac{(1 + u_{BMG}(x))^\lambda - (1 - u_{BMG}(x))^\lambda}{(1 + u_{BMG}(x))^\lambda + (1 - u_{BMG}(x))^\lambda}, \frac{2(v_{BNG}(x))^\lambda}{(2 - v_{BNG}(x))^\lambda + (v_{BNG}(x))^\lambda} \right) \right. \\ \left. \cdot \left(\frac{(1 + \alpha_{BMG})^\lambda - (1 - \alpha_{BMG})^\lambda}{(1 + \alpha_{BMG})^\lambda + (1 - \alpha_{BMG})^\lambda}, \frac{2(\beta_{BNG})^\lambda}{(2 - \beta_{BNG})^\lambda + (\beta_{BNG})^\lambda} \right) \right) \right). \quad (20)$$

Then,

$$\lambda(A_{LD} \otimes_\varepsilon B_{LD}) = \left(\left[s_{\lambda(A - \theta(X) + B - \theta(X))}, s_{\lambda(A - \tau(X) + B - \tau(X))} \right], \right. \\ \left(\frac{((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda - ((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda}{((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda + ((1 + u_{AMG}(x))(1 + u_{BMG}(x)))^\lambda}, \right. \\ \left. \cdot \frac{2(v_{ANG}(x) \cdot v_{BNG}(x))^\lambda}{(4 - 2v_{ANG}(x) - 2v_{BNG}(x) + v_{ANG}(x) \cdot v_{BNG}(x))^\lambda + (v_{ANG}(x) \cdot v_{BNG}(x))^\lambda} \right), \\ \left(\frac{((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda - ((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda}{((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda + ((1 + \alpha_{AMG})(1 + \alpha_{BMG}))^\lambda}, \frac{2(\beta_{ANG} \cdot \beta_{BNG})^\lambda}{(4 - 2\beta_{ANG} - 2\beta_{BNG} + \beta_{ANG} \cdot \beta_{BNG})^\lambda + (\beta_{ANG} \cdot \beta_{BNG})^\lambda} \right) \right). \quad (21)$$

Hence, we get $\lambda(A_{LD} \otimes_\varepsilon B_{LD}) = \lambda A_{LD} \otimes_\varepsilon \lambda B_{LD}$, $\lambda > 0$.
Similarly, we prove that part 4. Know, for part (5), we have

$$A_{LD}^{\lambda_1} = \left(\left[s_{A - \theta(X)^{\lambda_1}}, s_{A - \tau(X)^{\lambda_1}} \right], \left(\frac{2(u_{AMG}(x))^{\lambda_1}}{(2 - u_{AMG}(x))^{\lambda_1} + (u_{AMG}(x))^{\lambda_1}}, \frac{(1 + v_{ANG}(x))^{\lambda_1} - (1 - v_{ANG}(x))^{\lambda_1}}{(1 + v_{ANG}(x))^{\lambda_1} + (1 - v_{ANG}(x))^{\lambda_1}} \right) \right. \\ \left. \cdot \left(\frac{2(\alpha_{AMG})^{\lambda_1}}{(2 - \alpha_{AMG})^{\lambda_1} + (\alpha_{AMG})^{\lambda_1}}, \frac{(1 + \beta_{ANG})^{\lambda_1} - (1 - \beta_{ANG})^{\lambda_1}}{(1 + \beta_{ANG})^{\lambda_1} + (1 - \beta_{ANG})^{\lambda_1}} \right) \right), \quad (22)$$

$$A_{LD}^{\lambda_2} = \left(\left[s_{A - \theta(X)^{\lambda_2}}, s_{A - \tau(X)^{\lambda_2}} \right], \left(\frac{2(u_{AMG}(x))^{\lambda_2}}{(2 - u_{AMG}(x))^{\lambda_2} + (u_{AMG}(x))^{\lambda_2}}, \frac{(1 + v_{ANG}(x))^{\lambda_2} - (1 - v_{ANG}(x))^{\lambda_2}}{(1 + v_{ANG}(x))^{\lambda_2} + (1 - v_{ANG}(x))^{\lambda_2}} \right) \right. \\ \left. \cdot \left(\frac{2(\alpha_{AMG})^{\lambda_2}}{(2 - \alpha_{AMG})^{\lambda_2} + (\alpha_{AMG})^{\lambda_2}}, \frac{(1 + \beta_{ANG})^{\lambda_2} - (1 - \beta_{ANG})^{\lambda_2}}{(1 + \beta_{ANG})^{\lambda_2} + (1 - \beta_{ANG})^{\lambda_2}} \right) \right).$$

Then,

$$\begin{aligned}
A_{LD}^{\lambda_1} \otimes_E A_{LD}^{\lambda_2} = & \left([S_{A-\theta(X)}\lambda_1 + \lambda_2, S_{A-\tau(X)}\lambda_1 + \lambda_2], \right. \\
& \left(\frac{(u_{AMG}(x))^{\lambda_1+\lambda_2}}{(2 - u_{AMG}(x))^{\lambda_2} (u_{AMG}(x))^{\lambda_1} + (2 - u_{AMG}(x))^{\lambda_1} (u_{AMG}(x))^{\lambda_2}}, \right. \\
& \quad \cdot \frac{2(1 + v_{ANG}(x))^{\lambda_1+\lambda_2}}{((1 + v_{ANG}(x))^{\lambda_1} + (1 - v_{ANG}(x))^{\lambda_1})((1 + v_{ANG}(x))^{\lambda_2} + (1 - v_{ANG}(x))^{\lambda_2})}), \\
& \left. \left(\frac{(\alpha_{ANG})^{\lambda_1+\lambda_2}}{(2 - \alpha_{AMG})^{\lambda_2} (\alpha_{AMG})^{\lambda_1} + (2 - \alpha_{AMG})^{\lambda_1} (\alpha_{AMG})^{\lambda_2}}, \frac{2(1 + \beta_{ANG})^{\lambda_1+\lambda_2}}{((1 + \beta_{ANG})^{\lambda_1} + (1 + \beta_{ANG})^{\lambda_1})((1 + \beta_{ANG})^{\lambda_2} + (1 + \beta_{ANG})^{\lambda_2})}, \right) \right). \tag{23}
\end{aligned}$$

For the right hand of the part (5), we get

$$\begin{aligned}
A_{LD}^{\lambda_1+\lambda_2} = & \left([S_{A-\theta(X)}\lambda_1 + \lambda_2, S_{A-\tau(X)}\lambda_1 + \lambda_2], \right. \\
& \left(\frac{(u_{AMG}(x))^{\lambda_1+\lambda_2}}{(2 - u_{AMG}(x))^{\lambda_2} (u_{AMG}(x))^{\lambda_1} + (2 - u_{AMG}(x))^{\lambda_1} (u_{AMG}(x))^{\lambda_2}}, \right. \\
& \quad \cdot \frac{2(1 + v_{ANG}(x))^{\lambda_1+\lambda_2}}{((1 + v_{ANG}(x))^{\lambda_1} + (1 - v_{ANG}(x))^{\lambda_1})((1 + v_{ANG}(x))^{\lambda_2} + (1 - v_{ANG}(x))^{\lambda_2})}), \\
& \left. \left(\frac{(\alpha_{ANG})^{\lambda_1+\lambda_2}}{(2 - \alpha_{AMG})^{\lambda_2} (\alpha_{AMG})^{\lambda_1} + (2 - \alpha_{AMG})^{\lambda_1} (\alpha_{AMG})^{\lambda_2}}, \frac{2(1 + \beta_{ANG})^{\lambda_1+\lambda_2}}{((1 + \beta_{ANG})^{\lambda_1} + (1 + \beta_{ANG})^{\lambda_1})((1 + \beta_{ANG})^{\lambda_2} + (1 + \beta_{ANG})^{\lambda_2})}, \right) \right). \tag{24}
\end{aligned}$$

Hence, we get $A_{LD}^{\lambda_1} \otimes_E A_{LD}^{\lambda_2} = A_{LD}^{\lambda_1+\lambda_2}$, $\lambda_1, \lambda_2 > 0$. Similarly, we prove that part (6). $\square$

5. Some LDULFPE Operators

In this section, we combine the PA operator and Einstein operations to the LDUL environment, propose the LDULFPEA operator, LDULFPEWA operator, LDULFPEG operator, and LDULFPEWG operator, and discuss the properties of them.

Definition 14. For any LDULNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, the LDULFPEA operator is elaborated by

$$\begin{aligned}
& LDULFPEA(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) \\
& = \frac{\otimes_{i=1}^n (1 + T(A_{LD-i})) * A_{LD-i}}{\sum_{i=1}^n (1 + T(A_{LD-i}))} \tag{25} \\
& = \otimes_{i=1}^n \left(\frac{(1 + T(A_{LD-i})) * A_{LD-i}}{\sum_{i=1}^n (1 + T(A_{LD-i}))} \right),
\end{aligned}$$

where $T(A_{LD-i}) = \sum_{j=1}^n \sup(A_{LD-i}, A_{LD-j})$, and $\sup(A_{LD-i}, A_{LD-j})$ is the support for A_{LD-i} from A_{LD-j} .

Theorem 3. For any LDULFNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, then the result aggregated from Definition 14 is still an LDULFN:

$$\begin{aligned}
\text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-D}) = & \left(\left[S_{\sum_{i=1}^n ((A|\theta_i(1+T(A_{\text{LD}-i)))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}, S_{\sum_{i=1}^n (A-\tau_i(\theta_i(1+T(A_{\text{LD}-i)))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} \right], \right. \\
& \left(\frac{\prod_{i=1}^n (1+u_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} - \prod_{i=1}^n (1-u_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}}{\prod_{i=1}^n (1+u_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} + \prod_{i=1}^n (1-u_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}} \right. \\
& \left. \frac{2 \prod_{i=1}^n (v_{\text{ANG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}}{\prod_{i=1}^n (2-v_{\text{ANG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} + \prod_{i=1}^n (v_{\text{ANG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}} \right), \\
& \left(\frac{\prod_{i=1}^n (1+\alpha_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} - \prod_{i=1}^n (1-\alpha_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}}{\prod_{i=1}^n (1+\alpha_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} + \prod_{i=1}^n (1-\alpha_{\text{AMG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}} \right. \\
& \left. \frac{2 \prod_{i=1}^n (\beta_{\text{ANG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}}{\prod_{i=1}^n (2-\beta_{\text{ANG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} + \prod_{i=1}^n (\beta_{\text{ANG}-i})^{((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}} \right) \left. \right), \tag{26}
\end{aligned}$$

where $T(A_{\text{LD}-i}) = \sum_{j=1}^n \sup_{j \neq i} (A_{\text{LD}-i}, A_{\text{LD}-j})$, and $\sup_{j \neq i} (A_{\text{LD}-i}, A_{\text{LD}-j})$ is the support for $A_{\text{LD}-i}$ from $A_{\text{LD}-j}$.

Proof. To simplify equation (26), we suppose $c_i = ((1+T(A_{\text{LD}-i))) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))) (i = 1, 2, \dots, n)$; then, equation (26) can be elaborated as

$$\begin{aligned}
\text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-D}) = & \left(\left[S_{\sum_{i=1}^n A=\theta_i c_i}, S_{\sum_{i=1}^n A-\tau_i c_i} \right], \right. \\
& \left(\frac{\prod_{i=1}^n (1+u_{\text{AMG}-i})^{c_i} - \prod_{i=1}^n (1-u_{\text{AMG}-i})^{c_i}}{\prod_{i=1}^n (1+u_{\text{AMG}-i})^{c_i} + \prod_{i=1}^n (1-u_{\text{AMG}-i})^{c_i}}, \frac{2 \prod_{i=1}^n (v_{\text{ANG}-i})^{c_i}}{\prod_{i=1}^n (2-v_{\text{ANG}-i})^{c_i} + \prod_{i=1}^n (v_{\text{ANG}-i})^{c_i}} \right), \\
& \left(\frac{\prod_{i=1}^n (1+\alpha_{\text{AMG}-i})^{c_i} - \prod_{i=1}^n (1-\alpha_{\text{AMG}-i})^{c_i}}{\prod_{i=1}^n (1+\alpha_{\text{AMG}-i})^{c_i} + \prod_{i=1}^n (1-\alpha_{\text{AMG}-i})^{c_i}}, \frac{2 \prod_{i=1}^n (\beta_{\text{ANG}-i})^{c_i}}{\prod_{i=1}^n (2-\beta_{\text{ANG}-i})^{c_i} + \prod_{i=1}^n (\beta_{\text{ANG}-i})^{c_i}} \right) \left. \right). \tag{27}
\end{aligned}$$

The equation (26) can be proved by mathematical induction on n as

(2) Suppose when $n = k$, equation (26) is right, i.e.,

(1) When $n = 1$, equation (26) is right obviously

$$\begin{aligned}
\text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-k}) = & \left(\left[S_{\sum_{i=1}^k A=\theta_i c_i}, S_{\sum_{i=1}^k A-\tau_i c_i} \right], \right. \\
& \left(\frac{\prod_{i=1}^k (1+u_{\text{AMG}-i})^{c_i} - \prod_{i=1}^k (1-u_{\text{AMG}-i})^{c_i}}{\prod_{i=1}^k (1+u_{\text{AMG}-i})^{c_i} + \prod_{i=1}^k (1-u_{\text{AMG}-i})^{c_i}}, \frac{2 \prod_{i=1}^k (v_{\text{ANG}-i})^{c_i}}{\prod_{i=1}^k (2-v_{\text{ANG}-i})^{c_i} + \prod_{i=1}^k (v_{\text{ANG}-i})^{c_i}} \right), \\
& \left(\frac{\prod_{i=1}^k (1+\alpha_{\text{AMG}-i})^{c_i} - \prod_{i=1}^k (1-\alpha_{\text{AMG}-i})^{c_i}}{\prod_{i=1}^k (1+\alpha_{\text{AMG}-i})^{c_i} + \prod_{i=1}^k (1-\alpha_{\text{AMG}-i})^{c_i}}, \frac{2 \prod_{i=1}^k (\beta_{\text{ANG}-i})^{c_i}}{\prod_{i=1}^k (2-\beta_{\text{ANG}-i})^{c_i} + \prod_{i=1}^k (\beta_{\text{ANG}-i})^{c_i}} \right) \left. \right). \tag{28}
\end{aligned}$$

Then, when $n = k + 1$, we have

$$\begin{aligned}
& \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-k+1}) = \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-k}) \otimes_{\varepsilon} (c_{k+1} A_{\text{LD}-i}) \\
& = \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-k}) \otimes_{\varepsilon} ([S_{A-\theta_{k+1}c_{k+1}}, S_{A-\tau_{k+1}c_{k+1}}]), \\
& \left(\frac{(1 + u_{\text{AMG}-k+1})^{c_{k+1}} - (1 - u_{\text{AMG}-k+1})^{c_{k+1}}}{(1 + u_{\text{AMG}-k+1})^{c_{k+1}} + (1 - u_{\text{AMG}-k+1})^{c_{k+1}}}, \frac{2(v_{\text{ANG}-k+1})^{k+1}}{(2 - v_{\text{ANG}-k+1})^{k+1} + (v_{\text{ANG}-k+1})^{k+1}} \right), \\
& \left(\frac{(1 + \alpha_{\text{AMG}-k+1})^{c_{k+1}} - (1 - \alpha_{\text{AMG}-k+1})^{c_{k+1}}}{(1 + \alpha_{\text{AMG}-k+1})^{c_{k+1}} + (1 - \alpha_{\text{AMG}-k+1})^{c_{k+1}}}, \frac{2(\beta_{\text{ANG}-k+1})^{K+1}}{2 - \beta_{\text{ANG}-k+1}^{K+1} + (\beta_{\text{ANG}-k+1})^{K+1}} \right) \\
& = \text{LDULFPEA}(A_{L\ D-1}, A_{L\ D-2}, \dots, A_{L\ D-k+1}) = \left([S_{\sum_{i=1}^{k+1} A=\theta_i c_i}, S_{\sum_{i=1}^{k+1} A=\tau_i c_i}], \right. \\
& \left(\frac{\prod_{i=1}^{k+1} (1 + u_{\text{AMG}-i})^{c_i} - \prod_{i=1}^{k+1} (1 - u_{\text{AMG}-i})^{c_i}}{\prod_{i=1}^{k+1} (1 + u_{\text{AMG}-i})^{c_i} + \prod_{i=1}^{k+1} (1 - u_{\text{AMG}-i})^{c_i}}, \frac{2\prod_{i=1}^{k+1} (v_{\text{ANG}-i})^{c_i}}{\prod_{i=1}^{k+1} (2 - v_{\text{ANG}-i})^{c_i} + \prod_{i=1}^{k+1} (v_{\text{ANG}-i})^{c_i}} \right), \\
& \left. \left(\frac{\prod_{i=1}^{k+1} (1 + \alpha_{\text{AMG}-i})^{c_i} - \prod_{i=1}^{k+1} (1 - \alpha_{\text{AMG}-i})^{c_i}}{\prod_{i=1}^{k+1} (1 + \alpha_{\text{AMG}-i})^{c_i} + \prod_{i=1}^{k+1} (1 - \alpha_{\text{AMG}-i})^{c_i}}, \frac{2\prod_{i=1}^{k+1} (\beta_{\text{ANG}-i})^{c_i}}{\prod_{i=1}^{k+1} (2 - \beta_{\text{ANG}-i})^{c_i} + \prod_{i=1}^{k+1} (\beta_{\text{ANG}-i})^{c_i}} \right) \right).
\end{aligned} \tag{29}$$

$$\text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) = A. \tag{30}$$

So, when $n = k + 1$, equation (26) is also right. According to steps (1) and (2), we can get that equation (26) is right for n . $\square$

Theorem 4. For any LDULNs $A_{\text{LD}-j} = A$, $j = 1, 2, \dots, n$ and $A = ([s_{A-\theta}, s_{A-\tau}], (u_{\text{AMG}}, v_{\text{ANG}}), (\alpha_{\text{AMG}}, \beta_{\text{ANG}}))$,

Proof. Since $A_{\text{LD}-i} = A$, for all i , we have

$$\begin{aligned}
& \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) = \left([S_{\sum_{i=1}^n A|\theta_i(1+T(A))}, S_{\sum_{i=1}^n A|\tau_i(1+T(A))}], \right. \\
& \left(\frac{\prod_{i=1}^n (1 + u_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A)) - \prod_{i=1}^n (1 - u_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))}{\prod_{i=1}^n (1 + u_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A)) + \prod_{i=1}^n (1 - u_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))}, \right. \\
& \left. \frac{2 \prod_{i=1}^n (v_{\text{ANG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))}{\prod_{i=1}^n (2 - v_{\text{ANG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A)) + \prod_{i=1}^n (v_{\text{ANG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))} \right), \\
& \left(\frac{\prod_{i=1}^n (1 + \alpha_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A)) - \prod_{i=1}^n (1 - \alpha_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))}{\prod_{i=1}^n (1 + \alpha_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A)) + \prod_{i=1}^n (1 - \alpha_{\text{AMG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))}, \right. \\
& \left. \frac{2 \prod_{i=1}^n (\beta_{\text{ANG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))}{\prod_{i=1}^n (2 - \beta_{\text{ANG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A)) + \prod_{i=1}^n (\beta_{\text{ANG}})^{(1+T(A))} \sum_{i=1}^n (1 + T(A))} \right) \Big)
\end{aligned}$$

$$\begin{aligned}
&= \left(\left[S_{(A-\theta) \sum_{i=1}^n (1+T(A))/ \sum_{i=1}^n (1+T(A))}, S_{(A-\tau) \sum_{i=1}^n (1+T(A))/ \sum_{i=1}^n (1+T(A))} \right], \right. \\
&\left(\frac{(1+u_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))} - (1-u_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}}{(1+u_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))} + (1-u_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}}, \right. \\
&\quad \left. \cdot \frac{2(v_{\text{ANG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}}{2 - (v_{\text{ANG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))} + (v_{\text{ANG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}} \right), \\
&\left(\frac{(1+\alpha_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))} - (1-\alpha_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}}{(1+\alpha_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))} + (1-\alpha_{\text{AMG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}}, \right. \\
&\quad \left. \cdot \frac{2(\beta_{\text{ANG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}}{\left((2+\beta_{\text{ANG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))} \right) + (\beta_{\text{ANG}})^{(1+T(A))/ \sum_{i=1}^n (1+T(A))}} \right) \\
&= ([s_{A-\theta}, s_{A-\tau}], (u_{\text{AMG}}, v_{\text{ANG}}), (\alpha_{\text{AMG}}, \beta_{\text{ANG}})) = A.
\end{aligned} \tag{31}$$

□

Theorem 5. For any LDULNs $A_{\text{LD}-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{\text{AMG}-i}, v_{\text{ANG}-i}), (\alpha_{\text{AMG}-i}, \beta_{\text{ANG}-i})), i = 1, 2, \dots, n$, if LDULFPEA operator lies between the max and min operators,

$$\begin{aligned}
\min(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) &= \min A, \\
\max(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) &= \max A;
\end{aligned} \tag{32}$$

then,

$$\min A \leq \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) \leq \max A. \tag{33}$$

Proof. First, let $g(u_{\text{AMG}-i}) = (1 - (u_{\text{AMG}-i})) / (1 + (u_{\text{AMG}-i}))$, $(u_{\text{AMG}-i}) \in [0, 1]$, we can get $g'(u_{\text{AMG}-i}) = -2 / (1 + (u_{\text{AMG}-i}))^2 < 0$ by taking derivative, so $g(u_{\text{AMG}-i})$ is a decreasing function. Suppose $\min(u_{\text{AMG}}) \leq u_{\text{AMG}-i} \leq \max(u_{\text{AMG}})$ for all j , we can get $((1 - \max(u_{\text{AMG}})) / (1 + \max(u_{\text{AMG}}))) \leq ((1 - (u_{\text{AMG}-i})) / (1 + (u_{\text{AMG}-i}))) \leq ((1 - \min(u_{\text{AMG}})) / (1 + \min(u_{\text{AMG}})))$; then,

$$\begin{aligned}
&\left(\frac{1 - \max(u_{\text{AMG}})}{1 + \max(u_{\text{AMG}})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq \left(\frac{1 - (u_{\text{AMG}-i})}{1 + (u_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \\
&\leq \left(\frac{1 - \min(u_{\text{AMG}})}{1 + \min(u_{\text{AMG}})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\
&\prod_{i=1}^n \left(\frac{1 - \max(u_{\text{AMG}})}{1 + \max(u_{\text{AMG}})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq \prod_{i=1}^n \left(\frac{1 - (u_{\text{AMG}-i})}{1 + (u_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \\
&\leq \prod_{i=1}^n \left(\frac{1 - \min(u_{\text{AMG}})}{1 + \min(u_{\text{AMG}})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\
&\left(\frac{1 - \max(u_{\text{AMG}})}{1 + \max(u_{\text{AMG}})} \right)^{\sum_{i=1}^n (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \\
&\leq \prod_{i=1}^n \left(\frac{1 - (u_{\text{AMG}-i})}{1 + (u_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq \left(\frac{1 - \min(u_{\text{AMG}})}{1 + \min(u_{\text{AMG}})} \right)^{\sum_{i=1}^n (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))},
\end{aligned} \tag{34}$$

i.e.,

$$\begin{aligned}
\frac{1 - \max(u_{\text{AMG}})}{1 + \max(u_{\text{AMG}})} &\leq \prod_{i=1}^n \left(\frac{1 - (u_{\text{AMG}-i})}{1 + (u_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq \frac{1 - \min(u_{\text{AMG}})}{1 + \min(u_{\text{AMG}})}, \\
\frac{2}{1 + \max(u_{\text{AMG}})} &\leq 1 + \prod_{i=1}^n \left(\frac{1 - (u_{\text{AMG}-i})}{1 + (u_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq \frac{2}{1 + \min(u_{\text{AMG}})}, \\
\frac{1 + \max(u_{\text{AMG}})}{2} &\geq \frac{1}{1 + \prod_{i=1}^n ((1 - (u_{\text{AMG}-i})) / (1 + (u_{\text{AMG}-i))))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1 + T(A_{\text{LD}-i}))}} \geq \frac{(1 + \min(u_{\text{AMG}}))}{2}.
\end{aligned} \tag{35}$$

Thus,

$$\begin{aligned}
1 + \max(u_{\text{AMG}}) &\geq \frac{2}{1 + \prod_{i=1}^n ((1 - (u_{\text{AMG}-i})) / (1 + (u_{\text{AMG}-i))))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1 + T(A_{\text{LD}-i}))}} \geq 1 + \min(u_{\text{AMG}}), \\
\max(u_{\text{AMG}}) &\geq \frac{2}{1 + \prod_{i=1}^n ((1 - (u_{\text{AMG}-i})) / (1 + (u_{\text{AMG}-i))))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1 + T(A_{\text{LD}-i}))}} - 1 \geq \min(u_{\text{AMG}}).
\end{aligned} \tag{36}$$

Therefore,

$$\max(u_{\text{AMG}}) \geq \frac{\prod_{i=1}^n (1 + (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1 + T(A_{\text{LD}-i}))} - \prod_{i=1}^n (1 - (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1 + T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (1 + (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1 + T(A_{\text{LD}-i}))} + \prod_{i=1}^n (1 - (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1 + T(A_{\text{LD}-i}))}} \geq \min(u_{\text{AMG}}). \tag{37}$$

Second, let $f(v_{\text{AMG}-i}) = (2 - v_{\text{AMG}-i})/v_{\text{AMG}-i}$, $v_{\text{AMG}-i} \in [0, 1]$, and we can get $f(v_{\text{AMG}-i}) = -2/(v_{\text{AMG}-i})^2 < 0$ by taking derivative, so $f(v_{\text{AMG}-i})$ is a decreasing function.

Suppose $\min(v_{\text{AMG}}) \leq v_{\text{AMG}-i} \leq \max(v_{\text{AMG}})$ for all i , we can get $(2 - \min(v_{\text{AMG}}))/\min(v_{\text{AMG}}) \leq (2 - (v_{\text{AMG}-i}))/v_{\text{AMG}-i} \leq (2 - \max(v_{\text{AMG}}))/\max(v_{\text{AMG}})$; then, we have

$$\begin{aligned}
\left(\frac{2 - \min(v_{\text{AMG}})}{\min(v_{\text{AMG}})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} &\leq \left(\frac{2 - (v_{\text{AMG}-i})}{(v_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \\
&\leq \left(\frac{2 - \max(v_{\text{AMG}})}{\max(v_{\text{AMG}})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\
\left(\frac{2 - \min(v_{\text{AMG}})}{\min(v_{\text{AMG}})} \right)^{\sum_{i=1}^n ((1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} & \\
&\leq \prod_{i=1}^n \left(\frac{2 - (v_{\text{AMG}-i})}{(v_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq \left(\frac{2 - \max(v_{\text{AMG}})}{\max(v_{\text{AMG}})} \right)^{\sum_{i=1}^n ((1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))}, \\
\frac{2 - \min(v_{\text{AMG}})}{\min(v_{\text{AMG}})} &\leq \prod_{i=1}^n \left(\frac{2 - (v_{\text{AMG}-i})}{(v_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq \frac{2 - \max(v_{\text{AMG}})}{\max(v_{\text{AMG}})},
\end{aligned}$$

$$\begin{aligned}
\frac{2}{\min(v_{\text{AMG}})} &\leq \prod_{i=1}^n \left(\frac{2 - (v_{\text{AMG}-i})}{(v_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + 1 \leq \frac{2}{\max(v_{\text{AMG}})}, \\
\frac{\min(v_{\text{AMG}})}{2} &\geq \frac{1}{\prod_{i=1}^n ((2 - (v_{\text{AMG}-i})) / (v_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + 1} \geq \frac{\max(v_{\text{AMG}})}{2}, \\
\min(v_{\text{AMG}}) &\geq \frac{2}{\prod_{i=1}^n ((2 - (v_{\text{AMG}-i})) / (v_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + 1} \geq \max(v_{\text{AMG}}).
\end{aligned} \tag{38}$$

Therefore,

$$\min(v_{\text{AMG}}) \geq \frac{2 \prod_{i=1}^n (v_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (2 - (v_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + \prod_{i=1}^n (v_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}} \geq \max(v_{\text{AMG}}). \tag{39}$$

For α and β because $s_{A-\theta_{\min}} \leq s_{A-\theta_i} \leq s_{A-\theta_{\max}}$,
 $s_{A-\tau_{\min}} \leq s_{A-\tau_i} \leq s_{A-\tau_{\max}}$ for all i , then

$$\begin{aligned}
s_{(A-\theta_{\min})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} &\leq s_{(A-\theta_i)}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq s_{(A-\theta_{\max})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\
s_{(A-\tau_{\min})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} &\leq s_{(A-\tau_i)}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \leq s_{(A-\tau_{\max})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\
s_{\sum_{i=1}^n (A-\theta_{\min})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} &\leq s_{\sum_{i=1}^n (A-\theta_i)}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \\
&\leq s_{\sum_{i=1}^n (A-\theta_{\max})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\
s_{\sum_{i=1}^n (A-\tau_{\min})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} &\leq s_{\sum_{i=1}^n (A-\tau_i)}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \\
&\leq s_{\sum_{i=1}^n (A-\tau_{\max})}^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))},
\end{aligned} \tag{40}$$

i.e., $s_{A-\theta_{\min}} \leq s_{\sum_{i=1}^n ((A-\theta_i) (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} \leq s_{A-\theta_{\max}}$,
 $s_{A-\tau_{\min}} \leq s_{\sum_{i=1}^n ((A-\tau_i) (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i))))} \leq s_{A-\tau_{\max}}$.

If $\text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) = A = ([s_{A-\theta}, s_{A-\tau}], (u_{\text{AMG}}, v_{\text{ANG}}), (\alpha_{\text{AMG}}, \beta_{\text{ANG}}))$, we know that

$s_{A-\theta_{\min}} \leq s_{A-\theta} \leq s_{A-\theta_{\max}}$, $s_{A-\tau_{\min}} \leq s_{A-\tau} \leq s_{A-\tau_{\max}}$, $\min(u_{\text{AMG}}) \leq u_{\text{AMG}} \leq \max(u_{\text{AMG}})$, $\min(v_{\text{ANG}}) \leq v_{\text{ANG}} \leq \max(v_{\text{ANG}})$,
 $\min(\alpha_{\text{AMG}}) \leq \alpha_{\text{AMG}} \leq \max(\alpha_{\text{AMG}})$, and $\min(\beta_{\text{ANG}}) \leq \beta_{\text{ANG}} \leq \max(\beta_{\text{ANG}})$.

$$\begin{aligned}
s_{A-\theta_{\min}} + s_{A-\tau_{\min}} &\leq s_{A-\theta} + s_{A-\tau} \leq s_{A-\theta_{\max}} + s_{A-\tau_{\max}}, \\
\min(u_{\text{AMG}}) - \min(v_{\text{ANG}}) &\leq u_{\text{AMG}} - v_{\text{ANG}} \leq \max(u_{\text{AMG}}) - \max(v_{\text{ANG}}), \\
\min(\alpha_{\text{AMG}}) - \min(\beta_{\text{ANG}}) &\leq \alpha_{\text{AMG}} - \beta_{\text{ANG}} \leq \max(\alpha_{\text{AMG}}) - \max(\beta_{\text{ANG}}).
\end{aligned} \tag{41}$$

Therefore,

$$\min A \leq \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) \leq \max A. \quad (42)$$

□

Theorem 6. For any two LDULNs $A_{\text{LD}-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{\text{AMG}-i}, v_{\text{ANG}-i}), (\alpha_{\text{AMG}-i}, \beta_{\text{ANG}-i}))$ and $B_{\text{LD}-i} = ([s_{B-\theta_i}, s_{B-\tau_i}], (u_{\text{BMG}-i}, v_{\text{BNG}-i}), (\alpha_{\text{BMG}-i}, \beta_{\text{BNG}-i}))$ and if $s_{A-\theta_i} \leq s_{B-\theta_i}, s_{A-\tau_i} \leq s_{B-\tau_i}, u_{\text{AMG}-i} \leq u_{\text{BMG}-i}, v_{\text{ANG}-i} \leq v_{\text{BNG}-i}, \alpha_{\text{AMG}-i} \leq \alpha_{\text{BMG}-i}$ and $\beta_{\text{ANG}-i} \leq \beta_{\text{BNG}-i}$ for all $i = 1, 2, \dots, n$, then

$$\text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) \leq \text{LDULFPEA}(B_{\text{LD}-1}, B_{\text{LD}-2}, \dots, B_{\text{LD}-n}). \quad (43)$$

Proof. Since $s_{A-\theta_i} + s_{A-\tau_i} \leq s_{B-\theta_i} + s_{B-\tau_i}, u_{\text{AMG}-i} \leq u_{\text{BMG}-i}, v_{\text{ANG}-i} \leq v_{\text{BNG}-i}, \alpha_{\text{AMG}-i} \leq \alpha_{\text{BMG}-i}$ and $\beta_{\text{ANG}-i} \leq \beta_{\text{BNG}-i}$, for all i , we can get

$$\begin{aligned} & \sum_{i=1}^n \left(\frac{1 + T(A_{\text{LD}-i})}{\sum_{i=1}^n (1 + T(A_{\text{LD}-i}))} (s_{A-\theta_i} + s_{A-\tau_i}) \right) \leq \sum_{i=1}^n \left(\frac{1 + T(A_{\text{LD}-i})}{\sum_{i=1}^n (1 + T(A_{\text{LD}-i}))} (s_{B-\theta_i} + s_{B-\tau_i}) \right), \\ & s \sum_{i=1}^n (A_{-\theta_i}) ((1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))) + s \sum_{i=1}^n (A_{-\tau_i}) ((1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))) \\ & \leq s \sum_{i=1}^n (B_{-\theta_i}) ((1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))) + s \sum_{i=1}^n (B_{-\tau_i}) ((1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))) \end{aligned} \quad (44)$$

Since $((1 - u_{\text{BMG}-i}) / (1 + u_{\text{BMG}-i})) \leq ((1 - u_{\text{AMG}-i}) / (1 + u_{\text{AMG}-i})), i = 1, 2, \dots, n$, then

$$\begin{aligned} & \prod_{i=1}^n \left(\frac{1 - (u_{\text{BMG}-i})}{1 + (u_{\text{BMG}-i})} \right)^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))} \leq \prod_{i=1}^n \left(\frac{1 - (u_{\text{AMG}-i})}{1 + (u_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))} \\ & \frac{2}{1 + \prod_{i=1}^n ((1 - (u_{\text{BMG}-i)) / (1 + (u_{\text{BMG}-i))))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))}} \\ & \geq \frac{2}{1 + \prod_{i=1}^n ((1 - (u_{\text{AMG}-i)) / (1 + (u_{\text{AMG}-i))))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))}} \end{aligned} \quad (45)$$

So, we have

$$\begin{aligned} & \frac{\prod_{i=1}^n (1 + (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))} - \prod_{i=1}^n (1 - (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))}}{\prod_{i=1}^n (1 + (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))} + \prod_{i=1}^n (1 - (u_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i)))}} \\ & \geq \frac{\prod_{i=1}^n (1 + (u_{\text{BMG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))} - \prod_{i=1}^n (1 - (u_{\text{BMG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))}}{\prod_{i=1}^n (1 + (u_{\text{BMG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))} + \prod_{i=1}^n (1 - (u_{\text{BMG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i)))}} \end{aligned} \quad (46)$$

For $\alpha_{\text{AMG}-i}$, $\alpha_{\text{BMG}-i}$, we can get

$$\begin{aligned} & \frac{\prod_{i=1}^n (1 + \alpha_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} - \prod_{i=1}^n (1 - \alpha_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n ((1+T(A_{\text{LD}-i))))}}{\prod_{i=1}^n (1 + \alpha_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n ((1+T(A_{\text{LD}-i))))} + \prod_{i=1}^n (1 - \alpha_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n ((1+T(A_{\text{LD}-i))))}} \\ & \geq \frac{\prod_{i=1}^n (1 + \alpha_{\text{BMG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))} - \prod_{i=1}^n (1 - \alpha_{\text{BMG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))}}{\prod_{i=1}^n (1 + \alpha_{\text{BMG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))} + \prod_{i=1}^n (1 - \alpha_{\text{BMG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))}}. \end{aligned} \quad (47)$$

Since $((2 - v_{\text{BNG}-i})/v_{\text{BNG}-i}) \geq ((2 - v_{\text{ANG}-i})/v_{\text{ANG}-i})$, $i = 1, 2, \dots, n$, then

$$\begin{aligned} & \left(\frac{2 - (v_{\text{BMG}-i})}{(v_{\text{BMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} \geq \left(\frac{2 - (v_{\text{AMG}-i})}{(v_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\ & \prod_{i=1}^n \left(\frac{2 - (v_{\text{BMG}-i})}{(v_{\text{BMG}-i})} \right)^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))} \geq \prod_{i=1}^n \left(\frac{2 - (v_{\text{AMG}-i})}{(v_{\text{AMG}-i})} \right)^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}, \\ & \frac{1}{\prod_{i=1}^n \left(((2 - (v_{\text{BMG}-i})) / (v_{\text{BMG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))} \right) + 1} \\ & \leq \frac{1}{((2 - (v_{\text{AMG}-i})) / (v_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + 1}, \\ & \frac{2}{\prod_{i=1}^n \left(((2 - (v_{\text{BMG}-i})) / (v_{\text{BMG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))} \right) + 1} \\ & \leq \frac{2}{((2 - (v_{\text{AMG}-i})) / (v_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + 1}, \end{aligned} \quad (48)$$

so

$$\begin{aligned} & \frac{2 \prod_{i=1}^n (v_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (2 - (v_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + \prod_{i=1}^n (v_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}} \\ & \leq \frac{2 \prod_{i=1}^n (v_{\text{BMG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))}}{\prod_{i=1}^n (2 - (v_{\text{BMG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))} + \prod_{i=1}^n (v_{\text{BMG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))}}. \end{aligned} \quad (49)$$

For $\beta_{\text{ANG}-i}, \beta_{\text{BNG}-i}$, we can get

$$\begin{aligned} & \frac{2 \prod_{i=1}^n (\beta_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (2 - (\beta_{\text{AMG}-i}))^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))} + \prod_{i=1}^n (\beta_{\text{AMG}-i})^{(1+T(A_{\text{LD}-i})) / \sum_{i=1}^n (1+T(A_{\text{LD}-i}))}} \\ & \leq \frac{2 \prod_{i=1}^n (\beta_{\text{BNG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))}}{\prod_{i=1}^n (2 - (\beta_{\text{BNG}-i}))^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))} + \prod_{i=1}^n (\beta_{\text{BNG}-i})^{(1+T(B_{\text{LD}-i})) / \sum_{i=1}^n (1+T(B_{\text{LD}-i}))}}. \end{aligned} \quad (50)$$

Thus, we got that

$$\text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) \leq \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}). \quad (51)$$

Definition 15. For any LDULNs $A_{\text{LD}-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{\text{AMG}-i}, v_{\text{ANG}-i}), (\alpha_{\text{AMG}-i}, \beta_{\text{ANG}-i}))$, $i = 1, 2, \dots, n$, the LDULFPEWA operator is elaborated by

$$\text{LDULFPEWA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n}) = \frac{\otimes_{\varepsilon i=1}^n w_i (1 + T(A_{\text{LD}-i})) * A_{\text{LD}-i}}{\sum_{i=1}^n w_i (1 + T(A_{\text{LD}-i}))} = \otimes_{\varepsilon i=1}^n \left(\frac{w_i (1 + T(A_{\text{LD}-i})) * A_{\text{LD}-i}}{\sum_{i=1}^n w_i (1 + T(A_{\text{LD}-i}))} \right), \quad (52)$$

where $T(A_{\text{LD}-i}) = \sum_{j=1}^n \sup_{j \neq i} (A_{\text{LD}-i}, A_{\text{LD}-j})$, $\sup(A_{\text{LD}-i}, A_{\text{LD}-j})$ is the support for $A_{\text{LD}-i}$ from $A_{\text{LD}-j}$, and $w = (w_1, w_2, \dots, w_n)^T$ is the weighting vector of the $(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-n})$, such that $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$.

Theorem 7. For any LDULFNs $A_{\text{LD}-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{\text{AMG}-i}, v_{\text{ANG}-i}), (\alpha_{\text{AMG}-i}, \beta_{\text{ANG}-i}))$, $i = 1, 2, \dots, n$, the result aggregated from Definition 15 is still an LDULFN:

$$\begin{aligned} & \text{LDULFPEA}(A_{\text{LD}-1}, A_{\text{LD}-2}, \dots, A_{\text{LD}-D}) = \otimes_{\varepsilon i=1}^n \left(\frac{w_i (1 + T(A_{\text{LD}-i})) * A_{\text{LD}-i}}{\sum_{i=1}^n w_i (1 + T(A_{\text{LD}-i}))} \right) \\ & = \left(\left[S_{\sum_{i=1}^n (A-\theta_i w_i (1+T(A_{\text{LD}-i)})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i)}), S_{\sum_{i=1}^n (A-\tau_i w_i (\theta_i (1+T(A_{\text{LD}-i)})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i)}))} \right], \right. \\ & \left(\frac{\prod_{i=1}^n (1 + u_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))} - \prod_{i=1}^n (1 - u_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (1 + u_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))} + \prod_{i=1}^n (1 - u_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}} \right), \\ & \cdot \frac{2 \prod_{i=1}^n (v_{\text{ANG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (2 - v_{\text{ANG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))} + \prod_{i=1}^n (v_{\text{ANG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}} \right), \\ & \left(\frac{\prod_{i=1}^n (1 + \alpha_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))} - \prod_{i=1}^n (1 - \alpha_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (1 + \alpha_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))} + \prod_{i=1}^n (1 - \alpha_{\text{AMG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}} \right), \\ & \cdot \frac{2 \prod_{i=1}^n (\beta_{\text{ANG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}}{\prod_{i=1}^n (2 - \beta_{\text{ANG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))} + \prod_{i=1}^n (\beta_{\text{ANG}-i})^{w_i (1+T(A_{\text{LD}-i})) / \sum_{i=1}^n w_i (1+T(A_{\text{LD}-i}))}} \right) \Bigg), \end{aligned} \quad (53)$$

where $T(A_{LD-i}) = \sum_{j=1}^n \sup_{j \neq i} (A_{LD-i}, A_{LD-j})$, $\sup(A_{LD-i}, A_{LD-j})$ is the support for A_{LD-i} from A_{LD-j} , and $w = (w_1, w_2, \dots, w_n)^T$ is the weighting vector of the $(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n})$, such that $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$.

Proof. Trivial. $\square$

Theorem 8. For any LDULNs $A_{LD-j} = A$, $j = 1, 2, \dots, n$ and $A = ([s_{A-\theta}, s_{A-\tau}], (u_{AMG}, v_{ANG}), (\alpha_{AMG}, \beta_{ANG}))$, then

$$\text{LDULFPEWA}(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) = A. \quad (54)$$

Proof. Trivial. $\square$

Theorem 9. For any LDULNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, if LDULFPEWA operator lies between the max and min operators,

$$\begin{aligned} \min(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) &= \min A, \\ \max(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) &= \max A; \end{aligned} \quad (55)$$

then,

$$\min A \leq \text{LDULFPEWA}(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) \leq \max A. \quad (56)$$

Proof. Trivial. $\square$

Definition 16. For any LDULNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, the LDULFPEA operator is elaborated by

$$\begin{aligned} &\text{LDULFPEG}(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) \\ &= \otimes_{i=1}^n (A_{LD-i})^{1+T(A_{LD-i}) / \sum_{i=1}^n (1+T(A_{LD-i}))}, \end{aligned} \quad (57)$$

where $T(A_{LD-i}) = \sum_{j=1}^n \sup_{j \neq i} (A_{LD-i}, A_{LD-j})$, and $\sup(A_{LD-i}, A_{LD-j})$ is the support for A_{LD-i} from A_{LD-j} .

Theorem 10. For any LDULFNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, the result aggregated from Definition 16 is still an LDULFN:

$$\begin{aligned} \text{LDULFPEA}(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) &= \left(\left[S \prod_{i=1}^n A=\theta_i^{c_i}, S \prod_{i=1}^n A=\tau_i^{c_i} \right], \right. \\ &\left(\frac{2 \prod_{i=1}^n (u_{ANG-i})^{c_i}}{\prod_{i=1}^n (2 - u_{ANG-i})^{c_i} + \prod_{i=1}^n (u_{ANG-i})^{c_i}}, \frac{\prod_{i=1}^n (1 + v_{AMG-i})^{c_i} - \prod_{i=1}^n (1 - v_{AMG-i})^{c_i}}{\prod_{i=1}^n (1 + v_{AMG-i})^{c_i} + \prod_{i=1}^n (1 - v_{AMG-i})^{c_i}} \right), \\ &\left(\frac{2 \prod_{i=1}^n (\alpha_{AMG-i})^{c_i}}{\prod_{i=1}^n (2 - \alpha_{AMG-i})^{c_i} + \prod_{i=1}^n (\alpha_{AMG-i})^{c_i}}, \frac{\prod_{i=1}^n (1 + \beta_{AMG-i})^{c_i} - \prod_{i=1}^n (1 - \beta_{AMG-i})^{c_i}}{\prod_{i=1}^n (2 - \beta_{AMG-i})^{c_i} + \prod_{i=1}^n (1 - \beta_{AMG-i})^{c_i}} \right) \Bigg), \end{aligned} \quad (58)$$

where $c_i = (1 + T(A_{LD-i})) / \sum_{i=1}^n (1 + T(A_{LD-i}))$ ($i = 1, 2, \dots, n$), $T(A_{LD-i}) = \sum_{j=1}^n \sup_{j \neq i} (A_{LD-i}, A_{LD-j})$, and $\sup(A_{LD-i}, A_{LD-j})$ is the support for A_{LD-i} from A_{LD-j} .

Proof. Trivial. $\square$

Theorem 11. For any LDULNs $A_{LD-j} = A$, $j = 1, 2, \dots, n$ and $A = ([s_{A-\theta}, s_{A-\tau}], (u_{AMG}, v_{ANG}), (\alpha_{AMG}, \beta_{ANG}))$, then

$$\text{LDULFPEG}(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) = A. \quad (59)$$

Proof. Trivial. $\square$

Theorem 12. For any LDULNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, if LDULFPEG operator lies between the max and min operators,

$$\begin{aligned} \min(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) &= \min A, \\ \max(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) &= \max A; \end{aligned} \quad (60)$$

then,

$$\min A \leq \text{LDULFPEG}(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) \leq \max A. \quad (61)$$

Proof. Trivial. $\square$

Theorem 13. For any two LDULNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$ and $B_{LD-i} = ([s_{B-\theta_i}, s_{B-\tau_i}], (u_{BMG-i}, v_{BNG-i}), (\alpha_{BMG-i}, \beta_{BNG-i}))$ and if $s_{A-\theta_i} \leq s_{B-\theta_i}$, $s_{A-\tau_i} \leq s_{B-\tau_i}$, $\tau_i u_{AMG-i} \leq u_{BMG-i}$, $v_{ANG-i} \leq v_{BNG-i}$, $\alpha_{AMG-i} \leq \alpha_{BMG-i}$ and $\beta_{ANG-i} \leq \beta_{BNG-i}$, for all $i = 1, 2, \dots, n$, then

$$\text{LDULFPEG}(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) \leq \text{LDULFPEG}(B_{LD-1}, B_{LD-2}, \dots, B_{LD-n}). \quad (62)$$

Proof. Trivial. $\square$

Definition 17. For any LDULNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, the LDULFPEWG operator is elaborated by

$$LDULFPEWG(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) = \otimes_{i=1}^n (A_{LD-i})^{w_i (1+T(A_{LD-i})) / \sum_{i=1}^n w_i (1+T(A_{LD-i}))}, \quad (63)$$

where $T(A_{LD-i}) = \sum_{j=1}^n \sup_{j \neq i} (A_{LD-i}, A_{LD-j})$, $\sup(A_{LD-i}, A_{LD-j})$ is the support for A_{LD-i} from A_{LD-j} , and $w = (w_1, w_2, \dots, w_n)^T$ is the weighting vector of the $(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n})$, such that $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$.

Theorem 14. For any LDULFNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, then the result aggregated from Definition 17 is still an LDULFN:

$$LDULFPEA(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) = \otimes_{i=1}^n (A_{LD-i})^{w_i (1+T(A_{LD-i})) / \sum_{i=1}^n w_i (1+T(A_{LD-i}))} = \left(\left[S_{\prod_{i=1}^n A=\theta_i^{b_i}}, S_{\prod_{i=1}^n A=\tau_i^{b_i}} \right], \right. \\ \left(\frac{2 \prod_{i=1}^n (u_{ANG-i})^{b_i}}{\prod_{i=1}^n (2 - u_{ANG-i})^{b_i} + \prod_{i=1}^n (u_{ANG-i})^{b_i}}, \frac{\prod_{i=1}^n (1 + v_{AMG-i})^{b_i} - \prod_{i=1}^n (1 - v_{AMG-i})^{b_i}}{\prod_{i=1}^n (1 + v_{AMG-i})^{b_i} + \prod_{i=1}^n (1 - v_{AMG-i})^{b_i}} \right), \quad (64) \\ \left. \left(\frac{2 \prod_{i=1}^n (\alpha_{AMG-i})^{b_i}}{\prod_{i=1}^n (2 - \alpha_{AMG-i})^{b_i} + \prod_{i=1}^n (\alpha_{AMG-i})^{b_i}}, \frac{\prod_{i=1}^n (1 + \beta_{AMG-i})^{b_i} - \prod_{i=1}^n (1 - \beta_{AMG-i})^{b_i}}{\prod_{i=1}^n (1 + \beta_{AMG-i})^{b_i} + \prod_{i=1}^n (1 - \beta_{AMG-i})^{b_i}} \right) \right)$$

where $b_i = (w_i (1 + T(A_{LD-i})) / \sum_{i=1}^n w_i (1 + T(A_{LD-i})))$ ($i = 1, 2, \dots, n$), $T(A_{LD-i}) = \sum_{j=1}^n \sup_{j \neq i} (A_{LD-i}, A_{LD-j})$, and $\sup(A_{LD-i}, A_{LD-j})$ is the support for A_{LD-i} from A_{LD-j} . $w = (w_1, w_2, \dots, w_n)^T$ is the weighting vector of $(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n})$, such that $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$.

Proof. Trivial. $\square$

Theorem 15. For any LDULNs $A_{LD-j} = A$, $j = 1, 2, \dots, n$ and $A = ([s_{A-\theta}, s_{A-\tau}], (u_{AMG}, v_{ANG}), (\alpha_{AMG}, \beta_{ANG}))$,

$$LDULFPEWG(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) = A. \quad (65)$$

Proof. Trivial. $\square$

Theorem 16. For any LDULNs $A_{LD-i} = ([s_{A-\theta_i}, s_{A-\tau_i}], (u_{AMG-i}, v_{ANG-i}), (\alpha_{AMG-i}, \beta_{ANG-i}))$, $i = 1, 2, \dots, n$, if LDULFPEWG operator lies between the max and min operators,

$$\begin{aligned} \min(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) &= \min A, \\ \max(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) &= \max A; \end{aligned} \quad (66)$$

then,

$$\min A \leq LDULFPEWG(A_{LD-1}, A_{LD-2}, \dots, A_{LD-n}) \leq \max A. \quad (67)$$

Proof. Trivial. $\square$

6. MADM Technique for LDULSs Using PEAOs

In this part, we promote the examination of PEAOs of LDULNs in the MADM problem. We promote powered Einstein aggregation operators of LDULNs in the MADM problem. The impact of $\mathfrak{R}$ on the ranking results is investigated thoroughly.

MADM is a way of choosing the best alternative because of some finite attributes using the PE operators of LDULSs. The most favorable thing here is that the information is based on LDULNs which discuss the membership grades and non-membership grades of the information. Let the collection of alternatives be A_k (k is finite) and attributes be G_j (j is finite), which forms a decision matrix denoted by $D_{k \times j} = (T)_{k \times j} = ([s_{A-\theta_{kj}}, s_{A-\tau_{kj}}], (u_{AMG-kj}, v_{ANG-kj}), (\alpha_{AMG-kj}, \beta_{ANG-kj}))$, where the terms in triplet denote the membership grades, abstinence, and nonmembership grades of the information, where $w = (w_1, w_2, \dots, w_r)^T$ is the weight vector of $(A_{LD-1}, A_{LD-2}, \dots, A_{LD-r})$, $w_j \in [0, 1]$, $\sum_{j=1}^r w_j = 1$. A brief algorithm of the MADM process is illustrated in the following section.

6.1. Algorithm. The algorithm of MADM based on LDUL information and using LDULPEA, LDULPEWA, LDULPEG, and LDULFPEWG operators is proposed as follows:

Step 1: in this step, we collect the information about alternatives given by the decision-makers. The decision-makers gave the opinion about alternates in the

TABLE 1: Original decision matrix.

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	$([s_1, s_2], (0.4, 0.3), (0.3, 0.2))$	$([s_1, s_2], (0.41, 0.31), (0.31, 0.21))$	$([s_1, s_2], (0.42, 0.32), (0.32, 0.22))$	$([s_1, s_2], (0.43, 0.33), (0.33, 0.23))$
$\mathcal{A}_2$	$([s_1, s_3], (0.5, 0.4), (0.4, 0.3))$	$([s_1, s_3], (0.51, 0.41), (0.41, 0.31))$	$([s_1, s_3], (0.52, 0.42), (0.42, 0.32))$	$([s_1, s_3], (0.53, 0.43), (0.43, 0.33))$
$\mathcal{A}_3$	$([s_2, s_3], (0.6, 0.5), (0.2, 0.1))$	$([s_2, s_3], (0.61, 0.51), (0.21, 0.11))$	$([s_2, s_3], (0.62, 0.52), (0.22, 0.12))$	$([s_2, s_3], (0.63, 0.53), (0.23, 0.13))$
$\mathcal{A}_4$	$([s_0, s_1], (0.7, 0.4), (0.4, 0.5))$	$([s_0, s_1], (0.71, 0.41), (0.41, 0.51))$	$([s_0, s_1], (0.72, 0.42), (0.42, 0.52))$	$([s_0, s_1], (0.73, 0.43), (0.43, 0.53))$

TABLE 2: Aggregated values by using different operators.

	LDULPEA operator	LDULPEWA operator	LDULPEG operator	LDULPEWG operator
A_1	$([s_{1,4546}, s_{2,5757}], (0.4145, 0.3146), (0.3147, 0.2148))$	$([s_{1,5657}, s_{2,6868}], (0.5256, 0.4257), (0.4258, 0.3259))$	$([s_{1,3435}, s_{2,4646}], (0.3034, 0.2035), (0.2036, 0.1037))$	$([s_{1,2324}, s_{2,3535}], (0.2023, 0.1024), (0.1025, 0.0026))$
A_2	$([s_{1,4546}, s_{3,5757}], (0.5145, 0.4146), (0.4147, 0.3148))$	$([s_{1,5657}, s_{3,6868}], (0.6256, 0.5257), (0.5258, 0.4259))$	$([s_{1,3435}, s_{2,4646}], (0.4034, 0.3035), (0.3036, 0.2037))$	$([s_{1,2324}, s_{2,3535}], (0.3023, 0.2024), (0.2025, 0.1026))$
A_3	$([s_{2,4546}, s_{3,5757}], (0.6145, 0.5146), (0.2147, 0.1148))$	$([s_{2,5657}, s_{3,6868}], (0.7256, 0.6257), (0.3258, 0.2259))$	$([s_{1,3435}, s_{2,4646}], (0.5034, 0.4035), (0.1036, 0.0037))$	$([s_{1,2324}, s_{2,3535}], (0.4023, 0.3024), (0.0025, 0.0026))$
A_4	$([s_{0,4546}, s_{1,5757}], (0.7145, 0.4146), (0.4147, 0.5148))$	$([s_{0,5657}, s_{1,6868}], (0.8256, 0.5257), (0.5258, 0.6259))$	$([s_{1,3435}, s_{2,4646}], (0.6034, 0.3035), (0.3036, 0.4037))$	$([s_{1,2324}, s_{2,3535}], (0.5023, 0.2024), (0.2025, 0.3026))$

TABLE 3: Score values of the aggregated operators.

Scores	LDULPEA operator	LDULPEWA operator	LDULPEG operator	LDULPEWG operator
A_1	0.8209	0.9842	0.6698	0.5401
A_2	0.9216	1.0905	0.7651	0.6298
A_3	0.7201	0.8779	0.5746	0.4953
A_4	1.1231	1.3032	0.9554	0.8091

TABLE 4: Ranking values of the information in Table 3.

	Ranking analysis
LDULPEWA operator	$A_4 > A_2 > A_1 > A_3$
LDULPEWG operator	$A_4 > A_2 > A_1 > A_3$
LDULPEA operator	$A_4 > A_2 > A_1 > A_3$
LDULPEG operator	$A_4 > A_2 > A_1 > A_3$

TABLE 5: Comparative analysis of the elaborated operators and existing operators.

Method	Score values	Ranking values
Riaz and Hashmi [39]	Cannot be calculated	Cannot be calculated
Riaz and Hashmi [40]	Cannot be calculated	Cannot be calculated
Wang et al. [42]	Cannot be calculated	Cannot be calculated
LDULPEWA operator	0.9842, 1.0905, 0.8779, 1.3032	$A_4 > A_2 > A_1 > A_3$
LDULPEWG operator	0.5401, 0.6298, 0.4953, 0.8091	$A_4 > A_2 > A_1 > A_3$
LDULPEA operator	0.8209, 0.9216, 0.7201, 1.1231	$A_4 > A_2 > A_1 > A_3$
LDULPEG operator	0.6698, 0.7651, 0.5746, 0.9554	$A_4 > A_2 > A_1 > A_3$

form of LDULNs which leads to the formation of the decision matrix.

Step 2: if there exist attributes of cost type, we normalize the decision matrix by taking the complement of each triplet in the matrix. If not, we use the following LDULPEA, LDULPEWA, LDULPEG, and LDULPEWG operators to aggregate the data given in the decision matrix.

Step 3: in this step, we compute the scores of the aggregated information using Definition 3

Step 4: this step is based on the ranking of the alternatives

An illustrative example to see the viability of the proposed algorithm.

6.2. Illustrated Example. An example of technology commercialization is adapted from [30] where the selection of the most favorable software enterprise among a list of enterprises is carried out. Let us consider four software enterprises denoted by $A_{1 \leq k \leq 4}$, and the attributes are denoted by $G_{1 \leq j \leq 4}$, where G_1 represents the advancement of the technology, G_2 represents the market potential, G_3 represents the human resources and financial development, and G_4 represents the creating of employment and development of technology. The weight vector chosen in this case is $\omega = (0.4, 0.3, 0.2, 0.1)^T$. The selection of the weight vector is up to the decision-makers, and the information about the alternatives in terms of LDULNs is given in Table 1.

Step 1: in this step, the information given in Table 1 is aggregated using LDULPEA, LDULPEWA, LDULPEG, and LDULPEWG operators. The results of the aggregation are given in Table 2.

Step 2: in this step, the aggregated information obtained in Table 3 is targeted, and their scores are computed using Definition 3.

Step 3: based on the scores obtained in Table 4, the alternatives are ranked, and the ranking results are given in Table 5.

As shown above, we obtained the same ranking results, which are discussed in the form of Table 4. The best option is A_4 . The results portrayed in Table 3 are further described geometrically in Figure 2.

6.3. Comparative Analysis. Based on the investigated LDULPEA, LDULPEWA, LDULPEG, and LDULPEWG operators, we determined the reliability and consistency of the developed operators with the help of comparative analysis by using the information of Table 1 shown in 6.2. The information related to existing theories is as follows: Riaz and Hashmi [39] elaborated the aggregation operators for LDFS and their applications, Riaz and Hashmi [40] proposed some operators based on linear Diophantine sets, and Wang et al. [42] proposed interaction Hamacher operators for Pythagorean fuzzy sets. The reasonable analysis of the investigated operators and remaining operators are discussed in Table 5 by using the information of Table 1, by using the values of parameters $\bar{f} = \bar{h} = 1, r = 3$.

FIGURE 2: Geometrical expressions of the values in Table 3.

FIGURE 3: Geometrical expressions of the information in Table 5.

The results portrayed in Table 5 are further described geometrically in Figure 3.

As shown above, we chose the linear Diophantine uncertain linguistic types of information, so the operators investigated by Riaz and Hashmi [39] are not able to resolve it. However, if we decide the exiting types of information, then the investigated sort of information can cope with it. Therefore, the investigated operators based on LDULSs are more powerful to determine the rationality and consistency of the developed operators.

7. Conclusion

LDULS is a modified variety of the fuzzy set (FS) to manage problematic and inconsistent information in actual life troubles. LDULS covers the grade of truth, grade of falsity, and their reference parameters with ULT with $0 \leq \alpha_{AMG} u_{AMG}(x) + \beta_{ANG} v_{AMG}(x) \leq 1$, where $0 \leq \alpha_{AMG} + \beta_{ANG} \leq 1$. In this manuscript, the principle of LDULS and their useful laws are elaborated. Additionally, PEAO is a conventional sort of AO utilized in innovative decision-making troubles,

which is effective to aggregate the family of numerical elements. To determine the interrelationship between any numbers of arguments, we elaborate on the LDULPEA, LDULPEWA, LDULPEG, and LDULPEWG operators and discussed their useful results. Conclusively, a decision-making methodology is utilized for the MADM dilemma with elaborated information. A sensible illustration is specified to demonstrate the accessibility and rewards of the intended technique by comparison with certain prevailing techniques. The intended AOs are additional comprehensive than the prevailing ones to exploit the ambiguous and inaccurate knowledge. Numerous remaining operators are chosen as individual incidents of the suggested one. Ultimately, the supremacy and advantages of the elaborated operators are also discussed with the help of geometrical form to show the validity and consistency of explored operators.

In further research, considering the superiority of new LDULSs, we can extend them to some other spherical fuzzy sets [43], spherical linear Diophantine fuzzy sets [44], complex spherical fuzzy sets [45], aggregation operators [42], such as power mean aggregation operators, Bonferroni mean operators, Heronian mean operators, different methods [46–50], and so on [51].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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