Hindawi
Complexity
Volume 2021, Article ID 5543187, 15 pages
https://doi.org/10.1155/2021/5543187

*Research Article*

# Statistical Analysis of Joint Type-I Generalized Hybrid Censoring Data from Burr XII Lifetime Distributions

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Received 22 January 2021; Accepted 7 May 2021; Published 21 May 2021

Academic Editor: Ahmed Mostafa Khalil



The quality of the products coming from different lines of production requires some tests called comparative life tests. For lines having the same facility, the lifetime of the product is distributed by Burr XII, the lifetime distribution, and units are tested under type-I generalized hybrid censoring scheme. The observed censoring data are used under maximum likelihood and the Bayes method to estimate the model parameters. The theoretical results are discussed and assessed through data analysis and Monte Carlo simulation study. Finally, we reported some brief comments obtained from numerical computation.

## 1. Introduction

Statistical inference for the life products needs to put some units of product under test to get more information about the life products. Then, we design the life experiments to obtain the required data. Under consideration time and cost, data obtained may be complete or censored. The concept of complete data is used when the failure time of all units under the test is obtained. So far, the concept of censoring data is used when some but not all failure time of units are obtained. Censoring is available in different forms, the oldest ones are called type-I censoring scheme as well as type-II censoring scheme. When we need to run the experiments to prefixed time and the number of units fail is random, the type-I censoring scheme is a suitable scheme. But, when running the experiment to obtain a prefixed number of failure and the total test time is random, type-II censoring scheme is applied. The experimenter in some cases needs to run the experiment under joint cases of type-I and type-II censoring schemes, statistically known with hybrid censoring scheme (HCS).

Let the test time is denoted by $\tau^*$ and number of failure units needed to statistical inference is denoted by $m$, the experiment is removed in hybrid censoring scheme at the only one time of $(\tau^*, T_m)$. The HCS is combined with type-I and type-II censoring schemes to define type-I and type-II HCS. In type-I HCS, the experiment is removed from the test at the min $(\tau^*, T_m)$ [1, 2], but in type-II HCS, the experiment is removed from the test at the max $(\tau^*, T_m)$ [3]. All of these censoring schemes do not allow terminating units from the test other than the final point; then, it is generalized in progressive censoring scheme which helps us to terminate units at any stage of the experiment, and the key reference of the progressive censoring scheme is given in the study by Balakrishnan and Aggarwala [4]. The two types of censoring schemes are type-I and type-II or the hybrid case; type-I and type-II HCSs have the properties, smaller number of failure may be zero in type-I or total time of the test has a large time in type-II [5]. To overcome this problem [6], the two types of censoring schemes are generalized in generalized hybrid censoring scheme (GHCS) which is described as follows.

Type-I GHCS, for $n$ tested units, suppose prior integers $s$ and $m$ that satisfy $1 \leq s < m \leq n$ and prior ideal test time $\tau^* \in (0, \infty)$. The three cases are considered. If $T_s < \tau^*$, the test is terminated at min $(\tau^*, T_m)$, and in other cases, if $\tau^* < T_s < T_m$, the test is terminated at $T_s$, but if $t_{m;n} < \tau^*$, the test is terminated at $T_m$. The data in type-I GHCS satisfy the minimum number $s$ needing for statistical inference, and the data are summarized as follows:

(1) $\underline{t} = (t_{1;n} < t_{1;n} < \ldots < t_{s;n})$, if $\tau^* < T_s$
(2) $\underline{t} = (t_{1;n} < t_{1;n} < \ldots < t_{k;n} < \ldots < t_{r;n})$, if $\tau^* > T_s$ and $t_{m;n} > \tau^*$
(3) $\underline{t} = (t_{1;n} < t_{1;n} < \ldots < t_{m;n})$, if $t_{m;n} < \tau^*$

The scheme of the type-I GHCS can be formulated with the schematic diagram described by Figure 1.

Type-II GHCS, for $n$ tested units, suppose prior times $\tau_1^*$ and $\tau_2^* \in (0, \infty)$, such that $\tau_1^* < \tau_2^*$ and integer $m$ satisfies that $1 \leq m \leq n$. The three cases are considered. If $T_m < \tau_1^*$, the test is terminated at $\tau_1^*$, and in second case, if $\tau_1^* < T_m < \tau_2^*$, the test is terminated at $T_m$. If $\tau_2^* < T_m$, the test is terminated at $\tau_2^*$. The data in type-II GHCS satisfy the maximum time $\tau_2^*$, and the data are summarized as follows:

(1) $\underline{t} = (t_{1;n} < t_{1;n} < \ldots < t_{r;n})$, if $T_s < \tau_1^*$ or $\tau_2^* < T_m$
(2) $\underline{t} = (t_{1;n} < t_{1;n} < \ldots < t_{m;n})$, if $\tau_1^* < T_m < \tau_2^*$

The scheme of the type-II GHCS can be formulated with the schematic diagram described by Figure 2.

For manufactured products coming from different lines of production, the problem of determining the relative merits of life products has considerable attention through last view years. Practices, suppose two lines of production are denoted by $\Gamma_1$ and $\Gamma_2$ in competing duration and let two independent samples with sizes $N_1$ and $N_2$, respectively. The joint sample of $N_1$ and $N_2$ is put under life testing. This experiment is restricted under consideration of time and cost to terminate after fixed time or number of failure. The data obtained from type of censoring are called joint samples discussed early in [7, 8]. The exact likelihood inference with bootstrap technique under joint sample is presented in [9]. For progressive joint sample, refer studies by Rasouli A and Balakrishnan [10, 11] and recently by B. N. Al-Matrafi and G. A. Abd-Elmougod[12]. Also, for the accelerate model of Rayleigh distribution, refer studies by Faten A. Momenkhan and Abd-Elmougod [13, 14].

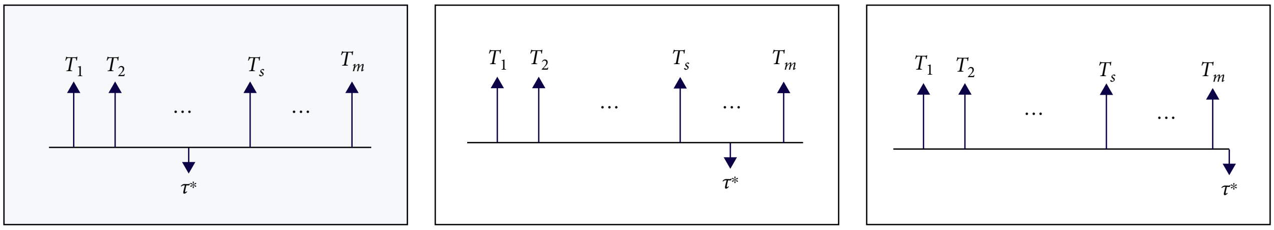
Type-I GHCS can save time and minimum number needing in statistical inference. So this study aims at development of statistical inference for life products in competing duration under considering type-I GHCS with jointly censoring scheme. Therefore, first, the model formulation under lifetime Burr XII distribution is under jointly type-I GHCS scheme. So far, parameters estimation of Burr XII distributions is carried out when jointly type-I GHCS samples are available. The maximum likelihood and Bayes methods are applied for the parameters estimation. The developed theoretical method assessed through the simulation study as well as illustration is reported with data analysis.

The study is planned as follows: the concept and model formulation are reported in Section 2. Estimation with maximum likelihood as point and interval estimators is discussed in Section 3. Bayesian approach for point and credible interval estimators with the help of the MCMC method is presented in Section 4. Data analysis is exposed in Section 5. The numerical computation is discussed through a simulation study in Section 6. Finally, some brief comments are reported in Section 7.

## 2. Model

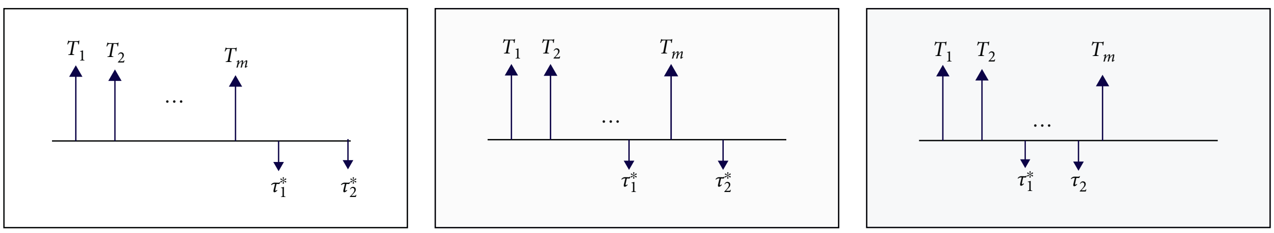
Consider that the product comes from two different lines of production $\Gamma_1$ and $\Gamma_2$ that have the same facility. Suppose, independent two samples of sizes $N_1$ and $N_2$ selected from $\Gamma_1$ and $\Gamma_2$ have independent and identical distributed (i.i.d) lifetimes $W_1, W_2, \ldots, W_{N_1}$ and $Z_1, Z_2, \ldots, Z_{N_2}$, respectively. The independent samples distributed with populations have $f_j(.)$ probability density functions (PDFs) and $F_j(.)$ cumulative distribution functions (CDFs) for $j = 1, 2$. The lifetime experiment begins with prior integers and ideal test time given by $(s, m, \tau^*)$. Through the experiment, the unit failure time, and its type, means from $\Gamma_1$ and $\Gamma_2$ are recorded. Then, the experiment is continual until $r^{\text{th}}$ failure is observed; if $\tau^* < T_s$, then $r = s$, but if $\tau^* > T_m$, then $r = m$, and in other cases, $s < r < m$. The vector of ordered sample $T = \{(T_2, \delta_1), (T_2, \delta_1), \ldots, (T_r, \delta_r)\}$ from the sample $\{W_1, W_2, \ldots, W_{N_{1r}}, Z_1, Z_2, \ldots, Z_{N2r}\}$ with $r = N_{1r} + N_{2r}$, and the integer $r$ is taken with $s$ and $m$, or integer that satisfies $s < r < m$ is called joint type-I GHSC. In joint type-I GHSC, $\delta_i, i = 1, 2, \ldots, r$, take the value 1 or 0 depending on failure unit from $\Gamma_1$ or $\Gamma_2$, respectively, and the two integers $m_1$ and $m_2$ denote the number of units fails from lines $\Gamma_1$ and $\Gamma_2$, respectively.

The joint likelihood function from observed joint type-I GHSC $t = \{(t_2, \delta_1), (t_2, \delta_1), \ldots, (t_r, \delta_r)\}$ and $m_1 = \sum_{i=1}^{r} \delta_i$, $m_2 = \sum_{i=1}^{r} (1 - \delta_i)$ is presented by




$\tau^* < T_s$ (a) | $\tau^* > T_s$ and $t_{m,n} > \tau^*$ (b) | $t_{m,n} < \tau^*$ (c)

Figure 1: Different types of type-I GHCS.




$t_{k,n} > \tau$ (a) | $t_{m,n} > \tau$ (b) | $t_{k,n} < \tau$ (c)

Figure 2: Different types of type-I GHCS.

$$f_{1,2,\ldots,r}(\underline{t}) = \frac{N_1!N_2![S_1(D)]^{N_1-m_1}[S_2(D)]^{N_2-m_2}}{(N_1-n_1)!(N_2-m_2)!}\prod_{i=1}^{r}[f_1(t_i)]^{\delta_i}[f_2(t_i)]^{1-\delta_i}, \quad (1)$$

where

$$D = \begin{cases} t_s, & \text{if } \tau^* < T_s \\ t_m, & \text{if } T_m < \tau^* \\ \tau^* & \text{if } T_s < \tau^* < T_m, \end{cases} \quad (2)$$

and survival functions $S_j(.)$, $j = 1, 2$.

Suppose that the lifetime $T$ has two parameters Burr XII populations with PDFs given by

$$f_l(t) = a_i b_i t^{b_i-1}\left(1+t^{b_i}\right)^{-(a_i+1)}, \quad t>0, \ (a_i>0,\ b_i>0), i=1,2, \quad (3)$$

and CDFs, survival functions $S_j(t)$, and failure rate function $h_i(t)$ are given by

$$\begin{aligned} F_i(t) &= 1-\left(1+t^{b_i}\right)^{-a_i}, \\ S_i(t) &= \left(1+t^{b_i}\right)^{-a_i}, \\ h_i(t) &= a_i b_i t^{b_i-1}\left(1+t^{b_i}\right)^{-1}, \quad t>0. \end{aligned} \quad (4)$$

The function (4) shows that the parameter $a_i$ does not affect the shape of failure rate function $h_i(t)$. Burr XII has been applied in areas of quality control, reliability studies, duration, and failure time modeling, see for example [15].

## 3. Estimations under the Maximum Likelihood Method

The joint likelihood function (1) under Burr XII populations (3) and (4) and observed joint type-I GHS sample $t = \{(t_2, \delta_1), (t_2, \delta_1), (t_r, \delta_r)\}$ is formed by

$$L(a_1,b_1,a_2,b_2|\underline{t}) \propto (a_1b_1)^{m_1}(a_2b_2)^{m_2}\exp\Bigg\{(b_1-1)\sum_{i=1}^{r}\delta_i\log t_i-(a_1+1)\sum_{i=1}^{r}\delta_i \times \log\left[\left(1+t_i^{b_1}\right)\right]-(a_1+1)(N_1-m_1)\log\left[1+D^{b_1}\right] + (b_2-1)\sum_{i=1}^{r}(1-\delta_i)\log t_i-(a_2+1)\sum_{i=1}^{r}(1-\delta_i)\log\left[\left(1+t_i^{b_1}\right)\right] - (a_2+1)(N_2-m_2)\log\left[1+D^{b_2}\right]\Bigg\}. \tag{5}$$

The natural logarithms of (5) without a normalized constant reduce to

$$\ell(a_1,b_1,a_2,b_2|\underline{t}) = m_1\log(a_1b_1)+m_2\log(a_2b_2)+(b_1-1)\sum_{i=1}^{r}\delta_i\log t_i-(a_1+1) \times\sum_{i=1}^{r}\delta_i\log\left[\left(1+t_i^{b_1}\right)\right]-(a_1+1)(N_1-m_1)\log\left[1+D^{b_1}\right] + (b_2-1)\sum_{i=1}^{r}(1-\delta_i)\log t_i-(a_2+1)\sum_{i=1}^{r}(1-\delta_i)\log\left[\left(1+t_i^{b_1}\right)\right] - (a_2+1)(N_2-m_2)\log\left[1+D^{b_2}\right]. \tag{6}$$

*3.1. MLEs.* Under partial derivative of the log-likelihood function with respect to model parameters and equating to zero, we obtain the likelihood equations as follows:

$$\frac{\partial\ell(a_1,b_1,a_2,b_2|\underline{t})}{\partial a_i}=0,\quad i=1,\,2, \tag{7}$$

and are reduced to

$$a_1(b_1)=\frac{m_1}{\sum_{i=1}^{r}\delta_i\log\left[\left(1+t_i^{b_1}\right)\right]+(N_1-m_1)\log\left[1+D^{b_1}\right]} \tag{8}$$

and

$$a_2(b_2)=\frac{m_2}{\sum_{i=1}^{r}(1-\delta_i)\log\left[\left(1+t_i^{b_2}\right)\right]+(N_2-m_2)\log\left[1+D^{b_2}\right]}. \tag{9}$$

Also,

$$\frac{\partial\ell(a_1,b_1,a_2,b_2|\underline{t})}{\partial b_i}=0,\quad i=1,2, \tag{10}$$

has presented that

$$\frac{m_1}{b_1}+\sum_{i=1}^{r}\delta_i\log t_i-(a_1+1)\left\{\sum_{i=1}^{r}\delta_i\frac{t_i^{b_1}\log t_i}{1+t_i^{b_1}}+(N_1-m_1)\frac{D^{b_1}\log D}{1+D^{b_1}}\right\}=0 \tag{11}$$

$$\frac{m_2}{b_2}+\sum_{i=1}^{r}(1-\delta_i)\log t_i-(a_2+1)\left\{\sum_{i=1}^{r}(1-\delta_i)\frac{t_i^{b_2}\log t_i}{1+t_i^{b_2}}+(N_2-m_2)\frac{D^{b_2}\log D}{1+D^{b_2}}\right\}=0. \tag{12}$$

Then, the nonlinear, equations (11) and (12) are reduced after replacing $a_1$ and $a_2$ from (8) and (9) to

$$\frac{m_1}{b_1} + \sum_{i=1}^{r} \delta_i \log t_i - \left( \frac{m_1}{\sum_{i=1}^{r} \delta_i \log\left[\left(1 + t_i^{b_1}\right)\right] + (N_1 - m_1)\log\left[1 + D^{b_1}\right]} + 1 \right) \left( \sum_{i=1}^{r} \delta_i \frac{t_i^{b_1} \log t_i}{1 + t_i^{b_1}} + (N_1 - m_1) \frac{D^{b_1} \log D}{1 + D^{b_1}} \right) = 0 \tag{13}$$

$$\frac{m_2}{b_2} + \sum_{i=1}^{r} (1 - \delta_i) \log t_i - \left( \frac{m_2}{\sum_{i=1}^{r} (1 - \delta_i) \log\left[\left(1 + t_i^{b_2}\right)\right] + (N_2 - m_2)\log\left[1 + D^{b_2}\right]} + 1 \right) \left( \sum_{i=1}^{r} (1 - \delta_i) \frac{t_i^{b_2} \log t_i}{1 + t_i^{b_2}} + (N_2 - m_2) \frac{D^{b_2} \log D}{1 + D^{b_2}} \right) = 0. \tag{14}$$

The two nonlinear equations presented by (13) and (14) present the likelihood equations of parameters $b_1$ and $b_2$, which are more simple to solve with Newton–Raphson or with fixed point iteration. After obtaining the values $\widehat{b}_1$ and $\widehat{b}_2$ from (13) and (14), the values $\widehat{a}_1$ and $\widehat{a}_2$ are obtained from (8) and (9). In some cases, if $m_1 = 0$ or $m_2 = 0$, the parameter values $a_1$ and $b_1$ or $a_2$ and $b_2$, respectively, are difficult to obtain [16].

*3.2. Approximate Interval Estimation.* The second partial derivatives of log-likelihood function (6) with respect to parameters vector $\omega = (a_1, b_1, a_2 . b_2)$ are given by

$$\begin{aligned}
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial a_i^2} &= \frac{-m_i}{a_i^2}, \quad i = 1, 2, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_1^2} &= \frac{-m_1}{b_1^2} - (a_1 + 1) \left\{ \sum_{i=1}^{r} \delta_i \frac{t_i^{b_1} (\log t_i)^2}{\left(1 + t_i^{b_1}\right)^2} + (N_1 - m_1) \frac{D^{b_1} (\log D)^2}{\left(1 + D^{b_1}\right)^2} \right\}, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_2^2} &= \frac{-m_2}{b_2^2} - (a_2 + 1) \left\{ \sum_{i=1}^{r} (1 - \delta_i) \frac{t_i^{b_2} (\log t_i)^2}{\left(1 + t_i^{b_2}\right)^2} + (N_2 - m_2) \frac{D^{b_2} (\log D)^2}{\left(1 + D^{b_2}\right)^2} \right\}, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial a_1 \partial b_1} &= \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_1 \partial a_1} = -\sum_{i=1}^{r} \delta_i \frac{t_i^{b_1} \log t_i}{1 + t_i^{b_1}} - (N_1 - m_1) \frac{D^{b_1} \log D}{1 + D^{b_1}}, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial a_1 \partial a_2} &= \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial a_2 \partial a_1} = \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial a_1 \partial b_2} = \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_2 \partial a_1} = 0, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_1 \partial a_2} &= \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial a_2 \partial b_1} = \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_1 \partial b_2} = \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_2 \partial b_1} = 0, \\
\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial a_2 \partial b_2} &= \frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial b_2 \partial a_2} = -\sum_{i=1}^{r} (1 - \delta_i) \frac{t_i^{b_2} \log t_i}{1 + t_i^{b_2}} - (N_2 - m_2) \frac{D^{b_2} \log D}{1 + D^{b_2}}.
\end{aligned} \tag{15}$$

The Fisher information matrix is defined by minus expectation of second partial derivative of the log-likelihood function. Practice, under a large sample, the Fisher information matrix can be approximate with approximate information matrix. Let $\eta$ denotes the Fisher information matrix defined by

$$\eta = -E\left(\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial\omega_i \partial\omega_j}\right), \quad i, j = 1, 2, 3, 4, \tag{16}$$

where $\omega = (a_1, b_1, a_2, b_2)$. Then, the approximate information matrix of $\eta$ denoted by $\eta_0$ is defined as

$$\eta_0 = -\left(\frac{\partial^2 \ell(a_1, b_1, a_2, b_2 | \underline{t})}{\partial\omega_i \partial\omega_j}\right)\Big|_{\widehat{\omega}=\left(\widehat{a}_1, \widehat{b}_1, \widehat{a}_2, \widehat{b}_2\right)}, \quad i, j = 1, 2, 3, 4. \tag{17}$$

Hence, under asymptotic normality distribution of MLEs $(\widehat{a}_1, \widehat{b}_1, \widehat{a}_2, \widehat{b}_2)$ with mean $(a_1, b_1, a_2, b_2)$ and variance covariance matrix $\eta_0^{-1}(\widehat{a}_1, \widehat{b}_1, \widehat{a}_2, \widehat{b}_2)$, $100(1-2\alpha)\%$ confidence intervals from the model parameter $\omega = (a_1, b_1, a_2, b_2)$ are given by

$$\begin{aligned} &\widehat{a}_1 \mp z_\alpha \sqrt{\epsilon_{11}}, \\ &\widehat{b}_1 \mp z_\alpha \sqrt{\epsilon_{22}}, \\ &\widehat{a}_2 \mp z_\alpha \sqrt{\epsilon_{33}}, \\ &\widehat{b}_2 \mp z_\alpha \sqrt{\epsilon_{44}}, \end{aligned} \tag{18}$$

and the vector $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{44})$ presents the diagonal of the covariance matrix $\eta_0^{-1}(\widehat{a}_1, \widehat{b}_1, \widehat{a}_2, \widehat{b}_2)$, and the value $z_\gamma$ is the percentile of the normal (0,1) with right-tail probability $\alpha$.

## 4. Bayesian Approach

In this section, we discuss the Bayes estimations of model parameters, point, and credible interval. Bayesian approach needs prior information about the model parameters, which we considered as independent gamma prior, described as follows:

$$\omega_i \propto \omega_i^{\rho_{i-1}} \exp\left[v_i \omega_i\right], \quad i = 1, 2, 3, 4, \tag{19}$$

where $\omega = (a_1, b_1, a_2, b_2)$ is the model parameter. The joint prior distribution is given by

$$\Psi^*(a_1, b_1, a_2, b_2) \propto \prod_{i=4}^{4} \omega_i^{\rho_{i-1}} \exp\left[v_i \omega_i\right], \quad i = 1, 2, 3, 4. \tag{20}$$

Generally, the posterior distribution from the model parameters $\Psi(a_1, b_1, a_2, b_2 | \underline{t})$ is given by

$$\Psi(a_1, b_1, a_2, b_2 | \underline{t}) = \frac{\Psi^*(a_1, b_1, a_2, b_2) \times L(a_1, b_1, a_2, b_2 | \underline{t})}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Psi^*(a_1, b_1, a_2, b_2) \times L(a_1, b_1, a_2, b_2 | \underline{t}) \mathrm{d}a_1 \mathrm{d}b_1 \mathrm{d}a_2 \mathrm{d}b_2}. \tag{21}$$

And the Bayes estimate under squared error loss function (SEL) of function $\Omega(a_1, b_1, a_2, b_2)$ is given by

$$\widehat{\Omega}_B = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Omega(a_1, b_1, a_2, b_2) \Psi(a_1, b_1, a_2, b_2 | \underline{t}) \mathrm{d}a_1 \mathrm{d}b_1 \mathrm{d}a_2 \mathrm{d}b_2. \tag{22}$$

The equation (22) has ratio of two integrals, generally cannot be obtained in a closed form. Then, numerical approximation will be used to solve this problem. One way is called numerical integration, and other way used can be called Lindley approximate. The important method which has considerable attention in the last year called the MCMC method is discussed as follows.

The proportional form of joint posterior distribution (21) with joint prior distribution (20) and the likelihood function (6) is given by

$$\begin{aligned} \Psi(a_1, b_1, a_2, b_2 | \underline{t}) \propto{}& a_1^{m_1+\rho_1-1} b_1^{m_1+\rho_2-1} a_2^{m_2+\rho_3-1} b_2^{m_2+\rho_4-1} \exp\{-v_1 a_1 - v_2 b_1 - v_3 a_2 - v_4 b_2 \\ &+ (b_1 - 1)\sum_{i=1}^r \delta_i \log t_i - (a_1 + 1)\sum_{i=1}^r \delta_i \log\left[\left(1 + t_i^{b_1}\right)\right] - (a_1 + 1) \times (N_1 - m_1)\log\left[1 + D^{b_1}\right] \\ &+ (b_2 - 1)\sum_{i=1}^r (1 - \delta_i)\log t_i - (a_2 + 1) \times \sum_{i=1}^r (1 - \delta_i)\log\left[\left(1 + t_i^{b_1}\right)\right] - (a_2 + 1)(N_2 - m_2)\log\left[1 + D^{b_2}\right]. \end{aligned} \tag{23}$$

The joint posterior distribution (23) reduced with the full-conditional probability distributions PDF's is given as follows:

$$\begin{aligned}\Psi_1(a_1|b_1,a_2,b_2,\underline{t}) &\propto \exp\left\{-a_1\left(v_1+\sum_{i=1}^{r}\delta_i\log\left[\left(1+t_i^{b_1}\right)\right]+(N_1-m_1)\log\left[1+D^{b_1}\right]\right)\right\}\\&\quad\times a_1^{m_1+\rho_1-1},\\ \Psi_2(a_2|a_1,b_1,b_2,\underline{t}) &\propto \exp\left\{-a_2\left(v_3+\sum_{i=1}^{r}(1-\delta_i)\log\left[\left(1+t_i^{b_2}\right)\right]+(N_2-m_2)\log\left[1+D^{b_2}\right]\right)\right\}\\&\quad\times a_2^{m_2+\rho_2-1},\\ \Psi_3(b_1|a_1,a_2,b_2,\underline{t}) &\propto b_1^{m_1+\rho_2-1}\exp\left\{-v_2b_1+b_1\sum_{i=1}^{r}\delta_i\log t_i-(a_1+1)\right.\\&\quad\left.\times\sum_{i=1}^{r}\delta_i\log\left[\left(1+t_i^{b_1}\right)\right](a_1+1)(N_1-m_1)\log\left[1+D^{b_1}\right]\right\}\end{aligned} \tag{24}$$

$$\begin{aligned}\Psi_4(b_2|a_1,a_2,b_1,\underline{t}) &\propto b_2^{m_2+\rho_4-1}\exp\left\{-v_4b_2+b_2\sum_{i=1}^{r}(1-\delta_i)\log t_i-(a_2+1)\right.\\&\quad\left.\times\sum_{i=1}^{r}(1-\delta_i)\log\left[\left(1+t_i^{b_2}\right)\right](a_2+1)(N_2-m_2)\log\left[1+D^{b_2}\right]\right\}.\end{aligned} \tag{25}$$

The concept of the MCMC method is dependent on the form of the full-conditional distributions and a suitable technique in variety type of MCMC schemes. The posterior distribution is reduced to two gamma distributions (24) as well as two functions (24) and (25), more similar to the normal distribution. Hence, we adopted Gibbs algorithms, and in more general cases, Metropolis Hasting (MH) under Gibbs [17] algorithms.

MH under Gibbs algorithms:

(1) The indicator $\kappa$ begins with the value 1 and initial parameters vector $\omega^{(0)} = (\hat{a}_1, \hat{b}_1, \hat{a}_2, \hat{b}_2)$

(2) The values $a_1^{(\kappa)}$ and $a_2^{(\kappa)}$ are generated from gamma distributions (24)

(3) MH algorithms with $N(b_1^{(\kappa-1)}, \epsilon_{22})$ and $N(b_2^{(\kappa-1)}, \epsilon_{44})$ proposal distributions are used to generate $b_1^{(\kappa)}$ and $b_2^{(\kappa)}$, respectively, as follows

(i) Begin with $\hat{b}_1$ and $\hat{b}_2$ as an arbitrary starting point $b_1^{(0)}$ and $b_1^{(0)}$ for (24) and (25)

(ii) At time $k$, sample a candidate point or proposal $b_1^{(*)}$, from $N(b_1^{(\kappa-1)}, \epsilon_{22})$, and $b_2^{(*)}$, from $N(b_2^{(\kappa-1)}, \epsilon_{44})$, the proposal distributions

(iii) Calculate the acceptance probability

$$\begin{aligned}\rho_1\left(b_1^{(k-1)}, b_1^*\right) &= \min\left[1, \frac{\Psi_3\left(b_1^{(*)}|a_1^{(k-1)}, a_2^{(k-1)}, b_2^{(k-1)}, \underline{t}\right)}{f\left(b_1^{(k-1)}|\underline{x}\right)}\right],\\ \rho_2\left(b_2^{(k-1)}, b_2^*\right) &= \min\left[1, \frac{\Psi_4\left(b_2^{(*)}|a_1^{(k-1)}, a_2^{(k-1)}, b_1^{(k-1)}, \underline{t}\right)}{f\left(b_2^{(k-1)}|\underline{x}\right)}\right].\end{aligned} \tag{26}$$

(iv) Generate $U_i \sim U(0,1), i = 1, 2$

(v) If $U_i \le \rho_i(b_i^{(k-1)}, b_i^*)$, accept the proposal and set $b_i^{(t)} = b_i^*$. Otherwise, reject the proposal and set $b_i^{(k)} = b_i^{(k-1)}$.

(4) Report the simulate parameters vector $\omega^{(\kappa)} = (a_1^{(\kappa)}, b_1^{(\kappa)}, a_2^{(\kappa)}, b_2^{(\kappa)})$

(5) Change $\kappa$ to $\kappa + 1$

(6) Repeat steps from 2 to 5 , $M$ times.

Bayes estimation with the MCMC method requires some measurements reported about the generation method and determine the number needed to reach the stationary distribution (burn-in) denoted by $M^*$. Then, the Bayes estimators are given by

$$\widehat{\Omega}_B = E_\Psi(\Omega|\underline{t}) = \frac{1}{M - M^*} \sum_{i=M^*+1}^{M} \Omega^{(i)}. \quad (27)$$

Also, the posterior variance of function $\Omega$ is given by

$$\widehat{V}(\Omega|\underline{t}) = \frac{1}{M - M^*} \sum_{i=M^*+1}^{M} \left(\Omega^{(i)} - \widehat{\Omega}_B\right)^2. \quad (28)$$

The credible interval is obtained with ordering the simulated vector $\Omega$; then, $100(1-2\alpha)\%$ credible interval of function $\Omega$ is given by

$$\left(\Omega_{\alpha(M-m^*)}, \Omega_{(1-\alpha)(M-M^*)}\right). \quad (29)$$

## 5. Illustrative Example

In this section, we summarized a simulated dataset from two Burr XII populations to check the theoretical results discussed through the study. The real parameters value is selected to satisfy prior information, so that with $(\rho_1, \rho_2, \rho_3, \rho_4) = (2, 3, 4, 2)$ and $(v_1, v_2, v_3, v_4) = (2, 2.5, 2, 2)$, the real parameter values are selected to satisfy $\omega_i \simeq E_{\Psi^*}(\omega_i) = (\rho_i/v_i)$. Then, the parameters values are selected to be $\omega = (a_1, b_1, a_2, b_2) = (1.82, 1.7, 1.2, 1.92)$. Under Burr XII distribution, parameter values (1.82, 1.7) generate a sample of size $N_1 = 30$ to present data from the line $\Gamma_1$ that are given by $W$={ 0.0604831, 0.167157, 0.268757, 0.284369, 0.336897, 0.389696, 0.402109, 0.434761, 0.539457, 0.548363, 0.569382, 0.584881, 0.604739, 0.607685, 0.640021, 0.719833, 0.747014, 0.752982, 0.760519, 0.789178, 0.848233, 0.885767, 0.892944, 0.912513, 0.931659, 0.939561, 1.21948, 1.49421, 1.50994, 1.59295}. Also, Burr XII distribution with parameters (1.2, 1.92) generates a sample of size $N_2 = 30$ to present data from the line $\Gamma_2$ that are given by $Z$={0.0459775, 0.156614, 0.293896, 0.337772, 0.397422, 0.446852, 0.466979, 0.469356, 0.470276, 0.475754, 0.499894, 0.596894, 0.761898, 0.766247, 0.824323, 0.922044, 0.940813, 0.968182, 0.975155, 0.975229, 1.0076, 1.18989, 1.21431, 1.62018, 1.74474, 1.82949, 1.98172, 2.59949, 2.76045, 5.17082}. Therefore, for given prior integers $(s, m) = (25, 35)$ and the prior time $\tau^* = 0.5$, the joint type-I GHC data are summarized in Table 1. In the last sections, we discussed two different methods of estimation, MLEs and Byes estimations. The point results of MLE and Bayes estimates as well as asymptotic confidence interval and credible interval are summarized in Table 2. The chen in MCMC methods is reported for 11000 iterations that contain the first 1000 samples as burn-in. Usually, it is not hard to construct a Markov chain with the desired properties. To determine how many steps are needed to converge to the stationary, more difficult problem is distribution within an acceptable error. Then, we can test if stationary distribution is reached quickly starting from an arbitrary position. The plot for the simulation number of the model parameters and the corresponding histogram shown in Figures 3–6 can be used to describe the convergence results in MCMC methods.

## 6. Simulation Studies

The quality of estimators dependent on some tolls or measures that are computed for a suitable numbers of generated samples from the populations with given parameter values is known by a simulation study. Then, we assess the theoretical estimation results of MLEs and Bayes estimators under discussing and computing average (AV) and MSEs for point estimate and coverage probability (CP) and average of interval length (AL) to the interval estimation. The simulation study is reported for different sample sizes $(N_1 + N_2)$ and different effected sample sizes $(m)$. Also, we consider different cases of min number $s$ and different ideal test time $(\tau^*)$. Also, we study the effect of parameters change with considering two sets of populations parameters, say $\omega = (a_1, b_1, a_2, b_2) \in$ {(1.0, 2.0, 2.0, 3.0), (0.4, 1.2, 1.5, 2.0)}. The prior parameters are selected to satisfy $E_{\Psi^*}(\omega_i) \simeq (\rho_i/v_i)$. Hence, in our simulation study, we proposed different two cases from prior information; one of them is expressed to noninformative prior (prior$_0$), in which the posterior distribution is proportional with likelihood function. The second case is expressed to informative prior (prior$_1$ and prior$_2$). The prior$_1$ is $(\rho_1, \rho_2, \rho_3, \rho_4) = (1, 3, 4, 5)$ and $(v_1, v_2, v_3, v_4) = (1, 2, 1.5, 2)$. The prior$_2$ is $(\rho_1, \rho_2, \rho_3, \rho_4) =$ (1, 2.5, 3, 4) and $(v_1, v_2, v_3, v_4) = (2.0, 2.0, 2.0, 1.5)$. For Bayesian approach, without loss of the generality for any loss function, all computations are reported under squared error loss function. The MCMC method is performed under 11000 chen with 1000 burn-in, and the results are reported in Tables 3–6.

## 7. Concluding Remarks

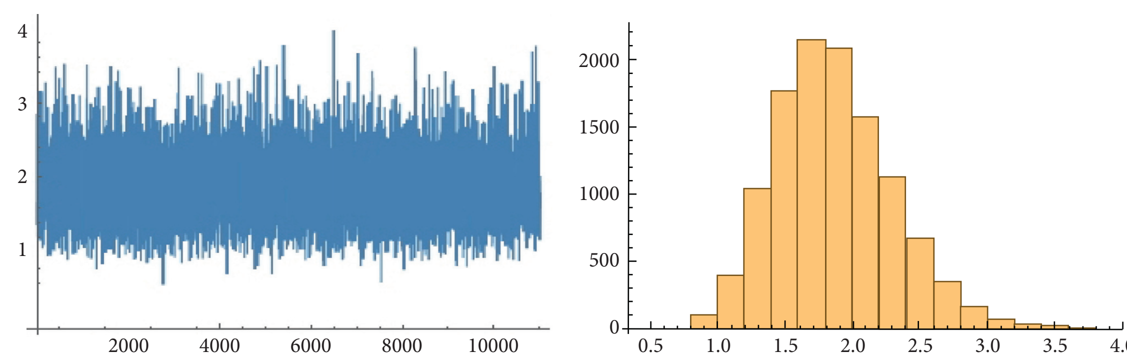
In the industrial field, the existing different lines of production have the same products under the same facility. The problem of measuring the relative merits of product in the competing duration has considerable attention in past view years. This problem has been discussed in this study for products distributed with Burr XII lifetime distribution. This problem presented in parameters estimation forms with ML and Bayesian estimations under joint type-I GHCS. Then, the developed method is assessed through the Monte Carlo simulation study. The results obtained from these studies show the following comments.

Table 1: The joint type-I GHS data with $(s, m) = (25, 35)$ and $\tau = 0.5$, $d = 25$.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0.0459775 | 0.0604831 | 0.156614 | 0.167157 | 0.268757 | 0.284369 | 0.293896 |
| 0 | 1 | 0 | 1 | 1 | 1 | 0 |
| 0.336897 | 0.337772 | 0.389696 | 0.397422 | 0.402109 | 0.434761 | 0.446852 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 |
| 0.466979 | 0.469356 | 0.470276 | 0.475754 | 0.499894 | 0.539457 | 0.548363 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 0.569382 | 0.584881 | 0.596894 | 0.604739 | 0.607685 | 0.640021 | 0.719833 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0.747014 | 0.752982 | 0.760519 | 0.761898 | 0.766247 | 0.789178 | 0.824323 |
| 1 | 1 | 1 | 0 | 0 | 1 | 0 |

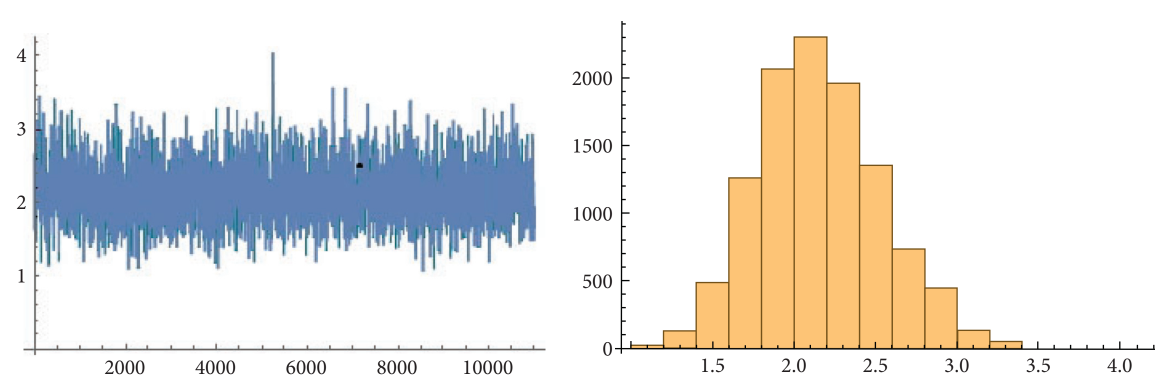
Table 2: The point and 95% confidence intervals (ACIs and CIs) of MLEs Bayes estimates.

| Pa.s | (.)ML | (.)BMCMC | 95% ACIs | Length | 95% CI | Length |
|---|---|---|---|---|---|---|
| $a_1$ =1.82 | 2.44789 | 1.86465 | (1.1898, 3.7060) | 2.5163 | (1.1225, 2.8007) | 1.6783 |
| $b_1$ =1.7 | 2.44643 | 2.14852 | (1.5600, 3.3329) | 1.7729 | (1.4706, 2.9120) | 1.4413 |
| $a_2$ =1.2 | 2.02458 | 1.39188 | (0.8110, 3.2382) | 2.4272 | (0.8109, 2.1501) | 1.3392 |
| $b_2$ =1.92 | 1.56413 | 1.82341 | (0.8581, 2.2702) | 1.4121 | (1.1559, 2.5984) | 1.4425 |



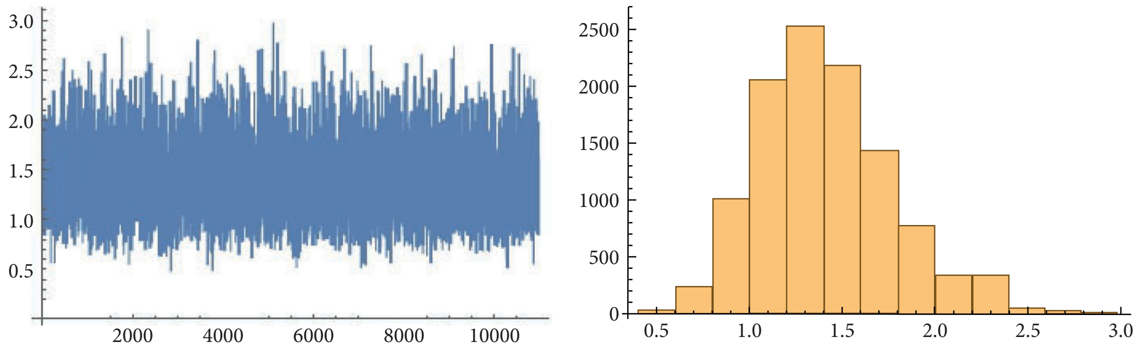

(a) (b)

Figure 3: Simulation number and the corresponding histogram of $a_1$ generated by the MCMC method.



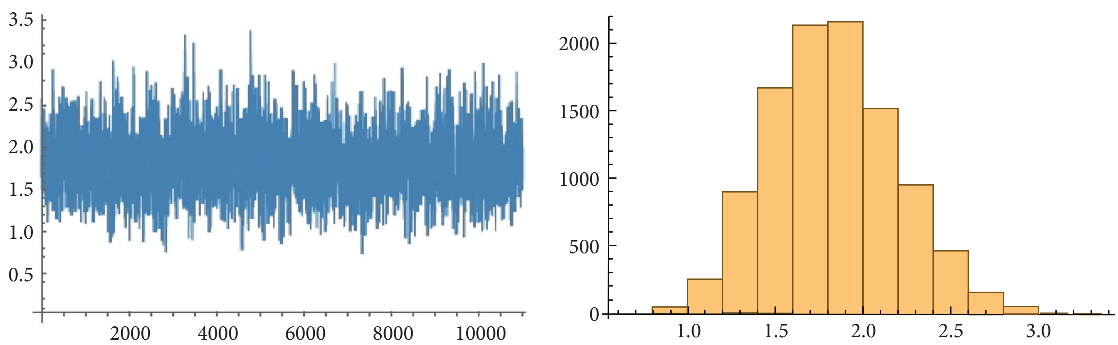

(a) (b)

Figure 4: Simulation number and the corresponding histogram of $b_1$ generated by the MCMC method.




(a) (b)

FIGURE 5: Simulation number and the corresponding histogram of $a_2$ generated by the MCMC method.




(a) (b)

FIGURE 6: Simulation number and the corresponding histogram of $a_2$ generated by the MCMC method.

TABLE 3: The AVs and MSEs of Burr XII populations with $\omega$ =(1.0, 2.0, 2.0, 3.0).

| $\tau^*$ | $(N_1, N_2)$ | $(k, m)$ | Pa. | MLEs | | Bayes (prior$_0$) | | Bayes (prior$_1$) | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | AVs | MSEs | AVs | MSEs | AVs | MSEs |
| 0.5 | (20, 20) | (15, 20) | $a_1$ | 1.232 | 0.324 | 1.241 | 0.327 | 1.210 | 0.210 |
| | | | $b_1$ | 2.213 | 0.410 | 2.230 | 0.407 | 2.200 | 0.332 |
| | | | $a_2$ | 2.188 | 0.433 | 2.201 | 0.399 | 2.214 | 0.341 |
| | | | $b_2$ | 3.241 | 0.621 | 3.245 | 0.618 | 3.214 | 0.540 |
| | | (15, 30) | $a_1$ | 1.201 | 0.300 | 1.222 | 0.309 | 1.202 | 0.198 |
| | | | $b_1$ | 2.200 | 0.389 | 2.215 | 0.398 | 2.187 | 0.301 |
| | | | $a_2$ | 2.174 | 0.414 | 2.198 | 0.401 | 2.201 | 0.317 |
| | | | $b_2$ | 3.215 | 0.598 | 3.232 | 0.600 | 3.220 | 0.525 |
| | | (25, 30) | $a_1$ | 1.187 | 0.265 | 1.182 | 0.263 | 1.179 | 0.154 |
| | | | $b_1$ | 2.199 | 0.365 | 2.200 | 0.363 | 2.155 | 0.277 |
| | | | $a_2$ | 2.142 | 0.401 | 2.151 | 0.395 | 2.188 | 0.298 |
| | | | $b_2$ | 3.202 | 0.575 | 3.225 | 0.580 | 3.202 | 0.478 |
| | (30, 30) | (25, 35) | $a_1$ | 1.160 | 0.230 | 1.177 | 0.225 | 1.161 | 0.115 |
| | | | $b_1$ | 2.188 | 0.332 | 2.190 | 0.337 | 2.141 | 0.238 |
| | | | $a_2$ | 2.130 | 0.384 | 2.135 | 0.377 | 2.175 | 0.265 |
| | | | $b_2$ | 3.189 | 0.532 | 3.195 | 0.527 | 3.175 | 0.422 |
| | | (25, 50) | $a_1$ | 1.144 | 0.218 | 1.152 | 0.221 | 1.149 | 0.100 |
| | | | $b_1$ | 2.146 | 0.311 | 2.155 | 0.314 | 2.122 | 0.219 |
| | | | $a_2$ | 2.122 | 0.373 | 2.130 | 0.369 | 2.165 | 0.240 |
| | | | $b_2$ | 3.170 | 0.511 | 3.188 | 0.514 | 3.164 | 0.401 |

Table 3: Continued.

| $\tau^*$ | $(N_1, N_2)$ | $(k, m)$ | Pa. | MLEs | | Bayes (prior$_0$) | | Bayes (prior$_1$) | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | AVs | MSEs | AVs | MSEs | AVs | MSEs |
| 1.2 | (20, 20) | (15, 20) | $a_1$ | 1.180 | 0.280 | 1.190 | 0.250 | 1.200 | 0.170 |
| | | | $b_1$ | 2.190 | 0.370 | 2.210 | 0.361 | 2.180 | 0.282 |
| | | | $a_2$ | 2.165 | 0.342 | 2.181 | 0.344 | 2.12 | 0.281 |
| | | | $b_2$ | 3.225 | 0.570 | 3.225 | 0.571 | 3.200 | 0.487 |
| | | (15, 30) | $a_1$ | 1.182 | 0.251 | 1.209 | 0.262 | 1.187 | 0.152 |
| | | | $b_1$ | 2.183 | 0.349 | 2.186 | 0.350 | 2.177 | 0.255 |
| | | | $a_2$ | 2.156 | 0.364 | 2.162 | 0.350 | 2.187 | 0.266 |
| | | | $b_2$ | 3.200 | 0.551 | 3.219 | 0.554 | 3.201 | 0.470 |
| | | (25, 30) | $a_1$ | 1.171 | 0.214 | 1.169 | 0.222 | 1.156 | 0.109 |
| | | | $b_1$ | 2.179 | 0.318 | 2.170 | 0.316 | 2.142 | 0.231 |
| | | | $a_2$ | 2.124 | 0.357 | 2.144 | 0.342 | 2.160 | 0.240 |
| | | | $b_2$ | 3.179 | 0.528 | 3.211 | 0.542 | 3.180 | 0.436 |
| | (30, 30) | (25, 35) | $a_1$ | 1.146 | 0.231 | 1.133 | 0.187 | 1.146 | 0.086 |
| | | | $b_1$ | 2.165 | 0.290 | 2.169 | 0.300 | 2.124 | 0.187 |
| | | | $a_2$ | 2.111 | 0.341 | 2.119 | 0.330 | 2.169 | 0.221 |
| | | | $b_2$ | 3.180 | 0.480 | 3.180 | 0.487 | 3.161 | 0.387 |
| | (25, 50) | | $a_1$ | 1.127 | 0.269 | 1.137 | 0.170 | 1.141 | 0.060 |
| | | | $b_1$ | 2.140 | 0.265 | 2.141 | 0.270 | 2.101 | 0.165 |
| | | | $a_2$ | 2.102 | 0.323 | 2.119 | 0.327 | 2.148 | 0.200 |
| | | | $b_2$ | 3.151 | 0.459 | 3.167 | 0.469 | 3.148 | 0.366 |

Table 4: The ALs and PCs of 95% CI of Burr XII populations with $\omega$ =(1.0, 2.0, 2.0, 3.0).

| $\tau^*$ | $(N_1, N_2)$ | $(k, m, )$ | Pa. | MLEs | | Bayes (prior$_0$) | | Bayes (prior$_1$) | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | ALs | PCs | ALs | PCs | ALs | PCs |
| 0.5 | (20, 20) | (15, 20) | $a_1$ | 2.513 | (90) | 2.490 | (91) | 2.122 | (91) |
| | | | $b_1$ | 4.124 | (89) | 4.115 | (90) | 3.562 | (90) |
| | | | $a_2$ | 4.002 | (90) | 4.011 | (91) | 3.562 | (92) |
| | | | $b_2$ | 4.865 | (90) | 4.800 | (91) | 4.142 | (91) |
| | | (15, 30) | $a_1$ | 2.475 | (90) | 2.462 | (91) | 2.086 | (92) |
| | | | $b_1$ | 4.100 | (91) | 4.085 | (92) | 3.501 | (93) |
| | | | $a_2$ | 3.992 | (91) | 4.001 | (91) | 3.539 | (93) |
| | | | $b_2$ | 4.851 | (90) | 4.730 | (93) | 4.103 | (94) |
| | | (25, 30) | $a_1$ | 2.438 | (90) | 2.433 | (92) | 2.010 | (92) |
| | | | $b_1$ | 4.050 | (93) | 4.040 | (92) | 3.470 | (92) |
| | | | $a_2$ | 3.943 | (91) | 3.925 | (90) | 3.500 | (93) |
| | | | $b_2$ | 4 .801 | (92) | 4.750 | (93) | 4.009 | (95) |
| | (30, 30) | (25, 35) | $a_1$ | 2.401 | (90) | 2.403 | (92) | 1.777 | (94) |
| | | | $b_1$ | 3.950 | (93) | 3.941 | (93) | 3.401 | (93) |
| | | | $a_2$ | 3.900 | (90) | 3.890 | (92) | 3.420 | (93) |
| | | | $b_2$ | 4.751 | (93) | 4.735 | (93) | 3.889 | (93) |
| | | (25, 50) | $a_1$ | 2.370 | (93) | 2.362 | (92) | 1.715 | (95) |
| | | | $b_1$ | 3.901 | (92) | 3.902 | (95) | 3.360 | (93) |
| | | | $a_2$ | 3.850 | (92) | 3.856 | (92) | 3.381 | (94) |
| | | | $b_2$ | 4.714 | (95) | 4.701 | (94) | 3.811 | (94) |

Table 4: Continued.

| $\tau^*$ | $(N_1, N_2)$ | $(k, m,)$ | Pa. | MLEs ALs | MLEs PCs | Bayes (prior$_0$) ALs | Bayes (prior$_0$) PCs | Bayes (prior$_1$) ALs | Bayes (prior$_1$) PCs |
|---|---|---|---|---|---|---|---|---|---|
| 1.2 | (20, 20) | (15, 20) | $a_1$ | 2.480 | (91) | 2.475 | (91) | 2.085 | (92) |
| | | | $b_1$ | 4.100 | (90) | 4.190 | (90) | 3.541 | (90) |
| | | | $a_2$ | 3.980 | (91) | 3.986 | (91) | 3.533 | (92) |
| | | | $b_2$ | 4.838 | (90) | 4.801 | (91) | 4.112 | (91) |
| | | (15, 30) | $a_1$ | 2.443 | (91) | 2.441 | (93) | 2.052 | (95) |
| | | | $b_1$ | 4.072 | (93) | 4.066 | (92) | 3.471 | (93) |
| | | | $a_2$ | 3.961 | (91) | 3.955 | (93) | 3.512 | (93) |
| | | | $b_2$ | 3.721 | (91) | 3.718 | (93) | 3.003 | (94) |
| | (30, 30) | (25, 30) | $a_1$ | 2.411 | (92) | 2.409 | (92) | 1.980 | (95) |
| | | | $b_1$ | 4.012 | (93) | 4.007 | (93) | 3.441 | (92) |
| | | | $a_2$ | 3.922 | (92) | 3.905 | (93) | 3.471 | (95) |
| | | | $b_2$ | 4.790 | (95) | 4.727 | (93) | 4.000 | (95) |
| | | (25, 35) | $a_1$ | 2.370 | (92) | 2.381 | (93) | 1.744 | (94) |
| | | | $b_1$ | 3.935 | (93) | 3.922 | (93) | 3.380 | (93) |
| | | | $a_2$ | 3.871 | (94) | 3.865 | (93) | 3.400 | (93) |
| | | | $b_2$ | 4.733 | (93) | 4.725 | (93) | 3.842 | (95) |
| | | (25, 50) | $a_1$ | 2.352 | (94) | 2.344 | (92) | 1.652 | (97) |
| | | | $b_1$ | 3.872 | (94) | 3.881 | (95) | 3.340 | (94) |
| | | | $a_2$ | 3.823 | (92) | 3.826 | (93) | 3.354 | (96) |
| | | | $b_2$ | 4.692 | (95) | 4.680 | (94) | 3.800 | (94) |

Table 5: The AVs and MSEs of Burr XII populations with $\omega$ =(0.4, 1.2, 1.5, 2.0).

| $\tau^*$ | $(N_1, N_2)$ | $(k, m,)$ | Pa. | MLEs AVs | MLEs MSEs | Bayes (prior$_0$) AVs | Bayes (prior$_0$) MSEs | Bayes (prior$_1$) AVs | Bayes (prior$_1$) MSEs |
|---|---|---|---|---|---|---|---|---|---|
| 0.4 | (20, 20) | (15, 20) | $a_1$ | 0.589 | 0.121 | 0.585 | 0.122 | 0.564 | 0.091 |
| | | | $b_1$ | 1.444 | 0.371 | 1.440 | 0.365 | 1.399 | 0.277 |
| | | | $a_2$ | 1.545 | 0.375 | 1.522 | 0.370 | 1.539 | 0.309 |
| | | | $b_2$ | 2.331 | 0.432 | 2.327 | 0.436 | 2.231 | 0.332 |
| | | (15, 30) | $a_1$ | 0.562 | 0.088 | 0.555 | 0.090 | 0.534 | 0.068 |
| | | | $b_1$ | 1.413 | 0.315 | 1.417 | 0.324 | 1.329 | 0.227 |
| | | | $a_2$ | 1.541 | 0.340 | 1.538 | 0.338 | 1.530 | 0.282 |
| | | | $b_2$ | 2.310 | 0.399 | 2.299 | 0.395 | 2.231 | 0.281 |
| | (30, 30) | (25, 30) | $a_1$ | 0.522 | 0.071 | 0.531 | 0.068 | 0.531 | 0.049 |
| | | | $b_1$ | 1.395 | 0.311 | 1.402 | 0.307 | 1.299 | 0.211 |
| | | | $a_2$ | 1.535 | 0.318 | 1.531 | 0.325 | 1.537 | 0.269 |
| | | | $b_2$ | 2.301 | 0.362 | 2.299 | 0.368 | 2.228 | 0.268 |
| | | (25, 35) | $a_1$ | 0.489 | 0.061 | 0.499 | 0.067 | 0.488 | 0.035 |
| | | | $b_1$ | 1.381 | 0.277 | 1.388 | 0.269 | 1.290 | 0.210 |
| | | | $a_2$ | 1.525 | 0.303 | 1.521 | 0.301 | 1.545 | 0.255 |
| | | | $b_2$ | 2.280 | 0.355 | 2.275 | 0.338 | 0.215 | 0.261 |
| | | (25, 50) | $a_1$ | 0.471 | 0.044 | 0.460 | 0.041 | 0.466 | 0.022 |
| | | | $b_1$ | 1.358 | 0.264 | 1.366 | 0.261 | 1.271 | 0.181 |
| | | | $a_2$ | 1.527 | 0.299 | 1.536 | 0.300 | 1.536 | 0.256 |
| | | | $b_2$ | 2.272 | 0.344 | 2.266 | 0.351 | 2.225 | 0.251 |

Table 5: Continued.

| $\tau^*$ | $(N_1, N_2)$ | $(k, m,)$ | Pa. | MLEs AVs | MLEs MSEs | Bayes (prior$_0$) AVs | Bayes (prior$_0$) MSEs | Bayes (prior$_1$) AVs | Bayes (prior$_1$) MSEs |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | (20, 20) | (15, 20) | $a_1$ | 0.562 | 0.085 | 0.555 | 0.082 | 0.532 | 0.061 |
| | | | $b_1$ | 1.421 | 0.331 | 1.412 | 0.328 | 1.385 | 0.238 |
| | | | $a_2$ | 1.524 | 0.345 | 1.511 | 0.341 | 1.518 | 0.284 |
| | | | $b_2$ | 2.312 | 0.410 | 2.315 | 0.418 | 0.210 | 0.299 |
| | | (15, 30) | $a_1$ | 0.541 | 0.076 | 0.538 | 0.072 | 0.520 | 0.049 |
| | | | $b_1$ | 1.399 | 0.302 | 1.402 | 0.307 | 1.318 | 0.211 |
| | | | $a_2$ | 1.522 | 0.321 | 1.521 | 0.318 | 1.500 | 0.265 |
| | | | $b_2$ | 2.291 | 0.384 | 2.288 | 0.378 | 0.213 | 0.272 |
| | | (25, 30) | $a_1$ | 0.501 | 0.057 | 0.505 | 0.058 | 0.518 | 0.028 |
| | | | $b_1$ | 1.378 | 0.287 | 1.400 | 0.290 | 1.287 | 0.199 |
| | | | $a_2$ | 1.517 | 0.300 | 1.519 | 0.298 | 1.513 | 0.251 |
| | | | $b_2$ | 2.288 | 0.345 | 2.278 | 0.339 | 0.211 | 0.247 |
| | (30, 30) | (25, 35) | $a_1$ | 0.479 | 0.042 | 0.480 | 0.038 | 0.477 | 0.017 |
| | | | $b_1$ | 1.366 | 0.260 | 1.370 | 0.255 | 1.277 | 0.175 |
| | | | $a_2$ | 1.512 | 0.298 | 1.511 | 0.285 | 1.530 | 0.241 |
| | | | $b_2$ | 2.267 | 0.332 | 2.240 | 0.338 | 0.210 | 0.247 |
| | | (25, 50) | $a_1$ | 0.455 | 0.037 | 0.445 | 0.029 | 0.454 | 0.012 |
| | | | $b_1$ | 1.345 | 0.247 | 1.336 | 0.239 | 1.259 | 0.164 |
| | | | $a_2$ | 1.510 | 0.284 | 1.530 | 0.277 | 1.520 | 0.241 |
| | | | $b_2$ | 2.267 | 0.332 | 2.240 | 0.338 | 0.210 | 0.239 |

Table 6: The ALs and PCs of 95% CI of Burr XII populations with $\omega$ =(0.4, 1.2, 1.5, 2.0).

| $\tau^*$ | $(N_1, N_2)$ | $(k, m,)$ | Pa. | MLEs ALs | MLEs PCs | Bayes (prior$_0$) ALs | Bayes (prior$_0$) PCs | Bayes (prior$_1$) ALs | Bayes (prior$_1$) PCs |
|---|---|---|---|---|---|---|---|---|---|
| 0.4 | (20, 20) | (15, 20) | $a_1$ | 1.752 | (89) | 1.744 | (89) | 1.610 | (90) |
| | | | $b_1$ | 3.245 | (90) | 3.260 | (90) | 2.452 | (91) |
| | | | $a_2$ | 3.745 | (88) | 3.749 | (89) | 3.541 | (90) |
| | | | $b_2$ | 4.452 | (89) | 4.445 | (90) | 4.223 | (90) |
| | | (15, 30) | $a_1$ | 1.700 | (90) | 1.702 | (89) | 1.564 | (91) |
| | | | $b_1$ | 3.201 | (90) | 3.212 | (92) | 2.403 | (92) |
| | | | $a_2$ | 3.700 | (90) | 3.703 | (90) | 3.500 | (92) |
| | | | $b_2$ | 4.411 | (91) | 4.402 | (90) | 4.185 | (92) |
| | | (25, 30) | $a_1$ | 1.640 | (92) | 1.652 | (90) | 1.511 | (93) |
| | | | $b_1$ | 3.153 | (92) | 3.280 | (92) | 2.379 | (92) |
| | | | $a_2$ | 3.640 | (92) | 3.632 | (92) | 3.470 | (94) |
| | | | $b_2$ | 4.382 | (91) | 4.379 | (93) | 4.162 | (93) |
| | (30, 30) | (25, 35) | $a_1$ | 1.612 | (95) | 1.624 | (94) | 1.480 | (93) |
| | | | $b_1$ | 3.122 | (92) | 3.145 | (92) | 2.344 | (96) |
| | | | $a_2$ | 3.611 | (92) | 3.608 | (96) | 3.444 | (94) |
| | | | $b_2$ | 4.324 | (94) | 4.333 | (93) | 4.141 | (94) |
| | | (25, 50) | $a_1$ | 1.514 | (95) | 1.502 | (94) | 1.432 | (93) |
| | | | $b_1$ | 3.100 | (92) | 3.104 | (92) | 2.312 | (96) |
| | | | $a_2$ | 3.581 | (92) | 3.575 | (96) | 3.402 | (94) |
| | | | $b_2$ | 4.284 | (94) | 4.279 | (93) | 4.113 | (94) |

Table 6: Continued.

| $\tau^*$ | $(N_1, N_2)$ | $(k, m,)$ | Pa. | MLEs ALs | MLEs PCs | Bayes (prior$_0$) ALs | Bayes (prior$_0$) PCs | Bayes (prior$_1$) ALs | Bayes (prior$_1$) PCs |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | (20, 20) | (15, 20) | $a_1$ | 1.532 | (89) | 1.539 | (90) | 1.425 | (90) |
| | | | $b_1$ | 3.001 | (90) | 0.297 | (90) | 2.254 | (91) |
| | | | $a_2$ | 3.521 | (90) | 3.535 | (91) | 3.324 | (92) |
| | | | $b_2$ | 4.213 | (89) | 4.225 | (90) | 4.001 | (91) |
| | | (15, 30) | $a_1$ | 1.512 | (90) | 1.518 | (91) | 1.411 | (90) |
| | | | $b_1$ | 2.985 | (91) | 2.978 | (90) | 2.233 | (91) |
| | | | $a_2$ | 3.503 | (90) | 3.501 | (91) | 3.303 | (92) |
| | | | $b_2$ | 4.188 | (91) | 4.177 | (91) | 3.987 | (92) |
| | | (25, 30) | $a_1$ | 1.480 | (92) | 1.475 | (93) | 1.324 | (92) |
| | | | $b_1$ | 2.966 | (91) | 2.950 | (91) | 2.270 | (94) |
| | | | $a_2$ | 3.480 | (92) | 3.475 | (93) | 3.275 | (93) |
| | | | $b_2$ | 4.125 | (93) | 4.135 | (92) | 3.944 | (93) |
| | (30, 30) | (25, 35) | $a_1$ | 1.435 | (90) | 1.422 | (93) | 1.300 | (96) |
| | | | $b_1$ | 2.945 | (93) | 2.231 | (92) | 2.217 | (94) |
| | | | $a_2$ | 3.444 | (93) | 3.442 | (93) | 3.241 | (95) |
| | | | $b_2$ | 4.109 | (93) | 4.100 | (94) | 3.901 | (92) |
| | | (25, 50) | $a_1$ | 1.401 | (92) | 1.398 | (94) | 1.241 | (94) |
| | | | $b_1$ | 2.911 | (93) | 2.215 | (93) | 2.200 | (94) |
| | | | $a_2$ | 3.414 | (95) | 3.411 | (93) | 3.223 | (93) |
| | | | $b_2$ | 4.045 | (93) | 4.055 | (91) | 3.840 | (93) |

(1) All results obtained in Tables 3–6 show that the developed method works well in all cases under joint type-I GHCS for Burr XII lifetime products

(2) The results under MLE and noninformative Bayes estimation are closed to itself

(3) The Bayes method performs better than the ML method under informative prior

(4) The MSEs and interval length are reduced under increases in effective sample size $(s, m)$

(5) The results perform better for the large value of test time $\tau^*$

(6) Under parameters change, the results of the simulation study show the validly of the results of all parameters chosen

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah (D-654-363-1441). The authors, therefore, gratefully acknowledge DSR's technical and financial support.

## References

[1] R. D. Gupta and D. Kundu, "Hybrid censoring schemes with exponential failure distribution," *Communications in Statistics-Theory and Methods*, vol. 27, no. 12, pp. 3065–3083, 1998.

[2] D. Kundu and B. Pradhan, "Estimating the parameters of the generalized exponential distribution in presence of hybrid censoring," *Communications in Statistics-Theory and Methods*, vol. 38, no. 12, pp. 2030–2041, 2009.

[3] A. Childs, B. Chandrasekar, N. Balakrishnan, and D. Kundu, "Exact likelihood inference based on Type-I and Type-II hybrid censored samples from the exponential distribution," *Annals of the Institute of Statistical Mathematics*, vol. 55, no. 2, pp. 319–330, 2003.

[4] N. Balakrishnan and R. Aggarwala, *Progressive Censoring – Theory, Methods, and Applications*, Birkhä User, Boston, MA, USA, 2000.

[5] M. G. M. Ghazal, "Prediction of exponentiated family distributions observables under Type-II hybrid CensorednData," *Journal of Statistics Applications and Probability*, vol. 7, no. 2, pp. 307–319, 2018.

[6] B. Chandrasekar, A. Childs, and N. Balakrishnan, "Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring," *Naval Research Logistics*, vol. 51, no. 7, pp. 994–1004, 2004.

[7] U. V. R. Rao, I. R. Savage, and M. Sobel, "Contributions to the theory of rank order statistics: the two-sample censored case," *Annals of Mathematical Statistics*, vol. 31, no. 2, pp. 415–426, 1960.

[8] K. G. Mehrotra and G. K. Bhattacharyya, "Confidence intervals with jointly Type-II censored samples from two exponential distributions," *Journal of the American Statistical Association*, vol. 77, no. 378, pp. 441–446, 1982.

[9] N. Balakrishnan and A. Rasouli, "Exact likelihood inference for two exponential populations under joint Type-II censoring," *Computational Statistics and Data Analysis*, vol. 52, no. 5, pp. 2725–2738, 2008.

[10] A. Rasouli and N. Balakrishnan, "Exact likelihood inference for two exponential populations under joint progressive Type-II censoring," *Communication in Statistics- Theory and Methods*, vol. 39, no. 12, pp. 2172–2191, 2010.

[11] A. R. Shafaya, N. Balakrishnanbc, and Y. Abdel-Atyd, "Bayesian inference based on a jointly Type-II censored sample from two exponential populations," *Journal of Statistical Computation and Simulation*, vol. 84, no. 11, pp. 2427–2440, 2014.

[12] B. N. Al-Matrafi and G. A. Abd-Elmougod, "Statistical inferences with jointly Type-II censored samples from two Rayleigh distributions," *Global Journal of Pure and Applied Mathematics*, vol. 13, no. 12, pp. 8361–8372, 2017.

[13] F. A. Faten A. Momenkhan and G. A. Abd-Elmougod, "Stimations in partially step-stress accelerate life tests with jointly Type-II censored samples from Rayleigh distributions," *Transylvanian Review*, vol. 28, pp. 7609–7616, 2018.

[14] A. Algarni, A. M. Almarashi, G. A. Abd-Elmougod, and Z. A. Abo-Eleneen, "Two compound Rayleigh lifetime distributions in analyses the jointly Type-II censoring samples," *Journal of Mathematical Chemistry*, vol. 58, pp. 950–966, 2019.

[15] E. K. AL-Hussaini and Z. F. Jaheen, "Bayesian prediction bounds for the Burr Type XII failure model," *Communications in Statistics-Theory and Methods*, vol. 24, no. 7, pp. 1829–1842, 1995.

[16] D. Kundu and A. Joarder, "Analysis of Type-II progressively hybrid censored data," *Computational Statistics and Data Analysis*, vol. 50, no. 10, pp. 2509–2528, 2006.

[17] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, "Equations of state calculations by fast computing machines," *The Journal of Chemical Physics*, vol. 21, no. 6, pp. 1087–1091, 1953.