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Research Article

Hypo-EP Matrices of Adjointable Operators on Hilbert C^*-Modules

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This paper introduces and studies hypo-EP matrices of adjointable operators on Hilbert C^*-modules, based on the generalized Schur complement. The necessary and sufficient conditions for some modular operator matrices to be hypo-EP are given, and some special circumstances are also analyzed. Furthermore, an application of the EP operator in operator equations is given.

1. Introduction and Preliminaries

The EP matrix, as an extension of the normal matrix, was proposed by Schwerdtfeger; a square matrix T over the complex field $\mathbb{C}$ is said to be an EP matrix if T and T^* share the same range [1, 2]. The notion of EP matrices was extended by Campbell and Meyer to operators with closed range on a Hilbert space in [3]. Let H be a complex Hilbert space and $\mathscr{B}(H)$ the collection of all bounded linear operators on H. Let $T \in \mathscr{B}(H)$. Recall that T is called an EP operator if its range, $\mathscr{R}(T)$, is closed, and $\mathscr{R}(T) = \mathscr{R}(T^*)$ [3]. It is well known that $\mathscr{R}(T)$ is closed if and only if the Moore–Penrose inverse $T^\dagger$ of T exists and that T is an EP operator if and only if $T^\dagger T = TT^\dagger$. Sharifi [4] provided a generalization of the result for EP operators on Hilbert C^*-modules. This has been studied by many other authors, see, e.g., [5–8] and references therein. More generally, T is said to be a hypo-EP operator if $T^\dagger T \geq TT^\dagger$ [9]. In fact, T is a hypo-EP operator if and only if $\mathscr{R}(T)$ is closed and $\mathscr{R}(T) \subseteq \mathscr{R}(T^*)$. It is also shown that T is a hypo-EP operator if and only if $T^\dagger T^2 T^\dagger = TT^\dagger$. The hypo-EP operator is our focus of attention in this paper, and it has been studied in [10, 11]. The EP operator can be applied to the solution of operator equations, see Section 3 of this article. The properties of hypo-EP and EP operators can find applications also in reverse order law [12] and core partial order [13] and will be useful in some other applied fields [14, 15]. In this note, we investigate the hypo-EP operators on Hilbert C^*-modules, and then we formulate some results of hypo-EP matrices of adjointable operators on Hilbert C^*-modules. As an application, the solvability conditions, and the general expression for the EP solution to the operator equations are given.

Since the finite-dimensional spaces, Hilbert spaces, and C^*-algebras can all be regarded as Hilbert C^*-modules, one can study hypo-EP modular operators in a unified way in the framework of Hilbert C^*-modules. Let us briefly recall some basic knowledge about Hilbert C^*-modules and adjointable operators. Throughout this paper, $\mathbb{A}$ is a C^*-algebra. A Hilbert $\mathbb{A}$-module $\mathscr{H}$ is a right $\mathbb{A}$-module equipped with an $\mathbb{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{A}$ such that $\mathscr{H}$ is complete with respect to the induced norm $\| \cdot \| = \| \langle \cdot, \cdot \rangle \|^{(1/2)}$. Suppose that $\mathscr{H}$ and $\mathscr{K}$ are Hilbert $\mathbb{A}$-modules, and let $\mathscr{L}(\mathscr{H}, \mathscr{K})$ be the set of all maps $T: \mathscr{H} \longrightarrow \mathscr{K}$ for which there is a map $T^*: \mathscr{K} \longrightarrow \mathscr{H}$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for $x \in \mathscr{H}$ and $y \in \mathscr{K}$. It is well known that an arbitrary element T of $\mathscr{L}(\mathscr{H}, \mathscr{K})$ must be a bounded linear operator, which is also $\mathbb{A}$-linear in the sense of $T(xa) = (Tx)a$ for any $x \in \mathscr{H}$ and $a \in \mathbb{A}$. We call $\mathscr{L}(\mathscr{H}, \mathscr{K})$ the set of adjointable operators from $\mathscr{H}$ to $\mathscr{K}$. We use $\mathscr{L}(\mathscr{H})$ to denote the C^*-algebra $\mathscr{L}(\mathscr{H}, \mathscr{H})$. Let $\mathscr{L}(\mathscr{H})_{sa}$ be the set of Hermitian elements of $\mathscr{L}(\mathscr{H})$. For $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$, the range and the null space of T are denoted by $\mathscr{R}(T)$ and $\mathscr{N}(T)$, respectively. An operator

$T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ is said to be regular if there is an operator $T^- \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ satisfying $TT^-T = T$; T^- is called an inner inverse (or $\{1\}$-inverse) of T. It is easy to prove that T is regular if and only if $\mathscr{R}(T)$ is closed. The $\{1\}$-inverse of T is not unique in general.

In this paper, we use the generalized inverse to the generalized Schur complement as defined in [16]. Suppose $M \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ is a modular operator matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

where $A \in \mathscr{L}(\mathscr{H}), B \in \mathscr{L}(\mathscr{K}, \mathscr{H}), C \in \mathscr{L}(\mathscr{H}, \mathscr{K}), D \in \mathscr{L}(\mathscr{K})$. Then, the generalized Schur complement of A in M is

$$\frac{M}{A} = D - CA^- B, \tag{2}$$

where A^- is an inner inverse of A. Similarly, the generalized Schur complement of D in M is

$$\frac{M}{D} = A - BD^- C, \tag{3}$$

where D^- is an inner inverse of D. The formulas (2) and (3) have previously appeared in papers dealing with generalized inverses of partitioned matrices (cf. [17–19]).

Definition 1 (see [20]). Let $T \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. The Moore–Penrose inverse $T^\dagger$ of T (if exists) is an element in $\mathscr{L}(\mathscr{K}, \mathscr{H})$ which satisfies

(a) $TT^\dagger T = T$
(b) $T^\dagger TT^\dagger = T^\dagger$
(c) $(TT^\dagger)^* = TT^\dagger$
(d) $(T^\dagger T)^* = T^\dagger T$

These equations imply that $T^\dagger$ will be uniquely determined if it exists, and $T^\dagger T$ and $TT^\dagger$ are both orthogonal projections. Moreover, $\mathscr{R}(T^\dagger) = \mathscr{R}(T^\dagger T)$, $\mathscr{R}(T) = \mathscr{R}(TT^\dagger)$, $\mathscr{N}(T) = \mathscr{N}(T^\dagger T)$, and $\mathscr{N}(T^\dagger) = \mathscr{N}(TT^\dagger)$. Clearly, the Moore–Penrose inverse $T^\dagger$ of T exists if and only if $\mathscr{R}(T)$ is closed; T is Moore–Penrose invertible if and only if T^* is Moore–Penrose invertible, and in this case, $(T^*)^\dagger = (T^\dagger)^*$. Obviously, the Moore–Penrose inverse $T^\dagger$ of T is one of inner inverses of T.

Similar to [21], Lemma 2.2.4, and [22], Lemma 2.2, we have the following conclusions on Hilbert C^*-modules.

Lemma 1. *Let $A \in \mathscr{L}(\mathscr{H}), B \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ and $C \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. If A has an inner inverse A^-, then*

(i) $\mathscr{N}(A) \subseteq \mathscr{N}(C)$ if and only if $C = CA^- A$
(ii) $\mathscr{N}(A^*) \subseteq \mathscr{N}(B^*)$ if and only if $B = AA^- B$

Lemma 2. *Let M be a modular operator matrix of form (1) with $\mathscr{N}(A) \subseteq \mathscr{N}(C)$ and $\mathscr{N}(A^*) \subseteq \mathscr{N}(B^*)$. If A has an inner inverse A^-, then M is regular if and only if M/A is regular, where $M/A = D - CA^- B$. In this case, the inner inverse of M is given by*

$$M^- = \begin{pmatrix} A^- + A^- B\left(\frac{M}{A}\right)^- CA^- & -A^- B\left(\frac{M}{A}\right)^- \\ -\left(\frac{M}{A}\right)^- CA^- & \left(\frac{M}{A}\right)^- \end{pmatrix}. \tag{4}$$

From Lemma 2, we can obtain the following corollary.

Corollary 1. *Let M be a modular operator matrix of form (1) with $\mathscr{N}(A) \subseteq \mathscr{N}(C)$, $\mathscr{N}(A^*) \subseteq \mathscr{N}(B^*)$, $\mathscr{N}(M/A) \subseteq \mathscr{N}(B)$, and $\mathscr{N}((M/A)^*) \subseteq \mathscr{N}(C^*)$. If $\mathscr{R}(A)$ and $\mathscr{R}(M/A)$ are closed, then the Moore–Penrose inverse $M^\dagger$ of M can be expressed as*

$$M^\dagger = \begin{pmatrix} A^\dagger + A^\dagger B\left(\frac{M}{A}\right)^\dagger CA^\dagger & -A^\dagger B\left(\frac{M}{A}\right)^\dagger \\ -\left(\frac{M}{A}\right)^\dagger CA^\dagger & \left(\frac{M}{A}\right)^\dagger \end{pmatrix}. \tag{5}$$

Remark 1. The preceding result given in [17], Theorem 1, was proved for finite matrices.

Lemma 3 (see [23]). *Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$, where $A \in \mathscr{L}(\mathscr{H}), B \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, and $D \in \mathscr{L}(\mathscr{K})$. If $\mathscr{R}(A)$ and $\mathscr{R}(D)$ are closed, then $M^\dagger = \begin{pmatrix} A^\dagger & -A^\dagger BD^\dagger \\ 0 & D^\dagger \end{pmatrix}$ if and only if $\mathscr{N}(D) \subseteq \mathscr{N}(B)$ and $\mathscr{N}(A^*) \subseteq \mathscr{N}(B^*)$.*

Proof. The proof is similar to that in [22], Corollary 12, for Hilbert space operators. □

Definition 2 (see [4]). Let $\mathscr{H}$ be a Hilbert $\mathbb{A}$-module. An operator $T \in \mathscr{L}(\mathscr{H})$ is called EP if $\overline{\mathscr{R}(T)} = \overline{\mathscr{R}(T^*)}$.

Definition 3. Let $\mathscr{H}$ be a Hilbert $\mathbb{A}$-module. An operator $T \in \mathscr{L}(\mathscr{H})$ is called hypo-EP if $\overline{\mathscr{R}(T)} \subseteq \overline{\mathscr{R}(T^*)}$.

Obviously, the range of an EP or a hypo-EP operator on Hilbert C^*-modules is not necessarily closed, and we further have the following properties.

Proposition 1 (see [4]). *Let $\mathscr{H}$ be a Hilbert $\mathbb{A}$-module and $T \in \mathscr{L}(\mathscr{H})$ with closed range. Then, the following conditions are equivalent:*

(i) T is an EP operator
(ii) $\mathscr{N}(T) = \mathscr{N}(T^*)$
(iii) T is Moore–Penrose invertible and $T^\dagger T = TT^\dagger$

Proposition 2. *Let $\mathscr{H}$ be a Hilbert $\mathbb{A}$-module and $T \in \mathscr{L}(\mathscr{H})$ with closed range. Then, the following conditions are equivalent:*

(i) T is a hypo-EP operator
(ii) $\mathscr{N}(T) \subseteq \mathscr{N}(T^*)$

(iii) T is Moore–Penrose invertible and $T^{\dagger}T^{2}T^{\dagger} = TT^{\dagger}$

Remark 2. The class of all hypo-EP operators contains the class of all EP operators on Hilbert $\mathbb{A}$-modules. Meanwhile, the EP operator with closed range is an extension of the invertible operator and the normal operator with closed range. In the case of finite dimensional situation, EP and hypo-EP are the same.

2. Main Results and Proofs

First, using generalized Schur complements, we study the hypo-EP property of matrices of adjointable operators on Hilbert C^{*}-modules.

Theorem 1. *Let M be a modular operator matrix of the form (1) with $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(A^{*}) \subseteq \mathcal{N}(B^{*})$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, and $\mathcal{N}((M/A)^{*}) \subseteq \mathcal{N}(C^{*})$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed. Then, the following conditions are equivalent:*

(i) M is a hypo-EP operator matrix with closed range

(ii) A and M/A are hypo-EP operators

Proof. Let M be a hypo-EP operator matrix with closed range. Since $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed, let us consider the operator matrices

$$L = \begin{pmatrix} I & 0 \\ CA^{-} & I \end{pmatrix},$$

$$R = \begin{pmatrix} I & B\left(\frac{M}{A}\right)^{-} \\ 0 & I \end{pmatrix}, \tag{6}$$

$$P = \begin{pmatrix} A & 0 \\ 0 & \frac{M}{A} \end{pmatrix},$$

where $M/A = D - CA^{-}B$. Obviously, L and R are invertible. By using Lemma 1 and by assumptions $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, it is clear that M can be factorized as $M = LRP$. Hence, $\mathcal{N}(P) = \mathcal{N}(M)$. Since M is a hypo-EP operator matrix with closed range, $\mathcal{N}(P) = \mathcal{N}(M) \subseteq \mathcal{N}(M^{*})$. By using Lemma 1 again, it is immediate that

$$M^{*} = M^{*}P^{-}P \tag{7}$$

holds for every inner inverse P^{-} of P. In particular, for

$$P^{-} = \begin{pmatrix} A^{-} & 0 \\ 0 & \left(\frac{M}{A}\right)^{-} \end{pmatrix}, \tag{8}$$

we have from relation (7) that

$$M^{*} = \begin{pmatrix} A^{*} & C^{*} \\ B^{*} & D^{*} \end{pmatrix} = \begin{pmatrix} A^{*} & C^{*} \\ B^{*} & D^{*} \end{pmatrix} \begin{pmatrix} A^{-}A & 0 \\ 0 & \left(\frac{M}{A}\right)^{-}\left(\frac{M}{A}\right) \end{pmatrix},$$

$$= \begin{pmatrix} A^{*}A^{-}A & C^{*}\left(\frac{M}{A}\right)^{-}\left(\frac{M}{A}\right) \\ B^{*}A^{-}A & D^{*}\left(\frac{M}{A}\right)^{-}\left(\frac{M}{A}\right) \end{pmatrix}. \tag{9}$$

Then, $A^{*} = A^{*}A^{-}A$ implies $\mathcal{N}(A) \subseteq \mathcal{N}(A^{*})$. Hence, A is a hypo-EP operator. Since $C^{*} = C^{*}(M/A)^{-}(M/A)$, substituting $D = (M/A) + CA^{-}B$ into

$$D^{*} = D^{*}(M/A)^{-}(M/A) \tag{10}$$

yields $(M/A)^{*} = (M/A)^{*}(M/A)^{-}(M/A)$. This implies $\mathcal{N}(M/A) \subseteq \mathcal{N}((M/A)^{*})$. Thus, M/A is a hypo-EP operator.

Conversely, according to the assumptions $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, $\mathcal{N}(A^{*}) \subseteq \mathcal{N}(B^{*})$, and $\mathcal{N}((M/A)^{*}) \subseteq \mathcal{N}(C^{*})$, the Moore–Penrose inverse $M^{\dagger}$ of M exists, and $M^{\dagger}$ is given by

$$M^{\dagger} = \begin{pmatrix} A^{\dagger} + A^{\dagger}B\left(\frac{M}{A}\right)^{\dagger}CA^{\dagger} & -A^{\dagger}B\left(\frac{M}{A}\right)^{\dagger} \\ -\left(\frac{M}{A}\right)^{\dagger}CA^{\dagger} & \left(\frac{M}{A}\right)^{\dagger} \end{pmatrix}, \tag{11}$$

by Corollary 1. Using $\mathcal{N}(A^{*}) \subseteq \mathcal{N}(B^{*})$ and $\mathcal{N}((M/A)^{*}) \subseteq \mathcal{N}(C^{*})$, by Lemma 1, $MM^{\dagger}$ is described as

$$MM^{\dagger} = \begin{pmatrix} AA^{\dagger} & 0 \\ 0 & \left(\frac{M}{A}\right)\left(\frac{M}{A}\right)^{\dagger} \end{pmatrix}. \tag{12}$$

Similarly, by using $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, and Lemma 1, it is given that

$$M^{\dagger}M = \begin{pmatrix} A^{\dagger}A & 0 \\ 0 & \left(\frac{M}{A}\right)^{\dagger}\left(\frac{M}{A}\right) \end{pmatrix}. \tag{13}$$

Then,

$$M^{\dagger}M^{2}M^{\dagger} = \left(M^{\dagger}M\right)\left(MM^{\dagger}\right) = \begin{pmatrix} A^{\dagger}A^{2}A^{\dagger} & 0 \\ 0 & \left(\frac{M}{A}\right)^{\dagger}\left(\frac{M}{A}\right)^{2}\left(\frac{M}{A}\right)^{\dagger} \end{pmatrix}. \tag{14}$$

Since A and M/A are hypo-EP operators with closed range,

$$A^{\dagger}A^{2}A^{\dagger} = AA^{\dagger},$$
$$\left(\frac{M}{A}\right)^{\dagger}\left(\frac{M}{A}\right)^{2}\left(\frac{M}{A}\right)^{\dagger} = \left(\frac{M}{A}\right)\left(\frac{M}{A}\right)^{\dagger}. \tag{15}$$

Thus, $M^{\dagger}M^{2}M^{\dagger} = MM^{\dagger}$. Therefore, M is a hypo-EP operator matrix with closed range.

The following conclusion is a natural extension of [10], Theorem 3.1, on Hilbert C^{*}-modules. □

Corollary 2. *Let $A, X \in \mathscr{L}(\mathscr{H})$ and $M = \begin{pmatrix} A & AX \\ X^{*}A & X^{*}AX \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$. If $\mathscr{R}(A)$ is closed, then M is a hypo-EP operator matrix with closed range if and only if A is a hypo-EP operator.*

If using the generalized Schur complement $M/D = A - BD^{-}C$ of D in M, similar to Theorem 1, one can get the following results.

Theorem 2. *Let M be a modular operator matrix of form (1) with $\mathscr{N}(D) \subseteq \mathscr{N}(B)$, $\mathscr{N}(D^{*}) \subseteq \mathscr{N}(C^{*})$, $\mathscr{N}(M/D) \subseteq \mathscr{N}(C)$, and $\mathscr{N}((M/D)^{*}) \subseteq \mathscr{N}(B^{*})$. Suppose that $\mathscr{R}(D)$ and $\mathscr{R}(M/D)$ are closed. Then, the following conditions are equivalent:*

(i) M is a hypo-EP operator matrix with closed range

(ii) D and M/D are hypo-EP operators

Corollary 3. *Let $D, X \in \mathscr{L}(\mathscr{H})$ and $M = \begin{pmatrix} X^{*}DX & X^{*}D \\ DX & D \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$. If $\mathscr{R}(D)$ is closed, then M is a hypo-EP operator matrix with closed range if and only if D is a hypo-EP operator.*

Next, using the properties of generalized inverses, we study upper triangular hypo-EP matrices of adjointable operators on Hilbert C^{*}-modules.

Theorem 3. *Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ with $\mathscr{N}(D) \subseteq \mathscr{N}(B)$ and $\mathscr{N}(A^{*}) \subseteq \mathscr{N}(B^{*})$, where $A \in \mathscr{L}(\mathscr{H})$, $B \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, and $D \in \mathscr{L}(\mathscr{K})$. If $\mathscr{R}(A)$ and $\mathscr{R}(D)$ are closed, then M is a hypo-EP operator matrix with closed range if and only if A and D are hypo-EP operators.*

Proof. Let M be a hypo-EP operator matrix with closed range. We write

$$L := \begin{pmatrix} I & BD^{\dagger} \\ 0 & I \end{pmatrix},$$
$$P := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}. \tag{16}$$

Obviously, L is invertible. By Lemma 1 and assumption $\mathscr{N}(D) \subseteq \mathscr{N}(B)$, it is clear that M can be decomposed as $M = LP$. Hence, $\mathscr{N}(M) = \mathscr{N}(P)$. Since M is a hypo-EP operator matrix with closed range, $\mathscr{N}(P) = \mathscr{N}(M) \subseteq \mathscr{N}(M^{*})$. By Lemma 1, it is immediate that $M^{*} = M^{*}P^{\dagger}P$, where $P^{\dagger}$ is given by

$$P^{\dagger} = \begin{pmatrix} A^{\dagger} & 0 \\ 0 & D^{\dagger} \end{pmatrix}. \tag{17}$$

This gives

$$M^{*} = \begin{pmatrix} A^{*} & 0 \\ B^{*} & D^{*} \end{pmatrix} = \begin{pmatrix} A^{*} & 0 \\ B^{*} & D^{*} \end{pmatrix}\begin{pmatrix} A^{\dagger}A & 0 \\ 0 & D^{\dagger}D \end{pmatrix},$$
$$= \begin{pmatrix} A^{*}A^{\dagger}A & 0 \\ B^{*}A^{\dagger}A & D^{*}D^{\dagger}D \end{pmatrix}. \tag{18}$$

Hence, $A^{*} = A^{*}A^{\dagger}A$ implies $\mathscr{N}(A) \subseteq \mathscr{N}(A^{*})$. Thus, A is a hypo-EP operator. From $D^{*} = D^{*}D^{\dagger}D$, it follows that $\mathscr{N}(D) \subseteq \mathscr{N}(D^{*})$. Therefore, D is a hypo-EP operator.

Conversely, suppose A and D are hypo-EP operators. Since $\mathscr{R}(A)$ and $\mathscr{R}(D)$ are closed and $\mathscr{N}(D) \subseteq \mathscr{N}(B)$ and $\mathscr{N}(A^{*}) \subseteq \mathscr{N}(B^{*})$, by Lemma 3, the Moore–Penrose inverse $M^{\dagger}$ of M exists and

$$M^{\dagger} = \begin{pmatrix} A^{\dagger} & -A^{\dagger}BD^{\dagger} \\ 0 & D^{\dagger} \end{pmatrix}. \tag{19}$$

Since $\mathscr{N}(A^{*}) \subseteq \mathscr{N}(B^{*})$, by Lemma 1, $MM^{\dagger}$ is described as

$$MM^{\dagger} = \begin{pmatrix} AA^{\dagger} & 0 \\ 0 & DD^{\dagger} \end{pmatrix}. \tag{20}$$

Similarly, by Lemma 1, $\mathscr{N}(D) \subseteq \mathscr{N}(B)$ leads to

$$M^{\dagger}M = \begin{pmatrix} A^{\dagger}A & 0 \\ 0 & D^{\dagger}D \end{pmatrix}. \tag{21}$$

Then,

$$M^{\dagger}M^{2}M^{\dagger} = \left(M^{\dagger}M\right)\left(MM^{\dagger}\right) = \begin{pmatrix} A^{\dagger}A^{2}A^{\dagger} & 0 \\ 0 & D^{\dagger}D^{2}D^{\dagger} \end{pmatrix}. \tag{22}$$

Since A and D are hypo-EP operators with closed range, $A^{\dagger}A^{2}A^{\dagger} = AA^{\dagger}$ and $D^{\dagger}D^{2}D^{\dagger} = DD^{\dagger}$. Thus, $M^{\dagger}M^{2}M^{\dagger} = MM^{\dagger}$. Therefore M is a hypo-EP operator matrix with closed range. □

Corollary 4. *Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{K})$ with $\mathscr{N}(D) \subseteq \mathscr{N}(B)$ and $\mathscr{N}(A^{*}) \subseteq \mathscr{N}(B^{*})$, where $A \in \mathscr{L}(\mathscr{H})$, $B \in \mathscr{L}(\mathscr{K}, \mathscr{H})$, and $D \in \mathscr{L}(\mathscr{K})$. If $\mathscr{R}(A)$ and $\mathscr{R}(D)$ are closed, then M is an EP operator matrix with closed range if and only if A and D are EP operators.*

Proof. Let M be an EP operator matrix with closed range. In view of Theorem 3, to prove the necessity, it is enough to show $\mathscr{N}(A^{*}) \subseteq \mathscr{N}(A)$ and $\mathscr{N}(D^{*}) \subseteq \mathscr{N}(D)$. Since M is an EP operator matrix with closed range, by the proof of Theorem 3, $\mathscr{N}(P) = \mathscr{N}(M) = \mathscr{N}(M^{*})$. Applying Lemma 1, we have $P^{*} = MM^{\dagger}P^{*}$, i.e.,

$$P^* = \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix} = \begin{pmatrix} AA^\dagger & 0 \\ 0 & DD^\dagger \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}, \\ = \begin{pmatrix} AA^\dagger A^* & 0 \\ 0 & DD^\dagger D^* \end{pmatrix}. \tag{23}$$

Hence, $A^* = AA^\dagger A^*$ and $D^* = DD^\dagger D^*$ imply $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(D^*) \subseteq \mathcal{N}(D)$, respectively.

Conversely, let A and D be EP operators. Since $\mathcal{R}(A)$and $\mathcal{R}(D)$ are closed, $\mathcal{N}(D) \subseteq \mathcal{N}(B)$, and $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$, we get $M^\dagger = \begin{pmatrix} A^\dagger & -A^\dagger BD^\dagger \\ 0 & D^\dagger \end{pmatrix}$. Thus,

$$M^\dagger M - MM^\dagger = \begin{pmatrix} A^\dagger A - AA^\dagger & 0 \\ 0 & D^\dagger D - DD^\dagger \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{24}$$

Therefore, M is an EP operator matrix with closed range. □

Corollary 5. *Let $A, X, D \in \mathcal{L}(\mathcal{H})$ and $M = \begin{pmatrix} A & AX \\ 0 & D \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. If $\mathcal{R}(A)$ and $\mathcal{R}(D)$ are closed, then M is a hypo-EP operator matrix with closed range if and only if A and D are hypo-EP operators.*

Corollary 6. *Let $A \in \mathcal{L}(\mathcal{H})$ and $M = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. If $\mathcal{R}(A)$ is closed, then M is a hypo-EP operator matrix with closed range if and only if A is a hypo-EP operator.*

Remark 3. The Hilbert space version of the preceding four conclusions is given by [10], and the conditions of closed range can be naturally omitted there. Moreover, the alternative proofs of the conclusions in Hilbert space setting can be found in section 3 of [10]. In addition, these results originated from the research of the EP property of block matrices, according to Hartwig [24].

Finally, the following are devoted to investigating the hypo-EP property of antitriangular block matrices of adjointable operators on Hilbert C^*-modules.

Lemma 4. *Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. If $\mathcal{R}(B)$ and $\mathcal{R}(C)$ are closed, then $M^\dagger = \begin{pmatrix} 0 & C^\dagger \\ B^\dagger & -B^\dagger AC^\dagger \end{pmatrix}$ if and only if $\mathcal{N}(C) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$.*

Proof. Sufficiency: since $\mathcal{R}(B)$ and $\mathcal{R}(C)$ are closed, B and C are Moore–Penrose invertible. From $\mathcal{N}(C) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$, it follows that $AC^\dagger C = A$ and $BB^\dagger A = A$. We write $X: = \begin{pmatrix} 0 & C^\dagger \\ B^\dagger & -B^\dagger AC^\dagger \end{pmatrix}$. A direct calculation shows that

$$\begin{aligned} MXM &= M, \\ XMX &= X, \\ (MX)^* &= MX, \\ (XM)^* &= XM. \end{aligned} \tag{25}$$

By Definition 1, $M^\dagger = X$ as desired.

Necessity: since

$$\begin{aligned} MM^\dagger &= \begin{pmatrix} BB^\dagger & AC^\dagger - BB^\dagger AC^\dagger \\ 0 & CC^\dagger \end{pmatrix}, \\ M^\dagger M &= \begin{pmatrix} C^\dagger C & 0 \\ B^\dagger A - B^\dagger AC^\dagger C & B^\dagger B \end{pmatrix} \end{aligned} \tag{26}$$

are self-adjoint, we have $AC^\dagger - BB^\dagger AC^\dagger = 0$ and $B^\dagger A - B^\dagger AC^\dagger C = 0$. From $MM^\dagger M = M$, we get $BB^\dagger A = A = AC^\dagger C$. Therefore, $\mathcal{N}(C) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$. □

Lemma 5. *Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ with $\mathcal{N}(C) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$. If B and C are hypo-EP operators with closed ranges, then M is a hypo-EP operator matrix with closed range.*

Proof. Since $\mathcal{R}(B)$ and $\mathcal{R}(C)$ are closed, $\mathcal{N}(C) \subseteq \mathcal{N}(A)$, and $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$, by Lemma 4, the Moore–Penrose inverse $M^\dagger$ of M is given by

$$M^\dagger = \begin{pmatrix} 0 & C^\dagger \\ B^\dagger & -B^\dagger AC^\dagger \end{pmatrix}. \tag{27}$$

Using $\mathcal{N}(C) \subseteq \mathcal{N}(A)$ and $\mathcal{N}(B^*) \subseteq \mathcal{N}(A^*)$, by Lemma 1, we have

$$\begin{aligned} MM^\dagger &= \begin{pmatrix} BB^\dagger & 0 \\ 0 & CC^\dagger \end{pmatrix}, \\ M^\dagger M &= \begin{pmatrix} C^\dagger C & 0 \\ 0 & B^\dagger B \end{pmatrix}. \end{aligned} \tag{28}$$

By Definition 1, we have $\mathcal{R}(MM^\dagger) = \mathcal{R}(M)$ and $\mathcal{R}(M^\dagger M) = \mathcal{R}(M^\dagger) = \mathcal{R}(M^*)$. Since B and C are hypo-EP operators with closed ranges, $\mathcal{R}(BB^\dagger) = \mathcal{R}(B) \subseteq \mathcal{R}(B^*) = \mathcal{R}(B^\dagger B)$ and $\mathcal{R}(CC^\dagger) = \mathcal{R}(C) \subseteq \mathcal{R}(C^*) = \mathcal{R}(C^\dagger C)$. Then,

$$\mathcal{R}(M) = \mathcal{R}\left(MM^\dagger\right) = \mathcal{R}\left(BB^\dagger\right) \oplus \mathcal{R}\left(CC^\dagger\right) \subseteq \mathcal{R}\left(B^\dagger B\right) \oplus \mathcal{R}\left(C^\dagger C\right) = \mathcal{R}\left(M^\dagger M\right) = \mathcal{R}(M^*). \tag{29}$$

Therefore, M is a hypo-EP operator with closed range. □

Corollary 7. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ *with* $\mathscr{N}(C) \subseteq \mathscr{N}(A)$ *and* $\mathscr{N}(B^*) \subseteq \mathscr{N}(A^*)$. *If* B *and* C *are EP operators with closed ranges, then* M *is an EP operator matrix with closed range.*

Proof. According to the assumption, as with Lemma 5, we have equation (28). Since B and C are EP operators with closed ranges, $\mathscr{R}(BB^\dagger) = \mathscr{R}(B) = \mathscr{R}(B^*) = \mathscr{R}(B^\dagger B)$ and $\mathscr{R}(CC^\dagger) = \mathscr{R}(C) = \mathscr{R}(C^*) = \mathscr{R}(C^\dagger C)$. Then,

$$\mathscr{R}(M) = \mathscr{R}\left(MM^\dagger\right) = \mathscr{R}\left(BB^\dagger\right) \oplus \mathscr{R}\left(CC^\dagger\right) = \mathscr{R}\left(B^\dagger B\right) \oplus \mathscr{R}\left(C^\dagger C\right) = \mathscr{R}\left(M^\dagger M\right) = \mathscr{R}(M^*). \tag{30}$$

Thus, M is an EP operator matrix with closed range. □

Theorem 4. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ *with* $\mathscr{N}(B) = \mathscr{N}(C) \subseteq \mathscr{N}(A)$ *and* $\mathscr{N}(B^*) \subseteq \mathscr{N}(A^*)$. *If* $\mathscr{R}(B)$ *and* $\mathscr{R}(C)$*are closed, then* M *is a hypo-EP operator matrix with closed range if and only if* B *and* C *are hypo-EP operators.*

Proof. The sufficiency is clear by Lemma 5. Now, we suppose that M is a hypo-EP operator matrix with closed range. We write

$$L := \begin{pmatrix} I & AC^\dagger \\ 0 & I \end{pmatrix}, \qquad P := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{31}$$

In the similar way as in the proof of Theorem 3,

we have $M = LP$, and hence, $\mathscr{N}(P) = \mathscr{N}(M) \subseteq \mathscr{N}(M^*)$, since M is a hypo-EP with closed range. This means $M^* = M^* P^\dagger P$ by Lemma 1, i.e.,

$$\begin{pmatrix} A^* & C^* \\ B^* & 0 \end{pmatrix} = M^* = M^* P^\dagger P = \begin{pmatrix} A^* C^\dagger C & C^* B^\dagger B \\ B^* C^\dagger C & 0 \end{pmatrix}. \tag{32}$$

Hence, $C^* = C^* B^\dagger B$, which together with $\mathscr{N}(C) = \mathscr{N}(B)$, implies $\mathscr{N}(C) \subseteq \mathscr{N}(C^*)$. Thus, C is a hypo-EP operator. Similarly, it follows from $B^* = B^* C^\dagger C$ and $\mathscr{N}(B) = \mathscr{N}(C)$ that $\mathscr{N}(B) \subseteq \mathscr{N}(B^*)$, and therefore, B is a hypo-EP operator. □

Corollary 8. *Let* $B, X, C \in \mathscr{L}(\mathscr{H})$ *and* $M = \begin{pmatrix} BXC & B \\ C & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ *with* $\mathscr{N}(B) = \mathscr{N}(C)$. *If* $\mathscr{R}(B)$ *and* $\mathscr{R}(C)$ *are closed, then* M *is a hypo-EP operator matrix with closed range if and only if* B *and* C *are hypo-EP operators.*

Corollary 9. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ *with* $\mathscr{N}(B) = \mathscr{N}(C) \subseteq \mathscr{N}(A)$ *and* $\mathscr{N}(B^*) \subseteq \mathscr{N}(A^*)$. *If* $\mathscr{R}(B)$ *and* $\mathscr{R}(C)$*are closed, then* M *is an EP operator matrix with closed range if and only if* B *and* C *are EP operators.*

Proof. By Corollary 7, we only need to show the necessity, which can be easily verified according to the proofs of Corollary 4 and Theorem 4. □

Corollary 10. *Let* $B, X, C \in \mathscr{L}(\mathscr{H})$ *and* $M = \begin{pmatrix} BXC & B \\ C & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ *with* $\mathscr{N}(B) = \mathscr{N}(C)$. *If* $\mathscr{R}(B)$ *and* $\mathscr{R}(C)$ *are closed, then* M *is an EP operator matrix with closed range if and only if* B *and* C *are EP operators.*

Remark 4. In Hilbert space case, the conditions of closed range in Theorem 4 and Corollary 9 can be naturally omitted in Theorem 3.8 and Theorem 3.9 of [10], and the alternative proofs of Theorem 4 and Corollary 9 can be, respectively, found in Theorem 3.8 and Theorem 3.9 of [10].

3. The Application of EP Operators

In this section, let $\mathscr{H}, \mathscr{K}$, and $\mathscr{G}$ be Hilbert spaces. We establish the solvability conditions and the general expression for the EP solution to the operator equations

$$AX = C, \qquad XB = D, \tag{33}$$

where $A, C \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, $B, D \in \mathscr{B}(\mathscr{G}, \mathscr{H})$, and $X \in \mathscr{B}(\mathscr{H})$.

Lemma 6 (see [25]). *Let* $T \in \mathscr{B}(\mathscr{H})$ *with closed range. Then, the operator* T *is EP if and only if there exist Hilbert spaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$, $U \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2, \mathscr{H})$ *unitary, and* $T_1 \in \mathscr{B}(\mathscr{H}_1)$ *isomorphism such that*

$$T = U(T_1 \oplus 0)U^*, \tag{34}$$

where $\mathscr{H}_1 \oplus \mathscr{H}_2 = \mathscr{H}$.

Lemma 7 (see [22]). *Let* $A, C \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ *and* $B, D \in \mathscr{B}(\mathscr{G}, \mathscr{H})$. *Suppose that* A *and* B *have closed ranges. Then, equation (33) has a common solution* $X \in \mathscr{B}(\mathscr{H})$ *if and only if*

$$\mathscr{N}(A^*) \subseteq \mathscr{N}(C^*), \qquad \mathscr{N}(B) \subseteq \mathscr{N}(D), \qquad AD = CB. \tag{35}$$

In this case, the general common solution is given by

$$X = A^{-}C + DB^{-} - A^{-}ADB^{-} + (I_{\mathscr{H}} - A^{-}A)Y(I_{\mathscr{H}} - BB^{-}), \tag{36}$$

where $Y \in \mathscr{B}(\mathscr{H})$ is arbitrary.

Now, we consider the EP solution to equation (33). By the Lemma 6, for the unitary operator $U \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2, \mathscr{H})$, the solution has the following factorization:

$$X = U\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}U^{*}. \tag{37}$$

Let $\mathscr{R}(A)$, $\mathscr{R}(B)$ be closed, and

$$\begin{aligned} AU &= (A_1 \ \ A_2), \\ CU &= (C_1 \ \ C_2), \\ U^{*}B &= \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \\ U^{*}D &= \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \end{aligned} \tag{38}$$

where $A_1, C_1 \in \mathscr{B}(\mathscr{H}_1, \mathscr{K})$, $A_2, C_2 \in \mathscr{B}(\mathscr{H}_2, \mathscr{K})$, $B_1, D_1 \in \mathscr{B}(\mathscr{G}_1, \mathscr{H})$, $B_2, D_2 \in \mathscr{B}(\mathscr{G}_2, \mathscr{H})$, and $\mathscr{R}(A_1)$, $\mathscr{R}(B_1)$ are closed. Then, equation (33) has an EP solution if and only if operator equations

$$\begin{aligned} A_1X_1 &= C_1, \\ X_1B_1 &= D_1, \\ C_2 &= 0, \\ D_2 &= 0 \end{aligned} \tag{39}$$

have a common solution. By Lemma 7, we have the following theorem.

Theorem 5. *Let $A, C \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $B, D \in \mathscr{B}(\mathscr{G}, \mathscr{H})$, and let $\mathscr{R}(A)$, $\mathscr{R}(B)$ be closed. Suppose that $U \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2, \mathscr{H})$ is unitary such that*

$$\begin{aligned} AU &= (A_1 \ \ A_2), \\ CU &= (C_1 \ \ C_2), \\ U^{*}B &= \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \\ U^{*}D &= \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \end{aligned} \tag{40}$$

where $A_1, C_1 \in \mathscr{B}(\mathscr{H}_1, \mathscr{K})$, $A_2, C_2 \in \mathscr{B}(\mathscr{H}_2, \mathscr{K})$, $B_1, D_1 \in \mathscr{B}(\mathscr{G}_1, \mathscr{H})$, $B_2, D_2 \in \mathscr{B}(\mathscr{G}_2, \mathscr{H})$, and $\mathscr{R}(A_1)$ and $\mathscr{R}(B_1)$ are closed. Then, equation (33) has an EP solution $X \in \mathscr{B}(\mathscr{H})$ if and only if

$$\begin{aligned} \mathscr{N}(A_1^{*}) &\subseteq \mathscr{N}(C_1^{*}), \\ \mathscr{N}(B_1) &\subseteq \mathscr{N}(D_1), \\ A_1D_1 &= C_1B_1, \\ C_2 &= D_2 = 0. \end{aligned} \tag{41}$$

In this case, the general EP solution of (33) is given by

$$X = U\begin{pmatrix} A_1^{-}C_1 + D_1B_1^{-} - A_1^{-}A_1D_1B_1^{-} + \left(I_{\mathscr{H}_1} - A_1^{-}A_1\right)Y_1\left(I_{\mathscr{H}_1} - B_1B_1^{-}\right) & 0 \\ 0 & 0 \end{pmatrix}U^{*}, \tag{42}$$

where $Y_1 \in \mathscr{B}(\mathscr{H}_1)$ is arbitrary.

4. Concluding Remarks

In this work, we have characterized hypo-EP and EP matrices of adjointable operators on Hilbert C^{*}-modules, based on the generalized Schur complement, and an application of EP operator in operator equations is presented. In addition, the properties of hypo-EP and EP operators may have potential applications in some fields involving mathematics and its applications. In our opinion, it is worth establishing the hypo-EP and EP matrices of bounded linear operators on Krein space and of the adjointable operators on Krein C^{*}-modules.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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