Research Article

Hypo-EP Matrices of Adjointable Operators on Hilbert C*-Modules

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1. Introduction and Preliminaries

The EP matrix, as an extension of the normal matrix, was proposed by Schwertfeger; a square matrix \( T \) over the complex field \( \mathbb{C} \) is said to be an EP matrix if \( T \) and \( T^* \) share the same range [1, 2]. The notion of EP matrices was extended by Campbell and Meyer to operators with closed range on a Hilbert space in [3]. Let \( H \) be a complex Hilbert space and \( \mathcal{B}(H) \) the collection of all bounded linear operators on \( H \). Let \( T \in \mathcal{B}(H) \). Recall that \( T \) is called an EP operator if its range, \( \mathcal{R}(T) \), is closed, and \( \mathcal{R}(T) = \mathcal{R}(T^*) \) [3]. It is well known that \( \mathcal{R}(T) \) is closed if and only if the Moore–Penrose inverse \( T^+ \) of \( T \) exists and that \( T \) is an EP operator if and only if \( T^+T = TT^+ \). Sharifi [4] provided a generalization of the result for EP operators on Hilbert \( C^* \)-modules. This has been studied by many other authors, see, e.g., [5–8] and references therein. More generally, \( T \) is said to be a hypo-EP operator if \( T^1T \geq TT^1 \) [9]. In fact, \( T \) is a hypo-EP operator if and only if \( \mathcal{R}(T) \) is closed and \( \mathcal{R}(T) \subseteq \mathcal{R}(T^*) \). It is also shown that \( T \) is a hypo-EP operator if and only if \( T^1T^2 = TT^1 \). The hypo-EP operator is our focus of attention in this paper, and it has been studied in [10, 11]. The EP operator can be applied to the solution of operator equations, see Section 3 of this article. The properties of hypo-EP and EP operators can find applications also in reverse order law [12] and core partial order [13] and will be useful in some other applied fields [14, 15]. In this note, we investigate the hypo-EP operators on Hilbert \( C^* \)-modules, and then we formulate some results of hypo-EP matrices of adjointable operators on Hilbert \( C^* \)-modules. As an application, the solvability conditions, and the general expression for the EP solution to the operator equations are given.

Since the finite-dimensional spaces, Hilbert spaces, and \( C^* \)-algebras can all be regarded as Hilbert \( C^* \)-modules, one can study hypo-EP modular operators in a unified way in the framework of Hilbert \( C^* \)-modules. Let us briefly recall some basic knowledge about Hilbert \( C^* \)-modules and adjointable operators. Throughout this paper, \( \mathbb{A} \) is a \( C^* \)-algebra. A Hilbert \( \mathbb{A} \)-module \( \mathcal{H} \) is a right \( \mathbb{A} \)-module equipped with an \( \mathbb{A} \)-valued inner product \( \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{A} \) such that \( \mathcal{H} \) is complete with respect to the induced norm \( \| \cdot \| = \| \langle \cdot, \cdot \rangle \|^{1/2} \). Suppose that \( \mathcal{H} \) and \( \mathcal{K} \) are Hilbert \( \mathbb{A} \)-modules, and let \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) be the set of all maps \( T: \mathcal{H} \rightarrow \mathcal{K} \) for which there is a map \( T^*: \mathcal{K} \rightarrow \mathcal{H} \) such that \( \langle Tx, y \rangle = \langle x, T^*y \rangle \) for \( x \in \mathcal{H} \) and \( y \in \mathcal{K} \). It is well known that an arbitrary element \( T \) of \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) must be a bounded linear operator, which is also \( \mathbb{A} \)-linear in the sense of \( T(ax) = (Tx)a \) for any \( x \in \mathcal{H} \) and \( a \in \mathbb{A} \). We call \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) the set of adjointable operators from \( \mathcal{H} \) to \( \mathcal{K} \). We use \( \mathcal{L}(\mathcal{H}) \) to denote the \( C^* \)-algebra \( \mathcal{L}(\mathcal{H}, \mathcal{H}) \). Let \( \mathcal{L}(\mathcal{H})_a \) be the set of Hermitian elements of \( \mathcal{L}(\mathcal{H}) \). For \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \), the range and the null space of \( T \) are denoted by \( \mathcal{R}(T) \) and \( \mathcal{N}(T) \), respectively. An operator...
where \( A \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \), \( B \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \), \( C \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \), and \( D \in \mathcal{L}(\mathcal{K}) \). Then, the generalized Schur complement of \( A \) in \( M \) is

\[
\frac{M}{A} = D - CA^{-1}B, \quad (2)
\]

where \( A^{-1} \) is an inner inverse of \( A \). Similarly, the generalized Schur complement of \( D \) in \( M \) is

\[
\frac{M}{D} = A - BD^{-1}C, \quad (3)
\]

where \( D^{-1} \) is an inner inverse of \( D \). The formulas (2) and (3) have previously appeared in papers dealing with generalized inverses of partitioned matrices (cf. [17–19]).

**Definition 1** (see [20]). Let \( T \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \). The Moore–Penrose inverse \( T^\dagger \) of \( T \) (if exists) is an element in \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \) which satisfies

\[
\begin{align*}
(a) \quad & TT^\dagger T = T \\
(b) \quad & T^\dagger TT^\dagger = T^\dagger \\
(c) \quad & (TT^\dagger)^* = TT^\dagger \\
(d) \quad & (T^\dagger T)^* = T^\dagger T
\end{align*}
\]

These equations imply that \( T^\dagger \) will be uniquely determined if it exists, and \( TT^\dagger \) and \( TT^\dagger \) are both orthogonal projections. Moreover, \( \mathcal{R}(T^\dagger) = \mathcal{R}(T^\dagger^*), \mathcal{R}(T) = \mathcal{R}(TT^\dagger), \mathcal{N}(T) = \mathcal{N}(T^\dagger), \) and \( \mathcal{N}(T^\dagger) = \mathcal{N}(TT^\dagger) \). Clearly, the Moore–Penrose inverse \( T^\dagger \) of \( T \) exists if and only if \( \mathcal{R}(T) \) is closed; \( T \) is Moore–Penrose invertible if and only if \( T^\dagger \) is Moore–Penrose invertible, and in this case, \( (T^\dagger)^* = (T^\dagger)^* \). Obviously, the Moore–Penrose inverse \( T^\dagger \) of \( T \) is one of inner inverses of \( T \).

Similar to [21], Lemma 2.2.4, and [22], Lemma 2.2, we have the following conclusions on Hilbert \( C^* \)-modules.

**Lemma 1.** Let \( A \in \mathcal{L}(\mathcal{H}, \mathcal{K}), B \in \mathcal{L}(\mathcal{K}, \mathcal{H}) \), and \( C \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \). If \( A \) has an inner inverse \( A^{-1} \), then

\[
\begin{align*}
(i) \quad & \mathcal{N}(A) \subseteq \mathcal{N}(C) \quad \text{if and only if} \quad C = CA^{-1}A \\
(ii) \quad & \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \quad \text{if and only if} \quad B = AA^{-1}B
\end{align*}
\]

**Lemma 2.** Let \( M \) be a modular operator matrix of form (1) with \( \mathcal{N}(A) \subseteq \mathcal{N}(C) \) and \( \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \). If \( A \) has an inner inverse \( A^{-1} \), then \( M \) is regular if and only if \( M/A \) is regular, where \( M/A = D - CA^{-1}B \). In this case, the inner inverse of \( M \) is given by

\[
M^{-1} = \begin{pmatrix}
A^{-1} + A^{-1}B\left(\frac{M}{A}\right)^{-}CA^{-1} - A^{-1}B\left(\frac{M}{A}\right)^{-} \\
-\left(\frac{M}{A}\right)^{-}CA^{-1} & \left(\frac{M}{A}\right)^{-}
\end{pmatrix}. \quad (4)
\]

From Lemma 2, we can obtain the following corollary.

**Corollary 1.** Let \( M \) be a modular operator matrix of form (1) with \( \mathcal{N}(A) \subseteq \mathcal{N}(C), \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \), \( \mathcal{N}(M/A) \subseteq \mathcal{N}(B) \), and \( \mathcal{N}(M/A^*) \subseteq \mathcal{N}(C^*) \). If \( \mathcal{R}(A) \) and \( \mathcal{R}(M/A) \) are closed, then the Moore–Penrose inverse \( M^\dagger \) of \( M \) can be expressed as

\[
M^\dagger = \begin{pmatrix}
A^\dagger + A^\dagger B\left(\frac{M}{A}\right)^{+}CA^\dagger - A^\dagger B\left(\frac{M}{A}\right)^{+} \\
-\left(\frac{M}{A}\right)^{+}CA^\dagger & \left(\frac{M}{A}\right)^{+}
\end{pmatrix}. \quad (5)
\]

**Remark 1.** The preceding result given in [17], Theorem 1, was proved for finite matrices.

**Lemma 3** (see [23]). Let \( M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K}), \) where \( A \in \mathcal{L}(\mathcal{H}), B \in \mathcal{L}(\mathcal{K}, \mathcal{H}), \) and \( D \in \mathcal{L}(\mathcal{K}) \). If \( \mathcal{R}(A) \) and \( \mathcal{R}(D) \) are closed, then \( M^\dagger = \begin{pmatrix} A^\dagger & -A^\dagger BD^\dagger \\ 0 & D^\dagger \end{pmatrix} \) if and only if \( \mathcal{N}(D) \subseteq \mathcal{N}(B) \) and \( \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \).

**Proof.** The proof is similar to that in [22], Corollary 12, for Hilbert space operators. \( \square \)

**Definition 2** (see [4]). Let \( \mathcal{H} \) be a Hilbert \( A \)-module. An operator \( T \in \mathcal{L}(\mathcal{H}) \) is called EP if \( \mathcal{R}(T) = \mathcal{R}(T^*) \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is called hypo-EP if \( \mathcal{R}(T) \subseteq \mathcal{R}(T^*) \). Obviously, the range of an EP or a hypo-EP operator on Hilbert \( C^* \)-modules is not necessarily closed, and we further have the following properties.

**Proposition 1** (see [4]). Let \( \mathcal{H} \) be a Hilbert \( A \)-module and \( T \in \mathcal{L}(\mathcal{H}) \) with closed range. Then, the following conditions are equivalent:

\[
(i) \quad T \text{ is an EP operator} \\
(ii) \quad \mathcal{N}(T) = \mathcal{N}(T^*) \\
(iii) \quad T \text{ is Moore–Penrose invertible and } T^*T = TT^*
\]

**Proposition 2.** Let \( \mathcal{H} \) be a Hilbert \( A \)-module and \( T \in \mathcal{L}(\mathcal{H}) \) with closed range. Then, the following conditions are equivalent:

\[
(i) \quad T \text{ is a hypo-EP operator} \\
(ii) \quad \mathcal{N}(T) \subseteq \mathcal{N}(T^*)
\]
(iii) $T$ is Moore–Penrose invertible and $T^*T^2T^+ = TT^+$

**Remark 2.** The class of all hypo-EP operators contains the class of all EP operators on Hilbert $A_\lambda$-modules. Meanwhile, the EP operator with closed range is an extension of the invertible operator and the normal operator with closed range. In the case of finite dimensional situation, EP and hypo-EP are the same.

### 2. Main Results and Proofs

First, using generalized Schur complements, we study the hypo-EP property of matrices of adjointable operators on Hilbert $C^*$-modules.

**Theorem 1.** Let $M$ be a modular operator matrix of the form (1) with $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$. Suppose that $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed. Then, the following conditions are equivalent:

(i) $M$ is a hypo-EP operator matrix with closed range

(ii) $A$ and $M/A$ are hypo-EP operators

**Proof.** Let $M$ be a hypo-EP operator matrix with closed range. Since $\mathcal{R}(A)$ and $\mathcal{R}(M/A)$ are closed, let us consider the operator matrices

\[
L = \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} - CA^* I
\]

\[
R = \begin{pmatrix}
I & B(M/A) \\
0 & I
\end{pmatrix},
\]

\[
P = \begin{pmatrix}
A & 0 \\
0 & M
\end{pmatrix},
\]

where $M/A = D - CA^* B$. Obviously, $L$ and $R$ are invertible. By using Lemma 1 and by assumptions $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, it is clear that $M$ can be factorized as $M = LRP$. Hence, $\mathcal{N}(P) = \mathcal{N}(M) \subseteq \mathcal{N}(M^*)$. By using Lemma 1 again, it is immediate that

\[
M^* = M^* P^- P
\]

holds for every inner inverse $P^-$ of $P$. In particular, for

\[
P^- = \begin{pmatrix}
A^- & 0 \\
0 & (M/A)^-
\end{pmatrix},
\]

we have from relation (7) that

\[
M^* = \begin{pmatrix}
A^* & C^* \\
B^* & D^*
\end{pmatrix} = \begin{pmatrix}
A^- A & 0 \\
0 & (M/A)^-
\end{pmatrix}
\]

\[
= \begin{pmatrix}
A^* A^- & C^* (M/A)^- \\
B^* A^- & D^* (M/A)^-
\end{pmatrix}.
\]

Then, $M^* = A^* A^- A$ implies $M(A) \subseteq M(A^*)$. Hence, $A$ is a hypo-EP operator. Since $C^* = C^* (M/A)^- (M/A)$, substituting $D = (M/A) + CA^* B$ into

\[
D^* = D^* (M/A)^- (M/A)
\]

yields $(M/A)^* = (M/A)^* (M/A)^- (M/A)$. This implies $\mathcal{N}(M/A) \subseteq \mathcal{N}((M/A)^*)$. Thus, $M/A$ is a hypo-EP operator.

Conversely, according to the assumptions $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$, the Moore–Penrose inverse $M^*$ of $M$ exists, and $M^*$ is given by

\[
M^* = \begin{pmatrix}
A^+ A^* B(M/A) + A^+ (M/A)^+ \\
(M/A)^+ CA^+ (M/A)^+
\end{pmatrix},
\]

by Corollary 1. Using $\mathcal{N}(A^*) \subseteq \mathcal{N}(B^*)$ and $\mathcal{N}((M/A)^*) \subseteq \mathcal{N}(C^*)$, by Lemma 1, $MM^*$ is described as

\[
MM^* = \begin{pmatrix}
AA^+ & 0 \\
0 & (M/A)^+ (M/A)^+
\end{pmatrix}.
\]

Similarly, by using $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, $\mathcal{N}(M/A) \subseteq \mathcal{N}(B)$, and Lemma 1, it is given that

\[
M^* M = \begin{pmatrix}
A^+ A & 0 \\
0 & (M/A)^+ (M/A)^+
\end{pmatrix}.
\]

Then,

\[
M^* M^+ = (M^* M)(MM^*) = \begin{pmatrix}
A^+ A^* A^* & 0 \\
0 & (M/A)^* (M/A)^+ (M/A)^+
\end{pmatrix}.
\]

Since $A$ and $M/A$ are hypo-EP operators with closed range.
\[ A^A A^2 A^+ = AA^+ , \]
\[ \left( \frac{M}{A} \right)^+ \left( \frac{M}{A} \right)^2 \left( \frac{M}{A} \right)^?= \left( \frac{M}{A} \right)^+ . \]  

Thus, \( M^+ M^2 M^+ = MM^+ \). Therefore, \( M \) is a hypo-EP operator matrix with closed range.

The following conclusion is a natural extension of [10], Theorem 3.1, on Hilbert \( C^* \)-modules.

**Corollary 2.** Let \( A, X \in \mathcal{L}(\mathcal{H}) \) and \( M = \begin{pmatrix} A & AX \\ X^* A & X^* AX \end{pmatrix} \) \( \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \). If \( \mathcal{R}(A) \) is closed, then \( M \) is a hypo-EP operator matrix with closed range if and only if \( A \) is a hypo-EP operator.

If using the generalized Schur complement \( M/D = A - BD^* C \) of \( D \) in \( M \), similar to Theorem 1, one can get the following results.

**Theorem 2.** Let \( M \) be a modular operator matrix of form (1) with \( \mathcal{N}(D) \subseteq \mathcal{N}(B) \), \( \mathcal{N}(D^*) \subseteq \mathcal{N}(C^*) \), \( \mathcal{N}(M/D) \subseteq \mathcal{N}(C) \), and \( \mathcal{N}(M/D^*) \subseteq \mathcal{N}(B^*) \). Suppose that \( \mathcal{R}(D) \) and \( \mathcal{R}(M/D) \) are closed. Then, the following conditions are equivalent:

(i) \( M \) is a hypo-EP operator matrix with closed range

(ii) \( D \) and \( M/D \) are hypo-EP operators

**Corollary 3.** Let \( D, X \in \mathcal{L}(\mathcal{H}) \) and \( M = \begin{pmatrix} X^* D X^* & X^* D X^* \\ D X^* & D D X^* \end{pmatrix} \) \( \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \). If \( \mathcal{R}(D) \) is closed, then \( M \) is a hypo-EP operator matrix with closed range if and only if \( D \) is a hypo-EP operator.

Next, using the properties of generalized inverses, we study upper triangular hypo-EP matrices of adjointable operators on Hilbert \( C^* \)-modules.

**Theorem 3.** Let \( M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \) \( \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) with \( \mathcal{N}(D) \subseteq \mathcal{N}(B) \) and \( \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \), where \( A \in \mathcal{L}(\mathcal{H}), B \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \), and \( D \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \). If \( \mathcal{R}(A) \) and \( \mathcal{R}(D) \) are closed, then \( M \) is a hypo-EP operator matrix with closed range if and only if \( A \) and \( D \) are hypo-EP operators.

**Proof.** Let \( M \) be a hypo-EP operator matrix with closed range. We write

\[
L := \begin{pmatrix} 1 & BD^* \\ 0 & I \end{pmatrix},
\]

\[
P := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.
\]

Obviously, \( L \) is invertible. By Lemma 1 and assumption \( \mathcal{N}(D) \subseteq \mathcal{N}(B) \), it is clear that \( M \) can be decomposed as \( M = LP \). Hence, \( \mathcal{N}(M) = \mathcal{N}(P) \). Since \( M \) is a hypo-EP operator matrix with closed range, \( \mathcal{N}(P) = \mathcal{N}(M) \subseteq \mathcal{N}(M^+) \). By Lemma 1, it is immediate that \( M^+ = M^+ P^+ P \), where \( P^+ \) is given by

\[
P^+ = \begin{pmatrix} A^+ & 0 \\ 0 & D^+ \end{pmatrix}.
\]

This gives

\[
M^* = \begin{pmatrix} A^+ & 0 \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A^+ A & 0 \\ B^* A & D^* D \end{pmatrix}.
\]

Hence, \( A^* = A^* A^+ A \) implies \( \mathcal{N}(A) \subseteq \mathcal{N}(A^*) \). Thus, \( A \) is a hypo-EP operator. From \( D^* = D^* D D^* \), it follows that \( \mathcal{N}(D) \subseteq \mathcal{N}(D^*) \). Therefore, \( D \) is a hypo-EP operator.

Conversely, suppose \( A \) and \( D \) are hypo-EP operators. Since \( \mathcal{R}(A) \) and \( \mathcal{R}(D) \) are closed and \( \mathcal{N}(D) \subseteq \mathcal{N}(B^*) \) and \( \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \), by Lemma 3, the Moore-Penrose inverse \( M^+ \) of \( M \) exists and

\[
M^+ = \begin{pmatrix} A^+ - A^+ BD^* \\ 0 & D^+ \end{pmatrix}.
\]

Since \( \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \), by Lemma 1, \( MM^+ \) is described as

\[
MM^+ = \begin{pmatrix} AA^+ & 0 \\ 0 & DD^* \end{pmatrix}.
\]

Similarly, by Lemma 1, \( \mathcal{N}(D) \subseteq \mathcal{N}(B) \) leads to

\[
M^+ M = \begin{pmatrix} A^+ A & 0 \\ 0 & D^+ D \end{pmatrix}.
\]

Then

\[
M^+ M^2 M^+ = (M^+ M)(MM^+) = \begin{pmatrix} A^+ A^2 A^+ & 0 \\ 0 & D^+ D^2 D^+ \end{pmatrix}.
\]

Since \( A \) and \( D \) are hypo-EP operators with closed range, \( A^+ A^2 A^+ = AA^+ \) and \( D^+ D^2 D^+ = DD^+ \). Thus, \( M^+ M^2 M^+ = MM^+ \). Therefore \( M \) is a hypo-EP operator matrix with closed range.

**Corollary 4.** Let \( M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \) \( \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) with \( \mathcal{N}(D) \subseteq \mathcal{N}(B) \) and \( \mathcal{N}(A^*) \subseteq \mathcal{N}(B^*) \), where \( A \in \mathcal{L}(\mathcal{H}), B \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \), and \( D \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \). If \( \mathcal{R}(A) \) and \( \mathcal{R}(D) \) are closed, then \( M \) is an EP operator matrix with closed range if and only if \( A \) and \( D \) are EP operators.

**Proof.** Let \( M \) be an EP operator matrix with closed range. In view of Theorem 3, to prove the necessity, it is enough to show \( \mathcal{N}(A^*) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(D^*) \subseteq \mathcal{N}(D) \). Since \( M \) is an EP operator matrix with closed range, by the proof of Theorem 3, \( \mathcal{N}(P) = \mathcal{N}(M) = \mathcal{N}(M^+) \). Applying Lemma 1, we have \( P^* = MM^+ P^* \), i.e.,
\[ P^* = \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix} = \begin{pmatrix} AA^* & 0 \\ 0 & DD^* \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix}. \]

(23)

Hence, \( A^* = AA^* \) and \( D^* = DD^* \) imply \( \mathcal{N}(A^*) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(D^*) \subseteq \mathcal{N}(D) \), respectively.

Corollary 5. Let \( A, X, D \in \mathcal{L}(H) \) and \( M = \begin{pmatrix} A & AX \\ 0 & D \end{pmatrix} \in \mathcal{L}(H \oplus H) \). If \( \mathcal{R}(A) \) and \( \mathcal{R}(D) \) are closed, then \( M \) is a hypo-EP operator matrix with closed range if and only if \( A \) and \( D \) are hypo-EP operators.

Corollary 6. Let \( A \in \mathcal{L}(H) \) and \( M = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix} \in \mathcal{L}(H \oplus H) \). If \( \mathcal{R}(A) \) is closed, then \( M \) is a hypo-EP operator matrix with closed range if and only if \( A \) is a hypo-EP operator.

Remark 3. The Hilbert space version of the preceding four conclusions is given by [10], and the conditions of closed range can be naturally omitted there. Moreover, the alternative proofs of the conclusions in Hilbert space setting can be found in section 3 of [10]. In addition, these results originated from the research of the EP property of block matrices, according to Hartwig [24].

Finally, the following are devoted to investigating the hypo-EP property of antitriangular block matrices of adjointable operators on Hilbert C*-modules.

Lemma 4. Let \( M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H) \). If \( \mathcal{R}(B) \) and \( \mathcal{R}(C) \) are closed, then \( M^* = \begin{pmatrix} 0 & C^* \\ B^* & -B^*A C^* \end{pmatrix} \) if and only if \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(B^*) \subseteq \mathcal{N}(A^*) \).

Proof. Sufficiency: since \( \mathcal{R}(B) \) and \( \mathcal{R}(C) \) are closed, \( B \) and \( C \) are Moore–Penrose invertible. From \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(B^*) \subseteq \mathcal{N}(A^*) \), it follows that \( AC^* = A \) and \( BB^*A = A \).

We write \( X = \begin{pmatrix} 0 & C^* \\ B^* & AC^* \end{pmatrix} \). A direct calculation shows that

\[ MM^* = M, \quad XMX = X, \quad (MM)^* = XM, \quad (XM)^* = XM. \]

By Definition 1, \( M^* = X \) as desired.

Necessity: since

\[ MM^* = \begin{pmatrix} BB^* & AC^* - BB^*AC^* \\ 0 & CC^* \end{pmatrix}, \]

\[ M^*M = \begin{pmatrix} C^*C & 0 \\ B^*A - B^*AC^*C & B^*B \end{pmatrix} \]

are self-adjoint, we have \( AC^* - BB^*AC^* = 0 \) and \( B^*A - B^*AC^*C = 0 \). From \( MM^* = M \), we get \( BB^*A = A = AC^* \). Therefore, \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(B^*) \subseteq \mathcal{N}(A^*) \).

Lemma 5. Let \( M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H) \) with \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(B^*) \subseteq \mathcal{N}(A^*) \). If \( B \) and \( C \) are hypo-EP operators with closed ranges, then \( M \) is a hypo-EP operator matrix with closed range.

Proof. Since \( \mathcal{R}(B) \) and \( \mathcal{R}(C) \) are closed, \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(B^*) \subseteq \mathcal{N}(A^*) \), by Lemma 4, the Moore–Penrose inverse \( M^* \) of \( M \) is given by

\[ M^* = \begin{pmatrix} 0 & C^* \\ B^* & -B^*AC^* \end{pmatrix}. \]

Using \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(B^*) \subseteq \mathcal{N}(A^*) \), by Lemma 1, we have

\[ MM^* = \begin{pmatrix} BB^* & 0 \\ 0 & CC^* \end{pmatrix}, \]

\[ M^*M = \begin{pmatrix} C^*C & 0 \\ 0 & B^*B \end{pmatrix}. \]

By Definition 1, we have \( \mathcal{R}(MM^*) = \mathcal{R}(M) \) and \( \mathcal{R}(M^*M) = \mathcal{R}(M^*) = \mathcal{R}(M^*) \). Since \( B \) and \( C \) are hypo-EP operators with closed ranges, \( \mathcal{R}(BB^*) = \mathcal{R}(B) \subseteq \mathcal{R}(B^*) = \mathcal{R}(B^*B) \) and \( \mathcal{R}(CC^*) = \mathcal{R}(C) \subseteq \mathcal{R}(C^*) = \mathcal{R}(C^*C) \). Then,

\[ \mathcal{R}(M) = \mathcal{R}(MM^*) = \mathcal{R}(BB^*) \oplus \mathcal{R}(CC^*) \subseteq \mathcal{R}(B^*B) \oplus \mathcal{R}(C^*C) = \mathcal{R}(M^*M) = \mathcal{R}(M^*). \]

(29)
Therefore, \( M \) is a hypo-EP operator with closed range.

**Corollary 7.** Let \( M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) with \( \mathcal{N}(B) \subseteq \mathcal{N}(A) \) and \( \mathcal{N}(B^*) \subseteq \mathcal{N}(A^*) \). If \( B \) and \( C \) are EP operators with closed ranges, then \( M \) is an EP operator matrix with closed range.

**Proof.** The sufficiency is clear by Lemma 5. Now, we suppose that \( M \) is a hypo-EP operator matrix with closed range. We write

\[
L := \begin{pmatrix} I & AC^* \\ 0 & I \end{pmatrix},
\]

\[
P := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.
\]

In the similar way as in the proof of Theorem 3, we have \( M = LP \), and hence, \( \mathcal{N}(P) = \mathcal{N}(M) \subseteq \mathcal{N}(M^*) \), since \( M \) is a hypo-EP operator matrix with closed range. This means \( M^* = M^* P^* P \) by Lemma 1, i.e.,

\[
\begin{pmatrix} A^* & C^* \\ B^* & 0 \end{pmatrix} = M^* = M^* P^* P = \begin{pmatrix} A^* C^* & C^* B^* B \\ B^* C^* & 0 \end{pmatrix}.
\]

Hence, \( C^* = C^* B^* B \), which together with \( \mathcal{N}(C) = \mathcal{N}(B) \), implies \( \mathcal{N}(C) \subseteq \mathcal{N}(C^*) \). Thus, \( C \) is a hypo-EP operator. Similarly, it follows from \( B^* = B^* C^* C \) and \( \mathcal{N}(B) = \mathcal{N}(C) \) that \( \mathcal{N}(B) \subseteq \mathcal{N}(B^*) \), and therefore, \( B \) is a hypo-EP operator.

**Corollary 8.** Let \( B, X, C \in \mathcal{L}(\mathcal{H}) \) and \( M = \begin{pmatrix} BXC & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) with \( \mathcal{N}(B) = \mathcal{N}(C) \). If \( \mathcal{R}(B) \) and \( \mathcal{R}(C) \) are closed, then \( M \) is a hypo-EP operator matrix with closed range if and only if \( B \) and \( C \) are EP operators.

**Proof.** By Corollary 7, we only need to show the necessity, which can be easily verified according to the proofs of Corollary 4 and Theorem 4.

**Corollary 10.** Let \( B, X, C \in \mathcal{L}(\mathcal{H}) \) and \( M = \begin{pmatrix} BXC & B \\ C & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}) \) with \( \mathcal{N}(B) = \mathcal{N}(C) \). If \( \mathcal{R}(B) \) and \( \mathcal{R}(C) \) are closed, then \( M \) is an EP operator matrix with closed range if and only if \( B \) and \( C \) are EP operators.

**Remark 4.** In Hilbert space case, the conditions of closed range in Theorem 4 and Corollary 9 can be naturally omitted in Theorem 3.8 and Theorem 3.9 of [10], and the alternative proofs of Theorem 4 and Corollary 9 can be, respectively, found in Theorem 3.8 and Theorem 3.9 of [10].

### 3. The Application of EP Operators

In this section, let \( \mathcal{H}, \mathcal{K}, \) and \( \mathcal{G} \) be Hilbert spaces. We establish the solvability conditions and the general expression for the EP solution to the operator equations

\[
AX = C,
\]

\[
XB = D,
\]

where \( A, C \in \mathcal{B}(\mathcal{H}, \mathcal{K}), B, D \in \mathcal{B}(\mathcal{G}, \mathcal{H}), \) and \( X \in \mathcal{B}(\mathcal{H}) \).

**Lemma 6** (see [25]). Let \( T \in \mathcal{B}(\mathcal{H}) \) with closed range. Then, the operator \( T \) is EP if and only if there exist Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), \( U \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}) \) unitary, and \( T_1 \in \mathcal{B}(\mathcal{H}_1) \) isomorphism such that

\[
T = U(T_1 \oplus 0)U^*,
\]

where \( \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H} \).

**Lemma 7** (see [22]). Let \( A, C \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \) and \( B, D \in \mathcal{B}(\mathcal{G}, \mathcal{H}) \). Suppose that \( A \) and \( B \) have closed ranges. Then, equation (33) has a common solution \( X \in \mathcal{B}(\mathcal{H}) \) if and only if

\[
\mathcal{N}(A^*) \subseteq \mathcal{N}(C^*),
\]

\[
\mathcal{N}(B) \subseteq \mathcal{N}(D),
\]

\[
AD = CB.
\]

In this case, the general common solution is given by
where $Y \in \mathcal{B}(\mathcal{H})$ is arbitrary.

Now, we consider the EP solution to equation (33). By the Lemma 6, for the unitary operator $U \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H})$, the solution has the following factorization:

$$X = U \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} U^*.$$  \hspace{1cm} (37)

Let $\mathcal{R}(A), \mathcal{R}(B)$ be closed, and

$$AU = (A_1 \ A_2),$$
$$CU = (C_1 \ C_2),$$
$$U^* B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$
$$U^* D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},$$  \hspace{1cm} (38)

where $A_1, C_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}), A_2, C_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}), B_1, D_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}), B_2, D_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}),$ and $\mathcal{R}(A_1)$ and $\mathcal{R}(B_1)$ are closed. Then, equation (33) has an EP solution if and only if operator equations

$$A_1 X_1 = C_1,$$
$$X_1 B_1 = D_1,$$
$$C_2 = 0,$$
$$D_2 = 0$$  \hspace{1cm} (39)

have a common solution. By Lemma 7, we have the following theorem.

**Theorem 5.** Let $A, C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ and $B, D \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$, and let $\mathcal{R}(A), \mathcal{R}(B)$ be closed. Suppose that $U \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H})$ is unitary such that

$$AU = (A_1 \ A_2),$$
$$CU = (C_1 \ C_2),$$
$$U^* B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$
$$U^* D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},$$

where $A_1, C_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}), A_2, C_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}), B_1, D_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}), B_2, D_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}),$ and $\mathcal{R}(A_1)$ and $\mathcal{R}(B_1)$ are closed. Then, equation (33) has an EP solution $X \in \mathcal{B}(\mathcal{H})$ if and only if

$$\mathcal{N}(A_1) \subseteq \mathcal{N}(C_1),$$
$$\mathcal{N}(B_1) \subseteq \mathcal{N}(D_1),$$
$$A_1 D_1 = C_1 B_1,$$
$$C_2 = D_2 = 0.$$  \hspace{1cm} (41)

In this case, the general EP solution of (33) is given by

$$X = U \begin{pmatrix} A_1 C_1 + D_1 B_1^* - A_1^* A_1 D_1 B_1^* + (I_{\mathcal{H}_1} - A_1^* A_1) Y_1 (I_{\mathcal{H}_1} - B_1 B_1^*) & 0 \\ 0 & 0 \end{pmatrix} U^*,$$  \hspace{1cm} (42)

where $Y_1 \in \mathcal{B}(\mathcal{H}_1)$ is arbitrary.

### 4. Concluding Remarks

In this work, we have characterized hypo-EP and EP matrices of adjointable operators on Hilbert $C^*$-modules, based on the generalized Schur complement, and an application of EP operator in operator equations is presented. In addition, the properties of hypo-EP and EP operators may have potential applications in some fields involving mathematics and its applications. In our opinion, it is worth establishing the hypo-EP and EP matrices of bounded linear operators on Krein $C^*$-modules.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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