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Research Article

The Numerical Investigation of Fractional-Order Zakharov–Kuznetsov Equations

Pongsakorn Sunthrayuth,[1] Farman Ali,[2] A. A. Alderremy,[3] Rasool Shah,[4] Shaban Aly,[5] Y. S. Hamed,[6] and Jeevan Katle[7]

[1]*Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala, University of Technology Thanyaburi (RMUTT), Pathumthani 12110, Thailand*
[2]*Department of Software, Sejong Unversity, Seoul 05006, Republic of Korea*
[3]*Department of Mathematics, Faculty of Science, King Khalid University, Abha 61413, Saudi Arabia*
[4]*Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Pakistan*
[5]*Department of Mathematics, Faculty of Science, AL-Azhar University, Assiut, Egypt*
[6]*Department of Mathematics and Statistics, College of Science, Taif University, P O. Box 11099, Taif 21944, Saudi Arabia*
[7]*Central Department of Mathematics, Tribhuvan University, Kritipur, Nepal*

Correspondence should be addressed to Jeevan Katle; jeevan.kafle@cdmath.tu.edu.np

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In this article, a modified method called the Elzaki decomposition method has been applied to analyze time-fractional Zakharov–Kuznetsov equations. In this method, the Adomian decomposition technique and Elzaki transformation are combined. Two problems are investigated to show and validate the efficiency of the suggested method. It is also shown that the solutions achieved from the current producer are in good contact with the exact solutions to show with the help of graphs and table. It is observed that the suggested technique is to be reliable, efficient, and straightforward to implement for many related models of engineering and science.

1. Introduction

Nonlinear fractional partial differential equations play important role in demonstrating different physical appearances identified with solid-state physics, fluid mechanics, chemical kinetics, population dynamics, plasma physics, nonlinear optics, protein chemistry, soliton theory, etc. These nonlinear problems, just as their scientific arrangements, are of tremendous enthusiasm for suitable subjects. In many above-discussed science and engineering areas, the nonlinear problems perform a key factor in many phenomena. Differential equations demonstrate several frameworks and the majority of them are nonlinear [1–4].

The Zakharov–Kuznetsov (ZK) equation is an extremely appealing model equation for investigating vortices in geophysical streams. The ZK problems show up in numerous regions of material science, implemented arithmetic, and designing. Specifically, it appears in the territory of quantum physics [5–9]. The ZK problems administer the conduct of feebly nonlinear particle acoustic plasma waves, including cold particles and hot isothermal electrons within sight of a smooth magnetic field [10, 11]. Solitary wave arrangements were produced by determining the nondirect higher order of broadened KdV conditions for the free surface removal [12]. By utilizing fractional strategy, the precise expository structures of some nonlinear advancement equations in numerical material science, to be specific, space time-fractional Zakharov–Kuznetsov and modified Zakharov–Kuznetsov equations, were obtained [13]. It has been investigated in the past decades by many with the techniques such as new iterative Sumudu transform method [14], homotopy perturbation transform method [15], extended direct algebraic method [16], natural decomposition method, and q-homotopy analysis transform method [17].

In the last three decades, fractional differential conditions have picked up importance and ubiquity, mostly since its exhibited uses in various fields of material science and design. Numerous significant phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science, likelihood and measurements, electrochemistry of erosion, concoction physical science, and sign preparation are depicted in fractional differential equations [18–23]. Consequently, special consideration has been given to discover solutions of fractional differential equations.

The investigation of these equations and their solutions has extraordinary enthusiasm for numerous specialists because of its different applications. To refer to a couple, Wazwaz [24] used the Adomian disintegration strategy as a dependable method for treating Schrodinger conditions. In Wazwaz [25], the variational emphasis technique was utilized to obtain specific solutions for both linear and nonlinear Schrodinger equations. Additionally, Shah et al. [26] utilized He's recurrence definition as a technique to look for Schrodinger equations arrangements. The arrangements decided to end up being in good concurrence with the outcomes decided in [24, 25]. Notwithstanding, we mean to couple the Elzaki transform built up as late by Elzaki [27] with the commended technique for the 80th Adomian decay strategy [28, 29]. Recently, many researchers obtained the results of FPDEs; interested readers can see [30–36].

In this present work, the Elzaki decomposition technique is applied to investigate the result of the fractional-order ZK equation. The fractional derivatives are defined by the Caputo operator. The result of the given problems shows the validity of the suggested method. The solutions of the suggested technique are analyzed and shown with the help of the table and figures. Applying the current method, the results of time-fractional equations and integral-order equations are investigated. The given method is very helpful in solving other fractional-order of PDEs.

2. Basic Definitions

2.1. Definition. The fractional-order Riemann–Liouville $\rho > 0$, of a function $f \in C_\imath$, $\rho \geq -1$, is given as [27]

$$\begin{aligned} J^\rho h(\xi) &= \frac{1}{\Gamma(\rho)} \int_0^\xi (\xi - 1)^{\rho-1} h(\eta)\partial\eta\ , \quad \rho, \xi > 0, \\ J^\rho h(\xi) &= h(\xi). \end{aligned} \tag{1}$$

The operator of some properties:
For $h \in C_\imath$, $\rho \geq -1$, $\rho, \beta \geq 0$, and $\rho > -1$,

$$\begin{aligned} J^\rho J^\beta h(\xi) &= J^{\rho+\beta} h(\xi), \\ J^\rho J^\beta h(\xi) &= J^\rho J^\beta h(\xi), \\ J^\beta \xi^\rho &= \frac{\Gamma(\rho+1)}{(\beta+\rho+1)} \xi^{\beta+\rho}. \end{aligned} \tag{2}$$

2.2. Lemma. If $\imath - 1 < \rho \leq \imath$, $\imath \in \mathrm{N}$ and $h \in C_\imath$, $\rho \geq -1$, then $D^\rho J^\rho h(\xi) = h(\xi)$ [18–20],

$$D^\rho J^\rho h(\xi) = h(\xi) - \sum_{\imath=0}^{m-1} h^{(\imath)}(0) \frac{\xi^\imath}{\imath!}, \quad \xi > 0. \tag{3}$$

The basic theory of the Elzaki transformation:

A new transform called the Elzaki transform defines the function exponential order that we found in the set A, define by [27]

$$A = \left\{ h(\mathfrak{I}) \colon \sum \middle| M, k_1, k_2 > 0, \middle| h(\mathfrak{I}) \right| < M e^{|\mathfrak{I}|/k_\imath}, \quad \text{if}\, (\mathfrak{I}) \in (-1)^\imath \times [0, \infty). \tag{4}$$

The finite number M must be constant, k_1 and k_2 of infinite or finite, for a specified function in the set. The Elzaki transformation is defined throughout the following integral problem:

$$\mathrm{E}[h(\mathfrak{I})] = T(s) = s\int_0^\infty h(\mathfrak{I})e^{-\mathfrak{I}/s}d\mathfrak{I}, \quad \mathfrak{I} \geq 0, k_1 \leq s \leq k_2. \tag{5}$$

We can obtain the next solution from the explanation and the basic investigation

$$\begin{aligned} E[\mathfrak{I}^n] &= m!s^{m+2}, \\ E[h'(\mathfrak{I})] &= \frac{T(s)}{s} - sh(0), \\ E[h''(\mathfrak{I})] &= \frac{T(s)}{s^2} - h(0) - sh'(0), \\ E\left[h^{(m)}(\mathfrak{I})\right] &= \frac{T(s)}{s^m} - \sum_{k=0}^{n-1} s^{2-m+k}h^{(k)}(0). \end{aligned} \tag{6}$$

2.3. Theorem. The Elzaki Riemann–Liouville transform of the derivative can be defined as given if $T(s)$ is the Elzaki transformation of $(\mathfrak{I})$ [27]:

$$E[D^\rho h(\mathfrak{I})] = s^{-\rho}\left[T(s) - \sum_{k=1}^{m}\left\{D^{\rho-k}h(0)\right\}\right], \quad -1 < m-1 \leq \rho < m. \tag{7}$$

proof. Taking the Laplace transformation of $h\prime(\mathfrak{I}) = d/d\mathfrak{I}h(\mathfrak{I})$, we have

$$\begin{aligned} L[D^\rho h(\mathfrak{I})] &= s^\rho T(s) - \sum_{k=0}^{m-1} s^k\left[D^{\rho-k-1}h(0)\right] \\ &= s^\rho T(s) - \sum_{k=0}^{m-1} s^{k-1}\left[D^{\rho-k}h(0)\right] = s^\rho T(s) - \sum_{k=0}^{m-1} s^{k-2}\left[D^{\rho-k}h(0)\right] \\ &= s^\rho T(s) - \sum_{k=0}^{m-1}\frac{1}{s^{-k+2}}\left[D^{\rho-k}h(0)\right] = s^\rho T(s) - \sum_{k=0}^{m-1}\frac{1}{s^{\rho-k+2-\rho}}\left[D^{\rho-k}h(0)\right] \\ &= s^\rho T(s) - \sum_{k=0}^{m-1} s^\rho\frac{1}{s^{\rho-k+2}}\left[D^{\rho-k}h(0)\right] \\ L[D^\rho h(\mathfrak{I})] &= s^\rho\left[T(s) - \sum_{k=0}^{m-1}\left(\frac{1}{s}\right)^{\rho-k+2}\left[D^{\rho-k}h(0)\right]\right]. \end{aligned} \tag{8}$$

Therefore, the fractional-order Elzaki transform of $h(\mathfrak{I})$ is

$$E[D^\rho h(\mathfrak{I})] = s^{-\rho}\left[T(s) - \sum_{k=0}^{m}(s)^{\rho-k+2}\left[D^{\rho-k}h(0)\right]\right]. \tag{9}$$

□

2.4. Definition. Using Theorem 1, the fractional Caputo ET is provided as [18–20]

$$\mathrm{E}\left[D_{\mathfrak{I}}^\rho g(\mathfrak{I})\right] = s^\rho \mathrm{E}[g(\mathfrak{I})] - \sum_{k=0}^{\imath-1} s^{2-\rho+k}g^{(k)}(0), \quad \text{where } \imath - 1 < \rho < \imath. \tag{10}$$

3. The General Implementation of Elzaki Decomposition Technique

In this section, we present the Elzaki decomposition technique producer for fractional partial differential equation.

$$D^{\rho}\Psi(\xi,\mathfrak{I}) + L\Psi(\xi,\mathfrak{I}) + N\Psi(\xi,\mathfrak{I}) = q(\xi,\mathfrak{I}), \quad \xi,\mathfrak{I} \geq 0, \imath - 1 < \rho < \imath, \tag{11}$$

and the initial condition is

$$\Psi(\xi,0) = k(\xi). \tag{12}$$

Applying the Elzaki transform to equation (11), we get

$$E[D^{\rho}\Psi(\xi,\mathfrak{I})] + E[L\Psi(\xi,\mathfrak{I}) + N\Psi(\xi,\mathfrak{I})] = E[q(\xi,\mathfrak{I})]. \tag{13}$$

Using the Elzaki transform differentiation property,

$$\begin{aligned}\frac{1}{s^{\rho}}E[\Psi(\xi,\mathfrak{I})] - s^{2-\rho}\Psi(\xi,0) &= E[q(\xi,\mathfrak{I})] - E[L\Psi(\xi,\mathfrak{I}) + N\Psi(\xi,\mathfrak{I})]\\ E[\Psi(\xi,\mathfrak{I})] &= s^{2}\Psi(\xi,0) + s^{\rho}E[q(\xi,\mathfrak{I})] - s^{\rho}E[L\Psi(\xi,\mathfrak{I}) + N\Psi(\xi,\mathfrak{I})].\end{aligned} \tag{14}$$

Now, $\Psi(\xi,0) = k(\xi)$ and hence

$$E[\Psi(\xi,\mathfrak{I})] = s^{2}k(\xi) + s^{\rho}E[q(\xi,\mathfrak{I})] - s^{\rho}E[L\Psi(\xi,\mathfrak{I}) + N\Psi(\xi,\mathfrak{I})], \tag{15}$$

where $\Psi(\xi,\mathfrak{I})$ is defined as

$$\Psi(\xi,\mathfrak{I}) = \sum_{\imath=0}^{\infty}\Psi_{\imath}(\xi,\mathfrak{I}). \tag{16}$$

The nonlinearity of Adomian polynomials terms N is defined as

$$N\Psi(\xi,\mathfrak{I}) = \sum_{\imath=0}^{\infty}A_{\imath}, \tag{17}$$

$$A_{\imath} = \frac{1}{\imath!}\left[\frac{d^{\imath}}{d\lambda^{\imath}}\left[N\sum_{\imath=0}^{\infty}(\lambda^{\imath}\Psi_{\imath})\right]\right]_{\lambda=0}, \quad \imath = 0,1,2,\ldots. \tag{18}$$

Putting equation (16) and (17) into (15), we have

$$E\left[\sum_{\imath=0}^{\infty}\Psi_{\imath}(\xi,\mathfrak{I})\right] = s^{2}k(\xi) + s^{\rho}E[q(\xi,\mathfrak{I})] - s^{\rho}E\left[L\sum_{\imath=0}^{\infty}\Psi_{\imath}(\xi,\mathfrak{I}) + \sum_{\imath=0}^{\infty}A_{\imath}\right]. \tag{19}$$

Now using EDM, we have

$$E[\Psi_{0}(\xi,\mathfrak{I})] = s^{2}k(\xi) + s^{\rho}E[q(\xi,\mathfrak{I})]. \tag{20}$$

Generally, we can write

$$E[\Psi_{\imath+1}(\xi,\mathfrak{I})] = -s^{\rho}E[L\Psi_{\imath}(\xi,\mathfrak{I}) + A_{\imath}], \quad \imath \geq 1. \tag{21}$$

Taking the inverse Elzaki transform of equation (21), we have

$$\begin{aligned}\Psi_{0}(\xi,\mathfrak{I}) &= k(\xi) + E^{-1}[s^{\rho}E[q(\xi,\mathfrak{I})]],\\ \Psi_{\imath+1}(\xi,\mathfrak{I}) &= -E^{-1}[s^{\rho}E[L\Psi_{\imath}(\xi,\mathfrak{I}) + A_{\imath}]].\end{aligned} \tag{22}$$

4. Main Results

Example 1. Consider the two-dimensional Zakharov–Kuznetsov equation as

$$D_{\mathfrak{I}}^{\rho}\Psi + (\Psi^{2})_{\xi} + \frac{1}{8}(\Psi^{2})_{\xi\xi\xi} + \frac{1}{8}(\Psi^{2})_{\xi\zeta\zeta} = 0, \quad 0 < \rho \leq 1, \tag{23}$$

and the initial condition is

$$\Psi(\xi,\zeta,0) = \frac{4}{3}\eta\sinh^{2}(\xi+\zeta), \tag{24}$$

where η is an arbitrary constant.

Taking Elzaki transform of equation (23), we have

$$\begin{aligned}E\left[\frac{\partial^{\rho}\Psi}{\partial\mathfrak{I}^{\rho}}\right] &= E\left[-(\Psi^{2})_{\xi} - \frac{1}{8}(\Psi^{2})_{\xi\xi\xi} - \frac{1}{8}(\Psi^{2})_{\xi\zeta\zeta}\right],\\ \frac{1}{s^{\rho}}E[\Psi(\xi,\zeta,\mathfrak{I})] - s^{2-\rho}[\Psi(\xi,\zeta,0)] &= E\left[-(\Psi^{2})_{\xi} - \frac{1}{8}(\Psi^{2})_{\xi\xi\xi} - \frac{1}{8}(\Psi^{2})_{\xi\zeta\zeta}\right].\end{aligned} \tag{25}$$

Applying the inverse Elzaki transform, we have

$$\Psi(\xi,\zeta,\mathfrak{J}) = E^{-1}\left(s^2\Psi(\xi,\zeta,0)\right) + E^{-1}\left[s^{\rho}E\left[-\left(\Psi^2\right)_{\xi} - \frac{1}{8}\left(\Psi^2\right)_{\xi\xi\xi} - \frac{1}{8}\left(\Psi^2\right)_{\xi\zeta\zeta}\right]\right],$$
$$\Psi(\xi,\zeta,\mathfrak{J}) = \frac{4}{3}\eta\sinh^2(\xi+\zeta) + E^{-1}\left[s^{\rho}E\left[-\left(\Psi^2\right)_{\xi} - \frac{1}{8}\left(\Psi^2\right)_{\xi\xi\xi} - \frac{1}{8}\left(\Psi^2\right)_{\xi\zeta\zeta}\right]\right]. \tag{26}$$

Using ADM procedure, we get

$$\sum_{\imath=0}^{\infty}\Psi_{\imath}(\xi,\zeta,\mathfrak{J}) = \frac{4}{3}\eta\ \sinh^2(\xi+\zeta) + E^{-1}\left[s^{\rho}E\left[-N(\Psi)_{\xi} - \frac{1}{8}N(\Psi)_{\xi\xi\xi} - \frac{1}{8}N(\Psi)_{\zeta\zeta\xi}\right]\right], \tag{27}$$

where the nonlinear terms can be defined by Adomian polynomials in the above equations.

$$N(\Psi) = \Psi^2 = \sum_{\imath=0}^{\infty}\mathscr{A}_{\imath}(\Psi), \tag{28}$$

Adomian polynomials are given as

$$\mathscr{A}_0 = \Psi_0^2,$$
$$\mathscr{A}_1 = 2\Psi_0\Psi_1,$$
$$\mathscr{A}_2 = 2\Psi_0\Psi_2 + \Psi_1^2,$$
$$\Psi_0(\xi,\zeta,\mathfrak{J}) = \frac{4}{3}\eta\sinh^2(\xi+\zeta),$$
$$\Psi_{\imath+1}(\xi,\zeta,\mathfrak{J}) = E^{-1}\left[s^{\rho}E\left[-\left(\sum_{\imath=0}^{\infty}\mathscr{A}_{\imath}(\Psi)\right)_{\xi} - \frac{1}{8}\left(\sum_{\imath=0}^{\infty}\mathscr{A}_{\imath}(\Psi)\right)_{\xi\xi\xi} - \frac{1}{8}\left(\sum_{\imath=0}^{\infty}\mathscr{A}_{\imath}(\Psi)\right)_{\zeta\zeta\xi}\right]\right], \tag{29}$$

for $\imath = 0, 1, 2, \ldots,$

$$\Psi_1(\xi,\zeta,\mathfrak{J}) = E^{-1}\left[s^{\rho}E\left[-\left(\Psi_0^2\right)_{\xi} - \frac{1}{8}\left(\Psi_0^2\right)_{\xi\xi\xi} - \frac{1}{8}\left(\Psi_0^2\right)_{\xi\zeta\zeta}\right]\right],$$
$$\Psi_1(\xi,\zeta,\mathfrak{J}) = \left(-\frac{224}{9}\eta^2\sinh^3(\xi+\zeta)\cosh(\xi+\zeta) - \frac{32}{3}\eta^2\sinh(\xi+\zeta)\cosh^3(\xi+\zeta)\right)E^{-1}\left(s^{\rho+2}\right)$$
$$\Psi_1(\xi,\zeta,\mathfrak{J}) = \left(-\frac{224}{9}\eta^2\sinh^3(\xi+\zeta)\cosh(\xi+\zeta) - \frac{32}{3}\eta^2\sinh(\xi+\zeta)\cosh^3(\xi+\zeta)\right)\frac{\mathfrak{J}^{\rho}}{\Gamma(\rho+1)}. \tag{30}$$

The few terms of the given methods are

$$\begin{aligned}
&\Psi_2(\xi,\zeta,\mathfrak{I}) = E^{-1}\left[s^{\rho}E\left[-(2\Psi_0\Psi_1)_{\xi} - \frac{1}{8}(2\Psi_0\Psi_1)_{\xi\xi\xi} - \frac{1}{8}(2\Psi_0\Psi_1)_{\xi\zeta\zeta}\right]\right],\\
&\Psi_2(\xi,\zeta,\mathfrak{I}) = \frac{128}{27}\eta^3\left(1200\cosh^6(\xi+\zeta) - 2080\cosh^4(\xi+\zeta) + 968\cosh^2(\xi+\zeta) - 79\right)\frac{\mathfrak{I}^{2\rho}}{\Gamma(2\rho+1)},\\
&\Psi_3(\xi,\zeta,\mathfrak{I}) = E^{-1}\left[s^{\rho}E\left[-\left(2\Psi_0\Psi_2+\Psi_1^2\right)_{\xi} - \frac{1}{8}\left(2\Psi_0\Psi_2+\Psi_1^2\right)_{\xi\xi\xi} - \frac{1}{8}\left(2\Psi_0\Psi_2+\Psi_1^2\right)_{\xi\zeta\zeta}\right]\right],\\
&\Psi_3(\xi,\zeta,\mathfrak{I}) = -\frac{2048}{81}\eta^4\sinh(\xi+\zeta)\cosh(\xi+\zeta)\left[88400\cosh^6(\xi+\zeta) - 160200\right.\\
&\qquad\left.\cosh^4(\xi+\zeta) + 85170\cosh^2(\xi+\zeta) - 11903\right]\frac{\mathfrak{I}^{3\rho}}{\Gamma(3\rho+1)}.
\end{aligned}\tag{31}$$

The EDM result is

$$\begin{aligned}
&\Psi(\xi,\zeta,\mathfrak{I}) = \Psi_0(\xi,\zeta,\mathfrak{I}) + \Psi_1(\xi,\zeta,\mathfrak{I}) + \Psi_2(\xi,\zeta,\mathfrak{I}) + \Psi_3(\xi,\zeta,\mathfrak{I}) + \cdots,\\
&\Psi(\xi,\zeta,\mathfrak{I}) = \frac{4}{3}\eta\sinh(\xi+\zeta) - \left(\frac{224}{9}\eta^2\sinh^3(\xi+\zeta)\cosh(\xi+\zeta) + \frac{32}{3}\eta^2\sinh(\xi+\zeta)\cosh^3(\xi+\zeta)\right)\frac{\mathfrak{I}^{\rho}}{\Gamma(\rho+1)}\\
&\quad + \frac{128}{27}\eta^3\left(1200\cosh^6(\xi+\zeta) - 2080\cosh^4(\xi+\zeta) + 968\cosh^2(\xi+\zeta) - 79\right)\frac{\mathfrak{I}^{2\rho}}{\Gamma(2\rho+1)} - \frac{2048}{81}\eta^4\sinh(\xi+\zeta)\\
&\quad \cosh(\xi+\zeta)\left[88400\cosh^6(\xi+\zeta) - 160200\cosh^4(\xi+\zeta) + 85170\cosh^2(\xi+\zeta) - 11903\right]\frac{\mathfrak{I}^{3\rho}}{\Gamma(3\rho+1)} + \cdots.
\end{aligned}\tag{32}$$

For $\rho = 1$, we have

$$\Psi(\xi,\mathfrak{I}) = \frac{4}{3}\eta\sinh^2(\xi+\zeta-\eta\mathfrak{I}).\tag{33}$$

In Figure 1, the exact and the EDM solutions of problem 1 at $\rho = 1$ are shown by Figures 1(a) and 1(b), respectively. From the given figures, it can be seen that both the EDM and exact results are in close contact with each other. Also, in Figures 1(c) and 1(d), the EDM solutions of problem 1 are investigated at different fractional-order $\rho = 0.8$ and 0.6. It is analyzed that time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order. In Figure 2, the first graph shows the two dimensions of exact and analytical solutions with respect to ξ and $\mathfrak{I}$ and second one shows the different fractional-order graph with respect to ξ and $\mathfrak{I}$. Table 1 shows the different fractional-order absolute error.

Example 2. Consider the three-dimensional Zakharov–Kuznetsov equation as

$$D_{\mathfrak{I}}^{\rho}\Psi + \left(\Psi^3\right)_{\xi} + 2\left(\Psi^3\right)_{\xi\xi\xi} + 2\left(\Psi^3\right)_{\xi\zeta\zeta} = 0,\quad 0<\rho\le 1,\tag{34}$$

and the initial condition is

$$\Psi(\xi,\zeta,0) = \frac{3}{2}\eta\sinh\left[\frac{1}{6}(\xi+\zeta)\right],\tag{35}$$

where η is an arbitrary constant.

Taking Elzaki transform of equation (34), we have

$$\begin{aligned}
&E\left[\frac{\partial^{\rho}\Psi}{\partial\mathfrak{I}^{\rho}}\right] = E\left[-\left(\Psi^3\right)_{\xi} - 2\left(\Psi^3\right)_{\xi\xi\xi} - 2\left(\Psi^3\right)_{\xi\zeta\zeta}\right],\\
&s^{\rho}E[\Psi(\xi,\zeta,\mathfrak{I})] - s^{2-\rho}[\Psi(\xi,\zeta,0)] = E\left[-\left(\Psi^3\right)_{\xi} - 2\left(\Psi^3\right)_{\xi\xi\xi} - 2\left(\Psi^3\right)_{\xi\zeta\zeta}\right].
\end{aligned}\tag{36}$$

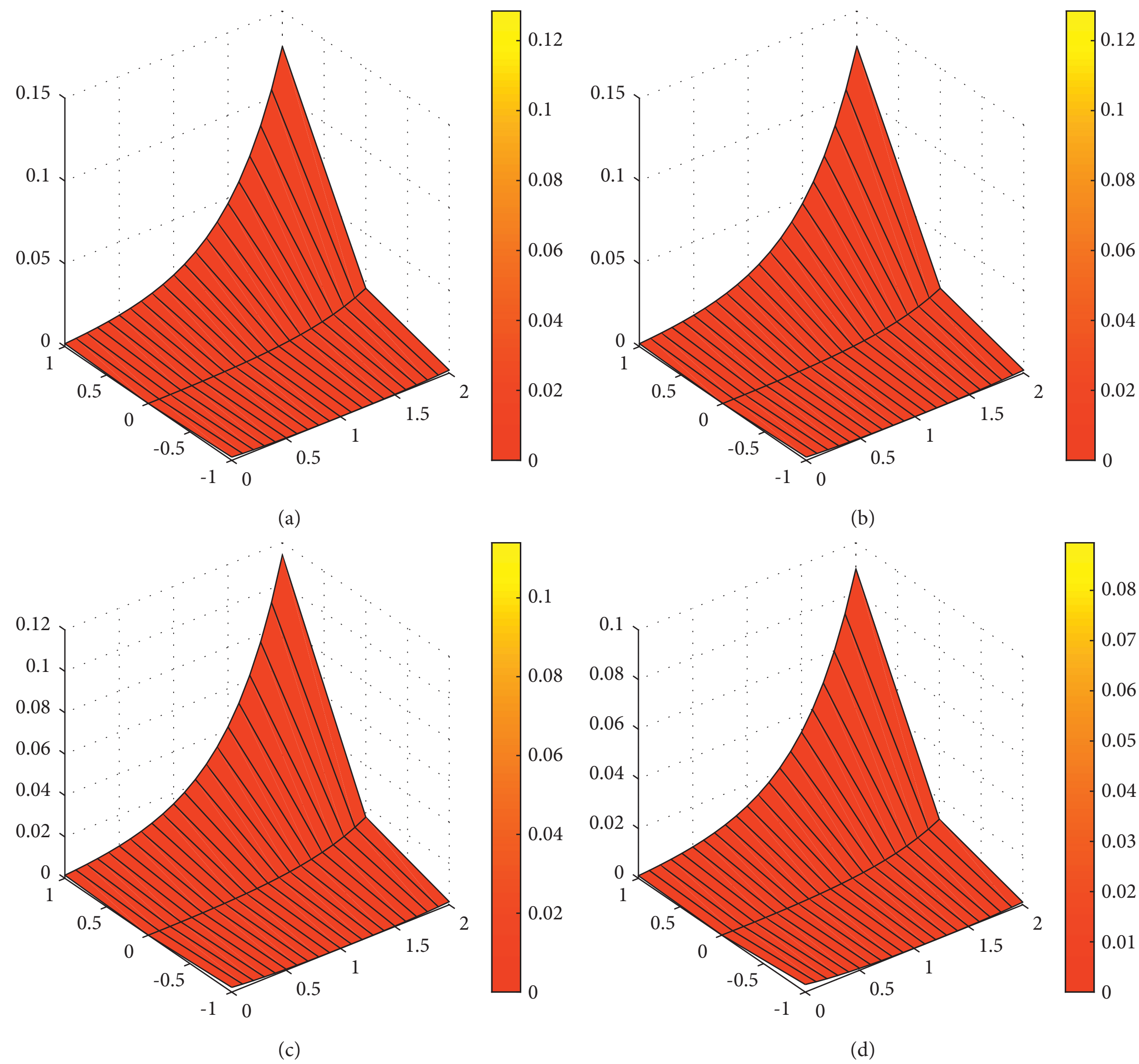


Figure 1: (a) The exact solution figure of $\Psi(\xi, \mathfrak{J})$ of Example 1. (b) The EDM solution figure of $\Psi(\xi, \mathfrak{J})$ of Example 1. (c) The graph EDM result of $\Psi(\xi, \mathfrak{J})$ at $\rho = 0.8$ problem 1. (d) The graph EDM result of $\Psi(\xi, \mathfrak{J})$ at $\rho = 0.6$ problem 1.

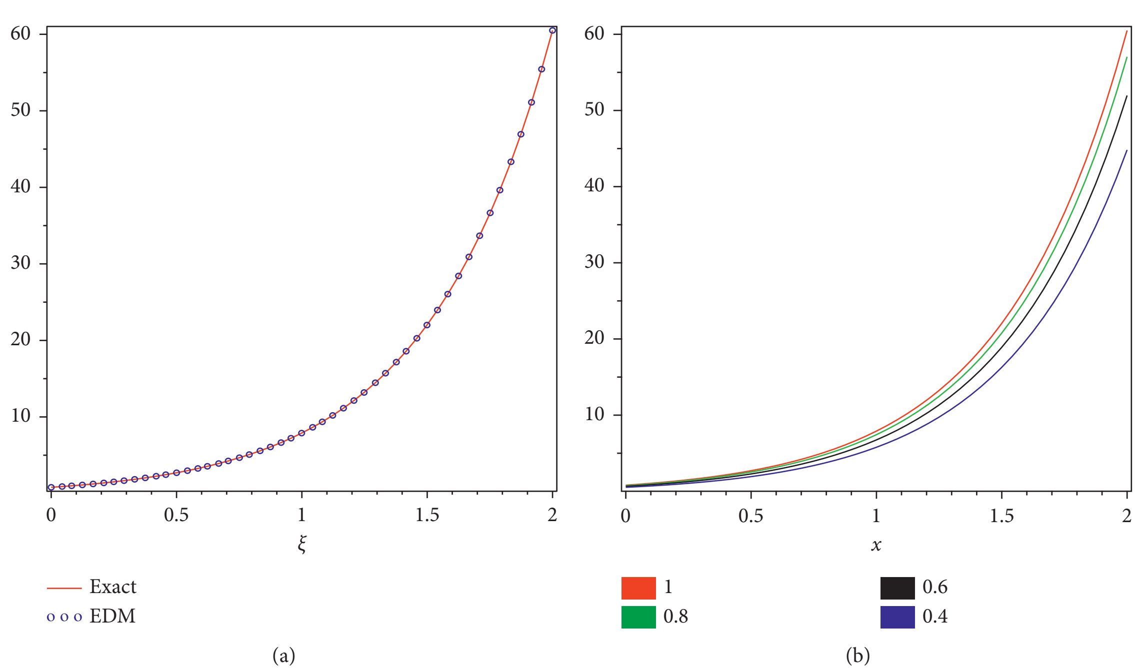
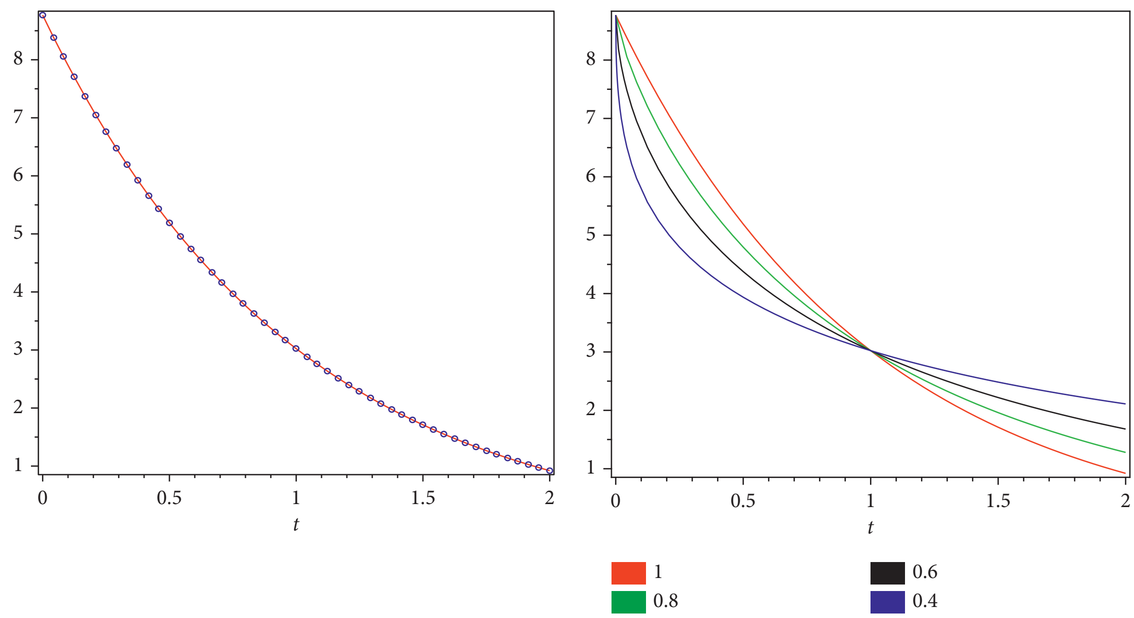


Figure 2: Continued.



(c) (d)

Figure 2: (a) The exact and EDM solution figure of $\Psi(\xi, \mathfrak{I})$ of Example 1. (b) The EDM solution figure of different fractional-order of ρ at $\Psi(\xi, \mathfrak{I})$ of Example 1. (c) The exact and EDM solution figure of $\Psi(\xi, \mathfrak{I})$ with respect to $\mathfrak{I}$ Example 1. (d) The EDM solution figure of different fractional-order of ρ with respect to $\mathfrak{I}$ of Example 1.

Table 1: EDM result for a different value of ρ when $\eta = 0.001$ and absolute error of Example 1.

			EDM				
ξ	ζ	$\mathfrak{I}$	$\rho = 0.6$	$\rho = 0.8$	EDM $(\rho = 1)$	Exact (E)	AE (E) $(\rho = 1)$
		0.1	$6.347977E-5$	$6.378034E-5$	$4.461135E-5$	4.488458-5	2.7323-7
0.2	0.2	0.3	$6.374898E-5$	$6.397867E-5$	$4.441928E-5$	4.499518-5	5.76800-7
		0.5	$6.329478E-5$	$6.358477E-5$	$4.393236E-5$	4.488588-5	8.92644-7
		0.1	$3.961810E-3$	$3.992936E-3$	$4.124545E-3$	4.148347-3	2.3802-5
0.5	0.5	0.3	$3.918638E-3$	$3.937942E-3$	$2.767287E-3$	2.868188-3	1.00902-4
		0.5	$2.851320E-3$	$3.986888E-3$	$3.014248E-3$	3.144685-3	1.31437-4
		0.1	$2.651681E-2$	$2.755798E-2$	$2.78556E-02$	2.864629-2	7.9069-4
1.0	1.0	0.3	$2.147458E-2$	$2.554775E-2$	$2.69884E-02$	2.863881-2	1.65041-3
		0.5	$3.686988E-3$	$3.286775E-2$	$2.55842E-02$	2.861074-2	3.02654-3

Using inverse Elzaki transformation,

$$\begin{aligned}\Psi(\xi,\zeta,\mathfrak{I}) &= E^{-1}\left[s^2\Psi(\xi,\zeta,0) + s^{\rho}E\left[-\left(\Psi^3\right)_{\xi} - 2\left(\Psi^3\right)_{\xi\xi\xi} - 2\left(\Psi^3\right)_{\xi\zeta\zeta}\right]\right],\\ \Psi(\xi,\zeta,\mathfrak{I}) &= \frac{3}{2}\eta\sinh\left[\frac{1}{6}(\xi+\zeta)\right] + E^{-1}\left[s^{\rho}E\left[-\left(\Psi^3\right)_{\xi} - 2\left(\Psi^3\right)_{\xi\xi\xi} - 2\left(\Psi^3\right)_{\xi\zeta\zeta}\right]\right].\end{aligned} \tag{37}$$

Applying the procedure of ADM, we get

$$\sum_{\imath=0}^{\infty}\Psi_{\imath}(\xi,\zeta,\mathfrak{I}) = \frac{3}{2}\eta\sinh\left[\frac{1}{6}(\xi+\zeta)t\right] + E^{-1}\left[s^{\rho}E\left[-N(\Psi)_{\xi} - 2N(\Psi)_{\xi\xi\xi} - 2N(\Psi)_{\xi\zeta\zeta}\right]\right], \tag{38}$$

where the nonlinear terms can be defined by Adomian polynomials in the above equations.

$$N(\Psi) = \Psi^3 = \sum_{\imath=0}^{\infty} \mathscr{B}_\imath(\Psi). \tag{39}$$

Adomian polynomials are given as

$$\begin{aligned}
\mathscr{B}_0 &= \Psi_0^3,\\
\mathscr{B}_1 &= 3\Psi_0^2\Psi_1,\\
\mathscr{B}_2 &= 3\Psi_0^2\Psi_2 + 3\Psi_0^2\Psi_1^2,\\
\Psi_0(\xi,\zeta,\mathfrak{T}) &= \frac{3}{2}\eta \sinh\left[\frac{1}{6}(\xi+\zeta)\right],\\
\Psi_{\imath+1}(\xi,\zeta,\mathfrak{T}) &= E^{-1}\left[s^\rho E\left[-\sum_{\imath=0}^{\infty}\mathscr{B}_\imath(\Psi)_\xi - 2\sum_{\imath=0}^{\infty}\mathscr{B}_\imath(\Psi)_{\xi\xi\xi} - 2\sum_{\imath=0}^{\infty}\mathscr{B}_\imath(\Psi)_{\xi\zeta\zeta}\right]\right],
\end{aligned} \tag{40}$$

for $\imath = 0, 1, 2, \ldots,$

$$\begin{aligned}
\Psi_1(\xi,\zeta,\mathfrak{T}) &= E^{-1}\left[s^\rho E\left[-\left(\Psi_0^3\right)_\xi - 2\left(\Psi_0^3\right)_{\xi\xi\xi} - 2\left(\Psi_0^3\right)_{\xi\zeta\zeta}\right]\right],\\
\Psi_1(\xi,\zeta,\mathfrak{T}) &= -\left[3\eta^3\sinh^2\left[\frac{1}{6}(\xi+\zeta)\right]\cosh\left[\frac{1}{6}(\xi+\zeta)\right] + \frac{3}{8}\eta^3\cosh^3\left[\frac{1}{6}(\xi+\zeta)\right]\right]E^{-1}\left(\frac{1}{s^{\rho+2}}\right),\\
\Psi_1(\xi,\zeta,\mathfrak{T}) &= -\left[3\eta^3\sinh^2\left[\frac{1}{6}(\xi+\zeta)\right]\cosh\left[\frac{1}{6}(\xi+\zeta)\right] + \frac{3}{8}\eta^3\cosh^3\left[\frac{1}{6}(\xi+\zeta)\right]\right]\frac{\mathfrak{T}^\rho}{\Gamma(\rho+1)}.
\end{aligned} \tag{41}$$

The subsequent terms are

$$\begin{aligned}
\Psi_2(\xi,\zeta,\mathfrak{T}) &= E^{-1}\left[s^\rho E\left[-\left(3\Psi_0^2\Psi_1\right)_\xi - 2\left(3\Psi_0^2\Psi_1\right)_{\xi\xi\xi} - 2\left(3\Psi_0^2\Psi_1\right)_{\xi\zeta\zeta}\right]\right],\\
&= \frac{3}{32}\eta^5\sinh\left[\frac{1}{6}(\xi+\zeta)\right]\left[765\cosh^4\left[\frac{1}{6}(\xi+\zeta)\right] - 729\cosh^2\left[\frac{1}{6}(\xi+\zeta)\right] + 91\right]\frac{\mathfrak{T}^{2\rho}}{\Gamma(2\rho+1)},\\
\Psi_3(\xi,\zeta,\mathfrak{T}) &= E^{-1}\left[s^\rho E\left[-\left(3\Psi_0^2\Psi_2 + 3\Psi_0^2\Psi_1^2\right)_\xi - 2\left(3\Psi_0^2\Psi_2 + 3\Psi_0^2\Psi_1^2\right)_{\xi\xi\xi} - 2\left(3\Psi_0^2\Psi_2 + 3\Psi_0^2\Psi_1^2\right)_{\xi\zeta\zeta}\right]\right],\\
&= -\frac{3}{128}\cosh\left[\frac{1}{6}(\xi+\zeta)\right]\left[171738\cosh^6\left[\frac{1}{6}(\xi+\zeta)\right] - 349884\cosh^4\left[\frac{1}{6}(\xi+\zeta)\right] + 215496\cosh^2\left[\frac{1}{6}(\xi+\zeta)\right]\right.\\
&\quad\left. - 36907\right]\frac{\mathfrak{T}^{2\rho}}{\Gamma(2\rho+1)}.
\end{aligned} \tag{42}$$

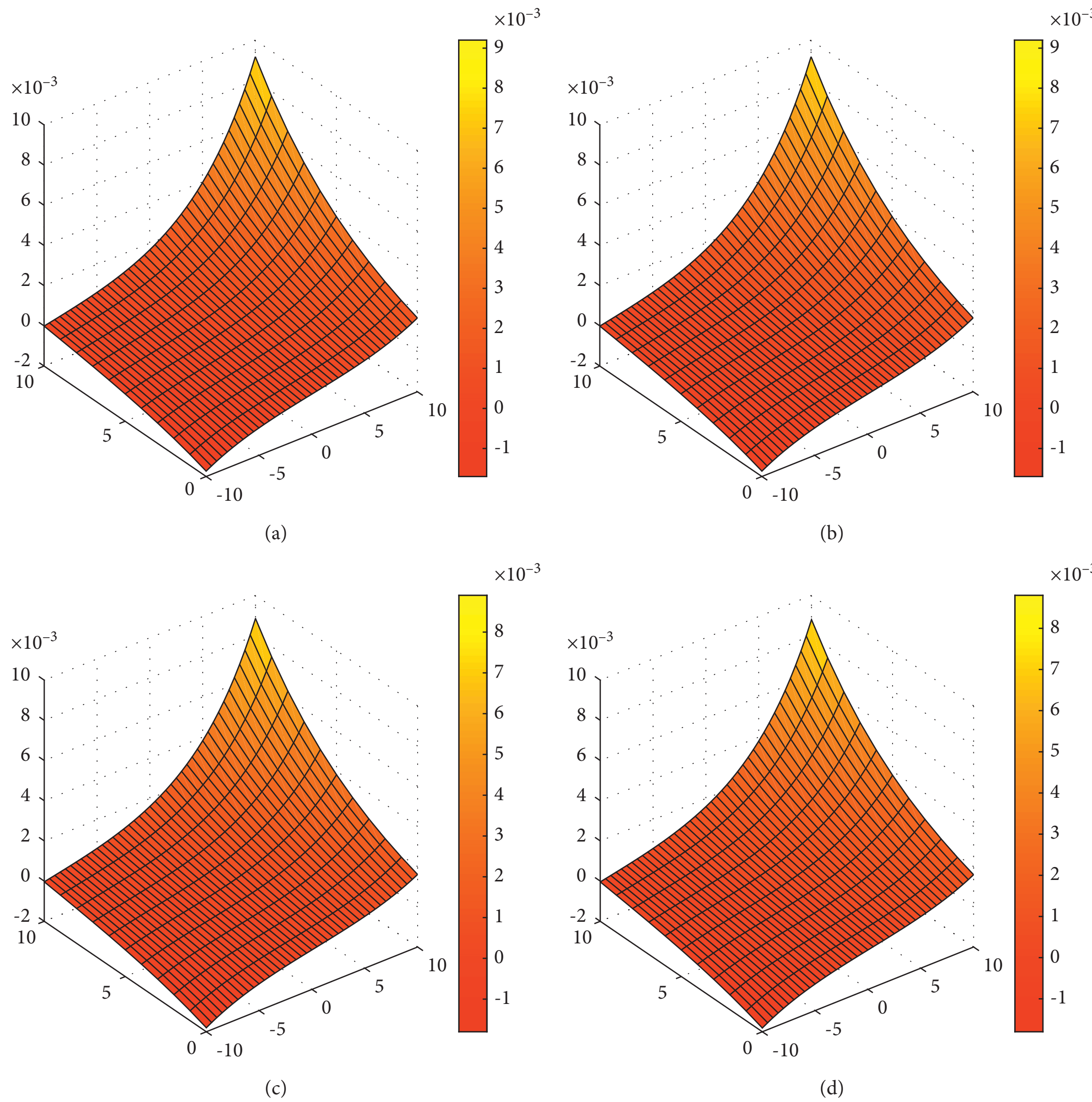


Figure 3: (a) The exact solution figure of $\Psi(\xi, \mathfrak{T})$ of Example 2. (b) The EDM solution figure of $\Psi(\xi, \mathfrak{T})$ of Example 2. (c) The EDM solution figure of $\Psi(\xi, \mathfrak{T})$ at $\rho = 0.8$ Example 2. (d) The EDM solution figure of $\Psi(\xi, \mathfrak{T})$ at $\rho = 0.6$ Example 2.

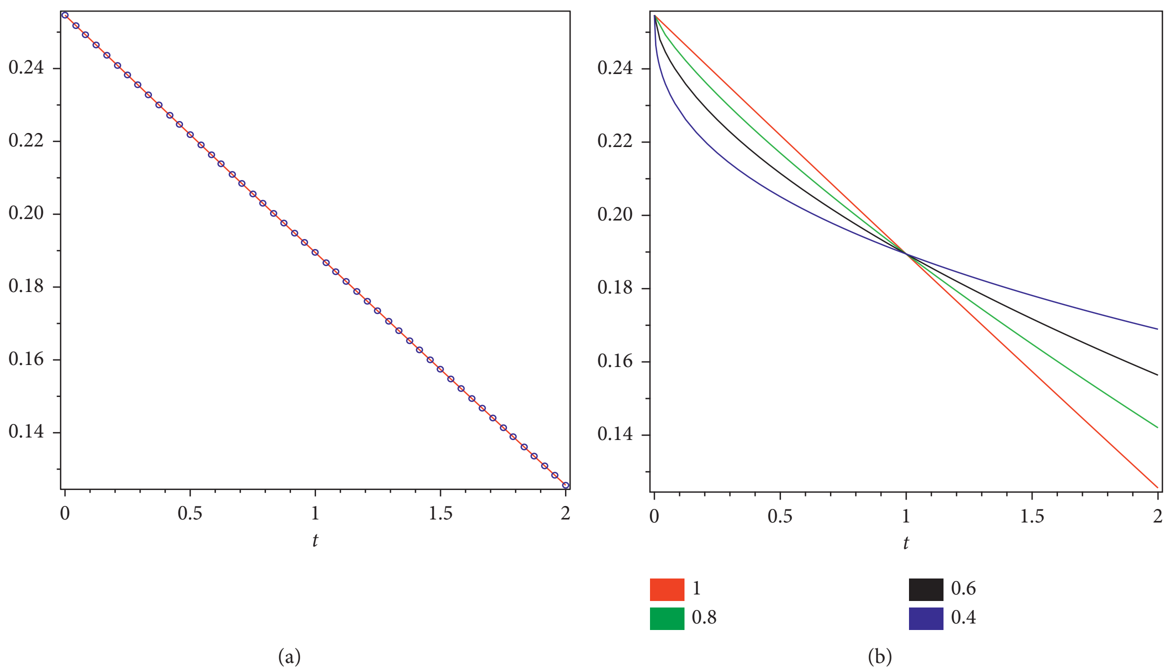


Figure 4: (a) The exact and EDM solution of $\Psi(\xi, \mathfrak{I})$ with respect to $\mathfrak{I}$ Example 1. (b) The EDM solution figure of different fractional-order of ρ with respect to $\mathfrak{I}$ of Example 1.

The EDM result is

$$\begin{aligned}
\Psi(\xi,\zeta,\mathfrak{I}) &= \Psi_0(\xi,\zeta,\mathfrak{I}) + \Psi_1(\xi,\zeta,\mathfrak{I}) + \Psi_2(\xi,\zeta,\mathfrak{I}) + \Psi_3(\xi,\zeta,\mathfrak{I}) + \cdots, \\
\Psi(\xi,\zeta,\mathfrak{I}) &= \frac{3}{2}\eta \sinh\left[\frac{1}{6}(\xi+\zeta)\right] - \left[3\eta^3\sinh^2\left[\frac{1}{6}(\xi+\zeta)\right]\cosh\left[\frac{1}{6}(\xi+\zeta)\right]\right. \\
&\quad \left. + \frac{3}{8}\eta^3\cosh^3\left[\frac{1}{6}(\xi+\zeta)\right]\right]\frac{\mathfrak{I}^{\rho}}{\Gamma(\rho+1)} + \frac{3}{32}\eta^5\sinh\left[\frac{1}{6}(\xi+\zeta)\right] \\
&\quad \left[765\cosh^4\left[\frac{1}{6}(\xi+\zeta)\right] - 729\cosh^2\left[\frac{1}{6}(\xi+\zeta)\right] + 91\right]\frac{\mathfrak{I}^{2\rho}}{\Gamma(2\rho+1)} \\
&\quad - \frac{3}{128}\cosh\left[\frac{1}{6}(\xi+\zeta)\right]\left[171738\cosh^6\left[\frac{1}{6}(\xi+\zeta)\right] - 349884\right. \\
&\quad \left.\cosh^4\left[\frac{1}{6}(\xi+\zeta)\right] + 215496\cosh^2\left[\frac{1}{6}(\xi+\zeta)\right] - 36907\right]\frac{\mathfrak{I}^{2\rho}}{\Gamma(2\rho+1)} + \cdots.
\end{aligned} \tag{43}$$

For $\rho = 1$,

$$\Psi(\xi,\mathfrak{I}) = \frac{3}{2}\eta \sinh\left[\frac{1}{6}(\xi+\zeta-\eta\mathfrak{I})\right]. \tag{44}$$

In Figure 3, the exact and the EDM solutions of Example 2 at $\rho = 1$ are shown by Figures 3(a) and 3(b), respectively. From the given figures, it can be seen that both the EDM and exact solutions are in close contact with each other. Also, in Figures 3(c) and 3(d), the EDM results of problem 2 are investigated at different fractional-order $\rho = 0.8$ and 0.6. It is analyzed that time-fractional problem results are convergent to an integer order effect as time-fractional analysis to integer order. In Figure 4, the first graph shows the two dimensions of exact and analytical solutions with respect to $\mathfrak{I}$ and the second one shows the different fractional-order graph with respect to $\mathfrak{I}$.

5. Conclusion

In this article, we investigated the time-fractional Zakharov–Kuznetsov equations using an Elzaki decomposition method. The given test examples illustrate the leverage and effectiveness of the suggested method. The obtained solutions are demonstrated by tables and graphs. The Elzaki decomposition method solution is in close contact with the actual result of the given problems. The figures show that the time-fractional solutions obtained have verified the convergence towards the integer order solutions. Moreover, the current technique is simple and straightforward as compared to other analytical techniques; the proposed method can solve other linear and nonlinear fractional-order partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Disclosure

Pongsakorn Sunthrayuth and Farman Ali are the co-first authors.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

Pongsakorn Sunthrayuth and Farman Ali contributed equally to this work.

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