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*Research Article*

# On the Boundedness of the Numerical Solutions' Mean Value in a Stochastic Lotka–Volterra Model and the Turnpike Property

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In this paper, we study some properties of the solutions of a stochastic Lotka–Volterra predator-prey model, namely, the boundedness in the mean of numerical solutions, the strong convergence for this kind of solutions, and the turnpike property of solutions of an optimal control problem in a population modelled by a Lotka–Volterra system with stochastic environmental fluctuations. Even though there are numerous results in the deterministic case, there are few results for the behavior of numerical solutions in a population dynamic with random fluctuations. First, we show, using the Euler–Maruyama scheme, that the boundedness of numerical solutions and the convergence of the scheme are preserved in the stochastic case. Second, we analyze a property of the long-term behavior of a Lotka–Volterra system with stochastic environmental fluctuations known as turnpike property. In optimal control theory, the optimal solutions dwell mostly in the neighborhood of a balanced equilibrium path, corresponding to the optimal steady-state solution. Our study shows, by means of the Stochastic Maximum Principle, that this turnpike property is preserved, when the noise in the system is small. Numerical simulations are implemented to support our results.

## 1. Introduction

In 1925, Vito Volterra and Alfred Lotka obtained simultaneously a mathematical model on population dynamics and competition systems [1, 2]. Their models are based on the increment, noted by Umberto D'Ancona [3], of fish population due to reduction in fishing during World War I and the subsequent growth of sharks after the war.

The Lotka–Volterra equations are the simplest predator-prey model of interaction between two populations. This model assumes that the prey population finds food all the time, that the food of the predator depends entirely on the prey population, and that the environment does not fluctuate so that it might influence the two populations. The model consists in a nonlinear ordinary differential equation system [4].

If we denote by $x_1(t)$ and $x_2(t)$ the differentiable functions meaning the density of the population of prey and predator, respectively, the deterministic model is given by

$$\begin{aligned} \mathrm{d}x_1(t) &= (\alpha x_1(t) - \beta x_1(t)x_2(t))\mathrm{d}t, \\ \mathrm{d}x_2(t) &= (-\gamma x_2(t) + \delta x_1(t)x_2(t))\mathrm{d}t, \end{aligned} \tag{1}$$

where $\alpha, \beta, \gamma$, and $\delta$ are positive constants, with $\alpha$ being the intrinsic growth rate of the prey population, $\gamma$ being the intrinsic death rate of the predator population, and $\beta$ and $\delta$ being the contact rates per unit of time between predator and prey, and vice versa, respectively.

There exists no explicit solution for this system [5], but numerical solutions can be found. It is very well known that this system has a stable solution in $(0, 0)$ and that it has a nonasymptotic solution and has a limit cycle [6, 7]. Also, there exist positive solutions, and there are existence and uniqueness of global positive solutions [8]. This deterministic model, however, does not take into account fluctuations in the environment, which play, in general, an important role in any real biological system, presenting disturbances caused by natural random variations in environmental conditions. In the study and modeling of the interaction between populations, it is also important to consider some stochastic factors that have impact on their growth, persistence, and extinction. Demographic and environmental factors, acting as disturbances or diffusion processes, can be modelled by suitable stochastic differential equations, which take into account the fact that independent individual stochastic events can affect each population, so they must contain different diffusion processes. The existence and uniqueness of a global positive solution of the system and the conditions for which extinction occurs for a stochastic prey-predator model have been extensively studied by various methods [9–11], including the Stochastic Maximum Principle (see [12]). In [13], the stochastic uniform boundedness of the solution and the existence of a globally unique positive solution are obtained, for a predator and prey model that incorporates disease invasion and sudden catastrophic shocks. Also, for stochastic predator-prey models with distributed delay and both white and telegraph noises, the existence at least of one positive T-periodic solution has been proved by constructing a stochastic Lyapunov function with regime switching [14]. In [15], a three-species model with time delays and Lévy jumps is investigated, given sufficient conditions for persistence in mean and extinction of three species, one-predator-two-prey. Unlike the present work, they use the discontinuous stochastic process to study some abrupt nature phenomena such as climate change and they do not introduce controls in the species, as we do in this analysis. Likewise, some mutualism systems (with the cooperation of two species) in random environments have been studied, obtaining positive solutions and their uniqueness [16]. However, there is limited mathematical literature on boundedness in the mean and strong convergence for numerical solutions and turnpike property for controlled stochastic systems [17, 18].

Sometimes, the appearance of random fluctuations in the environment alters the population dynamics of deterministic systems, causing the extinction process. Then, the corresponding model may lose the boundedness of the solutions, its stability, or its robustness and the numerical scheme may diverge. Also, when an environmental fluctuation occurs, the stability of the turnpike solution could be altered, causing loss of optimality. Hence, it is important to analyze the persistence of the former properties, because doing so allows us to describe the degeneracy of the properties of the system. A novelty of our analysis is the use of two controls in the populations and the extension of the work of [18] to a stochastic case, by using the Stochastic Maximum Principle. Thus, we have combined some techniques of the Geometric Control Theory with the property of exponential stability of the numerical solution of our stochastic model. We consider that a stochastic framework allows a more realistic study of the population dynamics and competition systems. This consideration is equally valid in other areas where the goal is to obtain the asymptotic stability of the trajectories at late time points [19], under different initial conditions, for example, in the aggregate-growth model in economics [20] or in the analysis and design of schemes of dynamic real-time optimization and economic-model predictive control [21].

To study the stochastic model, we take into account random fluctuations in $\alpha$ and $\beta$ and for the sake of simplicity and to place the equilibrium point of the system at $(1,1)$, we have selected $\alpha = \beta = \gamma = \delta = 1$, following [17]. In addition, we introduce controls $u_1(t)$ and $u_2(t)$ representing, by example, the hunting in each population and two independent random variations in each population, $W_1(t)$ and $W_2(t)$, given by standard Wiener process and defined over a probability space $(\Omega, F, P)$. We have modulated the effect of the controls and the fluctuations with the constants 0.4 and 0.2 for the controls and the coefficients $\alpha_1 \in (0,1]$ and $\alpha_2 \in (0,1]$, for the environmental fluctuations, on the prey and the predator populations, respectively. We obtain the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}x_1(t) &= \left(x_1(t) - x_1(t)x_2(t) - 0.4x_1(t)u_1(t)\right)\mathrm{d}t + \alpha_1 \mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) &= \left(-x_2(t) + x_1(t)x_2(t) - 0.2x_2(t)u_2(t)\right)\mathrm{d}t + \alpha_2 \mathrm{d}W_2(t), \end{aligned} \tag{2}$$

with the conditions

$$\begin{aligned} x_1(0) &= 0.7, \\ x_2(0) &= 0.5, \\ x_1(T) &= 0.7, \\ x_2(T) &= 0.5, \end{aligned} \tag{3}$$

where $0.4x_1(t)u_1(t)$ represents the moderate hunting of the prey, by a factor of 0.4, and $0.2x_2(t)u_2(t)$ represents the moderate hunting of the predator, by a factor of 0.2. The maximum possible value that these constants can take is 1, which corresponds to the total hunting of the species. Thus, we have the following stochastic optimal control problem: to find the controls $u_1(t)$ and $u_2(t)$ and the states $x_1(t)$ and $x_2(t)$ of system (2) which minimize the cost functional

$$J(u_1, u_2) = \mathbb{E}\left\{\frac{1}{2}\int_0^T \left(x_1^2(t) + x_2^2(t) + u_1^2(t) + u_2^2(t)\right)\mathrm{d}t\right\}. \tag{4}$$

Stochastic differential system (2) is of the general type

$$dx = f(t, x(t), u(t))dt + g(t, x(t), u(t))dW(t), \quad (5)$$

where $u(t) = (u_1(t), u_2(t))$, $f(t, x, u) = (f^1(t, x, u), \ldots, f^n(t, x, u))^\top$ is a measurable function defined for $(t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^m$-valued, known as the drift, $u: \mathbb{R} \longrightarrow \mathbb{R}^m$ is a measurable function called the control, and $g(t, x, u) = (g^1(t, x, u), \ldots, g^m(t, x, u))$, with $g^j(t, x, u) = (g^{1j}(t, x, u), \ldots, g^{nj}(t, x, u))^\top$, where $1 \leq j \leq m$, being a measurable function defined also on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^{n\times m}$-valued ($n \times m$-real matrix), called the diffusion coefficient. In this work,

$$f(t, x, u) = \begin{pmatrix} x_1(t) - x_1(t)x_2(t) - 0.4x_1(t)u_1(t) \\ -x_2(t) + x_1(t)x_2(t) - 0.2x_2(t)u_2(t) \end{pmatrix},$$
$$g(t, x, u) = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}. \quad (6)$$

To find the optimal control that minimizes (4) in this optimal control problem, we use the Pontryagin Maximum Stochastic Principle, setting forward differential stochastic equations (2) and the following backward differential stochastic equations or terminal value problem:

$$dp(t) = -\left\{ f_x(t, x, u)^\top p(t) + \sum_{j=1}^{m} g_x^j(t, x(t), u(t))^\top q_j(t) - (f_0(t, x(t), u(t)))_x \right\} dt + q(t)dW(t), \; p(T) = (0.5, 0.7)^\top, \quad (7)$$

where

$$f_0(t, x, u) = \frac{1}{2}\left(x_1^2(t) + x_2^2(t) + u_1^2(t) + u_2^2(t)\right). \quad (8)$$

$p$ is the adjoint variable and $q$ is a matrix given by

$$p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix},$$
$$q(t) = \begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix}, \quad (9)$$
$$dW(t) = \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}.$$

In our case, we have

$$\left.\begin{array}{l} dp_1(t) = (x_1(t) - p_1(t) + p_1(t)x_2(t) + 0.4p_1(t)u_1(t) - p_2(t)x_2(t))dt + q_{11}(t)dW_1(t) + q_{12}(t)dW_2(t) \\ dp_2(t) = (x_2(t) + p_2(t) + p_1(t)x_1(t) - p_2(t)x_1(t) + 0.2p_2(t)u_2(t))dt + q_{21}(t)dW_1(t) + q_{22}(t)dW_2(t) \\ p_1(T) = 0.5 \\ p_2(T) = 0.7 \end{array}\right\}. \quad (10)$$

Analytic explicit solutions are not known for coupled (2) and (10), and they must be solved numerically. Therefore, let us find the necessary conditions for the controls in terms of the adjoint variable. We define the following extended Hamiltonian equation:

$$H(t, x(t), p(t), q(t)) = \langle p(t), f(t, x, u)\rangle + tr[q(t)g(t)] - f_0(t, x, u). \quad (11)$$

We obtain

$$H = x_1(t)p_1(t) - x_1(t)x_2(t)p_1(t) - 0.4x_1(t)p_1(t)u_1(t) - x_2(t)p_2(t) + x_1(t)x_2(t)p_2(t) - 0.2x_2(t)p_2(t)u_2(t) + \alpha_1 q_{11}(t) + \alpha_2 q_{22}(t) - \frac{1}{2}\left(x_1^2(t) + x_2^2(t) + u_1^2(t) + u_2^2(t)\right). \quad (12)$$

Thus,

$$H_{u_1} = -0.4x_1(t)p_1(t) - u_1(t),$$
$$H_{u_2} = -0.2x_2(t)p_2(t) - u_2(t). \tag{13}$$

And, according to the necessary conditions of the Stochastic Maximum Principle, we have

$$\left.\begin{aligned} u_1(t) &= -0.4p_1(t)x_1(t) \\ u_2(t) &= -0.2p_2(t)x_2(t) \end{aligned}\right\}. \tag{14}$$

As we have already mentioned, systems (2), (10), and (14) can be solved by numerical methods, like the Euler–Maruyama scheme.

The Euler–Maruyama scheme, corresponding to (11), is the simplest effective computational method used in stochastic differential equations. The Euler–Maruyama approximation is a continuous time stochastic process $x$, obtained by truncating Itô's formula of the stochastic Taylor series after the first terms. For each $k \geq 1$, it computes the approximations $x_k \approx x(k\Delta t_k)$. We select a grid of $[0, T]$:

$$0 \leq t_0 < \cdots < t_N = T, \tag{15}$$

defining

$$\Delta t_k = t_{k+1} - t_k,$$
$$\Delta W_k = W_{k+1} - W_k. \tag{16}$$

We denote $x(t_k)$ by $x_k$ and $x_0$ (the initial value). Therefore, we have, for $k = 1, \ldots N$,

$$x_{k+1} = x_k + f(x_k, t_k, u_k)\Delta t_k + g(x_k, t_k, u_k)\Delta W_k, \tag{17}$$

where $x_k = x(t_k)$ and we can consider

$$\Delta W_k = \sqrt{\Delta t_k} s_k, \tag{18}$$

with $s_i$ being a random real number in the interval $[0, 1]$. In this paper, we will consider an equidistant discretization of the time and $\Delta W_k \leq l$, with $l$ constant.

## 2. Properties of Solutions of the Stochastic Lotka–Volterra Model

We are interested in the numerical solutions of system (2) and their properties, such as boundedness and convergence. Thus, we introduce the following assumptions:

(i) (H1) $f(x, t, u)$ and $g(x, t, u)$ satisfy the Lipschitz and linear growth conditions: there exist constants $C_1$ and $C_2$, such that

$$\|f(x, t, u) - f(y, t, u)\| \leq C_1\|x - y\|,$$
$$\|g(x, t, u) - g(y, t, u)\| \leq C_2\|x - y\|. \tag{19}$$

(i) (H2) There exists a constant $C_3$, such that $\|u_i(t)\| \leq C_3$, for $i = 1, 2$.

(ii) (H3) There exists a constant $C_4$, such that $E\|x_0\|^2 \leq C_4$, where $E\|x_0\|^2$ is the expected value of $\|x_0\|^2$.

(iii) (H4) By writing $x_k = (x_{k,1}, x_{k,2})$ and $p_k = (p_{k,1}, p_{k,2})$, there exist constants $b, c, d,$ and $e$, such that, for $\forall k \geq 1, |x_{k+1,1}| \leq b$, $|x_{k+1,2}| \leq c$, $|p_{k+1,1}| \leq d$, and $|p_{k+1,2}| \leq e$.

Hence, the Euler–Maruyama scheme can be expressed in our case by the following systems:

$$\left.\begin{aligned} x_{k,1} &= x_{k-1,1} + \left(x_{k,1} - x_{k,1}x_{k,2} - 0.4x_{k,1}u_{k,1}\right)\Delta t_k + \Delta W_k \\ x_{k,2} &= x_{k-1,2} + \left(-x_{k,2} + x_{k,1}x_{k,2} - 0.2x_{k,2}u_{k,2}\right)\Delta t_k + \Delta W_k \\ p_{k,1} &= p_{k-1,1} + \left(x_{k,1} - p_{k,1} + p_{k,1}x_{k,2} + 0.4p_{k,1}(t)u_{k,1} - p_{k,2}x_{k,2}\right)\Delta t_k + p_{k,1}\Delta W_k \\ p_{k,2} &= p_{k-1,2} + \left(x_{k,2} + p_{k,2} + p_{k,1}x_{k,1} + 0.2p_{k,2}u_{k,2}\Delta t_k + p_{k,2}\Delta W_k\right) \end{aligned}\right\}, \tag{20}$$

where

$$u_{k,1} = -0.4p_{k,1}x_{k,1},$$
$$u_{k,2} = -0.2p_{k,2}x_{k,2}. \tag{21}$$

We observe that our functions $f(x, t, u)$ and $g(y, t, u)$ satisfy assumption (H1), because, for $f_1(x, u, t)$, we have

$$\begin{aligned} \|f_1(x, u, t) - f_1(y, u, t)\| &= \|[1 - 0.4u_1(t)]x_1(t) - x_1(t)x_2(t) - [1 - 0,4u_1(t)]y_1(t) - y_1(t)y_2(t)\| \\ &\leq \|[1 - 0.4u_1(t)](x_1(t) - y_1(t)) + y_1(t)y_2(t) - x_1(t)x_2(t)\| \\ &\leq L_1\|x_1(t) - y_1(t)\| + \|y_1(t)y_2(t) - x_1(t)x_2(t)\| \\ &\leq L_1\|x_1(t) - y_1(t)\| + \|y_1(t) - x_1(t)\|\|y_2(t) - x_2(t)\| \\ &\leq L_1\|x_1(t) + y_1(t)\| + L_2\|x_2(t) - y_2(t)\| \\ &\leq C_1\left(\|x_1(t) - y_1(t)\| + \|x_2(t) - y_2(t)\|\right) \\ &\leq C_1\|x - y\|, \end{aligned} \tag{22}$$

for constants $L_1, L_2$, and $C_1$. Similarly, for $f_2(x,u,t)$, it is trivial for $g$.

Now, we have our first result.

**Theorem 1.** *Let $x_k$ and $p_k$ be the numerical solution of Euler–Maruyama scheme (20); then, under (H1)–(H4) assumptions, there exist positive constants $M_0$ and $N_0$, such that*

$$\begin{aligned} E|x_k|^2 &\le M_0, \\ E|p_k|^2 &\le N_0. \end{aligned} \tag{23}$$

*Proof.* Observing expressions (20), we can use the following inequality, for any real numbers $\gamma, \theta$, and $\eta$:

$$|\gamma+\theta+\eta|^2 \le 3\left(|\gamma|^2+|\theta|^2+|\eta|^2\right), \tag{24}$$

to obtain the constants $b$ and $c$ linked to assumption $H_4$, from additional constants $r$ and $l$ and constants $C_3$ and $C_4$ of assumptions $H_2$ and $H_3$. Furthermore,

$$\begin{aligned} E|x_{k+1,1}|^2 &\le 3E|x_{0,1}|^2 + 3E\left|\left(x_{k+1,1} - x_{k+1,1}x_{k+1,2} - 0.4x_{k+1,1}\right)u_{k+1,1}\Delta_{k+1}\right|^2 \\ &\le 3E|\Delta W_{k+1}|^2 + 3E\left[|x_{k+1,1} - x_{k+1,1}x_{k+1,2}|^2\right. \\ &\quad +|0.4x_{k+1,1}u_{k+1,1}|^2 + 3E|x_{0,1}|^2 + 3E|\Delta W_{k+1}| \\ &\quad \left.+2|x_{k+1,1} - x_{k+1,1}x_{k+1,2}||0.4x_{k+1,1}u_{k+1,1}|\Delta_{k+1}\right] \\ &\le 3E|x_{0,1}|^2 + 3r\left(E\left[|x_{k+1,1}|^2 + |x_{k+1,1}x_{k+1,2}|^2 + 2|x_{k+1,1}|^2|x_{k+1,2}|\right.\right. \\ &\quad \left.\left.+0.4|x_{k+1,1}||u_{k+1,1}|^2 + 2|x_{k+1,1}| + 2|x_{k+1,1}||x_{k+1,2}|\right]\right) + 3E|\Delta W_{k+1}|^2 \\ &\le 3C_4^2 + 3r\left(b^2 + 3b^2c^2 + 0.4b^2C_3^2 + 2b + 2bc\right) + 3l. \end{aligned} \tag{25}$$

Also, for $x_{k+1,2}$, we have

$$\begin{aligned} E|x_{k+1,2}|^2 &\le 3E|x_{0,2}|^2 + 3E\left[|-x_{k+1,2} + x_{k+1,1}x_{k+1,2}\right. \\ &\quad \left.-0.2x_{k+1,2}u_{k+1,2}|\Delta_{k+1}|^2\right] + 3E|\Delta W_{k+1}|^2 \\ &\le 3E|x_{0,2}|^2 + 3E\left[|-x_{k+1,2}|^2 + |x_{k+1,1}x_{k+1,2} - 0.2x_{k+1,2}u_{k+1,2}|^2\right. \\ &\quad \left.+2|-x_{k+1,2}||x_{k+1,1}x_{k+1,2} - 0.2x_{k+1,2}u_{k+1,2}|\right] + 3E|\Delta W_{k+1}|^2 \\ &\le 3E|x_{0,2}|^2 + 3r\left(E\left[|-x_{k+1,2}|^2 + |x_{k+1,1}x_{k+1,2}|^2\right.\right. \\ &\quad + 2\left(|x_{k+1,1}x_{k+1,2} + 0.2|x_{k+1,2}u_{k+1,2}|\right) + |-0.2x_{k+1,2}u_{k+1,2}|^2 \\ &\quad \left.\left.+2|x_{k+1,2}|\left(|x_{k+1,1}x_{k+1,2}| + 0.2|x_{k+1,2}u_{k+1,2}|\right)\right]\right) + 3E|\Delta W_{k+1}|^2 \\ &\le 3C_4^2 + 3r\left(b^2 + b^2c^2 + 2\left(bc + 0.2bC_3^2\right) + 0.2c^2C_3^2 - 2c^2b + 0.2cC_3^2\right) + 3l^2. \end{aligned} \tag{26}$$

Finally, defining

$$\begin{aligned} M_{1,0} &= 3C_4^2 + 3r\left(b^2 + 3b^2c^2 + 0.4b^2C_3^2 + 2b + 2bc\right) + 3l, \\ M_{2,0} &= 3C_4^2 + 3r\left(b^2 + b^2c^2 + 2\left(bc + 0.2bC_3^2\right) + 0.2c^2C_3^2 - 2c^2b + 0.2cC_3^2\right) + 3l^2, \end{aligned} \tag{27}$$

and $M_0 = (M_{1,0}, M_{2,0})$, we have in the lexicographic order

$$E|x_k|^2 \le M_0 = (M_{1,0}, M_{2,0}). \quad (28)$$

Similarly for $p_{k+1,1}$ and $p_{k+1,2}$, we find constants $N_{1,0}$ and $N_{2,0}$, such that

$$E|p_k|^2 \le N_0 = (N_{1,0}, N_{2,0}), \quad (29)$$

concluding the proof. □

**Theorem 2.** *Let $x_k$ and $p_k$ be the numerical solution of the Euler–Maruyama scheme (20); then, under assumptions (H1)–(H4), there exist positive constants $C_5$ and $C_6$, given by assumption (H1), and there exist a constant P, such that*

$$E|x_k - x(t_k)|^2 \le P\Delta t e^{(2T+2)(C_5+C_6)}. \quad (30)$$

*Proof.* We define the process

$$Z_1(t) = \sum_{k=0}^{\infty} x_k \mathbf{1}_{[k\Delta t,(k+1)\Delta t]}(t), \quad (31)$$

and we assume $x_0 \neq x(0)$. For $0 \le t_{k+1} \le T$, we have

$$x_{k+1} - x(t_{k+1}) = x_0 - x(0) + \int_0^{t_{k+1}} [f(Z_1(s)) - f(x(s))]ds + \int_0^{t_{k+1}} [g(Z_1(s)) - g(x(s))]dW(s). \quad (32)$$

Hence, using (24) and Schwarz inequality, it follows that

$$E|x_{k+1} - x(t_{k+1})|^2 \le 3\Delta t E|x_0 - x(0)|^2 + 3E\left|\int_0^{t_{k+1}} [f(Z_1(s) - f(x(s))]ds\right|^2 + 3E\left|\int_0^{t_{k+1}} [g(Z_1(s) - g(x(s))]dW(s)\right|^2. \quad (33)$$

By Itô isometry

$$E|x_{k+1} - x(t_{k+1})|^2 \le 3\Delta t E|x_0 - x(0)|^2 3(T)\int_0^{t_{k+1}} E|f(Z_1(s)) - f(x(s))|^2 ds + 3\int_0^{t_{k+1}} E|g(Z_1(s)) - g(x(s))|^2 ds. \quad (34)$$

And, by the Lipschitz assumption, we obtain

$$\begin{aligned} E|x_{k+1} - x(t_{k+1})|^2 &\le 3\Delta t E|x_0 - x(0)|^2 \\ &\quad + (3T+3)\int_0^{t_{k+1}} (C_5 + C_6)E|Z_1 - x(s)|^2 ds \\ &\le 3\Delta t E|x_0 - x(0)|^2 \\ &\quad + (3T+3)(C_5 + C_6)\sum_{j=0}^{k} E|x_j - x(t_j)|. \end{aligned} \quad (35)$$

Therefore, if $|x_0 - x(0)| \le P$, by the discrete case of Gronwall inequality, we obtain

$$E|x_{k+1} - x(t_{k+1})|^2 \le 3P\Delta t e^{(2T+2)(C_5+C_6)}. \quad (36)$$

Hence, for $\Delta t \longrightarrow 0$, we obtain the strong convergence to the exact solution. □

## 3. Stochastic Turnpike Property

In this section, we analyze the stability of optimal-trajectory turnpike property of the solutions of the stochastic Lotka–Volterra model. The turnpike property means that the most important fact about the behavior of solutions is the optimality criterion considered and the choice of time interval or the data used are irrelevant, for times far from the endpoints of the time interval. This property is a characteristic of the turnpike theory which was introduced by Samuelson in mathematical economics and recently has been reconsidered in Control Theory by the authors of [18, 22]. The turnpike property of a solution in an optimal control problem means that an optimal trajectory for most of the time could stay in a neighborhood of a balanced equilibrium path, corresponding to the optimal steady-state solution.

A general result on the turnpike property for nonlinear optimal control systems, positing that the optimal trajectory, the optimal control, and the corresponding adjoint vector remain exponentially close to a steady state, was proved in [18]. In particular, for a Lotka–Volterra system, the study in [17] reports this behaviour. In Figures 1–4, we can observe the turnpike property for state transfer $x_1$, $x_2$, and $(x_1, x_2)$, optimal controls $u_1$ and $u_2$, and adjoint states $p_1$ and $p_2$, respectively, in the numerical solution of system (1), for values of parameters $\alpha = \beta = \gamma = \delta = 1$, using the Euler–Maruyama scheme [17]. Here, we request that the turnpike property for numerical solutions should prevail in a Lotka–Volterra stochastic model, and for this purpose, we will use the approach used in [18].

We claim that the following theorem holds.

**Theorem 3.** *The solution of the optimal stochastic control problem*

,

$$\begin{aligned}
&\min\left\{J(u_1,u_2)=\mathbb{E}\left(\frac{1}{2}\int_0^T\left(x_1^2(t)+x_2^2(t)+u_1^2(t)+u_2^2(t)\right)\mathrm{d}t\right\}\text{a.s.}\\
&\mathrm{d}x_1(t)=\left(x_1(t)-x_1(t)x_2(t)-0.4x_1(t)u_1(t)\right)\mathrm{d}t+\alpha_1\mathrm{d}W_1(t),\\
&\mathrm{d}x_2(t)=\left(-x_2(t)+x_1(t)x_2(t)-0.2x_2(t)u_2(t)\right)\mathrm{d}t+\alpha_2\mathrm{d}W_2(t),\\
&x_1(0)=0.7,\\
&x_2(0)=0.5,\\
&x_1(T)=0.7,\\
&x_2(T)=0.5,
\end{aligned}\tag{37}$$

*satisfies the following property, called the turnpike property. Setting* $x=(x_1,x_2)$, $u=(u_1,u_2)$, *and* $p=(p_1,p_2)$, *there exist* $C_7$, $C_8$, *and* $C_9$ *positive constants, such that*

$$\left\|x_T(t)-\overline{x}\right\|+\left\|p_T(t)-\overline{p}\right\|+\left\|u_T(t)-\overline{u}\right\|\le(C_7+C_9)e^{-C_8t},\quad\forall t\in[0,T],\tag{38}$$

*where* $\overline{x},\overline{p}$, *and* $\overline{u}$ *are the static solutions of the optimal stochastic control problem, which means,*

$$\begin{aligned}
&\min\mathbb{E}\left\{\frac{1}{2}\left(x_1^2(t)+x_2^2(t)+u_1^2(t)+u_2^2(t)\right)\right\}\text{ a.s.}\\
&0=\left(x_1(t)-x_1(t)x_2(t)-0.4x_1(t)u_1(t)\right)\mathrm{d}t+\alpha_1\mathrm{d}W_1(t),\\
&0=\left(-x_2(t)+x_1(t)x_2(t)-0.2x_2(t)u_2(t)\right)\mathrm{d}t+\alpha_2\mathrm{d}W_2(t),
\end{aligned}\tag{39}$$

$$\begin{aligned}
\left\|u_1(t)\right\|&\le 1,\\
\left\|u_2(t)\right\|&\le 1.
\end{aligned}\tag{40}$$

*Proof.* System (37) has the following general form:

$$\mathrm{d}x=f(t,x(t),u(t))\mathrm{d}t+g(t,x(t),u(t))\mathrm{d}W(t),\tag{41}$$

with Hamiltonian equation (11). Following the idea in [18], we will keep this general form in our optimal stochastic control problem. We consider the solution to a static problem associated with (37), $(\overline{x},\overline{p},\overline{q},\overline{u})$, and the perturbation of variables $(x,p,q,u)$:

$$\begin{aligned}
x_T(t)&=\overline{x}(t)+\delta x(t),\\
p_T(t)&=\overline{p}(t)+\delta p(t),\\
u_T(t)&=\overline{u}(t)+\delta u(t),\\
q_T(t)&=\overline{q}(t)+\psi(t)\delta u(t),
\end{aligned}\tag{42}$$

where $\psi(t)$ is white noise, getting the Hamiltonian perturbed:

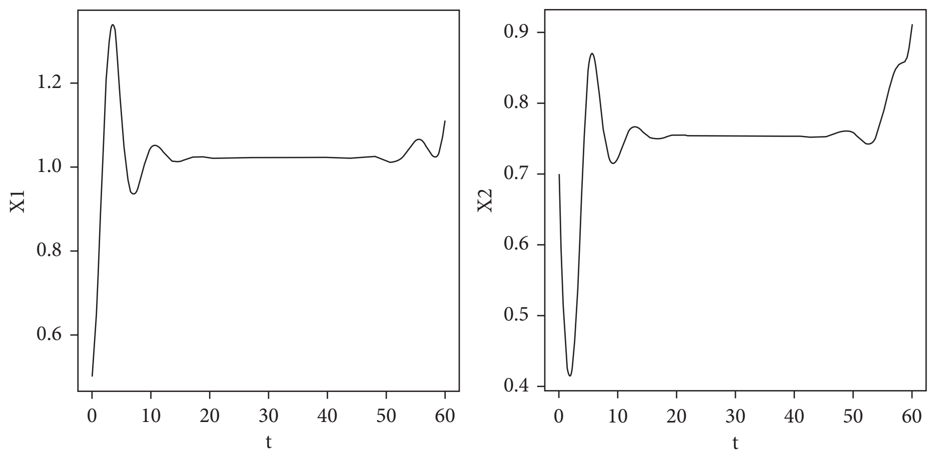


Figure 1: State transfer limit trajectory ($x_1(t)$, $x_2(t)$), using the Euler–Maruyama scheme, 16th iteration.

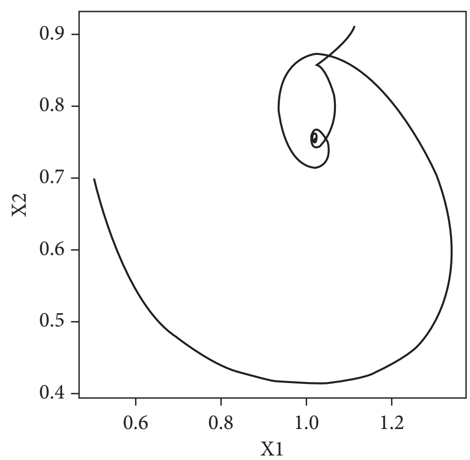


Figure 2: State transfer $(x_1, x_2)$-limit trajectory in the phase space, using the Euler–Maruyama scheme, 16th iteration.

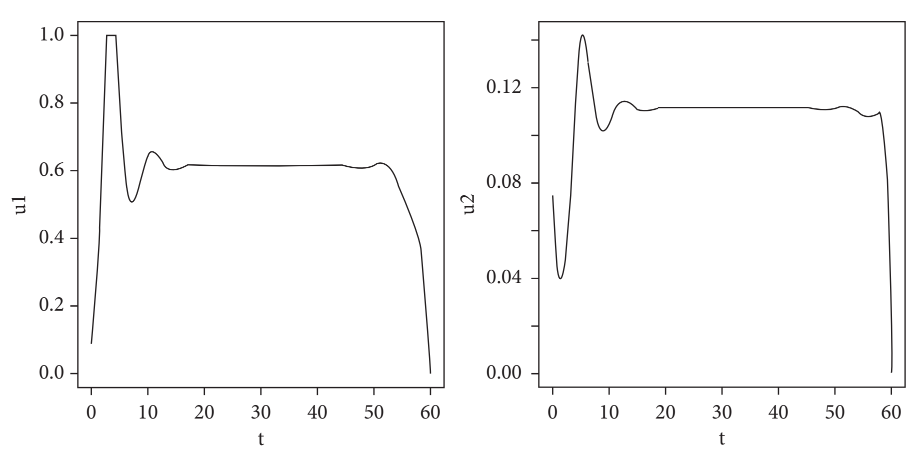


Figure 3: Optimal controls $u_1(t)$ and $u_1(t)$, using the Euler–Maruyama scheme, 16th iteration.

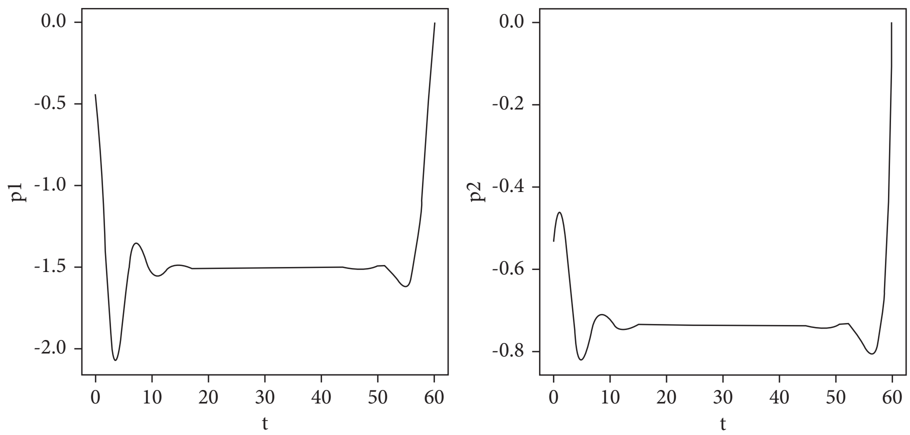


FIGURE 4: Adjoint states $p_1(t)$ and $p_2(t)$, using the Euler–Maruyama scheme, 16th iteration.

$$\delta H = H_x \delta x + H_u \delta u + H_p \delta p + H_q \psi(t) \delta q. \tag{43}$$

Thus, according to the Stochastic Maximum Principle,

$$\left.\begin{aligned} dx_T(t) &= \frac{\partial H}{\partial p} dt + \frac{\partial H}{\partial q} dW(t) \\ dp_T(t) &= -\frac{\partial H}{\partial x} dt + q(t) dW(t) \\ 0 &= \frac{\partial H}{\partial u} \end{aligned}\right\}, \tag{44}$$

which leads to

$$\begin{aligned} H_p(\overline{x}, \overline{p}, \overline{q}, \overline{u}) &= 0, \\ -H_x(\overline{x}, \overline{p}, \overline{q}, \overline{u}) &= 0, \\ H_q(\overline{x}, \overline{p}, \overline{q}, \overline{u}) &= 0, \\ H_u(\overline{x}, \overline{p}, \overline{q}, \overline{u}) &= 0, \end{aligned} \tag{45}$$

and then, by (43),

$$\delta u(t) = -(H_{uu})^{-1}\left(H_{ux}\delta x(t) + H_{up}\delta p(t) + H_{uq}\psi(t)\delta q(t)\right). \tag{46}$$

Hence, it follows that

$$\begin{aligned} \delta\left(\frac{dx}{dt}\right) &= \delta\left(\frac{\partial H}{\partial p}\right) + \delta\left(\frac{\partial H}{\partial q}\right) \\ &= -H_{px}\delta p - H_{ux}\left(-(H_{uu})^{-1}\left(H_{ux}\delta x + H_{up}\delta p + H_{uq}\psi(t)\delta q(t)\right)\right), \\ \delta\left(\frac{dp}{dt}\right) &= -\delta\left(\frac{\partial H}{\partial x}\right) \\ &= H_{xp}\delta x + H_{up}\left(-(H_{uu})^{-1}\left(H_{ux}\delta x + H_{up}\delta p + H_{uq}\psi(t)\delta q(t)\right)\right). \end{aligned} \tag{47}$$

Now, however,

$$\begin{aligned}
\delta H_x &= H_{xx}\delta x + H_{ux}\delta u + H_{px}\delta p + H_{qx}\delta q \\
&= H_{xx}\delta x + H_{ux}\left(-H_{uu}^{-1}\left[H_{xu}\delta x(t) + H_{pu}\delta p(t) + H_{qu}\delta q(t)\right]\right) \\
&\quad + H_{px}\delta p + H_{qx}\delta q \\
&= \left(H_{xx} - H_{ux}H_{uu}^{-1}H_{ux}\right)\delta x + \left(H_{px} - H_{ux}H_{uu}^{-1}H_{pu}\right)\delta p \\
&\quad + \left(H_{qx} - H_{ux}H_{uu}^{-1}H_{qu}\right)\delta q, \\
\delta H_p &= H_{xp}\delta x + H_{up}\delta u + H_{pp}\delta p + H_{qp}\delta q \\
&= H_{xp}\delta x + H_{up}\left(-(H_{uu})^{-1}\left[H_{xu}\delta x(t) + H_{pu}\delta p(t) + H_{qu}\delta q(t)\right]\right) \\
&\quad + H_{pp}\delta p + H_{qp}\delta q \\
&= \left(H_{xp} - H_{up}H_{uu}^{-1}H_{xu}\right)\delta x - \left(H_{up}H_{uu}^{-1}H_{pu}\right)\delta p \\
&\quad + \left(H_{qp} - H_{up}H_{uu}^{-1}H_{qu}\right)\delta q.
\end{aligned} \tag{48}$$

Therefore,

$$\begin{aligned}
\delta x(t) &= A\delta x + B\delta p + C\delta q, \\
\delta p(t) &= D\delta\ x + E\delta p + F\delta q,
\end{aligned} \tag{49}$$

where $A, B, C, D, E,$ and $F$ are the matrices

$$\begin{aligned}
A &= H_{xx} - H_{ux}H_{uu}^{-1}H_{ux}, \\
B &= H_{px} - H_{ux}H_{uu}^{-1}H_{pu}, \\
C &= H_{qx} - H_{ux}H_{uu}^{-1}H_{qu}, \\
D &= -H_{xp} - H_{up}H_{uu}^{-1}H_{xu}, \\
E &= -H_{up}H_{uu}^{-1}H_{pu}, \\
F &= -H_{qp} - H_{up}H_{uu}^{-1}H_{qu}.
\end{aligned} \tag{50}$$

Denoting $Z(t) = (\delta x(t), \delta p(t))^{\top}$ and considering $\mathrm{d}W = \psi(t)\delta q$, system (49) is equivalent to the following stochastic system:

$$\mathrm{d}Z(t) = MZ(t)\mathrm{d}t + Q\mathrm{d}W, \tag{51}$$

where $M = \begin{pmatrix} B & -CH_{uu}^{-1}C^{\dagger} \\ A & D \end{pmatrix}$ and $Q = \begin{pmatrix} E \\ F \end{pmatrix}$.

Therefore, writing $M = M_0 + F(Z(s))$, we consider the mild solution of (51):

$$Z(t) = e^{M_0(t-t_0)}Z_0 + \int_{t_0}^{t} e^{M_0(t-s)}F(Z(s))\mathrm{d}s + \int_{t_0}^{t} e^{M_0(t-s)}Q\mathrm{d}W(s), \tag{52}$$

for $M_0$ constant. By Schwarz inequality, it follows that

$$\begin{aligned}
E\|Z(t)\|^2 \leq 3\Bigg(& E\left\|e^{M_0(t-t_0)}Z(0)\right\|^2 \\
&+ E\left\|\int_{t_0}^{t} e^{M_0(t-s)}MZ(s)\mathrm{d}s\right\|^2 + E\left\|\int_{t_0}^{t} e^{M_0(t-s)}Q\mathrm{d}W(s)\right\|^2\Bigg).
\end{aligned} \tag{53}$$

Using Itô isometry,

$$\begin{aligned}
E\|Z(t)\|^2 \leq 3\Bigg(& E\left\|e^{C_8(t-t_0)}Z(0)\right\|^2 \\
&+ \left\|e^{C_8 t}\right\|^2 \int_{t_0}^{t} \left\|e^{-M_0 s}\right\|^2 E\|MZ(s)\|^2\mathrm{d}s + \left\|e^{C_8 t}\right\|^2 \int_{t_0}^{t} \left\|e^{-M_0 s}\right\|^2 E\|Q\|^2\mathrm{d}s\Bigg),
\end{aligned} \tag{54}$$

for some constant $C_8$. By Gronwall inequality,

$$E\|Z(t)\|^2 \le C_9 e^{C_2 t} \int_{t_0}^{t} \left( \|e^{-M_0 s}\|^2 E\|MZ(s)\|^2 + E\|Z(s)\|^2 \right) \mathrm{d}s, \tag{55}$$

for a constant $C_9$. Thus,

$$E\|Z(t)\|^2 \le C_9 e^{-C_8 t}. \tag{56}$$

On the other hand, for Hamiltonian (12), we have $H_{uq} = 0$, and then, using (46),

$$\delta u(t) = -(H_{uu})^{-1}\left(H_{ux}\delta x(t) + H_{up}\delta p(t)\right). \tag{57}$$

Also, we can estimate $\|\delta u\|^2$. Let $M$ and $K$ be constants defined by $L = \|H_{uu}^{-1}\|$ and $M = \max\left\{\|H_{up}\|, \|H_{ux}\|\right\}$, so

$$E\|\delta u(t)\|^2 \le L^2 M^2 C_9 e^{-C_8 t}, \tag{58}$$

and finally, for a constant $C_7$, we arrive at

$$E\|x(T) - \overline{x}\|^2 + E\|p(T) - \overline{p}\|^2 + E\|u(T) - \overline{u}\|^2 \le C_9 e^{-C_8 t} + C_7 e^{-C_8 t}. \tag{59}$$

Therefore, the solution of system (41) has the turnpike property. In particular, the numerical solution of system (37) has the turnpike property. □

*Observation 1.* Restriction (40) for the controls is related to the hunting of the species in the Lotka–Volterra model; however, in a more general context, it is possible not to consider it, since the theorem will remain valid.

*Example 1.* We consider the following stochastic optimal control problem system:

$$\begin{aligned} &\min J(u_1, u_2) = \mathbb{E}\left\{\frac{1}{2}\int_0^T \left(x_1^2(t) + x_2^2(t) + u_1^2(t) + u_2^2(t)\right)\mathrm{d}t\right\} \text{a.s.} \\ &\mathrm{d}x_1(t) = \left(x_1(t) - x_1(t)x_2(t) - 0.4x_1(t)u_1(t)\right)\mathrm{d}t + \alpha_1 \mathrm{d}W_1(t), \\ &\mathrm{d}x_2(t) = \left(-x_2(t) + x_1(t)x_2(t) - 0.2x_2(t)u_2(t)\right)\mathrm{d}t + \alpha_2 \mathrm{d}W_2(t). \end{aligned} \tag{60}$$

By applying the Stochastic Maximum Principle, we obtain the following adjoint system:

$$\begin{aligned} \mathrm{d}p_1(t) &= \left(x_1(t) - p_1(t) - p_1(t)x_2(t) + 0.4p_1(t)u_1(t) - p_2(t)x_2(t)\right)\mathrm{d}t + p_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}p_2(t) &= \left(x_2(t) + p_2(t) + p_1(t)x_1(t) - p_2(t)x_1(t) + 0.2p_2(t)u_2(t)\right)\mathrm{d}t + p_2(t)\mathrm{d}W_2(t), \end{aligned} \tag{61}$$

with the static problem associate

$$\begin{aligned} &\min \mathbb{E}\left\{\frac{1}{2}\left(x_1^2(t) + x_2^2(t) + u_1^2(t) + u_2^2(t)\right)\right\} \text{a.s. } \{x_1, x_2, u_1, u_2\}, \\ &0 = \left(x_1(t) - x_1(t)x_2(t) - 0.4x_1(t)u_1(t)\right)\mathrm{d}t + \alpha_1 \mathrm{d}W_1(t), \\ &0 = \left(-x_2(t) + x_1(t)x_2(t) - 0.2x_1(t)u_2(t)\right)\mathrm{d}t + \alpha_2 \mathrm{d}W_2(t), \\ &\left\|u_1(t)\right\| \le 1, \\ &\left\|u_2(t)\right\| \le 1, \end{aligned} \tag{62}$$

and we arrive at

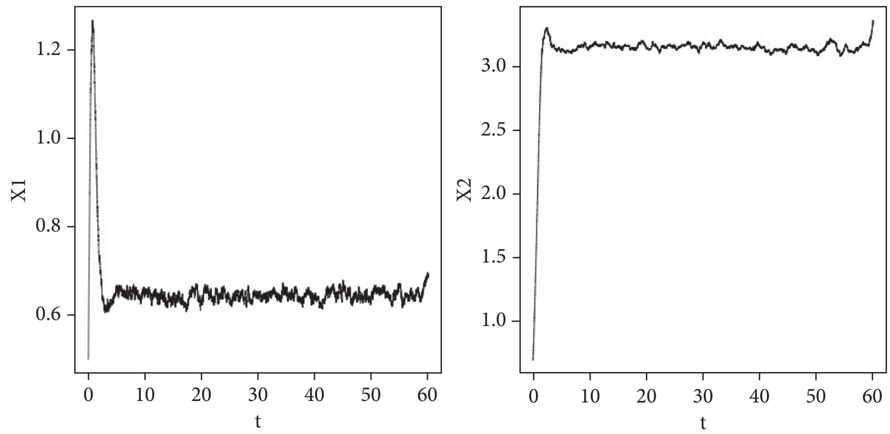


Figure 5: Stochastic state transfer limit trajectory $(x_1(t), x_2(t))$, using the Euler–Maruyama scheme, 12th iteration.

$$\begin{aligned} 0 &= x_1(1 - x_2 - 0.4u_1), \\ 0 &= x_2(-1 + x_1 - 0.2u_2), \\ 1 &\geq u_1, u_2. \end{aligned} \tag{63}$$

By using Lagrange multipliers method, we obtain

$$\begin{aligned} 0 &= p_1(1 - x_2 - 0.4u_1) + p_2x_2 - x_1, \\ 0 &= p_2(-1 + x_1 - 0.2u_2) - p_1x_1 - x_2, \\ 0 &= -u_1 - 0.4p_1x_1, \\ 0 &= -u_2 - 0.2p_2x_2. \end{aligned} \tag{64}$$

We calculate the solutions using ©Stephen Wolfram, LLC, obtaining the values: $\overline{x}_1 = 0.9615, \overline{x}_2 = 0.8621, \overline{p}_1 = -0.8966, \overline{p}_2 = 1.1154, \overline{u}_1 = 0.3448$, and $\overline{u}_2 = -0.1923$. Hence, it follows that $B = \begin{pmatrix} 0.13795 & 0 \\ 0 & 0.32306 \end{pmatrix}$, $C = 0$, and $B - CH_{uu}^{-1}C^{\dagger} = \begin{pmatrix} 0.13795 & 0 \\ 0 & 0.32306 \end{pmatrix}$. The eigenvalues of $B - CH_{uu}^{-1}C^{\dagger}$ are $\tau_1 = 0.13795$ and $\tau_2 = 0.32306$. Finally, we arrived at

$$E\|x(T) - \overline{x}\|^2 + E\|p(T) - \overline{p}\|^2 + E\|u(T) - \overline{u}\|^2 \leq C_3 e^{-C_2 t}, \tag{65}$$

with $C_1 = o(3)$ and $C_2 = 0.32306$, which proves that system (60) has the turnpike property.

## 4. Numerical Simulations

We have computed numerically the solutions of the systems

$$\begin{aligned} dx_1(t) &= (x_1(t) - x_1(t)x_2(t) - 0.4x_1(t)u_1(t))dt + \alpha_1 dW_1, \\ dx_2(t) &= (-x_2(t) + x_1(t)x_2(t) - 0.2x_2(t)u_2(t))dt + \alpha_2 dW_2, \\ x_1(0) &= 0.5, \\ x_2(0) &= 0.7, \\ dp_1(t) &= (x_1(t) - p_1(t) + p_1(t)x_2(t) + 0.4p_1(t)u_1(t) - p_2(t)x_2(t))dt + p_1(t)dW_1, \\ dp_2(t) &= (x_2(t) + p_2(t) + p_1(t)x_1(t) - p_2(t)x_1(t) + 0.2p_2(t)u_2(t))dt + p_2(t)dW_2, \\ p_1(60) &= 0.5, \\ p_2(60) &= 0.7, \end{aligned} \tag{66}$$

taking independent Wiener processes $dW_1(t)$ and $dW_2(t)$ and the parameters $\alpha_1$ and $\alpha_2$ equal to the following: $(\alpha_1, \alpha_2) = (0.15, 0.10)$, $(\alpha_1, \alpha_2) = (0.10, 0.15)$, and $(\alpha_1, \alpha_2) = (0.07, 0.03)$. We have programmed in ©R-Project, and the figures show the convergence of the algorithm in the 12th iteration. The program is very sensitive to the values of parameters $(\alpha_1, \alpha_2)$. Comparing the deterministic and stochastic cases, we note the prevalence of the turnpike property in the stochastic case, for a small stochastic perturbation of the deterministic case. This behavior is consistent both for the states (Figure 5 and Figure 6) for the controls (Figure 7) and the adjoint states (Figure 8).

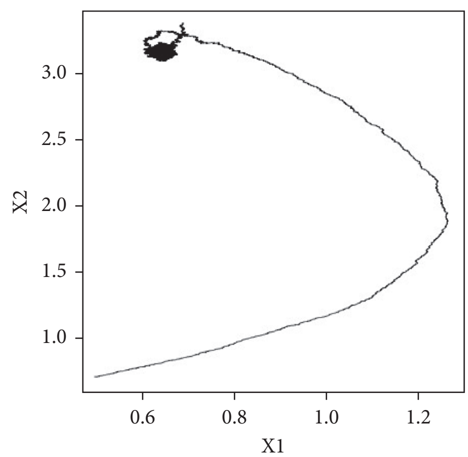


Figure 6: Stochastic state transfer $(x_1, x_2)$-limit trajectory in the phase space, using the Euler–Maruyama scheme, 12th iteration.

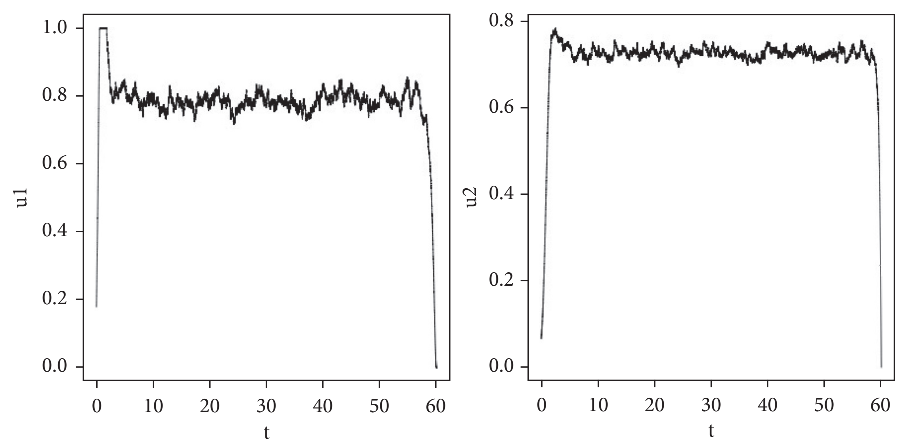


Figure 7: Stochastic optimal controls $u_1(t)$ and $u_2(t)$, using the Euler–Maruyama scheme, 12th iteration.

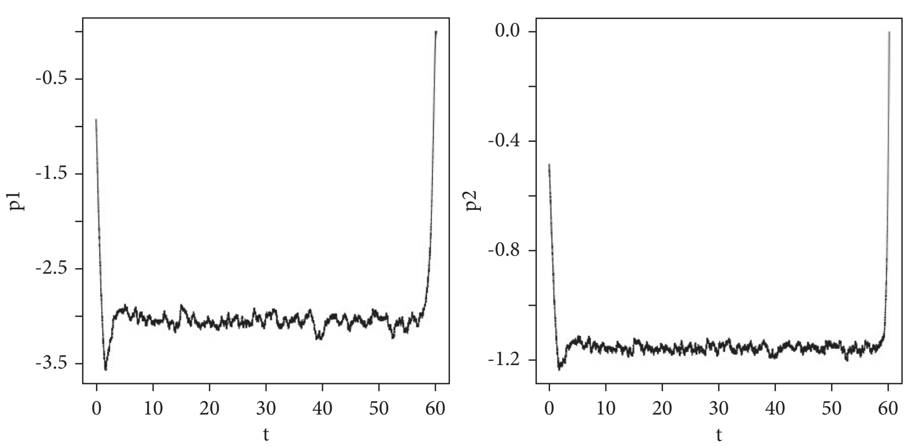


Figure 8: Stochastic adjoint states $p_1(t)$ and $p_2(t)$, using the Euler–Maruyama scheme, 12th iteration.

## 5. Conclusion

In this paper, we have considered the stochastic controlled Lotka–Volterra prey-predator model, introducing two noise-independent perturbations in prey and predator populations. Considering the numerical solutions via Euler–Maruyama scheme, we have proved the boundedness in the mean and strong convergence for this kind of solutions. On the other hand, for this stochastic model, we have combined some techniques of the Geometric Control Theory with the property of exponential stability of the numerical solution to prove the persistence in the turnpike property, under small stochastic perturbations of the deterministic Lotka–Volterra model, which is a novelty of our work. Finally, we have presented an example to illustrate our techniques, in the actual calculation of the estimations involved in the theorems. The numerical results are illustrated in the corresponding graphics. From our numerical simulations, we find that, if the noise perturbation is small ($\alpha_1$, $\alpha_1 < 0.20$), then the turnpike property is preserved in the stochastic model.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References


[1] A. J. Lotka, *Elements of Mathematical Biology. Unabridge Republication of: Elements of Physical Biology*, Dover, New Tork, 1956.

[2] V. Volterra, *Lecons sur la theorie mathematique de la lutte pour la vie*, Gauthier-Villars, Paris, 1931.

[3] U. D'Ancona, *The Struggle for Existence*, E. J. Brill, Leiden, 1954.

[4] L. Arnold, W. Horsthemke, and J. W. Stucki, "The influence of External real and white noise on the LOTKA-VOLTERRA model," *Biometrical Journal*, vol. 21, no. 5, pp. 451–471, 1979.

[5] D. G. Chapman, "Stochastics model in animal population ecology,"in *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley University of California Press, vol. 4, pp. 147–162, Berkeley, CA, USA, 1967.

[6] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, 1973.

[7] C. Pierce, "A new deterministic model for the interaction between predator and prey," *Biometrics*, vol. 26, pp. 387–392, 1970.

[8] J. Liu and W. Zhao, "Dynamic Analysis of Stochastic Lotka–Volterra Predator-Prey Model with Discrete Delays and Feedback Control," *Complexity*, vol. 2019, Article ID 4873290, 15 pages, 2019.

[9] A. Das and G. P. Samanta, "Stochastic prey-predator model with additional food for predator," *Physica A: Statistical Mechanics and Its Applications*, vol. 512, pp. 121–141, 2018.

[10] S.-R. Cheng, "Stochastic population systems," *Stochastic Analysis and Applications*, vol. 27, no. 4, pp. 854–874, 2009.

[11] G. Q. Cai and Y. K. Lin, "Stochastic analysis of predator-prey type ecosystems," *Ecological Complexity*, vol. 4, no. 4, pp. 242–249, 2007.

[12] X. D. Gu and W. Q. Zhu, "Stochastic optimal control of predator-prey ecosystem by using stochastic maximum principle," *Nonlinear Dynamics*, vol. 85, no. 2, pp. 1177–1184, 2016.

[13] T. Feng, X. Meng, T. Zhang, and Z. Qiu, "Analysis of the predator–prey interactions: a stochastic model incorporating disease invasion," *Qualitative Theory of Dynamical Systems*, vol. 19, p. 2, 2020.

[14] Q. Liu, D. Jiang, and T. Hayat, "Dynamics of stochastic predator–prey models with distributed delay and stage structure for prey," *International Journal of Biomathematics*, vol. 14, p. 4, 2021.

[15] T. Ma, X. Meng, and Z. Chang, "Dynamics and optimal Harvesting control for a stochastic one-predator-two-prey time delay system with jumps," *Complexity*, vol. 2019, 2019.

[16] G. Liu, H. Qi, Z. Chang, and X. Meng, "Asymptotic stability of a stochastic May mutualism system," *Computers & Mathematics with Applications*, vol. 79, no. 3, pp. 735–745, 2020.

[17] A. Ibañez, "Optimal control of the Lotka-Volterra system: turnpike property and numerical simulations," *Journal of Biological Dynamics*, vol. 11, no. 1, pp. 25–41, 2017.

[18] E. Trélat and E. Zuazua, "The turnpike property in finite-dimensional nonlinear optimal control," *Journal of Differential Equations*, vol. 258, no. 1, pp. 81–114, 2015.

[19] R. Z. Khasminskii and F. C. Klebaner, "Long term behavior of solutions of the Lotka–Volterra system under small random perturbations," *Annals of Applied Probability*, vol. 11, pp. 952–963, 2001.

[20] S. Joshi, "Turnpike theorems in Nonconvex Nonstationary environments," *International Economic Review*, vol. 38, no. 1, pp. 225–248, 1997.

[21] T. Faulwasser, M. Korda, C. N. Jones, and D. Bonvin, "Turnpike and dissipativity properties in dynamic real-time optimization and economic," *MPC 53rd IEEE Conference on Decision and Control*, pp. 2734–2739, 2014.

[22] A. Zaslavski, *Turnpike Properties in the Calculus of Variations and Optimal Control*, Springer, 2006.

[23] D. Barbu, "Local and global existence for mild solutions of stochastic differential equations," *Portugal. Math.*vol. 55, pp. 411–424, 1998.