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Research Article

Dynamics of the Exponential Population Growth System with Mixed Fractional Brownian Motion

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This paper examines the dynamics of the exponential population growth system with mixed fractional Brownian motion. First, we establish some useful lemmas that provide powerful tools for studying the stochastic differential equations with mixed fractional Brownian motion. We offer some explicit expressions and numerical characteristics such as mathematical expectation and variance of the solutions of the exponential population growth system with mixed fractional Brownian motion. Second, we propose two sufficient and necessary conditions for the almost sure exponential stability and the kth moment exponential stability of the solution of the constant coefficient exponential population growth system with mixed fractional Brownian motion. Furthermore, we conduct some large deviation analysis of this mixed fractional population growth system. To the best of the authors' knowledge, this is the first paper to investigate how the Hurst index affects the exponential stability and large deviations in the biological population system. It is interesting that the phenomenon of large deviations always occurs for addressed system when $(1/2) < H < 1$. Moreover, several numerical simulations are reported to show the effectiveness of the proposed approach.

1. Introduction

Many scholars recently have paid considerable attention to stochastic differential equations (SDEs), as they can be applied in many fields such as mathematics, physics, mechanics, biology, economics, complex networks, control engineering, multiagent systems, and financial markets [1–8]. However, these applications are largely dependent on the stability of the systems, namely, the long-time asymptotic behavior of the solutions to SDEs. In particular, in biology, engineering, complex networks, and control systems, it is most important to guarantee that the systems are stable, thus highlighting the need to investigate the stability of SDEs.

As is well known, there is a great amount of literature on stability analysis, for instance, see [2, 3, 9–17] and the references therein. It should be pointed out that the works in [2, 3, 9–14, 18–22] only considered the case of SDEs with Markovian noise such as Brownian motion, telegraph noise (or burst noise), Poisson noise, and Lévy noise. However, the stochastic process may fulfil the long-range dependence in some important fields like economics [17, 23–28], neural networks [5, 29], telecommunication networks [23, 26, 30], biology population [31, 32], and so forth.

The fractional Brownian motion (fBm) with the Hurst index $H \in (0, 1)$ is a Gaussian self-similar process with stationary increments. The concept of fBm was introduced by Kolmogorov [33] and Hurst [34] and then investigated in [23], where an integral is defined as the ordinary Brownian motion in the pointwise sense. fBm is one of the most important driving noises for stochastic systems, mainly as a result of its important properties, for example, long-term dependence and self-similarity features. These important properties make fBm have powerful memory effect and great potential applications [5, 16, 17, 24, 25, 27, 29, 31, 32, 35–38]. Unfortunately, because fBm is not a semimartingale or

Markov process, the theories of these processes cannot be applied to investigate fBm.

Population systems are often subject to environmental noise [2, 15, 18, 21, 39–41]. In particular, environmental Brownian noise can suppress explosions in the generalized Lotka–Volterra model investigated in [39]. Additionally, a predator-prey model with telegraph noise was considered in [40]. Shaikhet investigated the stability of a stochastic glassy-winged sharpshooter population [15], while Khodabin et al. [41] studied the interpolation solution of the population systems with Brownian motion. It should be noted that references [2, 15, 18, 39–41] only considered the case of population systems driven by Markovian noise. However, with regard to the analysis of the dynamics of population systems with non-Markovian processes, there has been little work in the literature [31, 32]. Therefore, it is significant to further reveal the influence of the non-Markovian process on the dynamics of biological population systems.

In this paper, we consider the dynamics of the exponential population growth system with mixed fractional Brownian motion (mfBm), which is a linear combination of independent Brownian motion and fBm. The mfBm is a non-Markovian, long-range dependent, mixed-self-similar, and correlated process [37, 42]. To the best of our knowledge, except for [31, 32], the extant literature heavily focuses on the dynamics of population systems with Brownian motion.

The almost sure exponential stability and the kth moment exponential stability are the most important stability problems. The relationship between them can be described by the theory of large deviations [11]. Several sufficient and necessary conditions for two types of exponential stability of the differential equations with Brownian motion were proposed [12]. However, the aforementioned two types of exponential stability of systems with mfBm have not been studied. Do the Hurst index and mfBm affect the exponential stability of systems? What are the large deviations, which are the rare events in biological population dynamics with mfBm? In this paper, we aim to give a positive answer to the exponential population growth system with mfBm.

In this paper, first, we will establish some useful lemmas that provide powerful tools for studying the stochastic differential equations with mfBm. We offer some explicit expressions and numerical characteristics such as mathematical expectation and variance of the solutions of the exponential population growth system with mfBm. Second, two sufficient and necessary conditions for the almost sure exponential stability and the kth moment exponential stability of the solution of constant coefficient exponential population growth system with mfBm are given. In view of the exponential stability of the system, we investigate the phenomenon of large deviations. In addition, it is indicated that the almost sure exponential stability and instability are clearly different when the Hurst index takes different values.

The rest of the paper is organized as follows. In Section 2, we give some notations and lemmas. In Sections 3 and 4, we present some explicit expressions, mathematical expectations, and variances of the solutions of the exponential population growth system with mfBm. We study the exponential stability in Section 5. In Section 6, we present the phenomenon of large deviations in biological population system. Finally, Section 7 summarizes the conclusions.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $|x|$ be the Euclidean norm of a vector x, $\mathbb{R} = (-\infty, +\infty)$ be the set of real numbers, and $\mathbb{R}_+ = [0, +\infty)$ be the set of positive real numbers. $S(\mathbb{R}_+)$ is the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}_+$, and $S(\mathbb{R}_+)$ is its dual space of tempered distributions.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and $L^2(\mathbb{P})$ be the shorthand notation for $L^2(\Omega, \mathscr{F}, \mathbb{P})$. $B^H(t)$ is a fBm with Hurst index $H \in (0,1)$, while $B(t)$ is a Brownian motion, with both being defined on the complete probability space. Assume that $B(t)$ and $B^H(t)$ are independent.

Let us define an operator M_H for $0 < H < 1$ on $\mathcal{S}(\mathbb{R}_+)$ by

$$\widehat{M_H} f(y) = C_H |y|^{(1/2)-H} \widehat{f}(y), \quad f \in S(\mathbb{R}_+), \tag{1}$$

where $\widehat{f}$ represents the Fourier transform of f and

$$C_H = \left[2\Gamma\left(H - \frac{1}{2}\right)\cos\left(\frac{\pi}{2}\left(H - \frac{1}{2}\right)\right)\right]^{-1} [\Gamma(2H+1)\sin(\pi H)]^{(1/2)}, \tag{2}$$

with Γ denoting the gamma function. $L_H^2(\mathbb{R}_+)$ is the closure of $\mathcal{S}(\mathbb{R}_+)$ with norm

$$\|f\|^2_{L_H^2(\mathbb{R}_+)} = \int_0^{+\infty} (Mf(x))^2 dx, \quad f \in S(\mathbb{R}_+). \tag{3}$$

We take $Mf(x) = f(x)$ for $H = (1/2)$, the identity map.

Definition 1 (see [25, 37]). Let H be a constant belonging to $(0,1)$. A fBm $B^H = \{B^H(t), t \in \mathbb{R}\}$ of Hurst index H is a centered Gaussian process with continuous sample paths and covariance

$$R_H(t,s) = \mathbb{E}\left[B^H(t)B^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right); \ t, \ s \in \mathbb{R}. \tag{4}$$

When $H = (1/2)$, the fBm is a standard Brownian motion.

We recall an Itô formula for the stochastic differential equation with fBm

$$dx(t) = \alpha(t)dt + \gamma(t)dB^H(t), \tag{5}$$

on $t \in \mathbb{R}_+$ with the initial value $x(0) = x_0 \in \mathbb{R}$, where $\alpha(t)$ and $\gamma(t)$ are deterministic continuous functions with $\alpha(t) \in L^1(\mathbb{R}_+; \mathbb{R})$, $\gamma(t) \in L_H^2(\mathbb{R}_+; \mathbb{R})$.

We now present a useful lemma of the fBm.

Lemma 1 (see [25, 35, 36]). *Let $Q(t, x(t)) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$ and the stochastic process $x(t)$ be described by (5). Assuming that the random variables*

$$Q(t,x(t)),$$
$$\int_0^t \frac{\partial Q}{\partial x}(s,x(s))dx(s), \tag{6}$$
$$\int_0^t s^{2H-1}\gamma^2(s)\frac{\partial^2 Q}{\partial x^2}(s,x(s))ds$$

all belong to $L^2(\mathbb{P})$ for all $t \in \mathbb{R}_+$; then,

$$Q(t,x(t)) = Q(0,x_0) + \int_0^t \frac{\partial Q}{\partial s}(s,x(s))ds + \int_0^t \frac{\partial Q}{\partial x}(s,x(s))dx(s) + H\int_0^t s^{2H-1}\gamma^2(s)\frac{\partial^2 Q}{\partial x^2}(s,x(s))ds. \tag{7}$$

We next state the mfBm-Itô lemma, which plays a critical role in what follows.

Consider a stochastic differential equation with mfBm

$$\begin{cases} dx(t) = \alpha(t)dt + \beta(t)dB(t) + \gamma(t)dB^H(t), & t \in \mathbb{R}_+, \\ x(0) = x_0, \end{cases} \tag{8}$$

where $\alpha(t) \in L^1(\mathbb{R}_+;\mathbb{R})$, $\beta(t) \in L^2(\mathbb{R}_+;\mathbb{R})$, and $\gamma(t) \in L_H^2(\mathbb{R}_+;\mathbb{R})$.

Lemma 2. *Let $Q(t,x(t)) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R};\mathbb{R})$ and the stochastic process $x(t)$ be described by (8). Assume that the random variables*

$$Q(t,x(t)),$$
$$\int_0^t \frac{\partial Q}{\partial x}(s,x(s))dx(s),$$
$$\int_0^t \beta^2(s)\frac{\partial^2 Q}{\partial x^2}(s,x(s))ds, \tag{9}$$
$$\int_0^t s^{2H-1}\gamma^2(s)\frac{\partial^2 Q}{\partial x^2}(s,x(s))ds$$

all belong to $L^2(\mathbb{P})$ for all $t \in \mathbb{R}_+$; then,

$$\begin{aligned} Q(t,x(t)) = Q(0,x_0) &+ \int_0^t \left(\frac{\partial Q}{\partial s}(s,x(s)) + \alpha(s)\frac{\partial Q}{\partial x}(s,x(s))\right)ds \\ &+ \int_0^t \left(\frac{1}{2}\beta^2(s) + Hs^{2H-1}\gamma^2(s)\right)\frac{\partial^2 Q}{\partial x^2}(s,x(s))ds \\ &+ \int_0^t \beta(s)\frac{\partial Q}{\partial x}(s,x(s))dB(s) + \int_0^t \gamma(s)\frac{\partial Q}{\partial x}(s,x(s))dB^H(s). \end{aligned} \tag{10}$$

Proof. Inspired by references [22, 43, 44], we here employ two methods to prove Lemma 2.

Method 1. Using fBm-Itô Lemma 1 to $Q(t,x(t))$, we have

$$\begin{aligned} dQ(t,x(t)) &= \left(\frac{\partial Q}{\partial t} + \frac{1}{2}\beta^2(t)\frac{\partial^2 Q}{\partial x^2} + Ht^{2H-1}\gamma^2(t)\frac{\partial^2 Q}{\partial x^2}\right)dt + \frac{\partial Q}{\partial x}dx(t), \\ &= \left(\frac{\partial Q}{\partial t} + \frac{1}{2}\beta^2(t)\frac{\partial^2 Q}{\partial x^2} + Ht^{2H-1}\gamma^2(t)\frac{\partial^2 Q}{\partial x^2} + \alpha(t)\frac{\partial Q}{\partial x}\right)dt + \beta(t)\frac{\partial Q}{\partial x}dB(t) + \gamma(t)\frac{\partial Q}{\partial x}dB^H(t). \end{aligned} \tag{11}$$

Then, integrating both sides of the above expression (11) from 0 to t gives (10).

Method 2. Using the well-known Taylor expansion formula leads to

$$dQ(t,x(t)) = \frac{\partial Q}{\partial t}dt + \frac{\partial Q}{\partial x}dx(t) + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}(dx(t))^2 + o(dt). \tag{12}$$

Because $\mathbb{E}[B(t)]^2 = t$ and $\mathbb{E}[B^H(t)]^2 = t^{2H}$, we can derive that

$$\begin{aligned}[\mathrm{d}B(t)]^2 &= \mathrm{d}t,\\ \left[\mathrm{d}B^H(t)\right]^2 &\approx \mathrm{Var}\left(\mathrm{d}B^H(t)\right)\\ &= \mathbb{E}\left(\mathrm{d}B^H(t)\right)^2 = \mathrm{d}t^{2H} = 2Ht^{2H-1}\mathrm{d}t.\end{aligned} \tag{13}$$

Moreover, the Itô formulas and independence of Brownian motion and fBm show that

$$\mathrm{d}t\cdot\mathrm{d}t = \mathrm{d}t\cdot\mathrm{d}B(t) = \mathrm{d}B(t)\cdot\mathrm{d}B^H(t) = \mathrm{d}t\cdot\mathrm{d}B^H(t) = 0. \tag{14}$$

Therefore,

$$\begin{aligned}(\mathrm{d}x(t))^2 &= \left(\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \gamma(t)\mathrm{d}B^H(t)\right)^2\\ &= \alpha^2(t)(\mathrm{d}t)^2 + \beta^2(t)(\mathrm{d}B(t))^2 + \gamma^2(t)\left(\mathrm{d}B^H(t)\right)^2 + 2\alpha(t)\beta(t)\mathrm{d}t\mathrm{d}B(t)\\ &\quad + 2\alpha(t)\gamma(t)\mathrm{d}t\mathrm{d}B^H(t) + 2\beta(t)\gamma(t)\mathrm{d}B(t)\mathrm{d}B^H(t)\\ &= \beta^2(t)\mathrm{d}t + 2Ht^{2H-1}\gamma^2(t)\mathrm{d}t + o(\mathrm{d}t),\end{aligned} \tag{15}$$

where

$$\begin{aligned}o(\mathrm{d}t) &= \alpha^2(t)(\mathrm{d}t)^2 + 2\alpha(t)\beta(t)\mathrm{d}t\mathrm{d}B(t)\\ &\quad + 2\alpha(t)\gamma(t)\mathrm{d}t\mathrm{d}B^H(t) + 2\beta(t)\gamma(t)\mathrm{d}B(t)\mathrm{d}B^H(t).\end{aligned} \tag{16}$$

By substituting (15) into (12) and letting $\mathrm{d}t \longrightarrow 0$ yields

$$\begin{aligned}\mathrm{d}Q(t,x(t)) &= \frac{\partial Q}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}\left(\beta^2(t)\mathrm{d}t + 2Ht^{2H-1}\gamma^2(t)\mathrm{d}t\right) + \frac{\partial Q}{\partial x}\left(\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \gamma(t)\mathrm{d}B^H(t)\right)\\ &= \frac{\partial Q}{\partial t}\mathrm{d}t + \alpha(t)\frac{\partial Q}{\partial x}\mathrm{d}t + \beta(t)\frac{\partial Q}{\partial x}\mathrm{d}B(t) + \gamma(t)\frac{\partial Q}{\partial x}\mathrm{d}B^H(t) + \frac{1}{2}\beta^2(t)\frac{\partial^2 Q}{\partial x^2}\mathrm{d}t + Ht^{2H-1}\gamma^2(t)\frac{\partial^2 Q}{\partial x^2}\mathrm{d}t.\end{aligned} \tag{17}$$

Hence, integrating both sides of the above expression from 0 to t yields (10). □

Lemma 3. *Assuming that the conditions of Lemma 2 hold, if $x(t)$ is a solution of system (8), then*

$$\begin{aligned}\mathrm{d}Q(t,x(t)) &= \frac{\partial Q}{\partial t}(t,x(t))\mathrm{d}t + \frac{\partial Q}{\partial x}(t,x(t))\mathrm{d}x(t)\\ &\quad + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2}(t,x(t))(\mathrm{d}x(t))^2,\end{aligned} \tag{18}$$

where $(dx(t))^2 = (dx(t))\cdot(dx(t))$ is computed by the rules

$$\begin{aligned}\mathrm{d}t\cdot\mathrm{d}t &= \mathrm{d}t\cdot\mathrm{d}B(t) = \mathrm{d}B(t)\cdot\mathrm{d}B^H(t) = \mathrm{d}t\cdot\mathrm{d}B^H(t) = 0,\\ \mathrm{d}B(t)\cdot\mathrm{d}B(t) &= \mathrm{d}t,\\ \mathrm{d}B^H(t)\cdot\mathrm{d}B^H(t) &= 2Ht^{2H-1}\mathrm{d}t.\end{aligned} \tag{19}$$

Remark 1. Lemmas 2 and 3 are called the Itô lemma of mfBm (mfBm-Itô lemma). It follows from (18) that the forms of the Itô formulas for Brownian motion and mfBm are highly unified. However, the computation rules of $(\mathrm{d}x(t))^2$ are slightly different. Moreover, when $H = (1/2)$, system (8) is driven by two independent Brownian motions. Therefore, Lemmas 2 and 3 also hold.

Remark 2. Several versions of the Itô lemma for fBm can be found in the literature (see [26–28, 36, 45–49]). Therefore, Lemma 2 or 3 is an extension of the fBm-Itô lemma with generalizability of those known results.

Based on Lemma 2, we will give some useful lemmas for mfBm.

Lemma 4. *Assuming that $u(t)$ and $v(t)$ are Borel measurable bounded scalar functions defined on $\mathbb{R}_+$, then*

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{\int_{t_0}^{t}\left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right\}\right]\\ &= \exp\left\{\int_{t_0}^{t}\left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right\}, \quad 0 \le t_0 \le t.\end{aligned} \tag{20}$$

Proof. Let $z(t) = \exp\left\{\int_{t_0}^{t}(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s))\right\}$, and apply Lemma 2.

$$\begin{aligned} \mathrm{d}z(t) = {} & \exp\left\{\int_{t_0}^{t}\left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right\}\left(u(t)\mathrm{d}B(t) + v(t)\mathrm{d}B^H(t)\right) \\ & + \frac{1}{2}\exp\left\{\int_{t_0}^{t}\left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right\}u^2(t)\mathrm{d}t \\ & + H\ \exp\left\{\int_{t_0}^{t}\left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right\}t^{2H-1}v^2(t)\mathrm{d}t. \end{aligned} \tag{21}$$

It is easy to obtain

$$\begin{aligned} z(t) = {} & z(t_0) + \int_{t_0}^{t}\exp\left\{\int_{t_0}^{s}\left(u(r)\mathrm{d}B(r) + v(r)\mathrm{d}B^H(r)\right)\right\}\left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right) \\ & + \frac{1}{2}\int_{t_0}^{t}\exp\left\{\int_{t_0}^{s}\left(u(r)\mathrm{d}B(r) + v(r)\mathrm{d}B^H(r)\right)\right\}u^2(s)\mathrm{d}s \\ & + H\int_{t_0}^{t}\exp\left\{\int_{t_0}^{s}\left(u(r)\mathrm{d}B(r) + v(r)\mathrm{d}B^H(r)\right)\right\}s^{2H-1}v^2(s)\mathrm{d}s. \end{aligned} \tag{22}$$

Taking mathematical expectation on both sides of (22) gives

$$\mathbb{E}[z(t)] = \mathbb{E}[z(t_0)] + \frac{1}{2}\int_{t_0}^{t}\mathbb{E}[z(s)]u^2(s)\mathrm{d}s + H\int_{t_0}^{t}\mathbb{E}[z(s)]s^{2H-1}v^2(s)\mathrm{d}s, \tag{23}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[z(t)] = \left(\frac{1}{2}u^2(t) + Ht^{2H-1}v^2(t)\right)\mathbb{E}[z(t)],\ \mathbb{E}[z(t_0)] = 1. \tag{24}$$

Consequently,

$$\mathbb{E}[z(t)] = \exp\left\{\int_{t_0}^{t}\left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right\}. \tag{25}$$

□

Lemma 5. *Assuming that $k > 0$, $u(t)$ and $v(t)$ are Borel measurable bounded scalar functions defined on $\mathbb{R}_+$, and*

$$z(t) = \exp\left\{\int_{t_0}^{t}\left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right\}, \tag{26}$$

then

$$\mathbb{E}\left[z^k(t)\right] = \exp\left\{k^2\int_{t_0}^{t}\left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right\},\quad 0 \le t_0 \le t. \tag{27}$$

Proof. By Lemma 2, we gain

$$\begin{aligned} z^k(t) = {} & z^k(t_0) + k\int_{t_0}^{t}u(s)z^k(s)\mathrm{d}B(s) \\ & + k\int_{t_0}^{t}v(s)z^k(s)\mathrm{d}B^H(s) \\ & + k^2\int_{t_0}^{t}\left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)z^k(s)\mathrm{d}s. \end{aligned} \tag{28}$$

Taking mathematical expectation on both sides of (28) gives

$$\begin{aligned} \mathbb{E}\left[z^k(t)\right] = {} & \mathbb{E}\left[z^k(t_0)\right] \\ & + k^2\int_{t_0}^{t}\left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathbb{E}\left[z^k(s)\right]\mathrm{d}s, \end{aligned} \tag{29}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[z^k(t)\right] = k^2\left(\frac{1}{2}u^2(t) + Ht^{2H-1}v^2(t)\right)\mathbb{E}\left[z^k(t)\right],\ \mathbb{E}\left[z^k(t_0)\right] = 1. \tag{30}$$

Consequently,

$$\mathbb{E}\left[z^k(t)\right] = \exp\left\{k^2 \int_{t_0}^{t} \left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right\}. \tag{31}$$

□

Lemma 6. *Assuming that $k > 0$, $u(t)$ and $v(t)$ are Borel measurable bounded scalar functions defined on $\mathbb{R}_+$, and*

$$z(t) = \exp\left(u(t)B(t) + v(t)B^H(t)\right), \tag{32}$$

then

$$\mathbb{E}\left[z^k(t)\right] = \exp\left[k^2\left(\frac{1}{2}u^2(t) + Ht^{2H-1}v^2(t)\right)\right]. \tag{33}$$

Proof. Using the mfBm-Itô Lemma 2 implies

$$\begin{aligned} z^k(t) &= z^k(0) + k\int_0^t u(s)z^k(s)\mathrm{d}B(s) \\ &\quad + k\int_0^t v(s)z^k(s)\mathrm{d}B^H(s) \\ &\quad + k^2\int_0^t \left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)z^k(s)\mathrm{d}s. \end{aligned} \tag{34}$$

Taking expectation on both sides of (34) yields

$$\mathbb{E}\left[z^k(t)\right] = \mathbb{E}\left[z^k(0)\right] + k^2\int_0^t \left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathbb{E}\left[z^k(s)\right]\mathrm{d}s, \tag{35}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[z^k(t)\right] = k^2\left(\frac{1}{2}u^2(t) + Ht^{2H-1}v^2(t)\right)\mathbb{E}\left[z^k(t)\right],\ \mathbb{E}\left[z^k(0)\right] = 1. \tag{36}$$

Consequently,

$$\mathbb{E}\left[z^k(t)\right] = \exp\left[k^2\left(\frac{1}{2}u^2(t) + Ht^{2H-1}v^2(t)\right)\right]. \tag{37}$$

Moreover, when $u(t) = u$ and $v(t) = v$ (u and v are constants, the same as below), one finds that

$$\mathbb{E}\left[z^k(t)\right] = \exp\left[k^2\left(\frac{1}{2}u^2t + Ht^{2H}v^2\right)\right]. \tag{38}$$

□

Remark 3. If $u(t) = 0$ or $v(t) = 0$, then Lemmas 4–6 are cases of the corresponding fBm or classical Brownian motion theory, respectively. Therefore, Lemmas 4–6 are generalizations of the classical Brownian motion and fBm theories.

Lemma 7 (law of the iterated logarithm for fBm [17, 25, 50]). *There exists a suitable constant $C_H > 0$ such that*

$$\limsup_{t\longrightarrow+\infty} \frac{B^H(t)}{t^H\sqrt{\log\log t}} = C_H, \tag{39}$$

and then

$$\limsup_{t\longrightarrow+\infty} \frac{B^H(t)}{t^H\sqrt{\log\log t}} = C_H + \varepsilon, \tag{40}$$

which indicates that

$$\limsup_{t\longrightarrow+\infty} \frac{B^H(t)}{t} = 0, \tag{41}$$

where $\epsilon > 0$ is a constant.

Remark 4. The conclusion of Lemma 7 also holds when $H = (1/2)$ and $C_{(1/2)} = \sqrt{2}$ (see [2], Theorem 4.2 on page 16), namely, the case of standard Brownian motion.

Lemma 8. *Assuming that $u(t)$ and $v(t)$ are Borel measurable bounded scalar functions defined on $\mathbb{R}_+$ and*

$$x(t) = \int_0^t u(s)\mathrm{d}B(s) + \int_0^t v(s)\mathrm{d}B^H(s), \quad \frac{1}{2} < H < 1, \tag{42}$$

then

$$x(t) \sim \mathbb{N}\left(0, \int_0^t u^2(s)\mathrm{d}s + H\int_0^t v^2(s)(t-s)^{2H-1}\mathrm{d}s\right). \tag{43}$$

Proof. We recall the basic property of fBm (see, e.g., [5], Remark 2.2, and [25], Definition 1.1.1 on page 5):

$$\mathbb{E}[x(t)] = \mathbb{E}\left[\int_0^t u(s)\mathrm{d}B(s)\right] + \mathbb{E}\left[\int_0^t v(s)\mathrm{d}B^H(s)\right] = 0. \tag{44}$$

It follows from the independence and isometry of Brownian motion and fBm that

$$\begin{aligned}\mathrm{Var}[x(t)] &= \mathbb{E}[x(t)-\mathbb{E}(x(t))]^2 = \mathbb{E}\left[\int_0^t u(s)\mathrm{d}B(s)+\int_0^t v(s)\mathrm{d}B^H(s)\right]^2\\ &= \mathbb{E}\left[\int_0^t u(s)\mathrm{d}B(s)\right]^2+\mathbb{E}\left[\int_0^t v(s)\mathrm{d}B^H(s)\right]^2\\ &= \int_0^t u^2(s)\mathrm{d}s+H(2H-1)\int_0^t\int_0^t v(\xi)v(\eta)|\xi-\eta|^{2H-2}\mathrm{d}\xi\mathrm{d}\eta\\ &= \int_0^t u^2(s)\mathrm{d}s+H(2H-1)\int_0^t\int_0^\xi v^2(s)(\xi-s)^{2H-2}\mathrm{d}s\mathrm{d}\xi\\ &= \int_0^t u^2(s)\mathrm{d}s+H(2H-1)\int_0^t\int_s^t v^2(s)(\xi-s)^{2H-2}\mathrm{d}\xi\mathrm{d}s\\ &= \int_0^t u^2(s)\mathrm{d}s+H\int_0^t v^2(s)(t-s)^{2H-1}\mathrm{d}s.\end{aligned} \tag{45}$$

□

Remark 5. Notably, Lemma 8 is not suitable for $0<H<(1/2)$, since the kernel $|\xi-\eta|^{2H-2}$ cannot be integrated over the diagonal.

3. The Exponential Population Growth System with mfBm

In this section, we discuss the exponential population growth system with mfBm.

We consider a simple population growth system

$$\begin{cases}\dfrac{\mathrm{d}P(t)}{\mathrm{d}t}=\mu(t)P(t),\\ P(0)=P_0,\end{cases} \tag{46}$$

where $P(t)$ and $\mu(t)$ are the size and relative growth rate of the population at time t, respectively. Let $\mu(t)=\theta(t)$ be a non-random function. Thus, we get that

$$P(t)=P_0\exp\left\{\int_0^t\theta(s)\mathrm{d}s\right\}. \tag{47}$$

In a special case, if $\theta(t)=\theta$, we obtain

$$P(t)=P_0\exp(\theta t). \tag{48}$$

It might happen that $\mu(t)$ is not completely known but subject to environmental noise. In other words,

$$\mu(t)=\theta(t)+\text{“noise”}. \tag{49}$$

Let “noise” $=u(t)W(t)+v(t)W^H(t)$; then,

$$\mu(t)=\theta(t)+u(t)W(t)+v(t)W^H(t), \tag{50}$$

where $W(t)=(\mathrm{d}B(t)/\mathrm{d}t)$ and $W^H(t)=(\mathrm{d}B^H(t)/\mathrm{d}t)$ are one-dimensional Gaussian white noise and fractional Gaussian noise, respectively. $B(t)$ and $B^H(t)$ are one-dimensional Brownian motion and fBm, respectively. $u(t)$ and $v(t)$ denote the intensities of the noise at t. Thus, this exponential population growth system with mfBm may be described as follows:

$$\begin{cases}\mathrm{d}P(t)=\theta(t)P(t)\mathrm{d}t+u(t)P(t)\mathrm{d}B(t)+v(t)P(t)\mathrm{d}B^H(t),\\ P(0)=P_0.\end{cases} \tag{51}$$

Remark 6. The exponential population growth system with mfBm (51) is a non-Markovian process because the stochastic perturbation is a mfBm. Notably system (51) reduces to the fractional exponential population growth system, if $u(t)=0, v(t)\neq 0$; it becomes a stochastic exponential population growth system, if $u(t)\neq 0, v(t)=0$; and it reduces to a deterministic exponential population growth system, if $u(t)=0, v(t)=0$. Thus, the exponential population growth system with mfBm (51) includes fractional, stochastic, and deterministic systems as special cases.

Theorem 1. *The explicit solution of system (51) is given by*

$$P(t)=P_0\exp\left(\int_0^t\left(\theta(s)-\frac{1}{2}u^2(s)-Hs^{2H-1}v^2(s)\right)\mathrm{d}s+\int_0^t u(s)\mathrm{d}B(s)+\int_0^t v(s)\mathrm{d}B^H(s)\right). \tag{52}$$

Proof. With the help of (51), we deduce

$$\int_0^t\frac{\mathrm{d}P(s)}{P(s)}=\int_0^t\theta(s)\mathrm{d}s+\int_0^t u(s)\mathrm{d}B(s)+\int_0^t v(s)\mathrm{d}B^H(s). \tag{53}$$

Using the mfBm-Itô Lemma 2 to $\ln x$, we obtain

$$\mathrm{d}(\ln P(t))=\frac{\mathrm{d}P(t)}{P(t)}-\left(\frac{1}{2}u^2(t)+Ht^{2H-1}v^2(t)\right)\mathrm{d}t, \tag{54}$$

or

$$\int_0^t \frac{\mathrm{d}P(s)}{P(s)} = \ln\frac{P(t)}{P_0} + \int_0^t \left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s. \tag{55}$$

By (53) and (55), we know that

$$\ln\frac{P(t)}{P_0} = \int_0^t \left(\theta(s) - \frac{1}{2}u^2(s) - Hs^{2H-1}v^2(s)\right)\mathrm{d}s + \int_0^t u(s)\mathrm{d}B(s) + \int_0^t v(s)\mathrm{d}B^H(s). \tag{56}$$

Consequently,

$$P(t) = P_0 \exp\left(\int_0^t \left(\theta(s) - \frac{1}{2}u^2(s) - Hs^{2H-1}v^2(s)\right)\mathrm{d}s + \int_0^t u(s)\mathrm{d}B(s) + \int_0^t v(s)\mathrm{d}B^H(s)\right). \tag{57}$$

Moreover, when $\theta(t) = \theta$, $u(t) = u$ and $v(t) = v$, one finds that

$$P(t) = P_0 \exp\left(\left(\theta - \frac{1}{2}u^2 - \frac{1}{2}t^{2H-1}v^2\right)t + uB(t) + vB^H(t)\right). \tag{58}$$

□

Theorem 2. *Assuming that P_0, $B(t)$ and $B^H(t)$ are independent random variables in (51), then*

$$\mathbb{E}[P(t)] = \mathbb{E}[P_0]\exp\left(\int_0^t \theta(s)\mathrm{d}s\right), \tag{59}$$

and

$$\mathrm{Var}[P(t)] = \exp\left(2\int_0^t \theta(s)\mathrm{d}s\right)\left\{\left(\mathrm{Var}[P_0] + \mathbb{E}^2[P_0]\right)\exp\left(\int_0^t \left(u^2(s) + 2Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right) - \mathbb{E}^2[P_0]\right\}. \tag{60}$$

Proof. From (52), one obtains

$$\mathbb{E}[P(t)] = \mathbb{E}[P_0]\exp\left(\int_0^t \left(\theta(s) - \frac{1}{2}u^2(s) - Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right) \times \mathbb{E}\left[\exp\left(\int_0^t \left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right)\right]. \tag{61}$$

Using Lemma 4, one knows that

$$\mathbb{E}\left[\exp\left(\int_0^t \left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right)\right] = \exp\left[\int_0^t \left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right]. \tag{62}$$

Substituting (62) into (61) results in

$$\mathbb{E}[P(t)] = \mathbb{E}[P_0]\exp\left(\int_0^t \theta(s)\mathrm{d}s\right). \tag{63}$$

Furthermore,

$$\mathbb{E}\left[P^2(t)\right] = \mathbb{E}\left[P_0^2\right]\exp\left(2\int_0^t \left(\theta(s) - \frac{1}{2}u^2(s) - Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right) \times \mathbb{E}\left[\exp\left(2\int_0^t \left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right)\right]. \tag{64}$$

According to Lemma 5, one has

$$\mathbb{E}\left[\exp\left(2\int_0^t \left(u(s)\mathrm{d}B(s) + v(s)\mathrm{d}B^H(s)\right)\right)\right] = \exp\left[4\int_0^t \left(\frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right], \tag{65}$$

which together with (64) leads to

$$\mathbb{E}\left[P^2(t)\right] = \mathbb{E}\left[P_0^2\right]\exp\left(2\int_0^t\left(\theta(s) + \frac{1}{2}u^2(s) + Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right). \tag{66}$$

Consequently,

$$\begin{aligned}\mathrm{Var}[P(t)] &= \exp\left(2\int_0^t \theta(s)\mathrm{d}s\right)\left\{\mathbb{E}\left[P_0^2\right]\exp\left(\int_0^t\left(u^2(s) - 2Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right) - \mathbb{E}^2[P_0]\right\}\\ &= \exp\left(2\int_0^t \theta(s)\mathrm{d}s\right)\left\{\left(\mathrm{Var}[P_0] + \mathbb{E}^2[P_0]\right)\exp\left(\int_0^t\left(u^2(s) - 2Hs^{2H-1}v^2(s)\right)\mathrm{d}s\right) - \mathbb{E}^2[P_0]\right\}.\end{aligned} \tag{67}$$

□

Corollary 1. *Assuming that P_0 is a non-random variable, then*

$$\mathbb{E}[P(t)] = [P_0]\exp\left(\int_0^t \theta(s)\mathrm{d}s\right). \tag{68}$$

In particular, if $\theta(t) = \theta$, we get

$$\mathbb{E}[P(t)] = P_0\exp(\theta t), \tag{69}$$

which is the same as (48).

Corollary 2. *Assuming that P_0 is a non-random variable, $\theta(t) = \theta$, $u(t) = u$, and $v(t) = v$, then*

$$\mathrm{Var}[P(t)] = P_0^2\exp(2\theta t)\left(\exp\left(u^2 + t^{2H-1}v^2\right)t - 1\right). \tag{70}$$

4. The Generalized Exponential Population Growth System with mfBm

The symbols $W(t) = (W_1(t), W_2(t), \ldots, W_n(t))$, $B(t) = (B_1(t), B_2(t), \ldots, B_n(t))$, $W^H(t) = (W_1^H(t), W_2^H(t), \ldots, W_n^H(t))$, and $B^H(t) = (B_1^H(t), B_2^H(t), \ldots, B_n^H(t))$ denote, respectively, the n-dimensional Gaussian white noise, Brownian motion, fractional Gaussian noise, and fBm.

By considering

$$\mu(t) = u(t) + \frac{1}{2}\sum_{i=1}^n u_i^2(t) + \sum_{i=1}^n v_i(t)W_i(t) + \sum_{i=1}^n w_i(t)W_i^H(t), \tag{71}$$

where $v_i(t)$ and $w_i(t)$ $(i = 1, \ldots, n)$ are the intensities of the noise at time t and $u_i(t)$ denote the error of the growth rate subject to randomly fluctuating environment source i at time t, then we obtain the generalized exponential population growth system with mfBm:

$$\begin{cases}\mathrm{d}P(t) = \left(\left(u(t) + \frac{1}{2}\sum_{i=1}^n u_i^2(t)\right)\mathrm{d}t + \sum_{i=1}^n v_i(t)\mathrm{d}B_i(t) + \sum_{i=1}^n w_i(t)\mathrm{d}B_i^H(t)\right)P(t),\\ P(0) = P_0.\end{cases} \tag{72}$$

Theorem 3. *The explicit solution of system (72) is given by*

$$P(t) = P_0\exp\left(\int_0^t\left(u(s) + \frac{1}{2}\sum_{i=1}^n\left(u_i^2(s) - v_i^2(s)\right) - Hs^{2H-1}\sum_{i=1}^n w_i^2(s)\right)ds + \sum_{i=1}^n\int_0^t v_i(s)dB_i(s) + \sum_{i=1}^n\int_0^t w_i(s)dB_i^H(s)\right). \tag{73}$$

Proof. It can be obtained from (72) that

$$\int_0^t \frac{\mathrm{d}P(s)}{P(s)} = \int_0^t \left(u(s) + \frac{1}{2}\sum_{i=1}^n u_i^2(s) \right) \mathrm{d}s + \sum_{i=1}^n \int_0^t v_i(s)\mathrm{d}B_i(s) + \sum_{i=1}^n \int_0^t w_i(s)\mathrm{d}B_i^H(s). \tag{74}$$

Applying mfBm-Itô Lemma 3 to $\ln x$, we get that

$$\mathrm{d}(\ln P(t)) = \frac{\mathrm{d}P(t)}{P(t)}(t) - \left(\frac{1}{2}\sum_{i=1}^n v_i^2(t) + Ht^{2H-1}\sum_{i=1}^n w_i^2(t) \right)\mathrm{d}t, \tag{75}$$

or

$$\int_0^t \frac{\mathrm{d}P(s)}{P(s)} = \ln\frac{P(s)}{P_0} + \int_0^t \left(\frac{1}{2}\sum_{i=1}^n v_i^2(s) + Hs^{2H-1}\sum_{i=1}^n w_i^2(s) \right)\mathrm{d}s. \tag{76}$$

According to (74) and (76), we show that

$$\ln\frac{P(t)}{P_0} = \int_0^t \left(u(s) + \frac{1}{2}\sum_{i=1}^n \left(u_i^2(s) - v_i^2(s)\right) - Hs^{2H-1}\sum_{i=1}^n w_i^2(s) \right)\mathrm{d}s + \sum_{i=1}^n \int_0^t v_i(s)\mathrm{d}B_i(s) + \sum_{i=1}^n \int_0^t w_i(s)\mathrm{d}B_i^H(s). \tag{77}$$

Consequently,

$$P(t) = P_0 \exp\left(\int_0^t \left(u(s) + \frac{1}{2}\sum_{i=1}^n \left(u_i^2(s) - v_i^2(s)\right) - Hs^{2H-1}\sum_{i=1}^n w_i^2(s) \right)\mathrm{d}s + \sum_{i=1}^n \int_0^t v_i(s)\mathrm{d}B_i(s) + \sum_{i=1}^n \int_0^t w_i(s)\mathrm{d}B_i^H(s). \right. \tag{78}$$

Moreover, when $u(t) = u$, $u_i(t) = u_i$, $v_i(t) = v_i$, and $w_i(t) = w_i (i = 1, \ldots, n)$, one finds that

$$P(t) = P_0 \exp\left[\left(u + \frac{1}{2}\sum_{i=1}^n \left(u_i^2 - v_i^2\right) - \frac{1}{2}t^{2H-1}\sum_{i=1}^n w_i^2 \right)t + \sum_{i=1}^n v_i B_i(t) + \sum_{i=1}^n w_i B_i^H(t) \right]. \tag{79}$$

□

Theorem 4. *Assuming that P_0, $B_i(t)$, and $B_i^H(t)(i = 1, \ldots, n)$ are independent random variables in system (72), then*

$$\mathbb{E}[P(t)] = \mathbb{E}[P_0]\exp\left(\int_0^t \left(u(s) + \frac{1}{2}\sum_{i=1}^n u_i^2(s) \right)\mathrm{d}s \right), \tag{80}$$

and

$$\mathrm{Var}[P(t)] = \exp\left(2\int_0^t \left(u(s) + \frac{1}{2}\sum_{i=1}^n u_i^2(s) \right)\mathrm{d}s \right) \times \left\{ \left(\mathrm{Var}[P_0] + \mathbb{E}^2[P_0]\right)\exp\left(\sum_{i=1}^n \int_0^t \left(v_i^2(s) + 2Hs^{2H-1}w_i^2(s)\right)\mathrm{d}s \right) - \mathbb{E}^2[P_0] \right\}. \tag{81}$$

Proof. From (73),

$$\mathbb{E}[P(t)] = \mathbb{E}[P_0]\exp\left(\int_0^t\left[u(s) + \frac{1}{2}\sum_{i=1}^n u_i^2(s) - v_i^2(s) - Hs^{2H-1}\sum_{i=1}^n w_i^2(s)\right]ds\right) \times \mathbb{E}\left[\exp\left(\sum_{i=1}^n\int_0^t v_i(s)dB_i(s) + \sum_{i=1}^n\int_0^t w_i(s)dB_i^H(s)\right)\right]. \tag{82}$$

Employing Lemma 4 and independent of the Brownian motion and fBm, one sees that

$$\mathbb{E}\left[\exp\left(\sum_{i=1}^n\int_0^t v_i(s)\mathrm{d}B_i(s) + \sum_{i=1}^n\int_0^t w_i(s)\mathrm{d}B_i^H(s)\right)\right] = \exp\left[\sum_{i=1}^n\int_0^t\left(\frac{1}{2}v_i^2(s) + Hs^{2H-1}w_i^2(s)\right)\mathrm{d}s\right]. \tag{83}$$

Substituting (83) into (82) leads to

$$\mathbb{E}[P(t)] = \mathbb{E}[P_0]\exp\left(\int_0^t\left(u(s) + \frac{1}{2}\sum_{i=1}^n u_i^2(s)\right)\mathrm{d}s\right). \tag{84}$$

Furthermore,

$$\mathbb{E}\left[P^2(t)\right] = \mathbb{E}\left[P_0^2\right]\exp\left[2\int_0^t\left(u(s) + \frac{1}{2}\sum_{i=1}^n\left(u_i^2(s) - v_i^2(s)\right) - Hs^{2H-1}\sum_{i=1}^n w_i^2(s)\right)\mathrm{d}s\right] \times \mathbb{E}\left[\exp\left(2\left(\sum_{i=1}^n\int_0^t v_i(s)\mathrm{d}B_i(s) + \sum_{i=1}^n\int_0^t w_i(s)\mathrm{d}B_i^H(s)\right)\right)\right]. \tag{85}$$

Applying Lemma 5, one derives that

$$\mathbb{E}\left[\exp\left(2\left(\sum_{i=1}^n\int_0^t v_i(s)\mathrm{d}B_i(s) + \sum_{i=1}^n\int_0^t w_i(s)\mathrm{d}B_i^H(s)\right)\right)\right] = \exp\left(4\left(\sum_{i=1}^n\int_0^t\left(\frac{1}{2}v_i^2(s) + Hs^{2H-1}w_i^2(s)\right)\mathrm{d}s\right)\right). \tag{86}$$

Hence,

$$\mathbb{E}\left[P^2(t)\right] = \mathbb{E}\left[P_0^2\right]\exp\left[2\int_0^t\left(u(s) + \frac{1}{2}\sum_{i=1}^n\left(u_i^2(s) + v_i^2(s)\right) + Hs^{2H-1}\sum_{i=1}^n w_i^2(s)\right)\mathrm{d}s\right]. \tag{87}$$

Therefore,

$$\begin{aligned}\mathrm{Var}[P(t)] &= \exp\left(2\int_0^t\left(u(s) + \frac{1}{2}\sum_{i=1}^n u_i^2(s)\right)\mathrm{d}s\right) \times \left\{\mathbb{E}\left[P_0^2\right]\exp\left(\sum_{i=1}^n\int_0^t\left(v_i^2(s) + 2Hs^{2H-1}w_i^2(s)\right)\mathrm{d}s\right) - \mathbb{E}^2[P_0]\right\} \\ &= \exp\left(2\int_0^t\left(u(s) + \frac{1}{2}\sum_{i=1}^n u_i^2(s)\right)\mathrm{d}s\right) \times \left\{\left(\mathrm{Var}[P_0] + \mathbb{E}^2[P_0]\right)\exp\left(\sum_{i=1}^n\int_0^t\left(v_i^2(s) + 2Hs^{2H-1}w_i^2(s)\right)\mathrm{d}s\right) - \mathbb{E}^2[P_0]\right\}.\end{aligned} \tag{88}$$

□

Corollary 3. *Assuming that P_0 is a non-random variable, $u(t) = u$, $u_i(t) = u_i$, $v_i(t) = v_i$, and $w_i(t) = w_i$ $(i = 1, \ldots, n)$, then*

$$\mathbb{E}[P(t)] = P_0 \exp\left(\left(u + \frac{1}{2}\sum_{i=1}^{n} u_i^2\right)t\right), \tag{89}$$

and

$$\mathrm{Var}[P(t)] = P_0^2 \exp\left(2\left(u + \frac{1}{2}\sum_{i=1}^{n} u_i^2\right)t\right)\left[\exp\sum_{i=1}^{n}\left(v_i^2 + t^{2H-1}w_i^2\right)t\right) - 1]. \tag{90}$$

5. Exponential Stability

For the sake of simplicity, in what follows, we discuss a constant coefficient exponential population growth system with mfBm:

$$\begin{cases} \mathrm{d}P(t) = \theta P(t)\mathrm{d}t + uP(t)\mathrm{d}B(t) + vP(t)\mathrm{d}B^H(t), \\ P(0) = P_0, \end{cases} \tag{91}$$

where the parameters are defined as before.

$P(t; t_0, P_0)$ represents the solution of system (91) at time t, with an initial value P_0 at time t_0. We then give the definition of sample (or simply) Lyapunov exponent and kth moment Lyapunov exponent for the solution of system (91).

Definition 2 (see [2, 14]). The two limits,

$$L := \limsup_{t\longrightarrow+\infty} \frac{1}{t}\log\left(\left|P(t; t_0, P_0)\right|\right), \tag{92}$$

and

$$L(k) := \limsup_{t\longrightarrow+\infty} \frac{1}{t}\log\mathbb{E}\left(\left|P(t; t_0, P_0)\right|\right)^k, \tag{93}$$

are called the sample Lyapunov exponent and the kth moment Lyapunov exponent, respectively.

5.1. The Almost Sure Exponential Stability. The following theorem presents a necessary and sufficient condition for system (91) to be almost surely exponentially stable.

Theorem 5

(i) When $0 < H < (1/2)$, the trivial solution of system (91) is almost surely exponentially stable if $\theta \le 0$ or $\theta < (1/2)u^2$ but unstable if $\theta > (1/2)u^2$.

(ii) When $H = (1/2)$, the trivial solution of system (91) is almost surely exponentially stable if $\theta \le 0$ or $\theta < (1/2)(u^2 + v^2)$ but unstable if $\theta > (1/2)(u^2 + v^2)$.

(iii) When $(1/2) < H < 1$, the trivial solution of system (91) is almost surely exponentially stable for all parameters θ, u and v.

Proof. From (41) and (58), we get

$$\begin{aligned} L &= \limsup_{t\longrightarrow+\infty} \frac{1}{t}\log\left[\left|P(t; 0, P_0)\right|\right] = \limsup_{t\longrightarrow+\infty} \frac{1}{t}\log\left[\left|P_0\right|\exp\left(\left(\theta - \frac{1}{2}u^2 - \frac{1}{2}t^{2H-1}v^2\right)t + uB(t) + vB^H(t)\right)\right] \\ &= \lim_{t\longrightarrow+\infty}\left(\theta - \frac{1}{2}u^2 - \frac{1}{2}t^{2H-1}v^2\right). \end{aligned} \tag{94}$$

Then, we obtain

$$L = \begin{cases} \theta - \frac{1}{2}u^2, & 0 < H < \frac{1}{2}, \\ \theta - \frac{1}{2}\left(u^2 + v^2\right), & H = \frac{1}{2}, \\ -\infty, & \frac{1}{2} < H < 1. \end{cases} \tag{95}$$

Consequently, the trivial solution is almost surely exponentially stable if and only if the sample Lyapunov exponent is negative. Hence, Theorem 5 must hold. □

Remark 7. Theorem 5 shows that the trivial solution of system (91) is almost surely exponentially stable for the arbitrary Hurst index $0 < H < 1$ when $\theta \le 0$. This means that the mfBm cannot destabilize the exponential population growth system (Figures 1–6).

5.2. kth Moment Exponential Stability. The following theorem gives a necessary and sufficient condition for the kth moment exponential stability of system (91).

Theorem 6

(i) When $0 < H < (1/2)$, the trivial solution of system (91) is kth moment exponentially stable if one of the following conditions holds: (1) $0 < k < 1$ and $\theta \leq 0$; (2) $0 < k < 1$ and $\theta < (1/2)(1-k)u^2$; (3) $k = 1$ and $\theta < 0$; or (4) $k > 1$, $\theta < 0$, and $(1/2)(k-1)u^2 < -\theta$. Moreover, it is kth moment exponentially unstable if one of the following conditions is satisfied: (1) $0 < k < 1$ and $\theta > (1/2)(1-k)u^2$; (2) $k = 1$ and $\theta > 0$; or (3) $k > 1$ and $\theta \geq 0$.

(ii) When $H = (1/2)$, the trivial solution of system (91) is kth moment exponentially stable if one of the following conditions holds: (1) $0 < k < 1$ and $\theta \leq 0$; (2) $0 < k < 1$ and $\theta < (1/2)(1-k)(u^2+v^2)$; (3) $k = 1$ and $\theta < 0$; or (4) $k > 1$, $\theta < 0$, and $(1/2)(k-1)(u^2+v^2) < -\theta$. Moreover, it is kth moment exponentially unstable if one of the following conditions is satisfied: (1) $0 < k < 1$ and $\theta > (1/2)(1-k)(u^2+v^2)$; (2) $k = 1$ and $\theta > 0$; or (3) $k > 1$ and $\theta \geq 0$.

(iii) When $(1/2) < H < 1$, the trivial solution of system (91) is kth moment exponentially stable if $0 < k < 1$ or $k = 1$ and $\theta < 0$, but it is kth moment exponentially unstable if $k > 1$ or $k = 1$ and $\theta > 0$.

Proof. According to Lemma 6 and (58), one deduces

$$\begin{aligned} L(k) &= \limsup_{t\longrightarrow+\infty} \frac{1}{t}\log\mathbb{E}\left[\left|P(t;0,P_0)\right|^k\right] = \limsup_{t\longrightarrow+\infty} \frac{1}{t}\log\left\{\left|P_0\right|^k\mathbb{E}\left[\exp\left(k\left(\theta-\frac{1}{2}u^2-\frac{1}{2}t^{2H-1}v^2\right)t+kuB(t)+kvB^H(t)\right)\right]\right\} \\ &= \lim_{t\longrightarrow+\infty}\left(k\theta+\frac{1}{2}k(k-1)u^2+\frac{1}{2}k(k-1)v^2t^{2H-1}\right). \end{aligned} \tag{96}$$

Therefore, one gets

$$L(k) = \begin{cases} k\theta+\frac{1}{2}k(k-1)u^2, & 0<H<\frac{1}{2}, \\ k\theta+\frac{1}{2}k(k-1)\left(u^2+v^2\right), & H=\frac{1}{2}, \\ -\infty, & \frac{1}{2}<H<1 \text{ and } 0<k<1, \\ \theta, & \frac{1}{2}<H<1 \text{ and } k=1, \\ +\infty, & \frac{1}{2}<H<1 \text{ and } k>1. \end{cases} \tag{97}$$

The trivial solution is kth moment exponentially stable if and only if the kth moment Lyapunp>ov exponent is negative. Consequently, the required assertion (iii) follows immediately from (97). When $H = (1/2)$, by considering $L(k) = k\theta + (1/2)k(k-1)(u^2+v^2) < 0$, we can get the following four cases: (1) $0 < k < 1$ and $\theta \leq 0$; (2) $0 < k < 1$ and $\theta < (1/2)(1-k)(u^2+v^2)$; (3) $k = 1$ and $\theta < 0$; and (4) $k > 1$, $\theta < 0$ and $(1/2)(k-1)(u^2+v^2) < -\theta$. Moreover, by considering $L(k) = k\theta + (1/2)k(k-1)(u^2+v^2) > 0$, we can obtain the following three cases: (1) $0 < k < 1$ and $\theta > (1/2)(1-k)(u^2+v^2)$; (2) $k = 1$ and $\theta > 0$; and (3) $k > 1$ and $\theta \geq 0$. Hence (ii) must hold. Similar to the proof in (ii), one can easily deduce (i). □

6. The Phenomenon of Large Deviations in Biological Population System

Large deviations [51] involve the probability of rare events in random processes. Such events can have dramatic consequences despite their scarcity, especially if one takes into account the behavior of systems in geology [52], population [53], and climate science [54] on long time scales. These large deviations are considered to be the link between the almost sure exponential stability and the kth moment exponential stability. In general, for sufficiently large $k > 0$, even for stable systems, the kth moment Lyapunov exponent is often positive due to the large deviations. Therefore, it is important to explore the phenomenon of large deviations in biological population systems such that $L < 0$ and $L(k) > 0$ for sufficiently large $k > 0$. When $L < 0$, for $P_0 \neq 0$, $|P(t;t_0,P_0)| \longrightarrow 0$ ($t \longrightarrow +\infty$) almost surely, and the random variable $P_{\max} = \sup\{|P(t;t_0,P_0)|: t \geq t_0\}$ is almost surely finite. In fact, if $P_{\max} \geq |P(t;t_0,P_0)| > 0$, then $\mathbb{E}[P_{\max}^k] < 0$ for all $k \leq 0$, and $\mathbb{E}[P_{\max}^k]$ is big for sufficiently large $k > 0$, resulting in the kth moment exponential instability, namely, the kth moment Lyapunov exponent $L(k) > 0$.

We now show the large deviations of system (91).

Theorem 7

(i) When $0 < H < (1/2)$, one of the following conditions fulfils the conditions $L = \theta - (u^2/2) < 0$ and $L(k) = k\theta + (1/2)k(k-1)u^2 > 0$:

(1) $0 < k < 1$ and $(1/2)(1-k)u^2 < \theta < (1/2)u^2$
(2) $k = 1$ and $0 < \theta < (1/2)u^2$

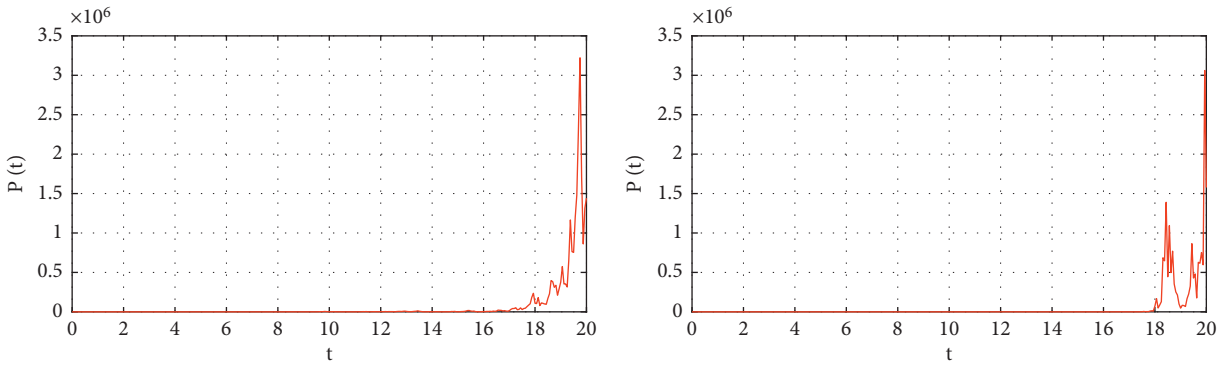

Figure 1: The trajectories of system (91) with $P_0 = 1, H = 0.3$. (a) $\theta = 1, u = 1, v = 1$. (b) $\theta = 1, u = 1, v = 2$. The trajectories of system (91) are almost surely exponentially unstable for both (a) and (b) in the sense of Theorem 5 (i).

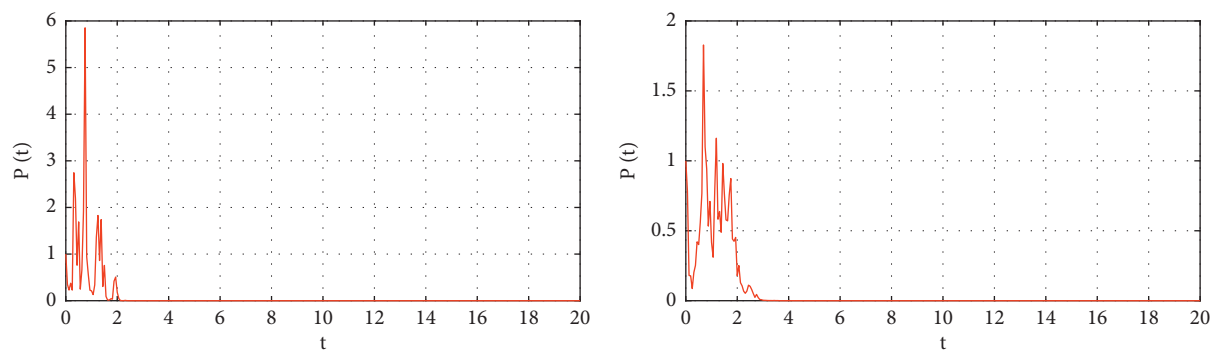

Figure 2: The trajectories of system (91) with $P_0 = 1, H = 0.3$. (a) $\theta = 1, u = 3, v = 2$. (b) $\theta = -1, u = 1, v = 1$. The trajectories of system (91) are almost surely exponentially stable for both (a) and (b) in the sense of Theorem 5 (i).

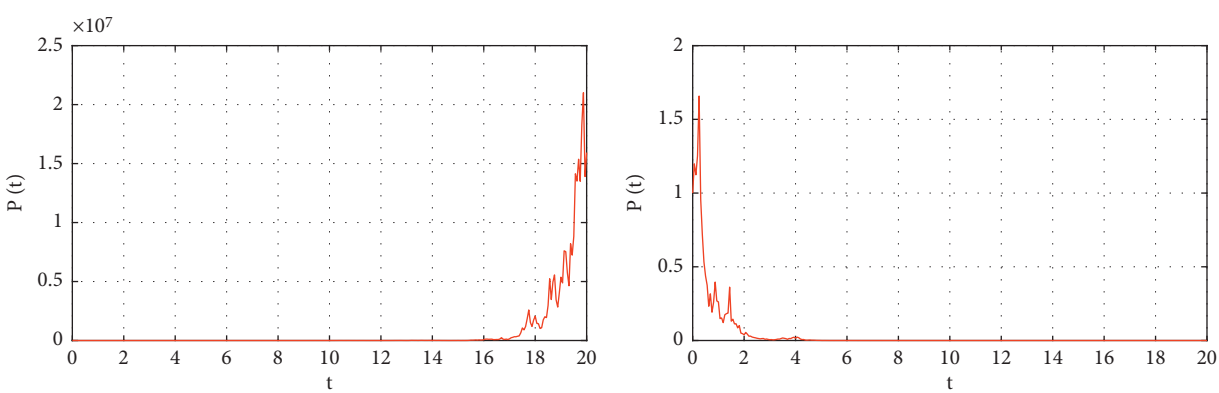

Figure 3: The trajectories of system (91) with $P_0 = 1, H = 0.5$. (a) $\theta = 2, u = 1, v = 1$. (b) $\theta = -1, u = 1, v = 1$. The trajectories of system (91) are almost surely exponentially unstable for (a) but stable for (b) in the sense of Theorem 5 (ii).

(3) $k > 1$ and $0 \le \theta < (1/2)u^2$

Thus, it holds for some sufficiently large k when $0 \le \theta < (1/2)u^2$. Therefore, the trivial solution of system (91) has large deviations for some special parameters.

(ii) When $H = (1/2)$, one of the following conditions fulfils the conditions $L = \theta - (1/2)(u^2 + v^2) < 0$ and $L(k) = k\theta + (1/2)k(k-1)(u^2 + v^2) > 0$:

(1) $0 < k < 1$ and $(1/2)(1-k)(u^2 + v^2) < \theta < (1/2)(u^2 + v^2)$
(2) $k = 1$ and $0 < \theta < (1/2)(u^2 + v^2)$
(3) $k > 1$ and. $0 \le \theta < (1/2)(u^2 + v^2)$

Thus, it holds for some sufficiently large k when $0 \le \theta < (1/2)(u^2 + v^2)$. Therefore, the trivial solution of system (91) has large deviations for some special parameters.

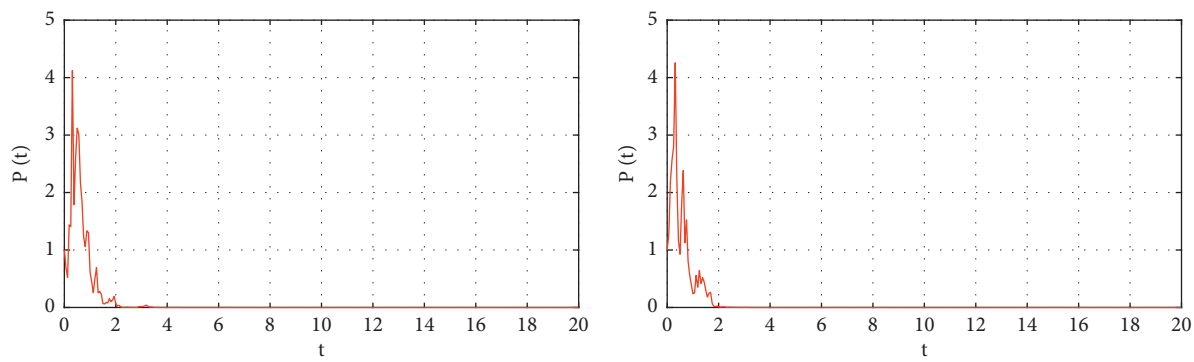

Figure 4: The trajectories of system (91) with $P_0 = 1, H = 0.5$. (a) $\theta = 1, u = 2, v = 1$. (b) $\theta = 1, u = 1, v = 2$. The trajectories of system (91) are almost surely exponentially stable for both (a) and (b) in the sense of Theorem 5 (ii).

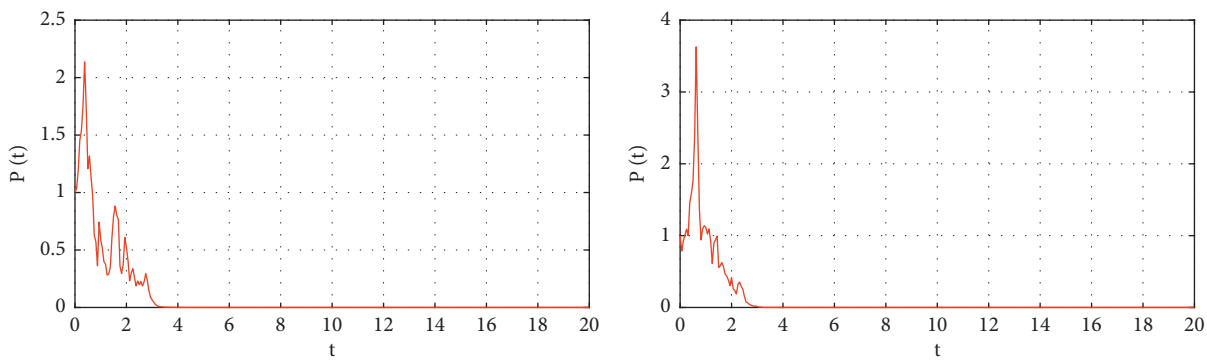

Figure 5: The trajectories of system (91) with $P_0 = 1, H = 0.8$. (a) $\theta = 1, u = 1, v = 1$. (b) $\theta = -1, u = 1, v = 1$. The trajectories of system (91) are almost surely exponentially stable for both (a) and (b) in the sense of Theorem 5 (iii).

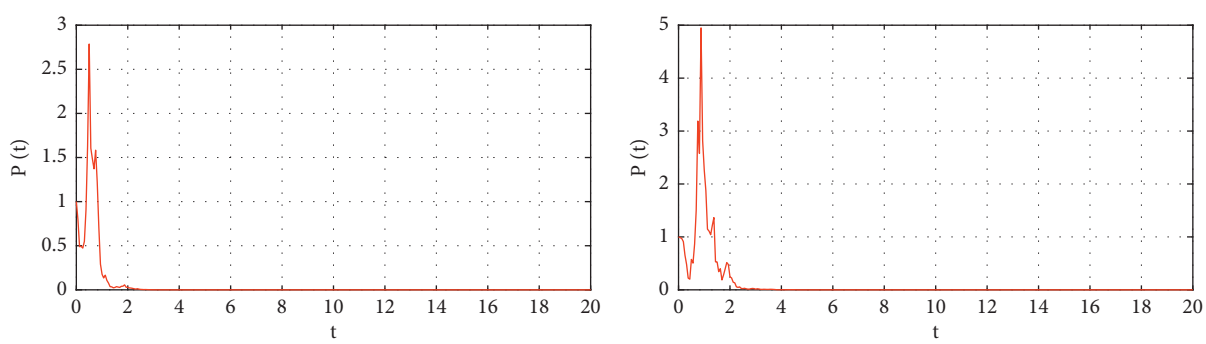

Figure 6: The trajectories of system (91) with $P_0 = 1, H = 0.8$. (a) $\theta = 1, u = 1, v = 2$. (b) $\theta = 1, u = 2, v = 1$. The trajectories of system (91) are almost surely exponentially stable for both (a) and (b) in the sense of Theorem 5 (iii).

(iii) When $(1/2) < H < 1$, we have $L = -\infty$ and $L(k) = -\infty$, θ, $+\infty$ for $0 < k < 1$, $k = 1$, $k > 1$, respectively. Therefore, the conditions $L < 0$ and $L(k) > 0$ hold for all parameters θ, u, v, and $k > 1$. Consequently, large deviations occur for all parameters θ, u, and v provided that $(1/2) < H < 1$.

Proof. Thus, we can prove this theorem using the expressions of L and $L(k)$. □

Remark 8. For the exponential population growth system with fBm (91) (i.e., $u = 0$), if $0 < H < (1/2)$, then $L = \theta$ and $L(k) = k\theta$, which implies that L and $L(k)$ have the same sign

for all $k > 0$. Thus, the trivial solution of the system is almost surely exponentially stable and kth moment exponentially stable for all $k > 0$, if $\theta < 0$. Thus, there are no large deviations of the trivial solution for the exponential population growth system with fBm, which is different from case (i). However, if $H = (1/2)$ and $(1/2) < H < 1$, the conclusions are similar to that in cases (ii) and (iii), respectively. Therefore, Theorems 5, 6, and 7 are extensions of those reported in [17, 38, 55].

Remark 9. Large deviations are the most concise formulas of the relation between the almost sure exponential stability and the kth moment exponential stability of system (91). This relation is tight for small k; however, for sufficiently large k, $L < 0$ and $L(k) > 0$. In other words, the following phenomenon may happen: for fixed t, $P(t; t_0, P_0) \longrightarrow 0$ $(t \longrightarrow +\infty)$ with sample Lyapunov exponent L with probability one, while for $|P(t; t_0, P_0)|$ is still large with small probability (see [17], Remark 5.2). This very rare event makes $\mathbb{E}[|P(t; t_0, P_0)|^k]$ big for sufficiently large k and gives rise to the kth moment exponential instability in system (91) due to the kth moment Lyapunov exponent $L(k) > 0$, which is a very interesting phenomenon in biological population systems.

7. Conclusion

This paper investigates the dynamics of the exponential population growth system with mfBm. We first present some lemmas of the SDEs with mfBm. Then, we offer some explicit expressions, mathematical expectations, and variances of the solutions of the exponential population growth system with mfBm. Furthermore, we provide two necessary and sufficient conditions that prove the exponential stability in the kth moment and the almost sure stability of the constant coefficient exponential population growth system with mfBm. We also obtain the large deviations for the system, showing that the Hurst index affects the exponential stability and the large deviations. Notably, the solution of the suggested system of large deviations always exists when $(1/2) < H < 1$ due to the long-range dependence, which is an important property of the fBm. Finally, the methods and lemmas presented in this paper can be applied to stochastic networked systems with mfBm, such as multiagent systems, complex networks, and neural networks, to study the dynamics of consensus, synchronization, and stability.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally.

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References

[1] Z. Zhang, S. Chen, and Y. Zheng, "Fully distributed scaled consensus tracking of high-order multiagent systems with time delays and disturbances," *IEEE Transactions on Industrial Informatics*, vol. 18, no. 1, pp. 305–314, 2022.

[2] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, UK, second edition, 2007.

[3] W. Ma, X. Luo, and Q. Zhu, "Practical exponential stability of stochastic age-dependent capital system with Lévy noise," *Systems & Control Letters*, vol. 144, Article ID 104759, 2020.

[4] H. Yang and X. Li, "Explicit approximations for nonlinear switching diffusion systems in finite and infinite horizons," *Journal of Differential Equations*, vol. 265, no. 7, pp. 2921–2967, 2018.

[5] W. Ma, B. Ding, H. Yang, and Q. Zhang, "Mean-square dissipativity of numerical methods for a class of stochastic neural networks with fractional Brownian motion and jumps," *Neurocomputing*, vol. 166, pp. 256–264, 2015.

[6] Y. Zheng, J. Ma, and L. Wang, "Consensus of hybrid multiagent systems," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 4, pp. 1359–1365, 2017.

[7] H. Su, H. Wu, and J. Lam, "Positive edge-consensus for nodal networks via output feedback," *IEEE Transactions on Automatic Control*, vol. 64, no. 3, pp. 1244–1249, 2019.

[8] H. Su, Y. Sun, and Z. Zeng, "Semiglobal observer-based nonnegative edge consensus of networked systems with actuator saturation," *IEEE Transactions on Cybernetics*, vol. 50, no. 6, pp. 2827–2836, 2020.

[9] W. Cao and Q. Zhu, "Razumikhin-type theorem for pth exponential stability of impulsive stochastic functional differential equations based on vector Lyapunov function p th exponential stability of impulsive stochastic functional differential equations based on vector Lyapunov function," *Nonlinear Analysis: Hybrid Systems*, vol. 39, Article ID 100983, 2021.

[10] X. Yang and Q. Zhu, "Stabilization of stochastic functional differential systems by steepest descent feedback controls," *IET Control Theory & Applications*, vol. 15, no. 6, pp. 805–813, 2021.

[11] L. Arnold, E. Oeljeklaus, and E. Pardoux, "Almost sure and moment stability for linear ito equations," in *Lyapunov Exponents*, L. Arnold and V. Wihstutz, Eds., Springer-Verlag, Berlin, Germany, 1986.

[12] R. Khasminskii, *Stochastic Stability of Differential Equations*, Springer, Berlin, Germany, second edition, 2011.

[13] W. Q. Zhu, "Lyapunov exponent and stochastic stability of quasi-non-integrable Hamiltonian systems," *International Journal of Non-linear Mechanics*, vol. 39, no. 4, pp. 569–579, 2004.

[14] J. Bao, X. Mao, and C. Yuan, "Lyapunov exponents of hybrid stochastic heat equations," *Systems & Control Letters*, vol. 61, no. 1, pp. 165–172, 2012.

[15] L. Shaikhet, "Stability of a positive equilibrium state for a stochastically perturbed mathematical model of glassy-

winged sharpshooter population," *Mathematical Biosciences and Engineering*, vol. 11, no. 5, pp. 1167–1174, 2014.

[16] X. Fan and C. Yuan, "Lyapunov exponents of PDEs driven by fractional noise with Markovian switching," *Statistics & Probability Letters*, vol. 110, pp. 39–50, 2016.

[17] C. Zeng, Y. Chen, and Q. Yang, "Almost sure and moment stability properties of fractional order Black-Scholes model," *Fractional Calculus and Applied Analysis*, vol. 16, no. 2, pp. 317–331, 2013.

[18] M. Liu and K. Wang, "Stochastic Lotka-Volterra systems with Lévy noise," *Journal of Mathematical Analysis and Applications*, vol. 410, no. 2, pp. 750–763, 2014.

[19] Y. Zheng, W. Chen, and L. Wang, "Finite-time consensus for stochastic multi-agent systems," *International Journal of Control*, vol. 84, no. 10, pp. 1644–1652, 2011.

[20] Y. Zheng, Q. Zhao, J. Ma, and L. Wang, "Second-order consensus of hybrid multi-agent systems," *Systems & Control Letters*, vol. 125, pp. 51–58, 2019.

[21] W. Guo, Q. Zhang, and L. Rong, "A stochastic epidemic model with nonmonotone incidence rate: sufficient and necessary conditions for near-optimality," *Information Sciences*, vol. 467, pp. 670–684, 2018.

[22] Z. Yang, "Default probability of American lookback option in a mixed jump-diffusion model," *Physica A: Statistical Mechanics and Its Applications*, vol. 540, Article ID 123242, 2020.

[23] B. B. Mandelbrot and J. W. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, no. 4, pp. 422–437, 1968.

[24] Y. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Springer, Berlin, Germany, 2008.

[25] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Springer, London, UK, 2008.

[26] T. E. Duncan, Y. Hu, and B. Pasik-Duncan, "Stochastic calculus for fractional brownian motion I. Theory," *SIAM Journal on Control and Optimization*, vol. 38, no. 2, pp. 582–612, 2000.

[27] Y. Hu, B. Øksendal, and D. M. Salopek, "Weighted local time for fractional brownian motion and applications to finance," *Stochastic Analysis and Applications*, vol. 23, pp. 1–30, 2005.

[28] Y. Hu, "Integral transformations and anticipative calculus for fractional Brownian motions," *Memoirs of the American Mathematical Society*, vol. 175, no. 825, 2005.

[29] W. Zhou, X. Zhou, J. Yang, J. Zhou, and D. Tong, "Stability analysis and application for delayed Neural Networks driven by fractional Brownian noise," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 5, pp. 1491–1502, 2018.

[30] M. Grigoriu, *Stochastic Calculus. Applications in Science and Engineering*, Birkhäuser, Boston, MA, USA, 2002.

[31] W.-J. Ma, Q.-M. Zhang, and C.-Z. Han, "Numerical analysis for stochastic age-dependent population equations with fractional Brownian motion," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 4, pp. 1884–1893, 2012.

[32] G. Jumarie, "New stochastic fractional models for Malthusian growth, the Poissonian birth process and optimal management of populations," *Mathematical and Computer Modelling*, vol. 44, no. 3-4, pp. 231–254, 2006.

[33] A. N. Kolmogorov, "Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum," *Academy URSS (N.S.)*, vol. 26, pp. 115–118, 1940, in German.

[34] H. E. Hurst, "Long-term storage capacity in reservoirs," *Transactions of the American Society of Civil Engineers*, vol. 116, pp. 400–410, 1951.

[35] L. Yan, "Maximal inequalities for the iterated fractional integrals," *Statistics & Probability Letters*, vol. 69, no. 1, pp. 69–79, 2004.

[36] C. Bender, "An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter," *Stochastic Processes and Their Applications*, vol. 104, no. 1, pp. 81–106, 2003.

[37] Y. Mishura and M. Zili, *Stochastic Analysis of Mixed Fractional Gaussian Processes*, Elsevier, Amsterdam, Netherlands, 2018.

[38] Q. Yang, C. Zeng, and C. Wang, "Fractional noise destroys or induces a stochastic bifurcation," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 23, no. 4, Article ID 043120, 2013.

[39] X. Mao, G. Marion, and E. Renshaw, "Environmental Brownian noise suppresses explosions in population dynamics," *Stochastic Processes and Their Applications*, vol. 97, no. 1, pp. 95–110, 2002.

[40] Y. Takeuchi, N. H. Du, N. T. Hieu, and K. Sato, "Evolution of predator-prey systems described by a Lotka-Volterra equation under random environment," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 938–957, 2006.

[41] M. Khodabin, K. Maleknejad, M. Rostami, and M. Nouri, "Interpolation solution in generalized stochastic exponential population growth model," *Applied Mathematical Modelling*, vol. 36, no. 3, pp. 1023–1033, 2012.

[42] P. Cheridito, "Mixed fractional Brownian motion," *Bernoulli*, vol. 7, no. 6, pp. 913–934, 2001.

[43] L. Wang, R. Zhang, L. Yang, Y. Su, and F. Ma, "Pricing geometric Asian rainbow options under fractional Brownian motion," *Physica A: Statistical Mechanics and Its Applications*, vol. 494, pp. 8–16, 2018.

[44] W.-G. Zhang, Z. Li, and Y.-J. Liu, "Analytical pricing of geometric Asian power options on an underlying driven by a mixed fractional Brownian motion," *Physica A: Statistical Mechanics and Its Applications*, vol. 490, pp. 402–418, 2018.

[45] E. Alòs, O. Mazet, and D. Nualart, "Stochastic calculus with respect to Gaussian processes," *Annals of Probability*, vol. 29, pp. 766–801, 2001.

[46] P. Carmona, L. Coutin, and G. Montseny, "Stochastic integration with respect to fractional Brownian motion," *Annales de l'Institut Henri Poincare (B): Probability and Statistics*, vol. 39, no. 1, pp. 27–68, 2003.

[47] L. Coutin, D. Nualart, and C. A. Tudor, "Tanaka formula for the fractional Brownian motion," *Stochastic Processes and Their Applications*, vol. 94, no. 2, pp. 301–315, 2001.

[48] L. Decreusefond and A. S. Üstünel, "Stochastic analysis of the fractional Brownian motion," *Potential Analysis*, vol. 10, pp. 177–214, 1998.

[49] N. Privault, "Skorohod stochastic integration with respect to non-adapted processes on Wiener space," *Stochastics and Stochastics Reports*, vol. 65, no. 1-2, pp. 13–39, 1998.

[50] M. A. Arcones, "On the law of the iterated Logarithm for Gaussian processes," *Journal of Theoretical Probability*, vol. 8, no. 4, pp. 877–903, 1995.

[51] H. Touchette, "The large deviation approach to statistical mechanics," *Physics Reports*, vol. 478, no. 1-3, pp. 1–69, 2009.

[52] P. E. Gretener, "Significance of the rare event in geology," *AAPG Bulletin*, vol. 51, no. 11, pp. 2197–2206, 1967.

[53] G. Ruget, "Large deviations and more or less rare events in population dynamics," in *Rhythms in Biology and Other Fields*

of Application: Deterministic and Stochastic Approaches, pp. 388–400, Springer-Verlag, Berlin, Germany, 1983.

[54] D. R. Easterling, G. A. Meehl, C. Parmesan, S. A. Changnon, T. R. Karl, and L. O. Mearns, "Climate extremes: observations, modeling, and impacts," *Science*, vol. 289, no. 5487, pp. 2068–2074, 2000.

[55] C. Zeng, Y. Chen, and Q. Yang, "Almost sure and moment stability properties of LTI stochastic dynamic systems driven by fractional Brownian motion," in *Proceedings of the 51st IEEE Conference on Decision and Control*, pp. 4733–4736, Maui, HI, USA, December 2012.