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Research Article

Marshall–Olkin Alpha Power Weibull Distribution: Different Methods of Estimation Based on Type-I and Type-II Censoring

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This paper introduces the new novel four-parameter Weibull distribution named as the Marshall–Olkin alpha power Weibull (MOAPW) distribution. Some statistical properties of the distribution are examined. Based on Type-I censored and Type-II censored samples, maximum likelihood estimation (MLE), maximum product spacing (MPS), and Bayesian estimation for the MOAPW distribution parameters are discussed. Numerical analysis using real data sets and Monte Carlo simulation are accomplished to compare various estimation methods. This novel model's supremacy upon some famous distributions is explained using two real data sets and it is shown that the MOAPW model can achieve better fits than other competitive distributions.

1. Introduction

In real-life phenomena, statistical distributions are widely used to describe these phenomena. For this reason, the theory of statistical distribution and generating new distributions are of great interest. Many authors studied and generated new distributions from old ones. In the last few years, many classes of generalized distributions have been developed and applied to describe different events in real life. These generalized distributions are preferred because they have more parameters and hence more flexibility to real-life model data.

Depending on a distribution with $g(x)$ as the probability density function (PDF) and $G(x)$ as the cumulative distribution function (CDF), several distributions have been generated using the PDF, the CDF, or also the survival functions as the base distribution to introduce novel ones. Marshall–Olkin's family of distributions has been presented by Marshall and Olkin [1], counting on adding a parameter to a family of distributions called extended distributions. If the CDF and PDF of a given random variable are $G(x)$ and $g(x)$, then the CDF and PDF of the Marshall–Olkin (MO) family are, respectively, given by

$$F(x) = \frac{G(x)}{\theta + (1-\theta)G(x)}, \quad \theta > 0,$$
$$f(x) = \frac{\theta g(x)}{[\theta + (1-\theta)G(x)]^2}. \quad (1)$$

The MO extended distribution suggests a broad range of behaviors than the basic distribution from which they are originated. For more details, information, and examples about this family, see the work of Ghitany [2], Ghitany et al. [3], Alice and Jose [4], Okasha and Kayid [5], Ahmad and Almetwally [6], and Ijaz and Asim [7].

Furthermore, alpha power (AP) transformation by adding a novel parameter to gain a family of distributions has been discussed by Mahdavi and Kundu [8]. If $G(x)$ is a CDF of any distribution, then $W(x)$ is a CDF that is defined by the following equation:

$$W(x)=\begin{cases}\dfrac{\alpha^{G(x)}-1}{\alpha-1}, & \text{if } \alpha>0, \alpha\neq 1\\ G(x), & \text{if } \alpha=1,\end{cases} \tag{2}$$

and the corresponding PDF has the form

$$w(x)=\begin{cases}\dfrac{\ln(\alpha)g(x)\alpha^{G(x)}}{\alpha-1}, & \text{if } \alpha>0, \alpha\neq 1,\\ g(x), & \text{if } \alpha=1.\end{cases} \tag{3}$$

Considerable work in distributions based on AP transformation had been done; for example, see the works of Nassar et al. [9], Elbatal et al. [10], Dey et al. [11, 12], Hassan et al. [13, 14], Basheer [15], and Almetwally and Ahmad [16].

The Marshall–Olkin alpha power (MOAP) family has been recently introduced by Nassar et al. [17]. This is a novel method to insert an extra parameter to a family of distributions for more flexibility. By combining the CDF of the AP family and the CDF of the MO family, the MOAP family CDF is defined as follows:

$$F_{\mathrm{MOAP}}(x)=\begin{cases}\dfrac{\alpha^{G(x)}-1}{(\alpha-1)\left(\theta+((1-\theta)/(\alpha-1))\left(\alpha^{G(x)}-1\right)\right)}, & \text{if } \alpha>0, \alpha\neq 1, \theta>0,\\ G(x) & \text{if } \alpha=1,\end{cases} \tag{4}$$

and its PDF can be expressed as

$$f_{\mathrm{MOAP}}(x)=\begin{cases}\dfrac{\theta\ln(\alpha)}{\alpha-1}\dfrac{g(x)\alpha^{G(x)}}{\left(\theta+((1-\theta)/(\alpha-1))\left(\alpha^{G(x)}-1\right)\right)^{2}}, & \text{if } \alpha>0, \alpha\neq 1, \theta>0,\\ g(x), & \text{if } \alpha=1.\end{cases} \tag{5}$$

The CDF and PDF of Weibull distribution with parameters β and λ are given, respectively, as

$$G(x;\beta,\lambda)=1-e^{-(x/\beta)^{\lambda}};\quad x\geq 0, \beta, \lambda>0, \tag{6}$$

$$g(x;\beta,\lambda)=\frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\lambda-1}e^{-(x/\beta)^{\lambda}}. \tag{7}$$

In life testing and reliability experiments, there are numerous situations where observations are missed or set aside from experimentation before failure. The tester may not secure complete information on failure times for all experimental observations. The handled data from all such mentioned trials are called censored data. Censored test has many types and the most important and used schemes are Type-I censored and Type-II censored; see, for example, the works of Balakrishnan and Ng [18], El-Morshedy et al. [19], and Almetwally et al. [20].

This paper's aim is to introduce a new lifetime distribution defined as Marshall–Olkin alpha power Weibull (MOAPW) distribution, depending on the MOAP family. Associated statistical properties of MOAPW distribution are shown. Parameter estimation for MOAPW distribution is discussed using MLE, MPS, and Bayesian methods. We also conduct the estimation for MOAPW distribution with Type-I and Type-II censored samples. Monte Carlo simulation is accomplished to evaluate the efficiency of the estimators. Two real data sets are elaborated to affirm the integrity of the model and the scheme.

This paper is organized as follows: In Section 2, the new distribution is introduced and described. In Section 3, reliability analysis is derived, while in Section 4, some statistical properties of MOAPW are discussed. Parameter estimation of MOAPW with complete, Type-I censored, and Type-II censored samples is presented in Section 5. Monte Carlo simulation study is conducted in Section 6. Applications with two real data sets are studied in Section 7. Finally, the conclusion of this study is discussed in Section 8.

2. Model Description and Notation

The MOAPW model and its submodels have been introduced in this section.

2.1. MOAPW Distribution. The MOAP family and Weibull distribution have been used to generate MOAPW distribution. It is represented by the random variable $X \sim \text{MOAPW}(\alpha, \beta, \theta, \lambda)$. By using equations (5)–(7), the PDF of MOAPW is given as

$$f_{\text{MOAPW}}(x, \Psi) = \frac{\theta \ln(\alpha)}{(\alpha - 1)} \frac{(\lambda/\beta)(x/\beta)^{\lambda-1} e^{-(x/\beta)^{\lambda}} \alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)}}{\left[\theta + (1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)} - 1\right)\right]^2}, \tag{8}$$

where $\Psi = (\alpha, \beta, \theta, \lambda)$, $\Psi > 0$, and $\alpha \neq 1$. By using equations (4) and (6), the CDF of MOAPW takes the form

$$F_{\text{MOAPW}}(x, \Psi) = \frac{\alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)} - 1}{(\alpha - 1)\left[\theta + ((1-\theta)/(\alpha-1))\left(\alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)} - 1\right)\right]}. \tag{9}$$

Figure 1 displays the PDF of the MOAPW for some parameters' values.

2.2. Submodels from MOAPW. Many submodels can be derived from MOAPW distribution, as is shown in Table 1.

3. Reliability Analysis

The following equation defines the survival function of MOAPW distribution:

$$S_{\text{MOAPW}}(x, \Psi) = \frac{\theta\left[\alpha - \alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)}\right]}{(\alpha - 1)\left[\theta + ((1-\theta)/(\alpha-1))\left(\alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)} - 1\right)\right]}, \tag{10}$$

while the following equation gives the hazard function of a MOAPW distribution:

$$h_{\text{MOAPW}}(x, \Psi) = \frac{\ln(\alpha)}{\alpha - \alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)}} \frac{(\lambda/\beta)(x/\beta)^{\lambda-1} e^{-(x/\beta)^{\lambda}} \alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)}}{\left[\theta + (1-\theta)(\alpha-1)^{-1}\left(\alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)} - 1\right)\right]}. \tag{11}$$

The hazard function of the MOAPW model for different parameter values is displayed in Figure 2.

By examining Figures 1 and 2, we conclude that the MOAPW distribution could be used as a compatible model for fitting skewed and different data, which may not be compatible with other popular distributions.

Equation (12) defines the reversed hazard (RH) function of MOAPW distribution:

$$\text{rh}_{\text{MOAPW}}(x, \Psi) = \frac{\theta \ln(\alpha)}{\alpha - 1} \frac{(\lambda/\beta)(x/\beta)^{\lambda-1} e^{-(x/\beta)^{\lambda}} \alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)}}{\left(\alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)} - 1\right)\left[\theta + (1-\theta)/(\alpha-1)\left(\alpha^{\left(1-e^{-(x/\beta)^{\lambda}}\right)} - 1\right)\right]}. \tag{12}$$

The reversed hazard function of the MOAPW distribution for some values of the parameters is displayed in Figure 3.

For the stress-strength reliability measure of MOAPW distribution, let X and Y be the independent strength and stress random variable, respectively, observed from

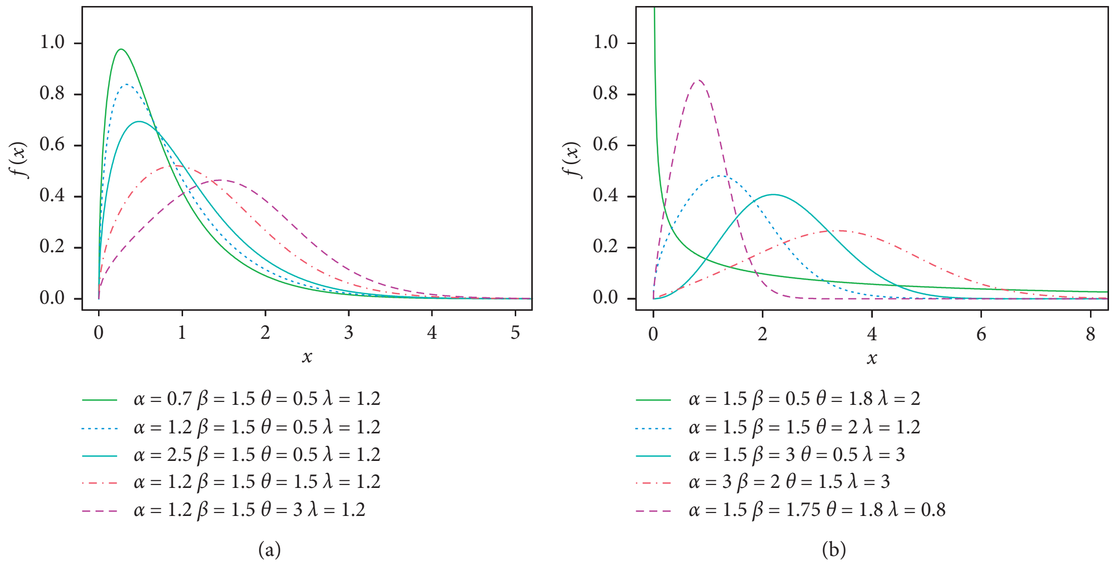


Figure 1: Plots of the PDF of the MOAPW with some values of parameters.

Table 1: Submodels from MOAPW distribution.

Models	α	β	θ	λ
Marshall–Olkin alpha power exponential (MOAPEx) distribution (Nassar et al. [17])	α	β	θ	1
Marshall–Olkin alpha power Rayleigh (MOAPR) distribution (new)	α	β	θ	2
Alpha power Weibull (APW) distribution (Nassar et al. [9])	α	β	1	λ
Marshall–Olkin Weibull (MOW) distribution (Ghitany et al. [2])	1	β	θ	λ
Weibull (W) distribution	1	β	1	λ
Exponential (Ex) distribution	1	β	1	1
Rayleigh (R) distribution	1	β	1	2

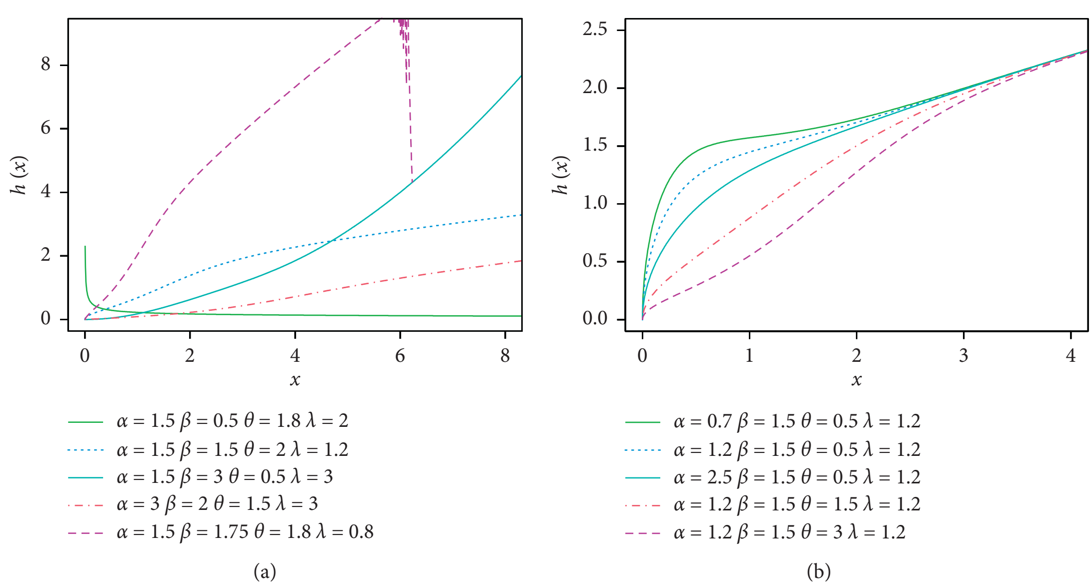


Figure 2: The hazard function of MOAPW under different parameters' values.

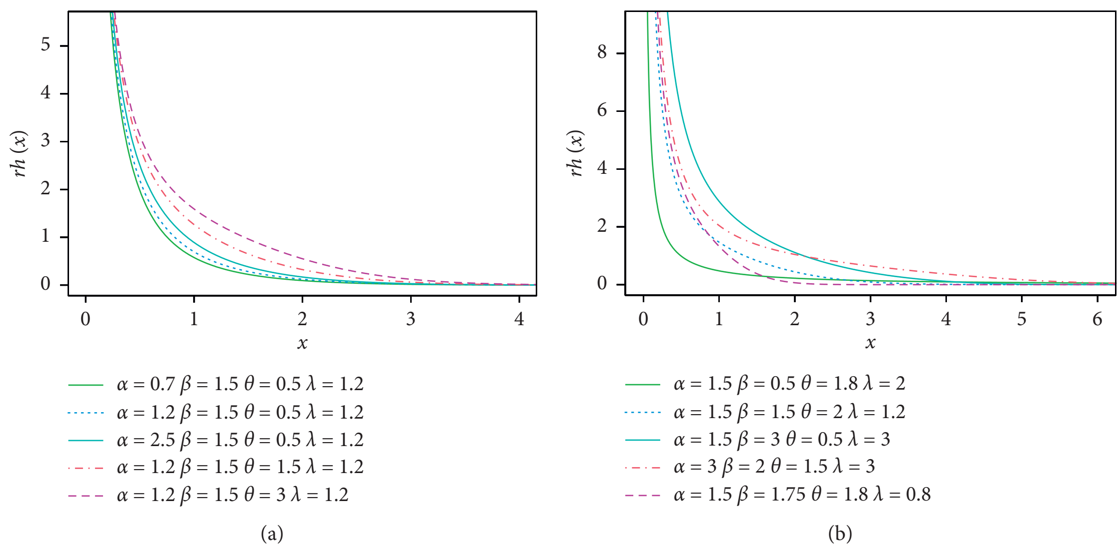


Figure 3: The reversed hazard of MOAPW with different values of the parameters.

MOAPW distribution; then, the stress-strength reliability R is calculated as

$$R = P(Y < X) = \int_{x=0}^{\infty} \left\{ \int_{y=0}^{x} f(y; \Psi_2) \mathrm{d}y \right\} f(x; \Psi_1) \mathrm{d}y \mathrm{d}x. \tag{13}$$

Integrating over y, we have

$$R = \frac{\theta_1 \lambda_1 \ln(\alpha_1)}{\beta_1 (\alpha_1 - 1)(\alpha_2 - 1)} \int_0^{\infty} \frac{(x/\beta_1)^{\lambda_1 - 1} e^{-(x/\beta_1)^{\lambda_1}} \alpha_1^{\left(1-e^{-(x/\beta_1)^{\lambda_1}}\right)} \left(\alpha_2^{\left(1-e^{-(x/\beta_1)^{\lambda_2}}\right)} - 1 \right)}{\left[\theta_1 + ((1-\theta_1)/(\alpha_1 - 1)) \left(\alpha_1^{\left(1-e^{-(x/\beta_1)^{\lambda_1}}\right)} - 1 \right) \right]^2 \left[\theta_2 + ((1-\theta_2)/(\alpha_2 - 1)) \left(\alpha_1^{\left(1-e^{-(x/\beta 2)^{\lambda_2}}\right)} - 1 \right) \right]} \mathrm{d}x. \tag{14}$$

It will be calculated numerically. Figure 4 displays plots of the stress-strength reliability measure for the MAOPW distribution for different values of the parameters. The higher the value of α_2 with other parameters remaining constant, the lower the reliability value. The greater the value of α_1 with other parameters remaining constant, the greater the reliability value, as shown in Figure 4.

4. Statistical Properties

This section is devoted to studying and obtaining some statistical properties of the MOAPW distribution, such as quantile, median, and mode.

4.1. The Quantile Function. By inverting the CDF equation in (9), we have the quantile function of MOAPW distribution as follows:

$$x_u = \beta \left(-\ln \left[1 - \frac{1}{\ln(\alpha)} \ln \left(1 + \frac{\theta u (\alpha - 1)}{1 - u(1 - \theta)} \right) \right] \right)^{1/\lambda}; \quad 0 < u < 1. \tag{15}$$

From equation (15), we can obtain the median (M) or the second quartile of MOAPW distribution when $u = 0.5$ as follows:

$$M = \beta \left(-\ln \left[1 - \frac{1}{\ln(\alpha)} \ln \left(\frac{1 + \alpha\theta}{1 + \theta} \right) \right] \right)^{1/\lambda}. \tag{16}$$

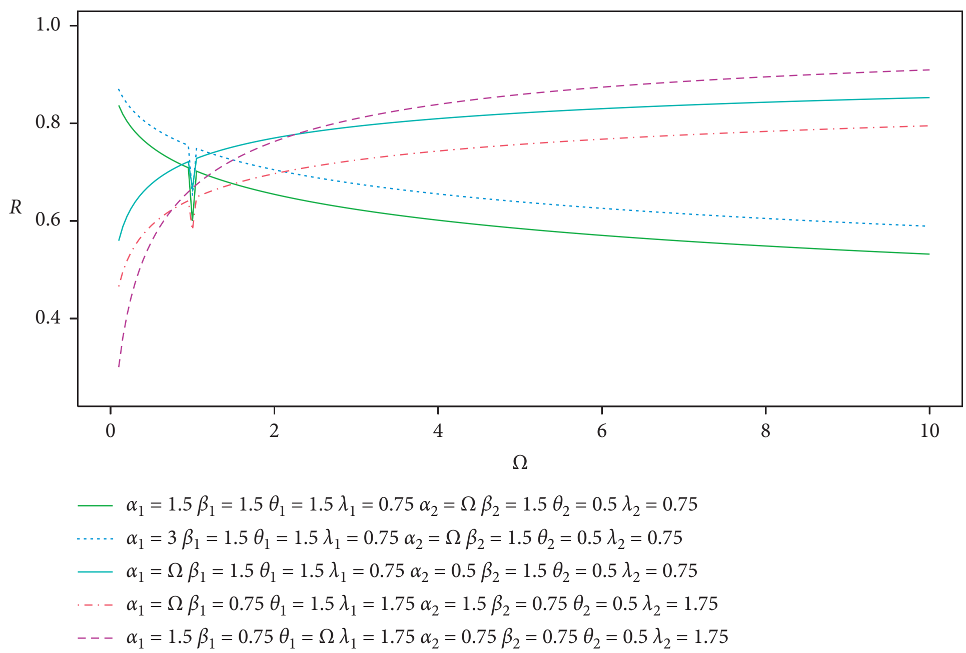


Figure 4: The stress-strength reliability measure for MAOPW.

We can obtain the first and third quartiles (q_1 and q_3) of MOAPW distribution when $u = 0.25$ and $u = 0.75$, respectively. From equations (15) and (16) first, second, and third quartiles of MOAPW distribution, we can obtain the Galton skewness (Sk), also known as Bowley's skewness, which is defined as

$$\mathrm{Sk} = \frac{q_3 - 2M + q_1}{q_3 - q_1}, \tag{17}$$

and the kurtosis (Ku) measure, which is given as

$$\mathrm{Ku} = \frac{x_{0.875} - x_{0.625} - x_{0.375} + x_{0.125}}{q_3 - q_1}. \tag{18}$$

4.2. The Mode of The MOAPW Distribution. From equation (8), the logarithm of MOAPW distribution is given by

$$\ln(f(x,\Psi)) \propto (\lambda - 1)\ln\left(\frac{x}{\beta}\right) - \left(\frac{x}{\beta}\right)^{\lambda} + \left(1 - e^{-(x/\beta)^{\lambda}}\right)\ln(\alpha) - 2\ln\left[\theta + \frac{(1-\theta)}{(\alpha - 1)}\left(\alpha^{\left(1 - e^{-(x/\beta)^{\lambda}}\right)} - 1\right)\right]. \tag{19}$$

By differentiating equation (19) with respect to x and equating to zero, we obtain

$$\frac{(\lambda - 1)}{x} - \frac{\lambda}{\beta}\left(\frac{x}{\beta}\right)^{\lambda - 1} + \frac{\lambda\ln(\alpha)}{\beta}\left(\frac{x}{\beta}\right)^{\lambda - 1} e^{-(x/\beta)^{\lambda}} - \frac{2\lambda\ln(\alpha)(1-\theta)}{\beta(\alpha - 1)} \frac{(x/\beta)^{\lambda - 1}\alpha^{\left(1 - e^{-(x/\beta)^{\lambda}}\right)} e^{-(x/\beta)^{\lambda}}}{\theta + ((1-\theta)/(\alpha - 1))\left(\alpha^{\left(1 - e^{-(x/\beta)^{\lambda}}\right)} - 1\right)} = 0. \tag{20}$$

The mode value of the MOAPW distribution can be obtained by solving numerically equation (20). Also, from Figure 4, we can note that the MOAPW distribution has one mode in most cases.

The first quartile, median, third quartile, skewness, kurtosis, and mode of MOAPW distribution with different values of the parameters are computed using the R program and displayed in Table 2. The results indicate that, for fixed β, θ, and λ, the first and third quartiles, median, and mode increase when α increases, while the skewness and kurtosis decrease when α increases. Also, for fixed α, β, and λ, but $\alpha < 1$, the first and third quartiles, median, and mode increase when θ increases, while the skewness and kurtosis decrease when θ increases. Likewise, for fixed β and λ, but $\alpha > 1$, the first and third quartiles and median increase with θ. It was also observed that the value of the model was

Table 2: The first and third quartile, median, skewness, kurtosis, and mode of MOAPW distribution with different values of the parameters.

α	β	θ	λ	$q1$	Median	$q3$	Sk	Kt	Mode
0.75	0.75	0.75	0.75	0.0856	0.2983	0.8316	0.4299	2.7266	0.1077
			3	0.4359	0.5956	0.7696	0.0432	2.3643	0.4617
		3	0.75	0.3999	1.0416	2.1093	0.2492	2.0719	0.4815
			3	0.6409	0.8142	0.9712	−0.0491	1.7949	0.6713
	3	0.75	0.75	0.3424	1.1930	3.3266	0.4299	3.1704	0.4308
			3	1.7438	2.3823	3.0785	0.0432	2.1755	1.8467
		3	0.75	1.5994	4.1663	8.4372	0.2492	1.6733	1.9259
			3	2.5635	3.2567	3.8850	−0.0491	1.2838	2.6853
3	0.75	0.75	0.75	0.1961	0.5840	1.3634	0.3354	2.6743	0.2410
			3	0.5363	0.7045	0.8709	−0.0056	1.7729	0.5647
		3	0.75	0.7468	1.6374	2.9146	0.1783	1.3558	0.8712
			3	0.7492	0.9117	1.0530	−0.0694	1.0403	0.7786
	3	0.75	0.75	0.7845	2.3359	5.4536	0.3354	2.6743	0.9642
			3	2.1453	2.8181	3.4835	−0.0056	1.7729	2.2588
		3	0.75	2.9871	6.5495	11.6583	0.1783	1.3558	3.4847
			3	2.9968	3.6466	4.2121	−0.0694	1.0403	3.1145

different in increase and decrease when α was greater than one and less than 1, while the skewness and kurtosis decrease when θ decreases. See Figure 1, which confirms the result of Table 2: the MOAPW has left-skewed, right-skewed, reversed-J, and symmetric shapes.

5. Parameter Estimation under Different Cases

In this section, parameter estimation for the MOAPW distribution using MLE, MPS, and Bayesian estimation methods in the presence of Type-I and Type-II censoring is discussed in detail.

5.1. MLE Based on Censored Samples. The general form for the likelihood function for Type-I and Type-II censoring is given as

$$L(\Psi) = \frac{n!}{(n-r)!}(1 - F(\wp; \Psi))^{n-r} \prod_{i=1}^{r} f(x_{i:n}; \Psi), \qquad (21)$$

where in Type-I censoring $\wp = T$ and in Type-II censoring $\wp = x_{r:n}$. For more information, see the work of Balakrishnan [21], El-Sherpieny et al. [22], Hassan and Abd-Allah [23], Abd El-Raheem et al. [24], Hafez et al. [25], and Hassan and Mohamed [26].

In Type-I censoring, we remove surviving units from a test at a prespecified time. The data consists of the observations $x_{1:n} < x_{2:n} < \cdots < \wp$ and the information that $(n-r)$ items survive beyond the time of termination T, where r is the number of the uncensored items. Assuming that we have n observation from MOAPW distribution which is found in life testing, the test is finished at a certain time T before the failure of all n observations. The number of failures r is random. In Type-II censoring, a life test is ended after a certain number of failures occur; here n and r are fixed and predetermined, but $\wp = T$ is random. The log-likelihood function of MOAPW distribution, depending on Type-I and Type-II censoring, is given by the following equation:

$$\begin{aligned} l(\Psi) = {} & r \ln\left(\frac{\theta \ln(\alpha)\lambda}{\beta(\alpha-1)}\right) + (\lambda - 1)\sum_{i=1}^{r} \ln\left(\frac{x_{i:n}}{\beta}\right) - \left(\frac{x_{i:n}}{\beta}\right)^{\lambda} + \ln(\alpha)\sum_{i=1}^{r}\left(1 - e^{-(x_{i:n}/\beta)^{\lambda}}\right) \\ & + 2\sum_{i=1}^{r} \ln\left[\theta + \frac{(1-\theta)}{(\alpha-1)}(\varphi(x_{i:n}, \alpha, \beta, \lambda) - 1)\right] + (n-r)\left[\ln(\theta) - \ln(\alpha - 1) + \ln\left(\alpha - \varphi\left(x_{\wp}, \alpha, \beta, \lambda\right)\right)\right. \\ & \left. - \ln\left(\theta + \frac{(1-\theta)}{(\alpha-1)}(\varphi(\wp, \alpha, \beta, \lambda) - 1)\right)\right]. \end{aligned} \qquad (22)$$

where $\varphi(x, \alpha, \beta, \lambda) = \alpha^{(1-e^{-(x/\beta)^{\lambda}})}$.

Equation (22) can be maximized directly by applying the R package by an optim function to solve the nonlinear likelihood equations obtained by differentiating equation (22) with respect to Ψ and equating to zero.

5.2. MPS under Censored Sample. The general form for the MPS function under Type-I censored and Type-II censored samples is given as

$$S(\Psi) = \frac{n!}{(n-r)!}(1 - F(\wp;\Psi))^{n-r} \prod_{i=1}^{r+1} (D_{i:n}(\Psi)), \tag{23}$$

where $D_{i:n}(\Psi) = \begin{cases} D_{1:n} = F(x_{1:n}, \Psi), \\ D_{i:n} = F(x_{i:n}, \Psi) - F(x_{(i-1):n}, \Psi),; i = \\ D_{(n+1):n} = 1 - F(x_{r:n}, \Psi), \end{cases}$ $2, \ldots, r$, where in Type-I censoring $\wp = T$ and in Type-II censoring $\wp = x_{r:n}$. For more information, see the work of Almetwally and Almongy [27, 28] and Alshenawy et al. [29, 30]. The natural logarithm of the product spacing function in the general form for the two different types of censored samples is given by

$$\begin{aligned} \ln G(\Psi) = (n-r)\left[\ln(\theta) - \ln(\alpha - 1) + \ln(\alpha - \varphi(\wp, \alpha, \beta, \lambda)) - \ln\left(\theta + \frac{(1-\theta)}{(\alpha - 1)}(\varphi(\wp, \alpha, \beta, \lambda) - 1)\right)\right] \\ + \ln(\varphi(x_{1:n}, \alpha, \beta, \lambda) - 1) - 2\ln(\alpha - 1) - \ln\left(\theta + \frac{(1-\theta)}{(\alpha - 1)}(\varphi(x_{1:n}, \alpha, \beta, \lambda) - 1)\right) \\ + \ln(\theta) + \ln(\alpha - \varphi(x_{r:n}, \alpha, \beta, \lambda)) - \ln\left(\theta + \frac{(1-\theta)}{(\alpha - 1)}(\varphi(x_{r:n}, \alpha, \beta, \lambda) - 1)\right) \\ + \sum_{i=2}^{r} \ln(F_{\text{MOAPW}}(x_{i:n}, \Psi) - F_{\text{MOAPW}}(x_{i-1:n}, \Psi)). \end{aligned} \tag{24}$$

The partial derivatives of MPS under censored samples with respect to the unknown parameters cannot be calculated directly. Hence, we utilize a numerical algorithm like the conjugate gradients method that can be used to count the MPS of Ψ.

5.3. Bayesian Estimation. We consider the Bayesian estimation of the unknown parameters $\Psi = (\alpha, \beta, \theta, \lambda)$ under Type-I censored and Type-II censored schemes. Bayesian estimation is considered under the assumption that the random variables Ψ have an independent and identical (*iid*) gamma prior, where $\alpha \sim \text{Gamma}(a_1, b_1), \beta \sim \text{Gamma}(a_2, b_2)$, $\theta \sim \text{Gamma}(a_3, b_3)$, and $\lambda \sim \text{Gamma}(a_4, b_4)$. The prior joint PDF of Ψ should be documented as

$$\begin{aligned} g(\Psi) &\propto \alpha^{a_1 - 1} e^{-(\alpha/b_1)} \beta^{a_2 - 1} e^{-(\beta/b_2)} \theta^{a_3 - 1} e^{-(\theta/b_3)} \lambda^{a_4 - 1} e^{-(\lambda/b_4)}; \\ g(\Psi) &\propto \prod_{j=1}^{4} \Psi_j^{a_j - 1} e^{-(\Psi_j/b_j)}, \quad a_j \text{ and } b_j > 0; j = 1, 2, \ldots, 4, \end{aligned} \tag{25}$$

where all the hyperparameters a_j and b_j, $j = 1, 2, \ldots, 3$, are known and nonnegative. According to Kundu and Howlader [31], the hyperparameters can be elected to fit the experimenter's prior belief in terms of the prior gamma distribution. By equating variance and mean of $\widehat{\Psi}_j^i$ with the mean and variance of the taken prior (Gamma priors), $j = 1, 2, \ldots, k$ and k is the number of samples available from the MOAPW distribution.

We have the following likelihood function:

$$\begin{aligned} L(\Psi) = \left(\frac{\theta\lambda\ln(\alpha)}{\beta(\alpha - 1)}\right)^r e^{-\sum_{i=1}^{r}(x_{i:n}/\beta)^{\lambda}} \prod_{i=1}^{r} \left(\frac{(x_i/\beta)^{\lambda - 1}\varphi(x_{i:n}, \alpha, \beta, \lambda)}{[\theta + ((1-\theta)/(\alpha - 1))(\varphi(x_{i:n}, \alpha, \beta, \lambda) - 1)]^2}\right) \\ \cdot \left(\frac{\theta[\alpha - \varphi(\wp, \alpha, \beta, \lambda)]}{(\alpha - 1)[\theta + ((1-\theta)/(\alpha - 1))(\varphi(\wp, \alpha, \beta, \lambda) - 1)]}\right)^{n-r}. \end{aligned} \tag{26}$$

In equation (25), the prior joint density is utilized in getting the joint posterior of MOAPW with Type-I censored and Type-II censored samples of MOAPW distribution with parameter Ψ as follows:

$$\pi(\Psi|x) = \mathfrak{M}\prod_{j=1}^{4}\left(\Psi_j^{a_j-1}e^{-\left(\Psi_j/b_j\right)}\right)\left(\frac{\theta\lambda\ln(\alpha)}{\beta(\alpha-1)}\right)^{r}e^{-\sum_{i=1}^{r}\left(x_{i:n}/\beta\right)^{\lambda}}\times\prod_{i=1}^{r}\left(\frac{\left(x_i/\beta\right)^{\lambda-1}\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)}{\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)-1\right)\right]^{2}}\right) \cdot\left(\frac{\theta\left[\alpha-\varphi(\wp,\alpha,\beta,\lambda)\right]}{(\alpha-1)\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi(\wp,\alpha,\beta,\lambda)-1\right)\right]}\right)^{n-r}, \tag{27}$$

where $\mathfrak{M}$ is the normalizing constant.

Markov Chain Monte Carlo (MCMC) is used in estimating the parameters. MCMC is specifically beneficial in Bayesian inference as a result of focusing on posterior distributions that are often hard to work with through analytic examination. MCMC allows the user to approximate the integrals for the posterior distributions that cannot be directly computed.

The conditional posterior densities of Ψ are as follows:

$$\begin{aligned}
\pi_1^*(\alpha|\beta,\theta,\lambda,x) &\propto \alpha^{a_1-1}e^{-(\alpha/b_1)}\left(\frac{\ln(\alpha)}{(\alpha-1)}\right)^{r}\prod_{i=1}^{r}\left(\frac{\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)}{\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)-1\right)\right]^{2}}\right)\\
&\quad\cdot\left(\frac{\left[\alpha-\varphi(\wp,\alpha,\beta,\lambda)\right]}{(\alpha-1)\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi(\wp,\alpha,\beta,\lambda)-1\right)\right]}\right)^{n-r},\\
\pi_2^*(\beta|\alpha,\theta,\lambda,x) &\propto \beta^{a_2-1}e^{-(\beta/b_2)}\left(\frac{1}{\beta}\right)^{r}e^{-\sum_{i=1}^{r}\left(x_{i:n}/\beta\right)^{\lambda}}\left(\frac{\left[\alpha-\varphi(\wp,\alpha,\beta,\lambda)\right]}{(\alpha-1)\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi(\wp,\alpha,\beta,\lambda)-1\right)\right]}\right)^{n-r}\\
&\quad\times\prod_{i=1}^{r}\left(\frac{\left(x_i/\beta\right)^{\lambda-1}\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)}{\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)-1\right)\right]^{2}}\right),\\
\pi_3^*(\theta|\alpha,\beta,\lambda,x) &\propto \theta^{a_3-1}e^{-(\theta/b_3)}(\theta)^{n}\prod_{i=1}^{r}\left(\frac{1}{\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)-1\right)\right]^{2}}\right)\\
&\quad\cdot\left(\frac{1}{\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi(\wp,\alpha,\beta,\lambda)-1\right)\right]}\right)^{n-r},\\
\pi_4^*(\lambda|\alpha,\beta,\theta,x) &\propto \pi(\Psi|x)\propto\prod_{i=1}^{r}\prod\left(\frac{\left(x_i/\beta\right)^{\lambda-1}\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)}{\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi\left(x_{i:n},\alpha,\beta,\lambda\right)-1\right)\right]^{2}}\right)\\
&\quad\cdot\left(\frac{\left[\alpha-\varphi(\wp,\alpha,\beta,\lambda)\right]}{\left[\theta+\left((1-\theta)/(\alpha-1)\right)\left(\varphi(\wp,\alpha,\beta,\lambda)-1\right)\right]}\right)^{n-r}\times\lambda^{a_4-1}e^{-(\lambda/b_4)}e_{i=1}^{-}\sum\nolimits_{r}\left(x_{i:n}/\beta\right)^{\lambda}
\end{aligned} \tag{28}$$

Therefore, to generate these parameters by this method, we used the Metropolis-Hastings algorithm introduced by Metropolis et al. [32] with normal proposal density function. For more details regarding the Metropolis-Hasting algorithm's implementation, we refer the readers to the works of Hafez et al. [25], Nassar et al. [33], Muhammed and. Almetwally [34], and Almetwally et al. [35].

6. Simulation Study

In this section, a Monte Carlo simulation is done to estimate MOAPW distribution parameters based on Type-I and Type-II censoring by using MLE, MPS, and Bayesian methods. Also, we used *R* packages for the following steps in simulation.

In simulation algorithm, Monte Carlo experiments were carried out under the following data generated from MOAPW distribution by using the quantile function in equation (15), where x is distributed as MOAPW distribution for different parameters $\Psi = (\alpha, \beta, \theta, \lambda, \delta)$ and the initial values of the parameters are as follows.

We have the three following cases: Case 1: $\alpha = 1.5; \beta = 1.5; \theta = 1.5; \lambda = 1.5$, Case 2: $\alpha = 3; \beta = 1.5;\ \theta = 0.75;\ \lambda = 1.5$, and Case 3: $\alpha = 0.75; \beta = 1.5; \theta = 3; \lambda = 1.5$, for different samples sizes $n = 50, 100$ and 200 and for different censored samples schemes, wherein in Type-II when $p = 0.7$ and 0.9 is ratio of sample size, $r = (35, 45), (70, 90)$, and $(140, 180)$ of sample size, while in Type-I, we used different times as 2 and 3.25. We can find the parameter estimation by using equations (22) and (24), optim function in *R* packages by using the Newton-Raphson algorithm, and 10,000 iterations.

We could define the best method as the scheme that minimizes bias and mean squared error (MSE), where $\text{MSE} = \text{Mean}\,(\Psi - \widehat{\Psi})^2$ and $\text{Bias} = \Psi - \widehat{\Psi}$, where $\widehat{\Psi}$ is the estimated value of Ψ.

By referring to the results in Tables 3–5 and Figure 5, we can conclude the following

(1) Bias and MSE decrease when n increases in all the estimates

(2) In Type-II censored sample, if the number of failures (r) increases, then the values of the Bias and MSE decrease for parameters of MOAPW distribution

(3) In Type-I censored sample, if the time (T) increases for censoring samples, then the values of the Bias and MSE decrease for the parameters of MOAPW distribution

(4) The Bayesian estimates have more relative efficiency than MLE and MPS for most parameters of MOAPW distribution

(5) In cases 1, 2, and 3, we note that MPS provides better estimation than MLE in most parameters in Type-I and Type-II according to the values of their MSE

7. Application to Real Data Sets

In this part, we applied two real data sets to illustrate the fitness of the MOAPW distribution in this section.

Firstly, we deal with the first data discussed by Nassar et al. [17], and these data refer to the fatigue times of 6061 T6, aluminium coupons. The data consist of 101 units with maximum stress per cycle 31,000 psi. Birnbaum and Saunders [36] used these data to illustrate the applications of their distribution. The data sets are as follows: 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 139, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, and 212. Bourguignon et al. [37] used these data to fit the Weibull-Burr XII (WBXII) distribution.

The MOAPW model has the highest p value and the lowest distance (D) of Kolmogorov-Smirnov (K-S) value when it is compared with all other models used here to fit the current data, which means that the new model fits the data better than the MOAPEx, WBXII, MOW, APW, and W models. The log-likelihood (LL) ratio test, Akaike information criterion, (AIC), correct Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) values for the five tested models are given for the first data set in Table 6, where k is the number of parameters in equation (29).

$$\begin{aligned}
\text{AIC} &= -2\mathscr{L}(\widehat{\Psi}|x) + 2k,\\
\text{CAIC} &= -2\mathscr{L}(\widehat{\Psi}|x) + 2k + 2\frac{k(k+1)}{n-k-1},\\
\text{HQIC} &= -2\mathscr{L}(\widehat{\Psi}|x) + 2\,k\log[\log(n)].
\end{aligned} \tag{29}$$

From Table 6, we find that the MOAPW distribution has the lowest AIC, CAIC, and HQIC values. Based on Figure 6, it is clear that the MOAPW model fits the first data.

Secondly, the second data set is obtained from Smith and Naylor [38]. The data include 63 observations of 1.5 cm glass fibers' strengths, measured at the National Physical Laboratory, England.

The MOAPW model has the highest p value and the lowest distance of K-S value when it is compared with the MOW, APW, WBXII, MOAPEx, and Weibull distributions used here to fit the current data, which means that the MOAPW fits the second data better than the MOAPEx, WBXII, MOW, APW, and W models. The LL, AIC, CAIC, and HQIC values for the five tested models are given for another data set. From Table 7, we find that the MOAPW distribution has the lowest LL, AIC, CAIC, and HQIC values. Based on Figure 7, it is clear that the MOAPW provides the best fitting among all competitive models according to the second data.

The relative histogram and fit of MOAPW distribution of both data sets are discussed in Figures 6 and 7, respectively. Also, the figures of the fitted MOAPW's PDF and empirical CDF of both data sets are displayed in Figures 6 and 7, respectively. Also, we present the Q-Q and P-P plot for the

Table 3: MLE, MPS, and Bayesian estimation methods under different censored sample in case 1.

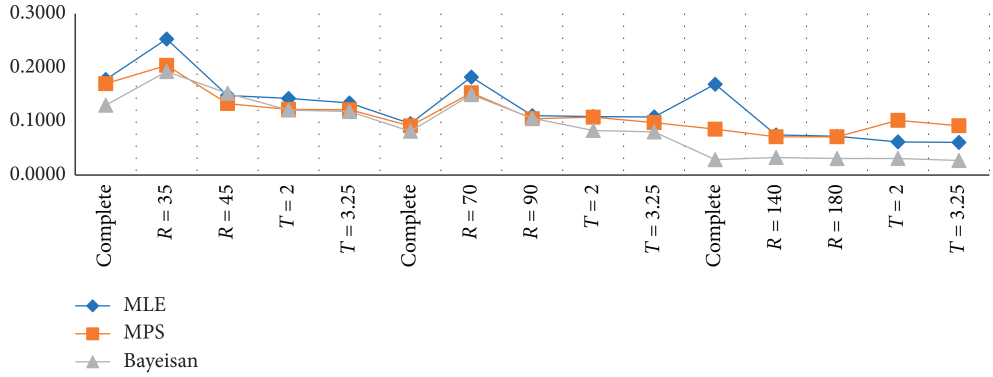
			$\alpha = 1.5; \beta = 1.5; \theta = 1.5; \lambda = 1.5$					
n	Scheme		MLE		MPS		Bayesian	
			Bias	MSE	Bias	MSE	Bias	MSE
50	Complete	$\hat{\alpha}$	−0.0360	0.1774	0.2175	0.1698	0.0419	0.1298
		$\hat{\beta}$	0.1846	0.1921	0.0603	0.1148	0.0577	0.0595
		$\hat{\theta}$	0.4359	0.8930	0.8054	0.7499	−0.0146	0.1780
		$\hat{\lambda}$	−0.1620	0.1278	−0.3004	0.1200	−0.0716	0.0520
	$r = 35$	$\hat{\alpha}$	−0.0232	0.2528	0.1915	0.2038	0.1838	0.1924
		$\hat{\beta}$	0.2013	0.2191	−0.0391	0.1241	0.2148	0.1802
		$\hat{\theta}$	−0.0207	0.9924	0.2372	0.8074	0.0421	0.3782
		$\hat{\lambda}$	0.2716	0.4938	0.0389	0.3475	0.1185	0.1309
	$r = 45$	$\hat{\alpha}$	0.0774	0.1479	0.1699	0.1328	0.0468	0.1520
		$\hat{\beta}$	0.1949	0.2041	0.2372	0.1956	0.2199	0.1142
		$\hat{\theta}$	0.0164	0.9007	0.0942	0.7554	−0.0403	0.2074
		$\hat{\lambda}$	−0.2474	0.2662	−0.2356	0.2361	−0.1024	0.1064
	$T = 2$	$\hat{\alpha}$	0.0823	0.1424	0.2635	0.1220	−0.0059	0.1208
		$\hat{\beta}$	0.5317	0.4697	0.2762	0.2208	0.2350	0.2028
		$\hat{\theta}$	0.6363	1.0312	0.7207	0.9805	−0.0635	0.1919
		$\hat{\lambda}$	−0.5251	0.3035	−0.6084	0.3020	−0.3654	0.1663
	$T = 3.25$	$\hat{\alpha}$	−0.0677	0.1339	0.1945	0.1214	0.0403	0.1181
		$\hat{\beta}$	0.2681	0.2086	0.0407	0.1143	0.1964	0.0879
		$\hat{\theta}$	0.3725	0.8488	0.3577	0.8394	0.0594	0.1908
		$\hat{\lambda}$	−0.1146	0.1113	−0.1661	0.1081	−0.0665	0.0387
100	Complete	$\hat{\alpha}$	0.0865	0.0955	0.1501	0.0913	0.0560	0.0813
		$\hat{\beta}$	0.1204	0.1225	−0.0354	0.0712	0.0587	0.0382
		$\hat{\theta}$	0.0092	0.7482	0.1817	0.5471	−0.0187	0.1522
		$\hat{\lambda}$	−0.1524	0.1055	0.0080	0.1120	0.0610	0.0518
	$r = 70$	$\hat{\alpha}$	0.0059	0.1821	0.2294	0.1528	−0.0328	0.1495
		$\hat{\beta}$	0.3972	0.2350	0.2505	0.1285	0.2346	0.0412
		$\hat{\theta}$	0.7158	0.8813	0.9267	0.8284	−0.0646	0.1760
		$\hat{\lambda}$	0.1761	0.3174	−0.5755	0.3049	−0.3818	0.1582
	$r = 90$	$\hat{\alpha}$	−0.0456	0.1104	0.1347	0.1046	0.0110	0.1055
		$\hat{\beta}$	0.2078	0.1125	0.0718	0.0864	0.2040	0.0391
		$\hat{\theta}$	0.5575	0.7737	0.4016	0.6121	0.0147	0.1609
		$\hat{\lambda}$	−0.1552	0.2859	−0.3288	0.1554	−0.1325	0.0936
	$T = 2$	$\hat{\alpha}$	0.1275	0.1087	0.2584	0.1076	−0.0005	0.0828
		$\hat{\beta}$	0.4158	0.2428	0.2697	0.1341	0.2336	0.1156
		$\hat{\theta}$	0.7382	0.9658	0.7934	0.9323	−0.0605	0.0953
		$\hat{\lambda}$	−0.5627	0.3295	−0.4616	0.3193	−0.4027	0.1732
	$T = 3.25$	$\hat{\alpha}$	−0.0463	0.1079	0.0914	0.0975	−0.0040	0.0801
		$\hat{\beta}$	0.1672	0.0880	0.0294	0.0526	0.1916	0.0635
		$\hat{\theta}$	0.5028	0.7441	0.4778	0.7156	0.0114	0.0920
		$\hat{\lambda}$	−0.1895	0.1109	−0.1293	0.1102	−0.0885	0.0274
200	Complete	$\hat{\alpha}$	0.0490	0.1688	0.1233	0.0853	0.0081	0.0287
		$\hat{\beta}$	0.0649	0.0667	−0.0389	0.0352	0.0119	0.0103
		$\hat{\theta}$	0.0367	0.5419	0.1535	0.3122	−0.0096	0.0226
		$\hat{\lambda}$	0.0957	0.1172	−0.0270	0.0887	0.0169	0.0097
	$r = 140$	$\hat{\alpha}$	0.1470	0.0745	0.1233	0.0711	0.0338	0.0327
		$\hat{\beta}$	0.3325	0.1433	0.2516	0.0927	0.2466	0.0742
		$\hat{\theta}$	0.8358	0.9530	0.9500	0.9412	−0.0677	0.0329
		$\hat{\lambda}$	−0.5513	0.3128	−0.5846	0.3050	−0.3789	0.1482
	$r = 180$	$\hat{\alpha}$	−0.0160	0.0720	0.0511	0.0710	0.0221	0.0307
		$\hat{\beta}$	0.1567	0.0523	0.0748	0.0301	0.1671	0.0404
		$\hat{\theta}$	0.6155	0.6336	0.7835	0.6388	0.0042	0.0243
		$\hat{\lambda}$	−0.2766	0.0997	−0.3367	0.0914	−0.1464	0.0276
	$T = 2$	$\hat{\alpha}$	0.1457	0.0616	0.2310	0.1017	−0.0200	0.0308
		$\hat{\beta}$	0.3627	0.1626	0.2764	0.1048	0.2124	0.0610
		$\hat{\theta}$	0.8223	0.8876	0.7968	0.8197	−0.0690	0.0306
		$\hat{\lambda}$	−0.5820	0.3448	−0.6162	0.3860	−0.3920	0.1579
	$T = 3.25$	$\hat{\alpha}$	−0.0594	0.0607	0.0715	0.0915	0.0047	0.0271
		$\hat{\beta}$	0.1333	0.0450	0.0528	0.0270	0.1430	0.0292
		$\hat{\theta}$	0.5418	0.5209	0.6965	0.7178	0.0034	0.0289
		$\hat{\lambda}$	−0.2187	0.0730	−0.2820	0.1038	−0.1017	0.0167

Table 4: MLE, MPS, and Bayesian estimation methods under different censored sample in case 2.

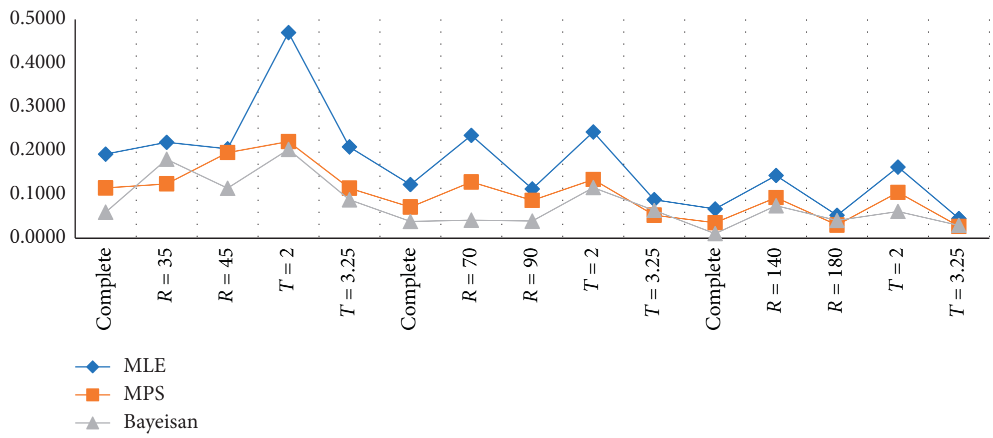
			$\alpha = 3; \beta = 1.5; \theta = 0.75; \lambda = 1.5$					
n	Scheme		MLE		MPS		Bayesian	
			Bias	MSE	Bias	MSE	Bias	MSE
50	Complete	$\hat{\alpha}$	−0.2255	0.3926	0.0956	0.0610	0.0431	0.0575
		$\hat{\beta}$	0.1382	0.2527	−0.0333	0.1338	0.0342	0.0560
		$\hat{\theta}$	0.1338	0.7024	0.3240	0.7186	0.1909	0.2318
		$\hat{\lambda}$	0.2241	0.4653	0.0858	0.4641	0.0645	0.1155
	$r = 35$	$\hat{\alpha}$	−0.1867	0.1341	−0.0448	0.0759	−0.0185	0.0698
		$\hat{\beta}$	0.3993	0.3219	0.1584	0.1540	0.2654	0.1309
		$\hat{\theta}$	0.8275	1.2105	1.1366	1.9224	−0.0753	0.0707
		$\hat{\lambda}$	−0.6171	0.4161	−0.7008	0.5274	−0.3388	0.1476
	$r = 45$	$\hat{\alpha}$	−0.1125	0.1337	−0.1038	0.1096	0.0056	0.0955
		$\hat{\beta}$	0.2095	0.1784	−0.0133	0.1068	0.1681	0.0771
		$\hat{\theta}$	0.6264	0.9327	0.9724	1.7041	0.0311	0.0603
		$\hat{\lambda}$	−0.2997	0.1919	−0.4358	0.2977	−0.1381	0.0502
	$T = 2$	$\hat{\alpha}$	−0.2269	0.1661	−0.0747	0.0973	−0.0228	0.0924
		$\hat{\beta}$	0.3579	0.2776	0.1249	0.1321	0.1135	0.1203
		$\hat{\theta}$	0.8357	1.2376	0.7502	1.1015	0.0282	0.1087
		$\hat{\lambda}$	−0.5831	0.3713	−0.6705	0.3482	−0.3044	0.1469
	$T = 3.25$	$\hat{\alpha}$	−0.1525	0.1344	−0.1504	0.1275	−0.0203	0.0901
		$\hat{\beta}$	0.1734	0.1616	−0.0430	0.1138	0.1687	0.0840
		$\hat{\theta}$	0.5107	0.8129	0.6842	0.8047	0.0947	0.1163
		$\hat{\lambda}$	−0.1792	0.1547	−0.3275	0.1452	−0.0550	0.0665
100	Complete	$\hat{\alpha}$	−0.1928	0.1134	0.0822	0.0421	−0.0337	0.0288
		$\hat{\beta}$	0.0434	0.1423	−0.0441	0.0910	−0.0037	0.0339
		$\hat{\theta}$	0.1034	0.6417	0.2664	0.5098	0.1368	0.1421
		$\hat{\lambda}$	0.1357	0.1635	0.0247	0.1531	0.0097	0.0741
	$r = 70$	$\hat{\alpha}$	−0.4302	0.3874	−0.2322	0.2279	−0.0184	0.0356
		$\hat{\beta}$	0.3004	0.1625	0.1560	0.0846	0.1611	0.0448
		$\hat{\theta}$	0.9529	1.2599	0.9418	1.1759	−0.1429	0.0417
		$\hat{\lambda}$	−0.6675	0.4623	−0.7230	0.4539	−0.3028	0.1081
	$r = 90$	$\hat{\alpha}$	−0.2125	0.1382	−0.0944	0.0630	−0.0125	0.0314
		$\hat{\beta}$	0.1335	0.0840	−0.0032	0.0563	0.1096	0.0254
		$\hat{\theta}$	0.6960	0.8407	0.5936	0.8299	−0.0251	0.0224
		$\hat{\lambda}$	−0.3760	0.1898	−0.4680	0.1727	−0.1302	0.0310
	$T = 2$	$\hat{\alpha}$	−0.4615	0.4541	−0.2527	0.3012	−0.0322	0.1221
		$\hat{\beta}$	0.2498	0.1223	0.1119	0.0624	0.1013	0.0514
		$\hat{\theta}$	0.6944	0.9279	0.6167	0.8798	0.0018	0.0534
		$\hat{\lambda}$	−0.6265	0.4081	−0.6853	0.4850	−0.3456	0.1397
	$T = 3.25$	$\hat{\alpha}$	−0.2270	0.1667	−0.1015	0.0793	−0.0201	0.0711
		$\hat{\beta}$	0.1025	0.0807	−0.0390	0.0661	0.1475	0.0487
		$\hat{\theta}$	0.5257	0.6447	0.5680	0.6140	0.0365	0.0593
		$\hat{\lambda}$	−0.2376	0.1293	−0.3516	0.1204	−0.0717	0.0336
200	Complete	$\hat{\alpha}$	−0.0825	0.1022	0.0640	0.0229	0.0146	0.0130
		$\hat{\beta}$	−0.0710	0.1285	−0.0251	0.0529	0.0038	0.0097
		$\hat{\theta}$	−0.0047	0.5160	0.1643	0.2726	0.0194	0.0139
		$\hat{\lambda}$	0.1043	0.1289	0.0064	0.1416	0.0082	0.0138
	$r = 140$	$\hat{\alpha}$	−0.3564	0.2818	−0.2216	0.1706	0.0066	0.0220
		$\hat{\beta}$	0.2494	0.0912	0.1718	0.0558	0.2151	0.0591
		$\hat{\theta}$	0.7983	0.8146	1.1072	0.7434	−0.1085	0.3310
		$\hat{\lambda}$	−0.6921	0.4856	−0.7224	0.4528	−0.3607	0.1388
	$r = 180$	$\hat{\alpha}$	−0.1564	0.0633	−0.0939	0.0415	−0.0052	0.0216
		$\hat{\beta}$	0.0852	0.3470	0.0061	0.0239	0.1581	0.0353
		$\hat{\theta}$	0.7078	0.6909	0.6846	0.6901	−0.0104	0.0166
		$\hat{\lambda}$	−0.4136	0.1943	−0.4694	0.1824	−0.1541	0.0328
	$T = 2$	$\hat{\alpha}$	−0.5039	0.4500	−0.3264	0.2498	−0.0108	0.0337
		$\hat{\beta}$	0.2073	0.0675	0.1305	0.0394	0.2213	0.0620
		$\hat{\theta}$	0.9640	0.7134	0.8097	0.6942	−0.0736	0.0237
		$\hat{\lambda}$	−0.6432	0.4205	−0.6773	0.4656	−0.3202	0.1112
	$T = 3.25$	$\hat{\alpha}$	−0.1813	0.0707	−0.1092	0.0378	0.0212	0.0319
		$\hat{\beta}$	0.0433	0.0606	−0.0273	0.0338	0.1245	0.0267
		$\hat{\theta}$	0.5239	0.5395	0.6971	0.6361	0.0190	0.0201
		$\hat{\lambda}$	−0.2700	0.1187	−0.3464	0.1615	−0.0898	0.0189

Table 5: MLE, MPS, and Bayesian estimation methods under different censored sample in case 3.

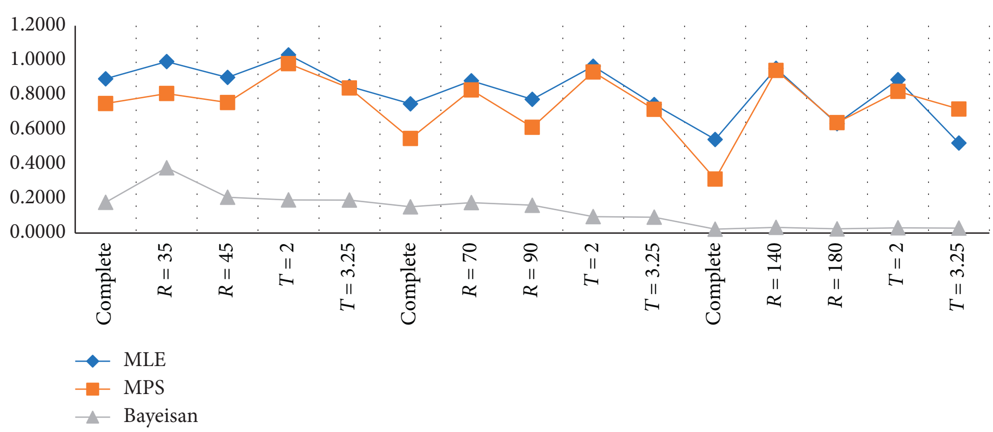
n	Scheme		$\alpha = 0.75; \beta = 1.5; \theta = 3; \lambda = 1.5$					
			MLE		MPS		Bayesian	
			Bias	MSE	Bias	MSE	Bias	MSE
50	Complete	$\hat{\alpha}$	0.0512	0.2843	0.1204	0.2456	0.0378	0.2365
		$\hat{\beta}$	0.1639	0.1279	−0.0561	0.0715	0.0534	0.0574
		$\hat{\theta}$	−0.1635	0.3660	0.2517	0.2201	−0.2279	0.2071
		$\hat{\lambda}$	0.2591	0.0732	0.0538	0.0692	0.0381	0.0598
	$r = 35$	$\hat{\alpha}$	0.3713	0.3574	0.5869	0.3613	0.2138	0.2400
		$\hat{\beta}$	0.5082	0.4102	0.2668	0.1952	0.4358	0.3002
		$\hat{\theta}$	0.1804	0.4252	0.1454	0.3620	−0.1546	0.1603
		$\hat{\lambda}$	−0.3757	0.1784	−0.3472	0.1611	−0.3341	0.1529
	$r = 45$	$\hat{\alpha}$	0.2722	0.3011	0.4501	0.2565	0.1397	0.2395
		$\hat{\beta}$	0.3083	0.2061	0.0930	0.0967	0.2408	0.0914
		$\hat{\theta}$	−0.0439	0.4594	0.1305	0.3059	−0.1547	0.2572
		$\hat{\lambda}$	−0.0860	0.0889	−0.2219	0.0913	−0.1003	0.0703
	$T = 2$	$\hat{\alpha}$	0.3552	0.3262	0.3548	0.3155	0.0625	0.1100
		$\hat{\beta}$	0.6355	0.5964	0.3683	0.2909	0.2894	0.1533
		$\hat{\theta}$	−0.0485	0.6034	0.3247	0.5875	−0.0454	0.1793
		$\hat{\lambda}$	−0.5173	0.2879	−0.5965	0.2379	−0.4294	0.2020
	$T = 3.25$	$\hat{\alpha}$	0.2787	0.2885	0.2516	0.2560	0.1192	0.1023
		$\hat{\beta}$	0.3194	0.2132	0.0963	0.0934	0.1931	0.0879
		$\hat{\theta}$	0.1837	0.3161	0.3996	0.3148	−0.1038	0.1750
		$\hat{\lambda}$	−0.1016	0.0767	−0.2429	0.0712	−0.1048	0.0356
100	Complete	$\hat{\alpha}$	0.0594	0.2035	0.1862	0.2033	0.0492	0.1692
		$\hat{\beta}$	0.0921	0.0693	−0.0531	0.0644	0.0282	0.0288
		$\hat{\theta}$	−0.0638	0.3419	0.2016	0.1389	−0.1159	0.1290
		$\hat{\lambda}$	0.1635	0.0621	0.0013	0.0615	0.0404	0.0442
	$r = 70$	$\hat{\alpha}$	0.3751	0.2465	0.5134	0.3871	0.2324	0.1827
		$\hat{\beta}$	0.4402	0.2618	0.3017	0.1504	0.4222	0.2422
		$\hat{\theta}$	0.0521	0.4199	0.2932	0.3902	−0.0520	0.3138
		$\hat{\lambda}$	−0.4086	0.1837	−0.4655	0.2331	−0.3407	0.1334
	$r = 90$	$\hat{\alpha}$	0.2969	0.2263	0.4724	0.4098	0.1215	0.1786
		$\hat{\beta}$	0.2441	0.1186	0.1102	0.0616	0.2560	0.1052
		$\hat{\theta}$	0.1918	0.2093	0.3412	0.2986	−0.1163	0.3028
		$\hat{\lambda}$	−0.1507	0.0651	−0.2437	0.0993	−0.0881	0.0347
	$T = 2$	$\hat{\alpha}$	0.4751	0.3949	0.6303	0.2592	−0.0459	0.0566
		$\hat{\beta}$	0.5061	0.3364	0.3499	0.1906	0.2457	0.0967
		$\hat{\theta}$	0.3453	0.3651	0.4948	0.4878	−0.0427	0.0647
		$\hat{\lambda}$	−0.5575	0.3221	−0.6094	0.3833	−0.4248	0.1924
	$T = 3.25$	$\hat{\alpha}$	0.3282	0.2529	0.5023	0.2152	0.0439	0.0507
		$\hat{\beta}$	0.2313	0.1043	0.0972	0.0517	0.1664	0.0518
		$\hat{\theta}$	0.0592	0.3084	0.3284	0.4415	−0.0140	0.0564
		$\hat{\lambda}$	−0.1651	0.0638	−0.2590	0.1024	−0.1195	0.0264
200	Complete	$\hat{\alpha}$	0.0314	0.1923	0.1407	0.1903	0.0371	0.0250
		$\hat{\beta}$	0.0640	0.0410	−0.0386	0.0256	−0.0007	0.0093
		$\hat{\theta}$	−0.0508	0.2378	0.1403	0.0736	−0.0162	0.0349
		$\hat{\lambda}$	0.1181	0.0614	−0.0148	0.0583	0.0026	0.0078
	$r = 140$	$\hat{\alpha}$	0.4520	0.2803	0.4553	0.2397	−0.0030	0.0261
		$\hat{\beta}$	0.3813	0.1767	0.2992	0.1192	0.2532	0.0762
		$\hat{\theta}$	0.3042	0.2124	0.3421	0.2031	−0.0261	0.0316
		$\hat{\lambda}$	−0.4417	0.2039	−0.4785	0.2014	−0.3435	0.1225
	$r = 180$	$\hat{\alpha}$	0.3164	0.1998	0.4313	0.2083	0.0305	0.0253
		$\hat{\beta}$	0.1991	0.0640	0.1210	0.0357	0.1660	0.0402
		$\hat{\theta}$	0.1133	0.1778	0.2635	0.1720	−0.0069	0.0289
		$\hat{\lambda}$	−0.1866	0.0954	−0.2442	0.0763	−0.1121	0.0198
	$T = 2$	$\hat{\alpha}$	0.4485	0.2599	0.5327	0.34	−0.0609	0.0304
		$\hat{\beta}$	0.4784	0.2676	0.3902	0.1873	0.2341	0.0698
		$\hat{\theta}$	0.1381	0.1634	0.274	0.2031	−0.059	0.0341
		$\hat{\lambda}$	−0.5603	0.3193	−0.5903	0.3537	−0.4174	0.1779
	$T = 3.25$	$\hat{\alpha}$	0.3228	0.1761	0.4383	0.279	0.0314	0.0265
		$\hat{\beta}$	0.1961	0.0603	0.1206	0.0339	0.1722	0.0407
		$\hat{\theta}$	0.2757	0.151	0.3633	0.1966	0.0054	0.0256
		$\hat{\lambda}$	−0.1892	0.0511	−0.2438	0.0743	−0.1218	0.0203



(a)

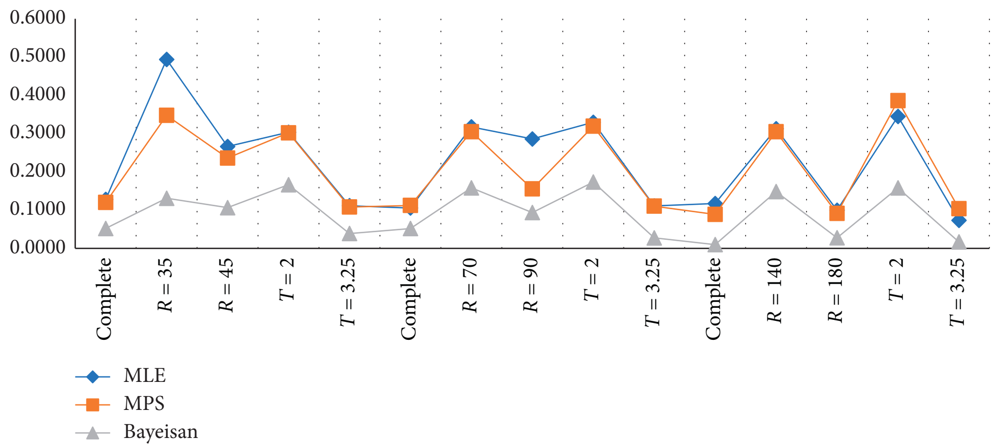


(b)



(c)

Figure 5: Continued.



(d)

FIGURE 5: MSE for estimation methods under different schemes for MAOPW: case 1. (a) MSE (α). (b) MSE (β). (c) MSE (θ). (d) MSE (λ).

TABLE 6: MLE, standard error (St. E), KS distance, p values, and different criteria for the first data set.

	MOAPW	MOAPEx	WBXII	MOW	APW	Weibull
α	187.2428 (30.4985)	225.635 (202.8139)	100.24 (191.960)	—	257.1313 (252.805)	—
β	1.1396 (0.091)	0.06031 (0.00298)	0.6383 (0.3306)	2.180439 (0.3621)	3.3823 (0.3482)	6.0859 (0.4238)
θ	21869.64 (1228.397)	503.0672 (191.592)	151.42 (12.817)	97.3626 (77.8902)	—	—
λ	18.7292 (2.8363)	—	0.0024 (0.0067)	66.57344 (12.7838)	107.656 (5.2713)	143.2649 (2.4816)
c	—	—	13.23 (5.6938)	—	—	—
D	0.0535	0.10482	0.05642	0.9948	0.06559	0.1006
p value	0.9342	0.2171	0.9053	0.000001	0.7776	0.2584
LL	455.6693	461.4193	455.1338	Not fitting	456.861	462.2396
AIC	919.3385	928.8386	920.268		919.729	928.4791
CAIC	919.7552	929.086	920.92		919.9693	928.6016
HQIC	922.5732	932.014	925.561		922.8979	930.5965

TABLE 7: MLE, St. E, K-S distance, p values, and different criteria for the second data set.

	MOAPW	MOAPEx	WBXII	MOW	APW	Weibull
α	1.0045 (0.5185)	251.6759 (426.8926)	0.0159 (0.01526)	—	10.9377 (12.6298)	—
β	3.22067 (0.9337)	5.2737 (0.4143)	1.87171 (1.3751)	3.222 (0.9333)	4.5126 (0.7583)	5.8099 (0.5784)
θ	16.98863 (27.502)	559.823 (356.482)	1.14334 (0.9228)	16.9565 (20.7189)	—	—
λ	1.119755 (0.2407)	—	2.04906 (1.3997)	1.1203 (0.2404)	1.4394 (0.0923)	1.626 (0.0368)
c	—	—	2.05501 1.652	—	—	—
D	0.0996	0.1552	0.13917	0.9948	0.1233	0.1537
p value	0.5589	0.0961	0.1742	0.000001	0.2939	0.1018
LL	11.5193	16.1572	13.9978	Not fitting	13.051	14.854
AIC	31.0385	38.3144	37.9956		32.1021	33.7081
CAIC	31.7282	38.7212	39.0482		32.5088	33.9081
HQIC	34.4102	40.8431	42.2101		34.6308	35.39391

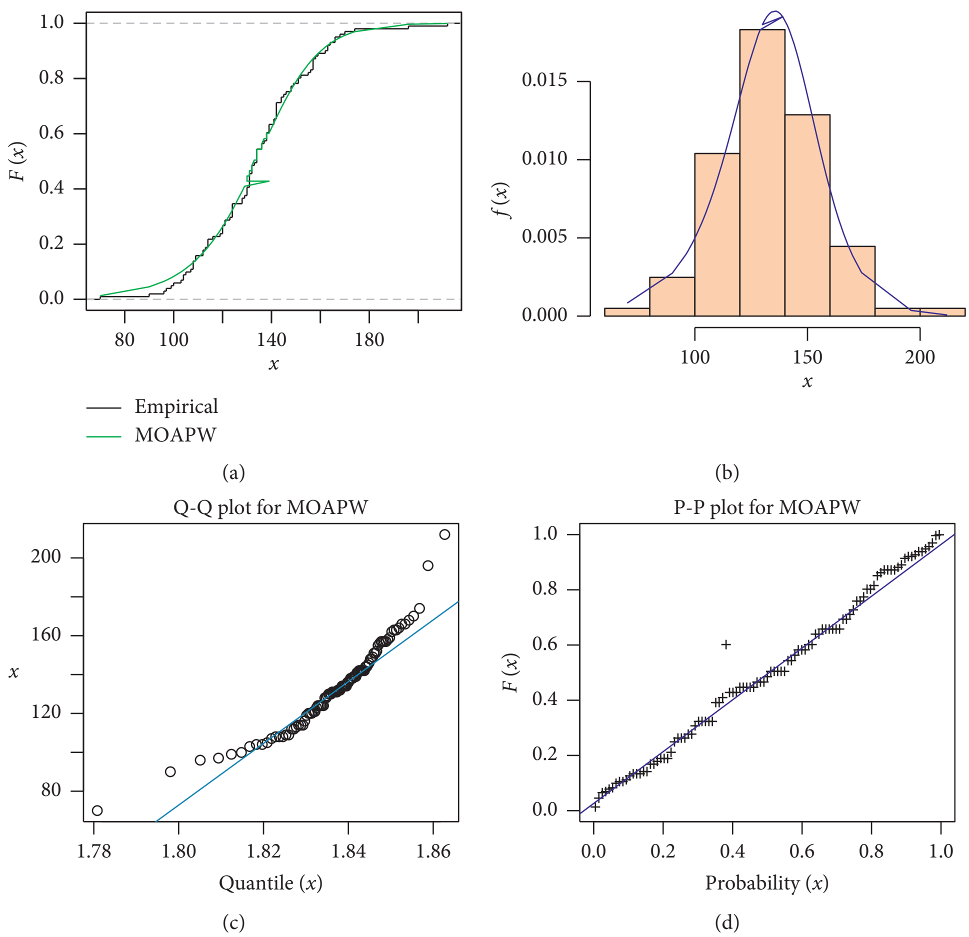


Figure 6: Cumulative function, empirical CDF, PDF, Q-Q, and P-P plots for the MOAPW distribution, for the first data set.

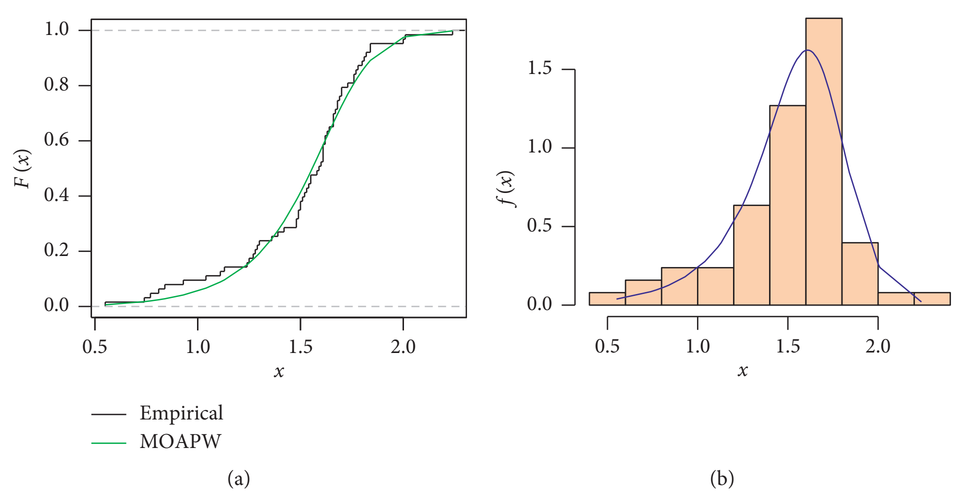


Figure 7: Continued.

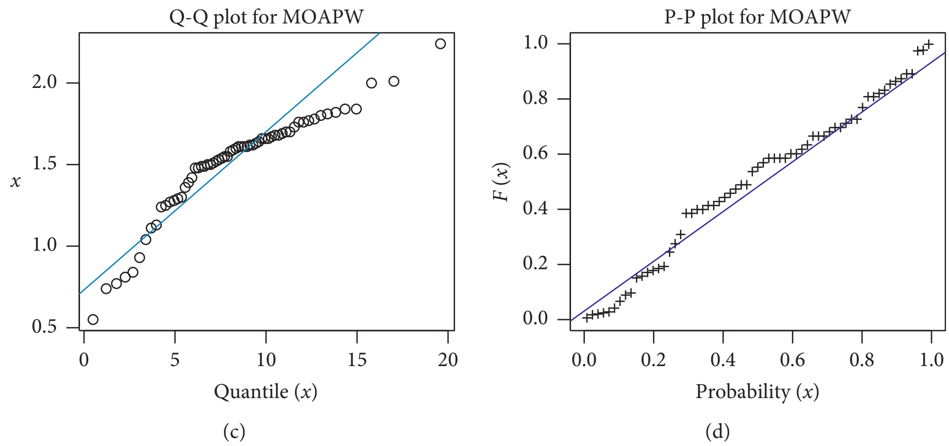


FIGURE 7: Cumulative function, empirical CDF, PDF, Q-Q, and P-P plots for the MOAPW distribution for the second data set.

first and second data, respectively, which allows us to compare the empirical distance of the data with the MOAPW.

8. Conclusion

We propose a new four-parameter model in this paper, called the Marshall–Olkin alpha power Weibull (MOAPW) distribution, which is an extension of Weibull distribution. The distribution of MOAPW is motivated by the wide use in the Weibull model's life testing and provides more flexibility to analyze lifetime data. Some statistical properties of the MOAPW distribution have been obtained, such as survival, hazard, reversed hazard, stress-strength reliability measure, quantile, median, and mode. The MOAPW distribution parameter estimation is derived by MLE, MPS, and Bayesian estimation methods. The estimation methods are used to estimate the MOAPW distribution parameters based on Type-I and Type-II censoring samples, and the simulation result is used to test the model's output. The Bayesian estimates have more relative efficiency than MLE and MPS for most of the parameters of MOAPW distribution. The two real-life types of data indicate that the MOAPW distribution proposed consistently provides better fit than the MOW, APW, WBXII, MOAPEx, and Weibull distributions.

Data Availability

All data are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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