Research Article

The Optimal Rubbling Number of Paths, Cycles, and Grids

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A pebbling move on a graph G consists of the removal of two pebbles from one vertex and the placement of one pebble on an adjacent vertex. Rubbling is a version of pebbling where an additional move is allowed, which is also called the strict rubbling move. In this new move, one pebble each is removed from u and v adjacent to a vertex w , and one pebble is added on w . The rubbling number of a graph G is the smallest number m , such that one pebble can be moved to each vertex from every distribution with m pebbles. The optimal rubbling number of a graph G is the smallest number m , such that one pebble can be moved to each vertex from some distribution with m pebbles. In this paper, we give short proofs to determine the rubbling number of cycles and the optimal rubbling number of paths, cycles, and the grid $P_2 \times P_n$; moreover, we give an upper bound of the optimal rubbling number of $P_m \times P_n$.

1. Introduction

Pebbling in graphs was first introduced by Chung [1]. It has its origin in number theory and also can be viewed as a model for the transportation of resources, starting from a pebble distribution on the vertices of a connected graph.

Let G be a simple connected graph; we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G , respectively. $d(u, v)$ is the distance of u and v , and we write $u \sim v$ if they are adjacent. $N(v) = \{u | u \sim v\}$ is the neighbour of v , and $d(v) = |N(v)|$ is the degree of v . Let H be a subgraph of G , and we use $d_H(v)$ to denote the degree of v in H . A pebble distribution D on G is a function $D: V(G) \rightarrow \mathbb{N}$ ($\mathbb{N}$ is the set of nonnegative integers), where $D(v)$ is the number of pebbles on v and $|D| = \sum_{v \in V(G)} D(v)$ is the size of D .

A pebbling move consists of the removal of two pebbles from a vertex and the placement of one pebble on an adjacent vertex. For two pebble distributions D and D' of G , we say that D contains D' if $D(v) \geq D'(v)$ for all $v \in V(G)$; and D' is reachable from D if we can obtain a distribution which contains D' from D by pebbling moves. For a graph G and a vertex v , we call v a root (or target vertex) if the goal is to place pebbles on v . If t pebbles can be moved to v from D by a

sequence of pebbling moves, then we say that D is t -fold v -solvable, and v is t -reachable from D . If D is t -fold v -solvable for every vertex v , we say that D is t -solvable. Let S be a subset of $V(G)$. If D is v -solvable for all $v \in S$, then we say D is S -solvable.

The t -pebbling number of a graph G , denoted by $f_t(G)$, is the smallest number m , such that every distribution with size m is t -solvable. While $t = 1$, we use $f(G)$ instead of $f_1(G)$, which is called the pebbling number of graph G . The optimal pebbling number $f_{\text{opt}}(G)$ of a graph G is the minimum number m such that there exists a solvable distribution with size m .

Rubbling is a version of pebbling where an additional move is allowed, which is also called the strict rubbling move. In this new move, one pebble each is removed from u and v adjacent to a vertex w , and one pebble is added on w . The t -rubbling number of a graph G , denoted by $\rho_t(G)$, is the smallest number m , such that every distribution with size m is t -solvable by rubbling moves. Similarly, while $t = 1$, we use $\rho(G)$ instead of $\rho_1(G)$, which is called the rubbling number of graph G . The optimal rubbling number $\rho_{\text{opt}}(G)$ of a graph G is the minimum number m such that there exists a solvable distribution with size m by rubbling moves.

There are many papers about pebbling on graphs; one can view the survey paper [2] written by Hurlbert. Rubbling is a new parameter with few results. The basic theory about rubbling and optimal rubbling was developed by Belford and Sieben [3]; they determined the rubbling numbers of trees and cycles, the optimal rubbling numbers of paths and cycles, and so on. The rubbling numbers of complete m -ary trees are studied in [4], the rubbling numbers of caterpillars are given in [5], the optimal rubbling numbers of ladders, prisms, and Möbius ladders are determined in [6], and in [7], they give the bounds for the rubbling numbers of diameter 2 graphs.

In this paper, we give some new proofs to determine the rubbling numbers of cycles and the optimal rubbling numbers of paths, cycles, ladders, prisms, and Möbius ladders. Also, we give an upper bound of the optimal rubbling number of $P_3 \times P_n$, from which we also get an upper bound of the optimal rubbling number of $P_m \times P_n$.

2. Main Result

Assume that $v, w \in N(u)$. A pebbling move from v to u is the removal of two pebbles from v and addition of one pebble on u , denoted by $(v, v \rightarrow u)$; a rubbling move from $\{v, w\}$ to u is the removal of one pebble from v and one pebble from w and addition of one pebble on u , denoted by $(v, w \rightarrow u)$.

The following lemma holds clearly.

Lemma 1 (see [3]). $2^{d(G)} \leq \rho(G) \leq f(G)$.

Definition 1 (see [3]). Given a multiset S of rubbling moves on G , the transition digraph $T(G, S)$ is a directed multigraph whose vertex set is $V(G)$, and each move $(v, w \rightarrow u)$ in S is represented by two directed edges (v, u) and (w, u) . The transition digraph of a rubbling sequence $S = (S_1, \dots, S_n)$ is $T(G, s) = T(G, S)$, where $S = \{s_1, \dots, s_n\}$ is the multiset of moves in s .

Lemma 2 (see [3, 8], no-cycle lemma). *Let S be a sequence of rubbling moves on G , reaching a distribution D . Then, there exists a sequence S^* of rubbling moves, reaching a distribution D^* , such that*

- (1) *On each vertex v , $D^*(v) \geq D(v)$.*
- (2) *$T(G, S^*)$ does not contain any directed cycles.*

A thread in a graph is a path containing vertices of degree 2. By Lemma 2, we can get the following.

Lemma 3 (see [3]). *Let P be a thread in G ; if vertex $x \notin V(P)$ is t -reachable from the pebble distribution D by a sequence of rubbling moves, then x is t -reachable from a rubbling sequence in which there is no strict rubbling move of the form $(v, w \rightarrow u)$ for $u \in V(P)$.*

Definition 2 (see [9]). For a given distribution D on $V(G)$, assume $d(v) = 2$ and $D(v) \geq 3$. A smoothing move from v changes D by removing two pebbles from v and adding one pebble on each neighbour of v .

Lemma 4 (see [9]). *Let D be a distribution on a graph G with distinct vertices u and v , where $d(v) = 2$, $D(v) \geq 3$. If u is t -reachable under D , then u is t -reachable under the distribution D' obtained by making a smoothing move from v .*

A distribution is *smooth*, if it has at most two pebbles on every vertex with degree 2. A vertex v is *unoccupied* under D if $D(v) = 0$.

Lemma 5 (smoothing lemma, [9]). *If G is a connected n -vertex graph, with $n \geq 3$, then G has a smooth optimal distribution with all leaves unoccupied.*

Lemma 6 (see [9]). *In a path with a smooth distribution D having at most two pebbles on each endpoint, let v be an unoccupied vertex. If v is an endpoint, then v is not 2-reachable under D . If v is an inner vertex, then no pebble can be moved out from v without using an edge in both directions.*

Definition 3 (see [9]). A graph H is a quotient of a graph G if the vertices of H correspond to the sets in a partition of $V(G)$, and distinct vertices of H are adjacent if at least one edge of G has endpoints in the sets corresponding to both vertices of H . In other words, each set in the partition of $V(G)$ collapses to a single vertex of H . If H is a quotient of G via the surjective map $\phi: V(G) \rightarrow V(H)$ and D is a distribution on G , then the quotient distribution D^* is the distribution on H defined by $D^*(u) = \sum_{v \in \phi^{-1}(u)} D(v)$.

Lemma 7 (collapsing lemma, [9]). *Let H be a quotient of G and D^* be the quotient distribution in H . Then, D is t -fold v -solvable by a sequence of pebbling moves $\Rightarrow D^*$ is t -fold $\phi(v)$ -solvable by a sequence of pebbling moves. In particular, $f_{opt}(G) \geq f_{opt}(H)$.*

Similarly, we can get the collapsing lemma on rubbling.

Lemma 8 *Let H be a quotient of G and D^* be the quotient distribution on H . Then, D is t -fold v -solvable by a sequence of rubbling moves $\Rightarrow D^*$ is t -fold $\phi(v)$ -solvable by a sequence of rubbling moves. In particular, $\rho_{opt}(G) \geq \rho_{opt}(H)$.*

Proof. The proof is similar to the proof of Lemma 7 in [9].

2.1. Rubbling and Optimal Rubbling in Paths and Cycles

Lemma 9 (see [10, 11]). *The t -pebbling numbers of the cycles C_{2n+1} and C_{2n} are*

$$f_t(C_{2n+1}) = 2\left\lfloor \frac{2^{n+1}}{3} \right\rfloor + (t-1)2^n + 1, \quad (1)$$

$$f_t(C_{2n}) = 2^n t.$$

In this section, we give short proofs about the rubbling number of cycles and the optimal rubbling numbers of paths and cycles, which are determined in [3].

Theorem 1 (see [3]). *The rubbling numbers of cycles are*

$$\begin{aligned}\rho(C_{2n}) &= 2^n, \\ \rho(C_{2n+1}) &= \left\lfloor \frac{7 \cdot 2^{n-1} - 2}{3} \right\rfloor + 1.\end{aligned}\quad (2)$$

Proof. For the even cycle C_{2n} , by Lemma 1, we have $2^n \leq \rho(C_{2n}) \leq f(C_{2n}) = 2^n$, over.

For the odd cycle C_{2n+1} , assume the target vertex is v_0 which is adjacent to v_1 and v_{-1} . Let D be a pebble distribution on C_{2n+1} , and $D(v_0) = 0$. We collapse the vertices $\{v_{-1}, v_0, v_1\}$ into one vertex x to get a new graph C_{2n-1} and the quotient distribution D^* on C_{2n-1} , with $D^*(x) = D(v_{-1}) + D(v_0) + D(v_1)$ and $D^*(y) = D(y)$ for $y \neq x$.

By Lemma 3, we may assume that there is no strict rubbing move of the form $(v, w \rightarrow u)$ where $u \neq v_0$. So, one pebble can be moved to v_0 from D on C_{2n+1} by a sequence of rubbing moves $\Leftrightarrow$ two pebbles can be moved to x from D^* on C_{2n-1} by a sequence of pebbling moves. Thus, $\rho(C_{2n+1}) = f_2(C_{2n-1}) = 2\lfloor 2^n/3 \rfloor + 2^{n-1} + 1$ (the last equality holds from Lemma 9). A simple calculation can show that $2\lfloor 2^n/3 \rfloor + 2^{n-1} + 1 = \lfloor (7 \cdot 2^{n-1} - 2)/3 \rfloor + 1$.

Lemma 10 (see [9]). *The optimal pebbling numbers of paths and cycles are*

$$f_{\text{opt}}(P_n) = f_{\text{opt}}(C_n) = \left\lceil \frac{2n}{3} \right\rceil. \quad (3)$$

Moreover, we can get the following lemma.

Lemma 11. *Let D be a pebble distribution with $2k$ pebbles on $P_{3k} = v_1, \dots, v_{3k}$. If D is solvable by pebbling moves, then $D(v_{3i+2}) = 2$ for $0 \leq i \leq k-1$, and $D(v) = 0$ otherwise.*

Proof. We show it by induction on k ; it holds for $k = 1$.

We can move $\lfloor D(v_1)/2 \rfloor$ pebbles from v_1 to $P_{3k-1} = v_2, \dots, v_{3k}$, so by induction, $2k - D(v_1) + \lfloor D(v_1)/2 \rfloor \geq f_{\text{opt}}(P_{3k-1}) = 2k$, and thus $D(v_1) = 0$. Similarly, $D(v_{3k}) = 0$.

We make smoothing moves on P_{3k} from D to obtain a smooth pebble distribution D^* .

Since $|D| = 2k < n = 3k$, there exist many vertices unoccupied under D^* . Assume v_i is a vertex unoccupied under D^* . Let $L_1 = v_1, \dots, v_i$ and $L_2 = v_{i+1}, \dots, v_{3k}$. By Lemma 6, we cannot move two pebbles to v_i using only one direction.

If v_i is solvable by $(v_{i-1}, v_{i-1} \rightarrow v_i)$, then we can get that $D^*|_{L_1}$ is L_1 -solvable, and thus $|D^*| \geq f_{\text{opt}}(P_i) + f_{\text{opt}}(P_{3k-i})$. Assume $i = 3j + r$ for $0 \leq r \leq 2$. If $r = 0$, then $|D^*| \geq 2j + 2(k-j) = 2k$. If $r = 1$, then $|D^*| \geq 2j + 1 + 2(k-j) = 2k + 1$, a contradiction to $|D| = 2k$. If $r = 2$, then $|D^*| \geq 2j + 2 + 2(k-j-1) + 1 = 2k + 1$, a contradiction to $|D| = 2k$, and thus $i = 3j$.

By a similar argument, if v_i is solvable by $(v_{i+1}, v_{i+1} \rightarrow v_i)$, then we can get $i = 3j + 1$.

Thus, we can always partition the path P_{3k} into two paths $L_1 = v_1, \dots, v_{3j}$ and $L_2 = v_{3j+1}, \dots, v_{3k}$, and $D^*|_{L_i}$ is solvable in L_i for $i = 1, 2$, respectively.

By induction, we know that $D^*(v_{3i+2}) = 2$ for $0 \leq i \leq k-1$, and $D^*(v) = 0$ otherwise.

Note that making a smooth move on the vertex v leaves at least one pebble on v , which means that if we can make a smooth move under D , then at least two adjacent vertices have pebbles under D^* , a contradiction to $D^*(v_{3i+2}) = 2$, and $D^*(v) = 0$ otherwise. So, $D = D^*$, which completes the proof.

In [3], Belford and Sieben determined the optimal rubbing number of paths and cycles; here we give a short proof.

Theorem 2 (see [3]). *The optimal rubbing number of path is $\rho_{\text{opt}}(P_n) = \lceil n + 1/2 \rceil$, and the optimal rubbing number of cycle is $\rho_{\text{opt}}(C_n) = \lceil n/2 \rceil$ for $n \geq 3$.*

Proof

Upper Bound. For $P_n = v_1, v_2, \dots, v_n$, let D be a distribution so that $D(v_i) = 1$ for i is odd or $i = n$, and $D(v_i) = 0$ otherwise; clearly D is solvable, and $|D| = \lceil n + 1/2 \rceil$. For $C_n = v_1, v_2, \dots, v_n$, let D be a distribution so that $D(v_i) = 1$ for i is odd, and $D(v_i) = 0$ otherwise; clearly D is solvable, and $|D| = \lceil n/2 \rceil$.

Lower Bound. First we show it for the path P_n . We use induction on n . By a simple calculation, one can show that it holds for $n \leq 3$.

If $n > 3$, assume it holds for P_j for all $j < n$. From the upper bound and Lemma 10, we know that $\rho_{\text{opt}}(P_n) < f_{\text{opt}}(P_n)$. Let D be a solvable pebble distribution with $\rho_{\text{opt}}(P_n)$ pebbles on P_n ; there must exist a vertex v_i , which is reachable only by the strict rubbing move $(v_{i-1}, v_{i+1} \rightarrow v_i)$. Let $P_1 = v_1, \dots, v_{i-1}$, $P_2 = v_{i+1}, \dots, v_n$ be two subpaths of P_n ; then, by Lemma 3, we do not need the strict rubbing move $(v_{i-1}, v_{i+1} \rightarrow v_i)$ to solve the vertices in P_i for $i = 1, 2$. That means $D|_{P_i}$ is v -solvable for all $v \in P_i$ ($i = 1, 2$), so $|D|_{P_i} \geq \rho_{\text{opt}}(P_i)$ for $i = 1, 2$. So, $|D| = |D|_{P_1} + |D|_{P_2} \geq \lceil i/2 \rceil + \lceil n - i + 1/2 \rceil \geq \lceil n + 1/2 \rceil$.

Now we show it for the cycle C_n ; similarly, by a simple calculation, one can show that it holds for $n = 3$.

If $n > 3$, from the upper bound and Lemma 10, we know that $\rho_{\text{opt}}(C_n) < f_{\text{opt}}(C_n)$. Thus, let D be an optimal pebble distribution with $\rho_{\text{opt}}(C_n)$ pebbles on C_n , and there must exist a vertex v_i , which is reachable only by a strict rubbing move $(v_{i-1}, v_{i+1} \rightarrow v_i)$. Then, by Lemma 3, we do not need the strict rubbing move $(v_{i-1}, v_{i+1} \rightarrow v_i)$ to solve the vertices in $C_n \setminus v_i$, and thus D is solvable on $P_{n-1} = C_n \setminus v_i$, so $|D| \geq \rho_{\text{opt}}(P_{n-1}) = \lceil n/2 \rceil$, which completes the proof.

3. Optimal Rubbling in Ladders and Prisms

Let G and H be simple connected graphs; we define the Cartesian product $G \times H$ to be the graph with vertex set $V(G \times H)$ and edge set the union of $\{(av, bv) | (a, b) \in E(G), v \in E(H)\}$ and $\{(ux, uy) | u \in V(G), (x, y) \in E(H)\}$. We call $P_n \times P_2$ a *ladder* and $C_n \times P_2$ a *prism*. It is clear that a prism can be obtained from a ladder by joining the 4 endvertices by two edges to form two vertex-disjoint C_n subgraphs. If the four endvertices are joined by

two new edges in a switched way to get a C_{2n} subgraph, then a Möbius ladder M_n is obtained.

The optimal rubbling numbers of ladders, prisms, and Möbius ladders are determined in [6]; here we give new proofs of these results.

Let $P_n = v_1, v_2, \dots, v_n$ and $P_2 = xy$. The vertices of $P_n \times P_2$ are denoted by $v_i x$ and $v_i y$ for $1 \leq i \leq n$. Let $H_n = (P_{n+1} \times P_2) \setminus v_{n+1} x$. Then, we have the following.

Theorem 3. Let $n = 3k + r$, $0 \leq r \leq 2$; the optimal rubbling number of the ladder $P_n \times P_2$ [6] is

$$\rho_{\text{opt}}(P_{3k+r} \times P_2) = \begin{cases} 2k+1 & \text{if } r = 0 \\ 2k+2 & \text{if } r = 1 = \lceil \frac{2(n+1)}{3} \rceil \\ 2k+2 & \text{if } r = 2 \end{cases} \quad (4)$$

The optimal rubbling number of H_n is

$$\rho(H_n) = \rho(P_{n-1} \times P_2) + 1. \quad (5)$$

Proof

Upper Bound. Let D be a distribution on $P_n \times P_2$ or $H_n \times P_2$.

If $r = 0$, then $D(v_i x) = 1$ for $i \equiv 1 \pmod{3}$; $D(v_i y) = 1$ for $i \equiv 2 \pmod{3}$ and $i \neq n$; $D(v_n y) = 2$ and $D(v) = 0$ otherwise.

If $r = 1$, then $D(v_i x) = 1$ for $i \equiv 1 \pmod{3}$ and $i \neq n$; $D(v_i y) = 1$ for $i \equiv 2 \pmod{3}$; $D(v_n y) = 2$ and $D(v) = 0$ otherwise.

If $r = 2$, for $P_n \times P_2$, then $D(v_i x) = 1$ for $i \equiv 1 \pmod{3}$; $D(v_i y) = 1$ for $i \equiv 2 \pmod{3}$; $D(v) = 0$ otherwise. For H_n , we add one more pebble on $v_{n+1} y$.

One can check that D is solvable for all $P_n \times P_2$ and H_n .

Lower Bound. Note that the diameter of $P_n \times P_2$ is n , and let R_i be the set of vertices with distance i from $v_1 x$ ($1 \leq i \leq n$), $R_0 = v_1 x$; if we collapse R_i into one vertex r_i for $0 \leq i \leq n$, then we can get a path $\tilde{P}_{n+1} = r_0, \dots, r_n$ (see Figure 1). By Lemma 10, the optimal pebbling number of $\tilde{P}_{n+1}$ is $f_{\text{opt}}(\tilde{P}_{n+1}) = \lceil 2(n+1)/3 \rceil$. By Lemma 8, $\rho_{\text{opt}}(H_n) \geq \rho_{\text{opt}}(P_n \times P_2)$. So, we only need to show that $\rho_{\text{opt}}(P_n \times P_2) \geq f_{\text{opt}}(\tilde{P}_{n+1})$ and $\rho_{\text{opt}}(H_{3k+2}) \geq 2k+3$.

We use induction on n for both H_n (only need to show while $n \equiv 2 \pmod{3}$) and $P_n \times P_2$; it holds for $n \leq 2$ clearly.

Assume it holds for $h < n$.

Case 1. First we consider $P_n \times P_2$, and let D be a solvable distribution on $P_n \times P_2$ with $\rho_{\text{opt}}(P_n \times P_2)$ pebbles; then, we collapse R_i into one vertex r_i for $0 \leq i \leq n$ to get a path $\tilde{P}_{n+1}$ with length n and the quotient distribution D^* on $\tilde{P}_{n+1}$. By Lemma 8, D^* is solvable in $\tilde{P}_{n+1}$. We will show that $\rho_{\text{opt}}(P_n \times P_2) \geq f_{\text{opt}}(\tilde{P}_{n+1})$.

Assume $\rho_{\text{opt}}(P_n \times P_2) < f_{\text{opt}}(\tilde{P}_{n+1})$; since D^* is a solvable distribution on $\tilde{P}_{n+1}$, there must exist some r_i of $\tilde{P}_{n+1}$ which can be reachable only by the strict rubbling move

FIGURE 1: $P_n \times P_2$.

$(r_{i-1}, r_{i+1} \rightarrow r_i)$. By Lemmas 3 and 8, we claim the following.

Claim 1. At most one pebble can be moved to r_{i-1} (r_{i+1}) under D^* and at most one pebble can be moved to R_{i-1} (R_{i+1}) under D .

Let $Q_1 = r_0, \dots, r_{i-1}$, $Q_2 = r_{i+1}, \dots, r_n$ be two subpaths of $\tilde{P}_{n+1}$ and W_1 and W_2 be the subgraphs of $P_n \times P_2$ which induce Q_1 and Q_2 , respectively.

Now we will show that $|D|_{W_j}| \geq \rho_{\text{opt}}(W_j)$ for $j = 1, 2$. Note that W_1 is isomorphic to H_{i-1} and W_2 is isomorphic to H_{n-i-1} .

Notation. For simplicity, a rubbling move $(R_{i-1}, R_{i+1} \rightarrow R_i)$ in $P_n \times P_2$ means a rubbling move $(v, u \rightarrow w)$ for some $v \in R_{i-1}$, $u \in R_{i+1}$ and $w \in R_i$. And similarly, we can define a rubbling move $(R_{i-1}, u \rightarrow R_i)$ and so on.

If $D|_{W_j}$ is W_j -solvable, then $|D|_{W_j}| \geq \rho_{\text{opt}}(W_j)$, and we are done. Assume $D|_{W_j}$ is not W_j -solvable for some j ; without loss of generality, assume $D|_{W_2}$ is not W_2 -solvable. If $i = n-1$, then $W_2 = \{v_n y\}$, and the strict rubbling move $(r_{n-2}, r_n \rightarrow r_{n-1})$ means that $D^*(r_n) \geq 1$, which means $D(v_n y) \geq 1$, a contradiction to $D|_{W_2}$ is not W_2 -solvable. Thus, $i < n-1$.

Since $D|_{W_2}$ is not W_2 -solvable, to solve some vertex of W_2 , we must use the pebbles on W_1 . Note that we can move at most one pebble on R_{i-1} from $D|_{W_1}$, and we must use the strict rubbling move $(R_{i-1}, R_{i+1} \rightarrow R_i)$ under D .

Consider the vertex $v_i y \in R_i$, $N(v_i y) = R_{i-1} \cup \{v_{i+1} y\}$. Since $v_i y$ is reachable from D , we can move two pebbles to $N(v_i y)$. From Claim 1, we can get that we can move one pebble to R_{i-1} and one pebble to $v_{i+1} y$ at the same time. Thus, $v_{i+1} y$ is reachable under $D|_{W_2}$.

Subcase 1. $D(v_{i+1} y) = 0$. Then, we can move two pebbles on $v_{i+2} y$ under $D|_{W_2}$ (for $v_{i+1} y$ is just adjacent to one vertex $v_{i+2} y$ of W_2). So, $v_{i+2} x$ is reachable under $D|_{W_2}$, too.

Note that the pebble moved on R_i cost at least two pebbles on $v_{i+2} y$ and we can move at most one pebble to R_{i+1} by using this pebble. We can use $(R_{i+2}, R_{i+2} \rightarrow R_{i+1})$ instead of the rubbling sequence which contains the strict rubbling move $(R_{i-1}, R_{i+1} \rightarrow R_i)$ and aim to move one pebble to R_{i+1} , so $D|_{W_2}$ is W_2 -solvable.

Subcase 2. $D(v_{i+1} y) = 1$. Then, at most one pebble can be moved to R_{i+2} under $D|_{W_2}$; otherwise, one more pebble can be moved from r_{i+2} to r_{i+1} , and then there are two pebbles on r_{i+1} , and one pebble can be moved to r_i under D^* , a contradiction.

By a similar argument, we can show there exists some vertex in W_2 so that we must use the strict rubbing move $(R_{i-1}, v_{i+1}y \rightarrow v_{i+1}x)$ to solve this vertex, and the next rubbing move must be $(R_{i+2}, v_{i+1}x \rightarrow v_{i+2}x)$. Since at most one pebble can be moved to R_{i+2} under $D|_{W_2}$, exactly one pebble can be moved to R_{i+2} under $D|_{W_2}$.

- (a) The vertex of R_{i+2} used in $(R_{i+2}, v_{i+1}x \rightarrow v_{i+2}x)$ is $v_{i+3}x$; then, if it continues, the rubbing move must be $(v_{i+3}y, v_{i+2}x \rightarrow v_{i+2}y)$, but then we can use $(v_{i+1}y, v_{i+3}y \rightarrow v_{i+2}y)$ instead, and thus we must use the strict rubbing move $(R_{i-1}, R_{i+1} \rightarrow R_i)$ to solve at most one vertex $v_{i+2}x$. Let $P_{\text{left}} = v_{i+3}, \dots, v_n$; then, the vertices of $P_{\text{left}} \times P_2$ are reachable without the strict rubbing moves and the pebble on $v_{i+1}y$. So, $|D|_{W_2}| - 1 \geq \rho_{\text{opt}}(P_{\text{left}} \times P_2)$, and by induction, $\rho_{\text{opt}}(P_{\text{left}} \times P_2) + 1 = \rho_{\text{opt}}(W_2)$, over.
- (b) The vertex of R_{i+2} used in $(R_{i+2}, v_{i+1}x \rightarrow v_{i+2}x)$ is $v_{i+2}y$; since at most one pebble can be moved to R_{i+2} , we may assume $D(v_{i+2}y) = 1$ (if not, then we can move two pebbles to $v_{i+3}y$, and then one pebble can be moved to $v_{i+3}x$; this is just (a)). Thus, we use one pebble each on R_{i-1} , $v_{i+1}y$, and $v_{i+2}y$ to move one pebble to $v_{i+2}x$. Then, we rearrange the distribution on $P_n \times P_2$ as follows: remove the pebble on $v_{i+1}y$, and add it on $v_{i+2}y$; one can view that we can still solve R_i and R_{i+1} . Thus, in the new distribution, we do not need the strict rubbing move $(R_{i-1}, R_{i+1} \rightarrow R_i)$ to solve W_2 , so $|D|_{W_2}| \geq \rho_{\text{opt}}(W_2)$.

Thus, $|D| = |D|_{W_1}| + |D|_{W_2}| \geq \rho_{\text{opt}}(W_1) + \rho_{\text{opt}}(W_2) = \rho_{\text{opt}}(H_{i-1}) + \rho_{\text{opt}}(H_{n-i-1})$.

Assume that $n = 3k + r$ and $i = 3j + s$ for $0 \leq r, s \leq 2$; by induction, we can get Table 1, where the lower bound is given by $\rho_{\text{opt}}(H_{i-1}) + \rho_{\text{opt}}(H_{n-i-1})$.

From Table 1, we can find $|D| \geq f_{\text{opt}}(L_{n+1})$, which is a contradiction to the assumption that $\rho_{\text{opt}}(P_n \times P_2) < f_{\text{opt}}(L_{n+1})$, and we are done.

Case 2. Now we consider H_{3k+2} . Note that the diameter of H_{3k+2} is $3k + 3$. Let R_i be the vertices set $\{v_i x, v_i y\}$ for $1 \leq i \leq 3k + 2$, and $R_{3k+3} = \{v_{3k+3} y\}$. If we collapse R_i into one vertex r_i for $1 \leq i \leq 3k + 3$, then we can get a path $\tilde{P}_{3k+3} = r_1 \dots r_{3k+3}$ (see Figure 2). By Lemma 10, the optimal pebbling number of $\tilde{P}_{3k+3}$ is $f_{\text{opt}}(\tilde{P}_{3k+3}) = \lceil 2(3k + 3)/3 \rceil = 2k + 2$. So, we only need to show that $\rho_{\text{opt}}(H_{3k+2}) > f_{\text{opt}}(\tilde{P}_{3k+3})$.

Let D be a solvable distribution with $\rho_{\text{opt}}(H_{3k+2})$ pebbles on H_{3k+2} and D^* be the quotient distribution on $\tilde{P}_{3k+3}$. So, D^* is solvable on $\tilde{P}_{3k+3}$ by Lemma 8.

Subcase 3. If $\rho_{\text{opt}}(H_{3k+2}) < f_{\text{opt}}(\tilde{P}_{3k+3})$, since D^* is a solvable distribution on $\tilde{P}_{3k+3}$, there must exist some vertex r_i of $\tilde{P}_{3k+3}$ which can be reachable. Assume that W_1 and W_2 be two disjoint subgraphs of H_{3k+2} which induce $Q_1 = r_1, \dots, r_{i-1}$ and $Q_2 = r_{i+1}, \dots, r_{3k+3}$, respectively.

We will show that $D|_{W_j}$ is W_j -solvable for $j = 1, 2$. Otherwise, we must use the strict rubbing move $(R_{i-1}, R_{i+1} \rightarrow R_i)$ under D to solve one vertex of W_i ; this move must be one of $(v_{i-1}x, v_{i+1}x \rightarrow v_i x)$ and $(v_{i-1}y, v_{i+1}y \rightarrow v_i y)$. Without loss of generality, assume the rubbing move is $(v_{i-1}x, v_{i+1}x \rightarrow v_i x)$; then, by Lemma 2, the target vertex of the next rubbing move using the pebble on $v_i x$ cannot be $v_{i+1}x$ or $v_{i-1}x$; if the target vertex is $v_i y$, we need one more pebble on R_{i-1} or R_{i+1} , a contradiction to the condition that r_i can be reachable only by the strict rubbing move $(r_{i-1}, r_{i+1} \rightarrow r_i)$.

So, $D|_{W_i}$ is W_i -solvable, and thus $|D|_{W_i}| \geq \rho_{\text{opt}}(W_i)$ for $i = 1, 2$. Note that W_1 is isomorphic to $P_{i-1} \times P_2$ and W_2 is isomorphic to H_{3k+2-i} . Thus, $|D| \geq \rho_{\text{opt}}(P_{i-1} \times P_2) + \rho_{\text{opt}}(H_{3k+2-i})$. Assume that $i = 3j + s$ for $0 \leq s \leq 2$; by induction, we can get Table 2, where the lower bound is given by $\rho_{\text{opt}}(P_{i-1} \times P_2) + \rho_{\text{opt}}(H_{3k+2-i})$.

From Table 2, we can find $\rho_{\text{opt}}(H_{3k+2}) > f_{\text{opt}}(L_{3k+3})$, a contradiction to the assumption $\rho_{\text{opt}}(H_{3k+2}) < f_{\text{opt}}(L_{3k+3})$, over.

Subcase 4. If $\rho_{\text{opt}}(H_{3k+2}) = f_{\text{opt}}(L_{3k+3}) = 2k + 2$, then D^* is solvable on $\tilde{P}_{3k+3}$. If there exists a vertex r_i which is reachable only by a strict rubbing move under D^* , then we can get a contradiction by a similar argument of Subcase 3.

So, we only need to consider the case that all vertices of $\tilde{P}_{3k+3}$ are reachable by only pebbling moves. By Lemma 11, we have $D^*(R_{3i+2}) = 2$ for $0 \leq i \leq k$, and $D^*(v) = 0$ otherwise. Thus, at least one of $\{v_1 x, v_1 y\}$ is not solvable under D , which completes the proof.

Theorem 4 (see [6]). *The optimal rubbing number of prism $C_n \times P_2$ is*

$$\rho_{\text{opt}}(C_{3k+r} \times P_2) = \begin{cases} 2k, & \text{if } r = 0, \\ 2k + 1, & \text{if } r = 1 = \lceil \frac{2n}{3} \rceil, \\ 2k + 2, & \text{if } r = 2, \end{cases} \quad (6)$$

except $\rho_{\text{opt}}(C_3 \times P_2) = 3$.

Proof. It holds for $n = 3$; we only need to show $\rho_{\text{opt}}(C_n \times P_2) = \rho_{\text{opt}}(P_{n-1} \times P_2)$ for $n \geq 4$.

Upper Bound. $P_{n-1} \times P_2$ can be viewed as a subgraph of $C_n \times P_2$; it is clear that a solvable distribution with $\rho_{\text{opt}}(P_{n-1} \times P_2)$ pebbles given in the proof of Theorem 3 is solvable on $C_n \times P_2$ for $n \geq 4$, and we are done.

Lower Bound. Let $C_n = v_1, \dots, v_n$, $P_2 = xy$. Assume D is a solvable distribution with $\rho_{\text{opt}}(C_n \times P_2)$ pebbles, and we collapse each set $\{v_i x, v_i y\}$ into one vertex R_i (see Figure 3); then, we get a cycle $C_n = R_1, \dots, R_n$ and the quotient distribution D^* . By Lemma 8, D^* is solvable on C_n . By Lemma 10, $f_{\text{opt}}(C_n) = \lceil 2n/3 \rceil$, and we only need to show $\rho_{\text{opt}}(C_n \times P_2) = f_{\text{opt}}(C_n)$.

Assume $\rho_{\text{opt}}(C_n \times P_2) < f_{\text{opt}}(C_n)$; then, there must exist a vertex R_i which is reachable under D^* only by the strict rubbing move $(R_{i-1}, R_{i+1} \rightarrow R_i)$. By a similar argument of

TABLE 1: Bounds on optimal rubbing of $P_n \times P_2$.

(r, s)	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)	(2, 0)	(2, 1)	(2, 2)
$\rho_{\text{opt}}(H_{i-1})$	$2j+1$	$2j+1$	$2j+2$	$2j+1$	$2j+1$	$2j+2$	$2j+1$	$2j+1$	$2j+2$
$\rho_{\text{opt}}(H_{n-i-1})$	$2(k-j)+1$	$2(k-j)$	$2(k-j)-1$	$2(k-j)+1$	$2(k-j)+1$	$2(k-j)$	$2(k-j)+2$	$2(k-j)+1$	$2(k-j)+1$
Lower bound	$2k+2$	$2k+1$	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$2k+3$	$2k+2$	$2k+3$
$f_{\text{opt}}(L_{n+1})$	$2k+1$	$2k+1$	$2k+1$	$2k+2$	$2k+2$	$2k+2$	$2k+2$	$2k+2$	$2k+2$

FIGURE 2: H_{3k+2} .TABLE 2: Bounds on optimal rubbing of H_{3k+2} .

s	0	1	2
$\rho_{\text{opt}}(P_{i-1} \times P_2)$	$2j+1$	$2j+2$	$2j+2$
$\rho_{\text{opt}}(H_{3k+2-i})$	$2(k-j)+2$	$2(k-j)+1$	$2(k-j)+1$
Lower bound	$2k+3$	$2k+3$	$2k+3$
$f_{\text{opt}}(L_{3k+3})$	$2k+2$	$2k+2$	$2k+2$

FIGURE 3: $C_n \times P_2$.

Case 3 in the proof of Theorem 3, if we remove $\{v_i^x, v_i^y\}$ from $C_n \times P_2$ (which is just isomorphic to $P_{n-1} \times P_2$), then D is still solvable. So, $|D| \geq \rho_{\text{opt}}(P_{n-1} \times P_2) = \lceil 2n/3 \rceil$, which is a contradiction to the assumption $\rho_{\text{opt}}(C_n \times P_2) < f_{\text{opt}}(C_n)$, and we are done.

Theorem 5 (see [6]). *The optimal rubbing number of the Möbius ladder M_n is*

$$\rho_{\text{opt}}(M_n) = \rho_{\text{opt}}(C_n \times P_2). \quad (7)$$

Proof. The proof is similar to the proof of prisms.

4. Bounds on the Optimal Rubbling of $P_m \times P_n$

In [12], the optimal pebbling number of $P_3 \times P_n$ is determined.

Theorem 6 (see [12]). *The optimal pebbling number of $P_3 \times P_n$ is $n+1$.*

In [13], the lower bound of the optimal pebbling number of $P_m \times P_n$ is given.

Theorem 7 (see [13]). *The optimal pebbling number of $P_m \times P_n$ is no less than $(2/13)mn$ when $m, n \geq 5$.*

Here we give an upper bound of the optimal rubbing number of $P_m \times P_n$ by giving a solvable distribution on $P_m \times P_n$.

Theorem 8. $\rho_{\text{opt}}(P_3 \times P_n) \leq \lceil 5n + 1/6 \rceil + 1$.

Proof. We give a solvable distribution D_n on $P_3 \times P_n$ with exactly $\lceil 5n + 1/6 \rceil + 1$ pebbles. While $n \leq 7$, the solvable distribution D_n is given in Figure 4.

While $n > 7$, assume $n \equiv r \pmod{6}$ for $0 \leq r \leq 5$. Then, the solvable distribution D_n on $P_3 \times P_n$ can be obtained from D_{n-6} on $P_3 \times P_{n-6}$ while $r \neq 1$ by placing five more pebbles on $P_3 \times P_6$ which is added on the left side of $P_3 \times P_{n-6}$ (see Figure 5(a)). So, $|D_n| = |D_{n-6}| + 5$ for $n \equiv r \pmod{6}$ with $r \neq 1$.

Assume $n > 7$ and $n \equiv 1 \pmod{6}$. Then, the solvable distribution D_n on $P_3 \times P_n$ can be obtained from D_{n-5} on $P_3 \times P_{n-5}$ by placing four more pebbles on $P_3 \times P_5$ which is added on the left side of $P_3 \times P_{n-5}$ (see Figure 5(b)). So, $|D_n| = |D_{n-5}| + 4$ for $n \equiv 1 \pmod{6}$.

FIGURE 4: A solvable distribution D_n on $P_3 \times P_n$ for $n \leq 7$. FIGURE 5: A solvable distribution D_n from D_{n-6} or D_{n-5} . (a) D_n from D_{n-6} . (b) D_n from D_{n-5} .

By a simple calculation, we have $|D_n| = \lceil 5n + 1/6 \rceil + 1$, and we are done.

One can check that the equality holds for $n \leq 7$, and here we give a conjecture. $\square$

Conjecture 1. $\rho_{opt}(P_3 \times P_n) = \lceil 5n + 1/6 \rceil + 1$.

Now we can get a bound of the optimal rubbling number of $P_m \times P_n$. We give a partition of P_m , for example, $m = x + 1 + y$; then, we can get three disjoint subgraphs of $P_m \times P_n$ called $H_1 = P_x \times P_n$, $H_2 = P_1 \times P_n$ and $H_3 = P_y \times P_n$, and then we put some pebbles on H_1 and H_3 , so that the pebbles on H_i can solve H_i for $i = 1, 3$. Clearly, this is a solvable distribution of $P_m \times P_n$. Thus, we have the following.

Theorem 9. $\rho_{opt}(P_4 \times P_n) \leq \rho_{opt}(P_n) + \rho_{opt}(P_2 \times P_n)$,
 $\rho_{opt}(P_5 \times P_n) \leq \rho_{opt}(P_2 \times P_n) + \rho_{opt}(P_2 \times P_n)$.

Theorem 10. If $m = 4k + r \geq 6$, $\rho_{opt}(P_{4k} \times P_n) \leq (k-1)\rho_{opt}(P_3 \times P_n) + \rho_{opt}(P_2 \times P_n) + \rho_{opt}(P_n)$,
 $\rho_{opt}(P_{4k+1} \times P_n) \leq (k-1)\rho_{opt}(P_3 \times P_n) + 2\rho_{opt}(P_2 \times P_n)$,
 $\rho_{opt}(P_{4k+2} \times P_n) \leq k\rho_{opt}(P_3 \times P_n) + \rho_{opt}(P_2 \times P_n)$,
 $\rho_{opt}(P_{4k+3} \times P_n) \leq (k+1)\rho_{opt}(P_3 \times P_n)$.

Proof. If $m = 4k$, then $m = 3 + 1 + 3 + 1 + \dots + 3 + 1 + 2 + 1 + 1$ which gives a partition of $P_m = P_3 + P_1 + P_3 + P_1 + \dots + P_3 + P_1 + P_2 + P_1 + P_1$. We put the pebbles on $P_i \times P_n$ except the inner subpaths $P_1 \times P_n$, so that the pebbles on each $P_i \times P_n$ can solve themselves, and thus we need $(k-1)\rho_{opt}(P_3 \times P_n) + \rho_{opt}(P_2 \times P_n) + \rho_{opt}(P_n)$ pebbles,

which is a solvable distribution. Similarly, we can get the other inequalities, which completes the proof. $\square$

Corollary 1. If $6 \leq m \leq n$, then $\rho_{opt}(P_m \times P_n) \leq (5/24)mn + (1/3)n + (1/2)m + (1/2)$.

Proof. From Theorems 2, 3, 8, and 10, we can get the result by a simple calculation. $\square$

Data Availability

No data were used to support the findings of the study.

Disclosure

This manuscript has been released as a pre-print at <https://arxiv.org/pdf/1909.01703v1.pdf>.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of the article.

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