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Research Article

Dynamics Analysis of a Stochastic Plant Disease Model with Continuous Cultural Control Strategy

Haisu Zhang and Tongqian Zhang

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

Correspondence should be addressed to Tongqian Zhang; zhangtongqian@sdust.edu.cn

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In this paper, a new stochastic plant disease model with continuous control strategy is proposed and analyzed. The dynamics of the system are explored under white noise disturbance. We prove that if $R_1 > 1$, then the disease is persistent; moreover, if $R_2 > 1$, then the solutions of the system have a stationary distribution. For the special case, we prove that if $R_1 < 1$, then the disease will eventually disappear. Finally, some numerical simulations were implemented to illustrate the theoretical results.

1. Introduction

Plant disease is an important constraint on global crop production and severely damages the yield and quality of crops. The main sources of infections that cause crop disease are viruses, bacteria, fungi, nematodes, and plant disease caused by parasitic plants. Plant viruses are a common source of infection that cause plant disease. Plant virologists have selected 10 plant viruses that harm plant growth [1]. Plant viruses have caused a large reduction in crop production. For example, in the late 1940s, tobacco mosaic virus caused an average loss of approximately 40 million pounds of tobacco in the United States each year, accounting for 2%-3% of tobacco production [2]. For tomato growers in Florida, the cost of producing an acre of tomatoes is about 6000–7000 US dollars, and nearly 25% of the expenses are used to control the tomato yellow leaf curl virus [3]. In the past ten years, cassava in East Africa and Central Africa has lost 47% of its production due to cassava brown streak virus and cassava mosaic virus [4, 5].

Plant infectious diseases are mainly infectious diseases caused by pathogens, which are generally passively spread by external forces, which can cause the occurrence and prevalence of disease. The main modes of transmission include air current transmission, rain transmission, insect and other biological transmission, and human factor transmission [6, 7]. Farmers have taken a variety of measures to control the harm of plant viruses, such as developing new plants with stronger resistance, cultural, physical, chemical, and biological control measures, and so on. However, a single control measure cannot control the disease well because it generally only plays an important role in the early stage of disease development. Facts have showed that the comprehensive utilization of several different control measures can achieve a good effect of disease control. This combined control strategy is often called integrated disease management (IDM) [8]. Researchers have established many mathematical models to explain the implementation process of IDM [9–14]. Recently, considering continuous replanting and roguing or removing diseased plants, a simple plant-virus disease model by ordinary differential equations has been proposed:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \sigma\phi - \beta S(t)I(t) - \eta S(t), \\[2ex] \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \sigma(1-\phi) + \beta S(t)I(t) - (\eta+\omega)I(t), \end{cases} \tag{1}$$

where S, I are the densities of susceptible and infected plants, respectively, σ is the total rate of the plants, $1-\phi\,(0<\phi\le 1)$ is the proportion of infected plants in the replanted plants, β

represents the transmission rate of diseased plants, η is the death rate of the plant, and ω represents the roguing (or removal) rate for the infected plants. For detailed parameter meaning and model explanation, we refer the readers to [11, 15]. Obviously, model (1) is different from the general infectious disease model because it considers the characteristics of plant disease transmission, that is, the disease can be transmitted by infecting seedlings or seeds, and it can also be transmitted by infecting mature seedlings. For system (1), if $0 < \phi < 1$, simple mathematical analysis shows that system (1) always has a globally asymptotically stable disease equilibrium $E(S^*, I^*)$ and no disease-free equilibrium; this means that plant disease will eventually persist. If $\phi = 1$, system (1) always has a disease-free equilibrium $E(S_0, 0)$, and it is globally asymptotically stable for $R < 1$ and unstable for $R > 1$; if $R > 1$, system (1) also has a globally asymptotically stable disease equilibrium $E_1(S^1, I^1)$, where $R = \sigma\beta/\eta(\eta + \omega)$.

As is known, biological systems are inevitably disturbed by environmental noise [16–21]. The spread of disease is also easily disturbed by environmental noise [22–25]; many researchers have thoroughly studied the impact of environmental noise on the spread of disease by establishing epidemic models by stochastic differential equations [26–29]. In the real world, because the occurrence, development, and spread of plant diseases is a very complex process, it will inevitably be disturbed by various unpredictable random factors, such as environmental conditions (temperature, moisture, and light), soil texture, pH value, and so on. For example, under the influence of high temperature, certain viruses (bacteria) may die, leading to the extinction of diseases; due to the scarcity of water, the growth cycle of certain viruses (bacteria) becomes longer, and the spread of diseases slows down; poor soil will cause the activity of some viruses (bacteria) to weaken, and disease infection is reduced [30, 31]. In this paper, on the basis of model (1), we assume that the environmental white noise is proportional to the variables S and I, respectively, and propose a simple plant disease model under stochastic perspective as follows:

$$\begin{cases} dS(t) = (\sigma\phi - \beta S(t)I(t) - \eta S(t))dt + \delta_1 S(t)dB_1(t), \\ dI(t) = (\sigma(1-\phi) + \beta S(t)I(t) - (\eta + \omega)I(t))dt + \delta_2 I(t)dB_2(t), \end{cases} \tag{2}$$

where $B_i(t)(i = 1, 2)$ are independent standard Brownian motions with $B_i(0) = 0(i = 1, 2)$ and $\delta_i > 0(i = 1, 2)$ represent the intensities of the white noise on the susceptible and infected plants, respectively. In the whole paper, we let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. The function $B_i(t)(i = 1, 2)$ is defined on $(\Omega, \mathscr{F}, \mathbb{P})$, and we denote $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2 : x_i > 0, 1 \leq i \leq 2\}$.

Our purpose is to investigate the impact of environmental noise on plant disease. The structure of the paper is as follows. The well-posedness of solutions of the system is discussed in Section 2. The persistence of the plant disease is discussed in Section 3. In Section 4, we explore the existence of the stationary distribution of the solution of the system. In Section 5, numerical simulations are implemented to illustrate the theoretical results. A brief conclusion is given in Section 6.

2. The Existence and Uniqueness of Global Positive Solution of System (2)

Theorem 1. *The solution of system (2) that satisfies the initial value $(S(0), I(0)) \in \mathbb{R}^2_+$, is unique and positive; moreover, the solution will remain in $\mathbb{R}^2_+$ with probability one.*

Proof. Firstly, let $\hbar_e$ be the explosion time, and we claim that for any initial value $(S(0), I(0)) \in \mathbb{R}^2_+$ there exists a unique local solution $(S(t), I(t)) \in \mathbb{R}^2_+$ on $t \in [0, \hbar_e)$ a.s. In fact, it is easy to get from the local Lipschitz property of the coefficients of system (2).

Secondly, we prove $\hbar_e = \infty$ a.s. This means that the solution is global. Let $n_0 \geq 1$ be sufficiently large such that $(S(0), I(0))$ all lie within the interval $[(1/n_0), n_0]$. For each integer $n \geq n_0$, define the stopping time

$$\hbar_n = \inf\left\{t \in [0, \hbar_e) : S(t) \notin \left(\frac{1}{n}, n\right) \text{ or } I(t) \notin \left(\frac{1}{n}, n\right)\right\}. \tag{3}$$

In the whole paper, we denote $\inf\varnothing = \infty$ (the empty set). Obviously, $\hbar_n$ is increasing as $n \longrightarrow \infty$. Let $\hbar_\infty = \lim_{n\longrightarrow\infty} \hbar_n$, whence $\hbar_\infty \leq \hbar_e$ a.s. If $\hbar_\infty = \infty$ a.s. is true, then $\hbar_e = \infty$ and $(S(t), I(t)) \in \mathbb{R}^2_+$ a.s. for all $t \geq 0$. Then, in order to achieve our purpose, we only need to show $\hbar_\infty = \infty$ a.s. If this affirmation is not true, then there exist two constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}\{\hbar_\infty \leq T\} > \varepsilon. \tag{4}$$

Thus, for an integer $n_1 \geq n_0$, we have

$$\mathbb{P}\{\hbar_n \leq T\} > \varepsilon, \ \forall n \geq n_1. \tag{5}$$

Define a C^2-function $V : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ as follows:

$$V = S - a_1 - a_1 \ln \frac{S}{a_1} + I - 1 - \ln I, \tag{6}$$

where a_1 is a positive constant to be determined later.

From Itô's formula, one gets

$$dV(S, I) = \mathscr{L}V(S, I)dt + \left(1 - \frac{a_1}{S}\right)\delta_1 S dB_1(t) + \left(1 - \frac{1}{I}\right)\delta_2 I dB_2(t), \tag{7}$$

where $\mathscr{L}V\colon \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ is defined by

$$\begin{aligned}\mathscr{L}V &= \left(1-\frac{a_1}{S}\right)(\sigma\phi-\beta SI-\eta S)+\left(1-\frac{1}{I}\right)(\sigma(1-\phi)+\beta SI-(\eta+\omega)I)+\frac{a_1\delta_1^2}{2}+\frac{\delta_2^2}{2}\\ &= \sigma\phi-\beta SI-\eta S-\frac{a_1\sigma\phi}{S}+a_1\beta I+a_1\eta+\sigma(1-\phi)+\beta SI\\ &\quad -(\eta+\omega)I-\frac{\sigma(1-\phi)}{I}-\beta S+\eta+\omega+\frac{a_1\delta_1^2}{2}+\frac{\delta_2^2}{2}\\ &\le \sigma\phi+a_1\eta+\sigma(1-\phi)+\eta+\omega+(a_1\beta-(\eta+\omega))I+\frac{a_1\delta_1^2}{2}+\frac{\delta_2^2}{2}.\end{aligned} \tag{8}$$

Choose $a_1 = (\eta+\omega)/\beta$. Then, we obtain

$$\mathscr{L}V \le \sigma\phi+\frac{\eta(\eta+\omega)}{\beta}+\sigma(1-\phi)+\eta+\omega+\frac{(\eta+\omega)\delta_1^2}{2\beta}+\frac{\delta_2^2}{2}K_1, \tag{9}$$

where K_1 is a positive constant. So, we obtain

$$\mathrm{d}V(S,I) \le K_1\mathrm{d}t+\left(1-\frac{\eta+\omega}{\beta S}\right)\delta_1 S\mathrm{d}B_1(t)+\left(1-\frac{1}{I}\right)\delta_1 I\mathrm{d}B_2(t). \tag{10}$$

Integrating the above formula from 0 to $\hbar_n\wedge T=\min\{\hbar_n, T\}$ and then taking the expectation on both sides, we have

$$\mathbb{E}V(S(\hbar_n\wedge T), I(\hbar_n\wedge T)) \le V(S(0), I(0))+K_1\mathbb{E}(\hbar_n\wedge T). \tag{11}$$

Hence,

$$\mathbb{E}V(S(\hbar_n\wedge T), I(\hbar_n\wedge T)) \le V(S(0), I(0))+K_1T. \tag{12}$$

Let $\Omega_n=\{\omega\in\Omega\colon \hbar_n=\hbar_n(\omega)\le T\}$ for $n\ge n_1$ and in view of (5), we obtain $\mathbb{P}(\Omega_n)\ge\varepsilon$. Note that for every $\omega\in\Omega_n$, there exists $S(\hbar_n,\omega)$ or $I(\hbar_n,\omega)$ equals either n or $1/n$. Therefore, $V(S(\hbar_n,\omega), I(\hbar_n,\omega))$ is no less than either

$$n-1-\ln n \text{ or } \frac{1}{n}-1-\ln\frac{1}{n}. \tag{13}$$

Therefore, we have

$$V(S(\hbar_n,\omega), I(\hbar_n,\omega)) \ge (n-1-\ln n)\wedge\left(\frac{1}{n}-1+\ln n\right). \tag{14}$$

In view of (12), we obtain

$$V(S(0), I(0))+K_1T \ge \mathbb{E}\left[1_{\Omega_n(\omega)}V(S(\hbar_n,\omega), I(\hbar_n,\omega))\right] \ge \varepsilon(n-1-\ln n)\wedge\left(\frac{1}{n}-1+\ln n\right), \tag{15}$$

where 1_{Ω_n} denotes the indicator function of Ω_n. Letting $n\longrightarrow\infty$ leads to the contradiction

$$\infty > V(S(0), I(0))+K_1T=\infty. \tag{16}$$

So, we must have $\hbar_\infty=\infty$ a.s. This completes the proof. □

3. Persistence in Mean

System (2) is said to be persistent in the mean if $\liminf_{t\longrightarrow\infty}(1/t)\int_0^t I(s)\mathrm{d}s>0$ a.s. For convenience, we define $\langle S(t)\rangle=(1/t)\int_0^t S(s)\mathrm{d}s$, $\langle I(t)\rangle=(1/t)\int_0^t I(s)\mathrm{d}s$.

The following lemma is essentially the same as that in [32], so we omit it.

Lemma 1. *Assume $\eta>(\delta_1^2\vee\delta_2^2)/2$. Let $(S(t), I(t))$ be the solution of system (2) with any initial value $(S(0), I(0))\in\mathbb{R}_+^2$; then,*

$$\begin{aligned}&\lim_{t\longrightarrow\infty}\frac{\int_0^t S(r)\mathrm{d}B_1(r)}{t}=0,\\ &\lim_{t\longrightarrow\infty}\frac{\int_0^t I(r)\mathrm{d}B_2(r)}{t}=0 \text{ a.s.}\end{aligned} \tag{17}$$

Lemma 2. *Let $(S(t), I(t))$ be the solution of system (2) with initial value $(S(0), I(0))\in\mathbb{R}_+^2$. Then,*

$$\begin{aligned}&\lim_{t\longrightarrow\infty}\frac{S(t)}{t}=0,\\ &\lim_{t\longrightarrow\infty}\frac{I(t)}{t}=0 \text{ a.s.}\end{aligned} \tag{18}$$

Proof. Let $w(t)=S(t)+I(t)$. Define

$$V(w)=(1+w)^\zeta, \tag{19}$$

where ζ is a positive constant to be determined later. Then,

$$\mathrm{d}V(w) = LV\mathrm{d}t + \zeta(1+w)^{\zeta-1}[\delta_1 S(t)\mathrm{d}B_1(t) + \delta_2 I(t)\mathrm{d}B_2(t)], \tag{20}$$

where

$$\begin{aligned} LV(w) &= \zeta(1+w)^{\zeta-1}[\sigma - \eta S - \eta I - \omega I] + \frac{\zeta(\zeta-1)}{2}(1+w)^{\zeta-2}\left(\delta_1^2 S^2 + \delta_2^2 I^2\right) \\ &= \zeta(1+w)^{\zeta-2}\left\{(1+w)[\sigma - \eta S - \eta I - \omega I] + \frac{\zeta-1}{2}\left(\delta_1^2 S^2 + \delta_2^2 I^2\right)\right\} \\ &\leq \zeta(1+w)^{\zeta-2}\left\{(1+w)[\sigma - \eta w] + \frac{\zeta-1}{2}\left(\delta_1^2 S^2 + \delta_2^2 I^2\right)\right\} \\ &\leq \zeta(1+w)^{\zeta-2}\left\{(1+w)[\sigma - \eta w] + \frac{\zeta-1}{2}\left(\delta_1^2 \vee \delta_2^2\right)w^2\right\} \\ &= \zeta(1+w)^{\zeta-2}\left\{-\left[\eta - \frac{\zeta-1}{2}\left(\delta_1^2 \vee \delta_2^2\right)\right]w^2 + (\sigma-\eta)w + \sigma\right\}. \end{aligned} \tag{21}$$

Choose $\zeta > 0$ such that

$$\eta - \left(\frac{\zeta-1}{2} \vee 0\right)\left(\delta_1^2 \vee \delta_2^2\right) := \chi > 0, \tag{22}$$

so

$$LV(w) \leq \zeta(1+w)^{\zeta-2}\left[-\chi w^2 + (\sigma-\eta)w + \sigma\right], \tag{23}$$

$$\begin{aligned} \mathrm{d}V(w) \leq{}& \zeta(1+w)^{\zeta-2}\left[-\chi w^2 + (\sigma-\eta)w + \sigma\right] \\ &+ \zeta(1+w)^{\zeta-1}[\delta_1 S(t)\mathrm{d}B_1(t) + \delta_2 I(t)\mathrm{d}B_2(t)]. \end{aligned} \tag{24}$$

For $0 < k < \zeta\chi$, we have

$$\mathrm{d}\left[e^{kt}V(w(t))\right] = L\left[e^{kt}V(w(t))\right]\mathrm{d}t + e^{kt}\zeta(1+w(t))^{\zeta-1}[\delta_1 S(t)\mathrm{d}B_1(t) + \delta_2 I(t)\mathrm{d}B_2(t)], \tag{25}$$

so

$$E\left[e^{kt}V(w(t))\right] = V(w(0)) + E\int_0^t L\left(e^{ks}V(w(s))\right)\mathrm{d}s, \tag{26}$$

where

$$\begin{aligned} L\left[e^{kt}V(w)\right] &= ke^{kt}V(w) + e^{kt}LV(w) \leq \zeta e^{kt}(1+w)^{\zeta-2}\left\{\frac{k}{\zeta}(1+w)^2 - \chi W^2 + (\sigma-\eta)w + \sigma\right\} \\ &= \zeta e^{kt}(1+w)^{\zeta-2}\left[-\left(\chi - \frac{k}{\zeta}\right)w^2 + \left(\sigma - \eta + \frac{2k}{\zeta}\right)w + \left(\sigma + \frac{k}{\zeta}\right)\right] \leq \zeta e^{kt}Q, \end{aligned} \tag{27}$$

and

$$Q := \sup_{w\in\mathbb{R}_+}(1+w)^{\zeta-2}\left[-\left(\chi - \frac{k}{\zeta}\right)w^2 + \left(\sigma - \eta + \frac{2k}{\zeta}\right)w + \sigma + \frac{k}{\zeta}\right] + 1. \tag{28}$$

Then, from (26), we get

$$E\left[e^{kt}\left(1+w(t)^{\zeta}\right)\right] \le (1+w(0))^{\zeta} + \frac{\zeta Q}{k}e^{kt}. \tag{29}$$

Consequently, we have

$$\limsup_{t\longrightarrow\infty} E\left[(1+w(t))^{\zeta}\right] \le \frac{\zeta Q}{k} \coloneqq Q_0 \text{ a.s.} \tag{30}$$

By the continuity of function $w(t)$, there exists a constant $M>0$ such that

$$E\left[(1+w(t))^{\zeta}\right] \le M,\ t\ge 0. \tag{31}$$

From (24), for sufficiently small $\varepsilon>0$, $k=1,2,\ldots$, we have

$$E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(t))^{\zeta}\right] \le E\left[(1+w(k\varepsilon))^{\zeta}\right] + I_1 + I_2 \le M + I_1 + I_2, \tag{32}$$

where

$$\begin{aligned} I_1 &= E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}\left|\int_{k\varepsilon}^{t}\zeta(1+w(r))^{\zeta-2}\left[-\chi w^2(r)+(\sigma-\eta)w(r)+\sigma\right]\mathrm{d}r\right|\right] \\ &\le c_1 E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}\left|\int_{k\varepsilon}^{t}(1+w(r))^{\zeta}\mathrm{d}r\right|\right] \le c_1 E\left[\int_{k\varepsilon}^{(k+1)\varepsilon}(1+w(r))^{\zeta}\mathrm{d}r\right] \\ &\le c_1\varepsilon E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(t))^{\zeta}\right], \end{aligned} \tag{33}$$

$$\begin{aligned} I_2 &= E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}\left|\int_{k\varepsilon}^{t}\zeta(1+w(r))^{\zeta-1}\left(\delta_1 S(r)\mathrm{d}B_1(r)+\delta_2 I(r)\mathrm{d}B_2(r)\right)\right|\right] \\ &\le \sqrt{32}E\left[\int_{k\varepsilon}^{(k+1)\varepsilon}\zeta^2(1+w(r))^{2(\zeta-1)}\left(\delta_1^2 S^2(r)+\delta_2^2 I^2(r)\right)\mathrm{d}r\right]^{1/2} \\ &\le \sqrt{32}E\zeta\left(\delta_1^2\vee\delta_2^2\right)^{1/2}\varepsilon^{1/2}E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(r))^{2\zeta}\right]^{1/2} \\ &\le \sqrt{32}E\zeta\left(\delta_1^2\vee\delta_2^2\right)^{1/2}\varepsilon^{1/2}E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(r))^{\zeta}\right]. \end{aligned} \tag{34}$$

By Burkholder–Davis–Gundy inequality [33], we obtain

$$E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(t))^{\zeta}\right] \le E\left[(1+w(k\varepsilon))^{\zeta}\right] + \left[c_1\varepsilon + \sqrt{32}\,\zeta\left(\delta_1^2\vee\delta_2^2\right)^{1/2}\varepsilon^{1/2}\right]E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(t))^{\zeta}\right]. \tag{35}$$

In particular, let $\varepsilon>0$ such that $c_1\varepsilon + \sqrt{32}\zeta\left(\delta_1^2\vee\delta_2^2\right)^{1/2}\varepsilon^{1/2} \le (1/2)$, and then

$$E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(t))^{\zeta}\right] \le 2E\left[(1+w(k\varepsilon))^{\zeta}\right] \le 2M. \tag{36}$$

For sufficiently small $\varepsilon_w>0$ and using Chebyshev's inequality, we have

$$\begin{aligned} &P\left\{\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(t))^{\zeta} > (k\varepsilon)^{1+\varepsilon_w}\right\} \\ &\quad\le \frac{E\left[\sup_{k\varepsilon\le t\le (k+1)\varepsilon}(1+w(t))^{\zeta}\right]}{(k\varepsilon)^{1+\varepsilon_w}} \\ &\quad\le \frac{2M}{(k\varepsilon)^{1+\varepsilon_w}},\ k=1,2,\ldots. \end{aligned} \tag{37}$$

According to Borel–Cantelli lemma [33], we know that for almost all $\omega \in \Omega$,

$$\sup_{k\varepsilon\le t\le (k+1)\varepsilon} (1+w(t))^{\zeta} \le (k\varepsilon)^{1+\varepsilon_w} \tag{38}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega \in \Omega$, for which (38) holds whenever $k \ge k_0$. Therefore, for almost all $\omega \in \Omega$, if $k \ge k_0$ and $k\varepsilon \le t \le (k+1)\varepsilon$,

$$\frac{\log\ (1+w(t))^{\zeta}}{\log t} \le \frac{(1+\varepsilon_w)\log(k\varepsilon)}{\log(k\varepsilon)} = 1+\varepsilon_w. \tag{39}$$

Taking the limit superior, we have

$$\limsup_{t\longrightarrow\infty} \frac{\log\ (1+w(t))^{\zeta}}{\log t} = 1+\varepsilon_w \text{ a.s.} \tag{40}$$

Let $\varepsilon_w \longrightarrow 0$, and we get

$$\limsup_{t\longrightarrow\infty} \frac{\log\ (1+w(t))^{\zeta}}{\log t} \le 1 \text{ a.s.} \tag{41}$$

For $1<\zeta<1+(2\eta/\delta_1^2\vee\delta_2^2)$, i.e., $\eta > ((\zeta-1)(\delta_1^2\delta_2^2)/2)$, we have

$$\limsup_{t\longrightarrow\infty} \frac{\log w(t)}{\log t} \le \limsup_{t\longrightarrow\infty} \frac{\log\ (1+w(t))^{\zeta}}{\log t} \le \frac{1}{\zeta} \text{ a.s.} \tag{42}$$

In other words, for arbitrarily small $0<\rho<1-(1/\zeta)$, there exist constant $T=T(\omega)$ and a set Ω_ρ such that $P(\Omega_\rho) \ge 1-\rho$, and for $t \ge T$, $\omega \in \Omega_\rho$,

$$\log w(t) \le \left(\frac{1}{\zeta}+\rho\right)\log t, \tag{43}$$

$$\limsup_{t\longrightarrow\infty} \frac{w(t)}{t} \le \limsup_{t\longrightarrow\infty} \frac{t^{(1/\zeta)+\rho}}{t} = 0. \tag{44}$$

According to Theorem 1, we get

$$\lim_{t\longrightarrow\infty} \frac{w(t)}{t} = \lim \frac{S(t)+I(t)}{t} = 0 \text{ a.s.}, \tag{45}$$

$$\begin{aligned} &\lim \frac{S(t)}{t} = 0, \\ &\lim \frac{I(t)}{t} = 0 \text{ a.s.} \end{aligned} \tag{46}$$

The proof of Lemma 2 is completed.

Denote

$$R_1 = \frac{\sigma\beta}{\eta\left(\eta+\omega+\left(\delta_2^2/2\right)\right)} > 1. \tag{47}$$

□

Theorem 2. *If $R_1 > 1$, for the solution $(S(t), I(t))$ of system (2) with any initial value $(S(0), I(0)) \in \mathbb{R}_+^2$, we have*

$$\limsup_{t\longrightarrow\infty} \langle I(t)\rangle \ge \frac{\eta\left(\eta+\omega+\left(\delta_2^2/2\right)\right)(R_1-1)}{(\eta+\omega)\beta} > 0 \text{ a.s.}, \tag{48}$$

i.e., the disease is persistent in mean.

Proof. Integrate and add the two equations of system (2) on both sides to get

$$\frac{S(t)-S(0)}{t} + \frac{I(t)-I(0)}{t} = \sigma - \eta\langle S(t)\rangle - (\eta+\omega)\langle I(t)\rangle + \frac{\delta_1\int_0^t S(r)\mathrm{d}B_1(r)}{t} + \frac{\delta_2\int_0^t I(r)\mathrm{d}B_2(r)}{t}. \tag{49}$$

Further calculation yields

$$\begin{aligned} \langle S(t)\rangle &= \frac{\sigma}{\eta} - \frac{(\eta+\omega)}{\eta}\langle I(t)\rangle + \omega_1 + \omega_2 \\ &\quad - \frac{S(t)-S(0)}{\eta t} - \frac{I(t)-I(0)}{\eta t}, \end{aligned} \tag{50}$$

where

$$\begin{aligned} \omega_1 &= \frac{\delta_1\int_0^t S(r)\mathrm{d}B_1(r)}{\eta t}, \\ \omega_2 &= \frac{\delta_2\int_0^t I(r)\mathrm{d}B_2(r)}{\eta t}. \end{aligned} \tag{51}$$

By using Itô's formula, one gets

$$\mathrm{d}(\ln I) = \left[\frac{\sigma(1-\phi)}{I} + \beta S - (\eta+\omega) - \frac{\delta_2^2}{2}\right]\mathrm{d}t + \delta_2 \mathrm{d}B_2(t), \quad (52)$$

$$\begin{aligned}
\frac{\ln I(t) - \ln I(0)}{t} &= \frac{\sigma(1-\phi)}{t}\int_0^t \frac{1}{I(r)}\mathrm{d}r + \beta\langle S(t)\rangle - \left(\eta+\omega+\frac{\delta_2^2}{2}\right) + \frac{\delta_2 B_2(t)}{t} \\
&\geq \beta\langle S(t)\rangle - \left(\eta+\omega+\frac{\delta_2^2}{2}\right) + \frac{\delta_2 B_2(t)}{t} \\
&= \beta\left[\frac{\sigma}{\eta} - \frac{(\eta+\omega)}{\eta}\langle I(t)\rangle + \omega_1 + \omega_2 - \frac{S(t)-S(0)}{\eta t} - \frac{I(t)-I(0)}{\eta t}\right] \\
&\quad - \left(\eta+\omega+\frac{\delta_2^2}{2}\right) + \frac{\delta_2 B_2(t)}{t} \\
&= \frac{\sigma\beta}{\eta} - \frac{(\eta+\omega)\beta}{\eta}\langle I(t)\rangle + \beta\omega_1 + \beta\omega_2 - \frac{\beta(S(t)-S(0))}{\eta t} \\
&\quad - \frac{\beta(I(t)-I(0))}{\eta t} - \left(\eta+\omega+\frac{\delta_2^2}{2}\right) + \frac{\delta_2 B_2(t)}{t}.
\end{aligned} \quad (53)$$

Solving the above inequality, we have

$$\begin{aligned}
\langle I(t)\rangle &\geq \frac{\eta}{(\eta+\omega)\beta}\frac{\ln I(0)-\ln I(t)}{t} + \frac{\sigma}{\eta+\omega} + \frac{\eta}{\eta+\omega}\omega_1 + \frac{\eta}{\eta+\omega}\omega_2 - \frac{S(t)-S(0)}{(\eta+\omega)t} \\
&\quad - \frac{I(t)-I(0)}{(\eta+\omega)t} - \frac{\eta\left(\eta+\omega+\left(\delta_2^2/2\right)\right)}{(\eta+\omega)\beta} + \frac{\eta\delta_2 B_2(t)}{(\eta+\omega)\beta t} \\
&= \frac{\eta}{(\eta+\omega)\beta}\frac{\ln I(0)-\ln I(t)}{t} + \frac{\eta}{\eta+\omega}\omega_1 + \frac{\eta}{\eta+\omega}\omega_2 - \frac{S(t)-S(0)}{(\eta+\omega)t} \\
&\quad - \frac{I(t)-I(0)}{(\eta+\omega)t} + \frac{\sigma\beta - \eta\left(\eta+\omega+\left(\delta_2^2/2\right)\right)}{(\eta+\omega)\beta} + \frac{\eta\delta_2 B_2(t)}{(\eta+\omega)\beta t} \\
&= \frac{\eta\left(\eta+\omega+\left(\delta_2^2/2\right)\right)(R_1-1)}{(\eta+\omega)\beta} + \frac{\eta}{\eta+\omega}\omega_1 + \frac{\eta}{\eta+\omega}\omega_2 - \frac{S(t)-S(0)}{(\eta+\omega)t} \\
&\quad - \frac{I(t)-I(0)}{(\eta+\omega)t} + \frac{\eta}{(\eta+\omega)\beta}\frac{\ln I(0)-\ln I(t)}{t} + \frac{\eta\delta_2 B_2(t)}{(\eta+\omega)\beta t}.
\end{aligned} \quad (54)$$

By taking the limit inferior of both sides of (54) and using Lemmas 1 and 2, we can obtain

$$\limsup_{t\longrightarrow\infty}\langle I(t)\rangle \geq \frac{\eta\left(\eta+\omega+\left(\delta_2^2/2\right)\right)(R_1-1)}{(\eta+\omega)\beta}. \quad (55)$$

This finishes the proof of Theorem 2. □

4. Existence of a Stationary Distribution

In this section, we explore existence of a stationary distribution for the solutions of system (2) under the condition $R_1 > 1$.

Let $Y(t)$ be a homogenous Markov process in l-dimensional Euclidean space $\mathbb{E}_l$, and $Y(t)$ satisfies

$$\mathrm{d}Y(t) = b(Y)\mathrm{d}t + \sum_{r=1}^{k} h_r(Y)\mathrm{d}B_r(t). \quad (56)$$

The diffusion matrix is defined as follows:

$$\begin{aligned}
A(x) &= \left(a_{ij}(y)\right), \\
a_{ij}(x) &= \sum_{r=1}^{k} h_r^i(y)h_r^j(y).
\end{aligned} \quad (57)$$

Lemma 3 (see [34]). *The Markov process $Y(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded domain $U \subset \mathbb{E}_l$ with regular boundary Γ and*

(A.1) There is a positive number M such that

$$\sum_{i,j=1}^{n} a_{ij}(y)\zeta_i\zeta_j \geq M|\zeta|^2,\ y \in U,\ \zeta \in \mathbb{E}_l. \tag{58}$$

(A.2) There exists a nonnegative C^2-function V such that $\mathscr{L}V$ is negative for any $\mathbb{E}_l \setminus U$. Then,

$$\mathbb{P}_y\left\{\lim_{\tau\longrightarrow\infty}\frac{1}{\tau}\int_0^\tau g(Y(t))\mathrm{d}t = \int_{\mathbb{E}_n} g(y)p(\mathrm{d}y)\right\} = 1, \tag{59}$$

for all $y \in \mathbb{E}_l$, where $g(\cdot)$ is a function integrable with respect to the measure p.

Theorem 3. *For any given initial value $(S(0), I(0)) \in \mathbb{R}_+^2$, if*

$$R_2 = \frac{\sigma\phi\beta}{\left(\left(\delta_1^2/2\right)+\eta\right)\left(\eta+\omega+\left(\delta_2^2/2\right)\right)} > 1 \tag{60}$$

holds, then system (2) has a unique ergodic stationary distribution.

Proof. According to Lemma 3, we need to verify the conditions of the lemma one by one. Firstly, we get the diffusion matrix of system (2) with the form

$$\widetilde{A} = \begin{pmatrix} \delta_1^2 S^2 & 0 \\ 0 & \delta_2^2 I^2 \end{pmatrix}. \tag{61}$$

Then, let $M = \min_{(S,I)\in\overline{U}}\left\{\delta_1^2 S^2, \delta_2^2 I^2\right\}$, and we have

$$\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j = \delta_1^2 S^2\xi_1^2 + \delta_2^2 I^2 \xi_2^2 \geq M|\xi|^2,\ (S,I) \in \overline{U},\ \xi \in \mathbb{R}^2. \tag{62}$$

Thus, condition (A.1) in Lemma 3 is satisfied.

In order to verify (A.2), we will construct a function $V \in C^2(\mathbb{R}_+^2; \mathbb{R})$ and a compact set $U_\epsilon \subset \mathbb{R}_+^2$ such that

$$\mathscr{L}V(S,I) \leq -1 \text{ on } (S,I) \in \mathbb{R}_+^2 \setminus U_\varepsilon. \tag{63}$$

In the following, let us construct a C^2-function V: $\mathbb{R}_+^2 \longrightarrow \mathbb{R}$ with the form

$$\overline{V}(S,I) = M(-a\ln S - \ln I) - \ln S + \frac{1}{\theta+1}(S+I)^{\theta+1}, \tag{64}$$

where θ and c are positive constants satisfying $0 < \theta < (2\eta/\delta_1^2 \vee \delta_2^2), a = (\sigma\phi\beta/((\delta_1^2/2)+\eta)^2)$. Let us choose $M > 0$ large enough such that

$$-M\lambda_2 + D \leq -2, \tag{65}$$

where

$$\lambda_2 := \left(\eta+\omega+\frac{\delta_2^2}{2}\right)(R_2 - 1). \tag{66}$$

Obviously,

$$\liminf_{w\longrightarrow(S,I)\in\mathbb{R}_+^2\setminus U_w} \overline{V}(S,I) = \infty, \tag{67}$$

where $U_w = ((1/w), w) \times ((1/w), w)$. Because of the continuity of function $\overline{V}(S,I)$, there exists a unique point $(\underline{S^*}, \underline{I^*})$ in $\mathbb{R}_+^2$ which is the minimum point of $\overline{V}(S,I)$. Thus, consider a positive-definite C^2-function V: $\mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ with the form

$$\begin{aligned} V(S,I,C) &= \overline{V}(S,I) - \overline{V}(\underline{S^*}, \underline{I^*}) \\ &= M(-a\ln S - \ln I) - \ln S \\ &\quad + \frac{1}{\theta+1}(S+I)^{\theta+1} - \overline{V}(\underline{S^*}, \underline{I^*}) \\ &= MV_1 + V_2 + V_3, \end{aligned} \tag{68}$$

where $V_1 = -a\ln S - \ln I$, $V_2 = -\ln S$, $V_3 = (1/\theta+1)(S+I)^{\theta+1} - \overline{V}(\underline{S^*}, \underline{I^*})$. From Itô's formula, we have

$$\begin{aligned} \mathscr{L}V_1 &= -\frac{a\sigma\phi}{S} + a\beta I + a\eta + \frac{a\delta_1^2}{2} - \frac{\sigma(1-\phi)}{I} - \beta S + \eta + \omega + \frac{\delta_2^2}{2} \\ &\leq -\frac{a\sigma\phi}{S} - \beta S + a\beta I + \eta + \omega + a\eta + \frac{a\delta_1^2}{2} + \frac{\delta_2^2}{2} \\ &\leq -2\sqrt{a\sigma\phi\beta} + a\beta I + \eta + \omega + a\left(\frac{\delta_1^2}{2}+\eta\right) + \frac{\delta_2^2}{2} \\ &\leq -\frac{\sigma\phi\beta}{\left(\left(\delta_1^2/2\right)+\eta\right)} + \eta + \omega + \frac{\delta_2^2}{2} + a\beta I \\ &\leq -\left(\eta+\omega+\frac{\delta_2^2}{2}\right)(R_2-1) + a\beta I \\ &:= -\lambda_2 + a\beta I. \end{aligned} \tag{69}$$

By using Itô's formula, we obtain

$$\mathscr{L}V_2 = -\frac{\sigma\phi}{S} + \beta I + \eta + \frac{\delta_1^2}{2}. \tag{70}$$

Similarly,

$$\begin{aligned} \mathscr{L}V_3 &= (S+I)^\theta(\sigma - \eta S - (\eta+\omega)I) + \frac{\theta}{2}(S+I)^{\theta-1}\left(\delta_1^2 S^2 + \delta_2^2 I^2\right) \\ &\leq \sigma(S+I)^\theta - \eta(S+I)^{\theta+1} + \frac{\theta}{2}\left(\delta_1^2 \vee \delta_2^2\right)(S+I)^{\theta+1} \\ &= \sigma(S+I)^\theta - \left[\eta - \frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right](S+I)^{\theta+1} \\ &\leq B - \frac{1}{2}\left[\eta - \frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right](S+I)^{\theta+1}, \end{aligned} \tag{71}$$

where

$$B=\sup_{(S,I)\in\mathbb{R}_+^2}\left\{\sigma(S+I)^{\theta}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right](S+I)^{\theta+1}\right\}<\infty. \tag{72}$$

Therefore,

$$\begin{aligned}\mathscr{L}V&\leq-M\lambda_2+aM\beta I+\beta I-\frac{\sigma\phi}{S}+\eta+\frac{\delta_1^2}{2}+B-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right](S+I)^{\theta+1}\\&\leq-M\lambda_2+\beta(1+aM)I-\frac{\sigma\phi}{S}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}.\end{aligned} \tag{73}$$

Consider a compact subset U:

$$U=\left\{\varepsilon\leq S\leq\frac{1}{\varepsilon},\varepsilon\leq I\leq\frac{1}{\varepsilon}\right\}, \tag{74}$$

where ε is a constant small enough that

$$-\frac{\sigma\phi}{\varepsilon}+C\leq-1, \tag{75}$$

$$-M\lambda_2+\beta(1+aM)\varepsilon+D\leq-1, \tag{76}$$

$$-\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]\frac{1}{\varepsilon^{\theta+1}}+E\leq-1, \tag{77}$$

$$-\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]\frac{1}{\varepsilon^{\theta+1}}+F\leq-1, \tag{78}$$

where C, D, E, and F are positive constants with the form in equations (82), (84), (86), and (88), respectively. Then,

$$\mathbb{R}_+^2\backslash U=U_1\cup U_2\cup U_3\cup U_4, \tag{79}$$

with

$$\begin{aligned}U_1&=\left\{(S,I)\in\mathbb{R}_+^2\,|\,0<S<\varepsilon\right\},\\U_2&=\left\{(S,I)\in\mathbb{R}_+^2\,|\,0<I<\varepsilon\right\},\\U_3&=\left\{(S,I)\in\mathbb{R}_+^2\,|\,S>\frac{1}{\varepsilon}\right\},\\U_4&=\left\{(S,I)\in\mathbb{R}_+^2\,|\,I>\frac{1}{\varepsilon}\right\}.\end{aligned} \tag{80}$$

Next, we discuss the negative of $\mathscr{L}V$ for any $(S,I)\in\mathbb{R}_+^2\backslash U$ in different regions.

Case I. If $(S,I)\in U_1$, we get

$$\begin{aligned}\mathscr{L}V&\leq-\frac{\sigma\phi}{S}+\beta(1+aM)I-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\\&\leq-\frac{\sigma\phi}{\varepsilon}+C,\end{aligned} \tag{81}$$

where

$$C=\sup_{(S,I)\in\mathbb{R}_+^2}\left\{\beta(1+aM)I-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\right\}<\infty. \tag{82}$$

According to (75), we can have $\mathscr{L}V\leq-1$ for all $(S,I)\in U_1$.

Case II. If $(S,I)\in U_2$, we can obtain that

$$\begin{aligned}\mathscr{L}V&\leq-M\lambda_2+\beta(1+aM)I-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\\&\leq-M\lambda_2+\beta(1+aM)\varepsilon+D,\end{aligned} \tag{83}$$

where

$$D = \sup_{(S,I)\in\mathbb{R}_+^2}\left\{-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\right\}<\infty. \tag{84}$$

According to (76), we can have $\mathscr{L}V\leq -1$ for all $(S,I)\in U_2$.

Case III. If $(S,I)\in U_3$, we have

$$\begin{aligned}\mathscr{L}V &\leq -\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}+\beta(1+aM)I-\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}\\ &\quad -\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\\ &\leq -\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}+E\\ &\leq -\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]\frac{1}{\varepsilon^{\theta+1}}+E,\end{aligned} \tag{85}$$

where

$$E = \sup_{(S,I)\in\mathbb{R}_+^2}\left\{\beta(1+aM)I-\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\right\}<\infty. \tag{86}$$

According to (77), we can get that $\mathscr{L}V\leq -1$ for all $(S,I)\in U_3$.

Case IV. If $(S,I)\in U_4$, it follows that

$$\begin{aligned}\mathscr{L}V &\leq -\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+\beta(1+aM)I-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}\\ &\quad -\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\\ &\leq -\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+F\\ &\leq -\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]\frac{1}{\varepsilon^{\theta+1}}+F,\end{aligned} \tag{87}$$

where

$$F = \sup_{(S,I)\in\mathbb{R}_+^2}\left\{\beta(1+aM)I-\frac{1}{2}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]S^{\theta+1}-\frac{1}{4}\left[\eta-\frac{\theta}{2}\left(\delta_1^2\vee\delta_2^2\right)\right]I^{\theta+1}+B+\eta+\frac{\delta_1^2}{2}\right\}<\infty. \tag{88}$$

Table 1: Model parameters, their interpretation, and default value.

Parameter	Description	Default value	Ref.
η	Harvest rate	0.002	Holt and Chancellor [35]
σ	Planting rate	0.0015	Gibson and Aritua [36]
ϕ	Fraction planted from in vitro propagated, virus free, material	1.0–0.0	Feng et al. [37]
ω	Roguing rate parameter	0.005	Fondong et al. [38]
β	Transmission rate parameter	0.025	Gibson et al. [39]

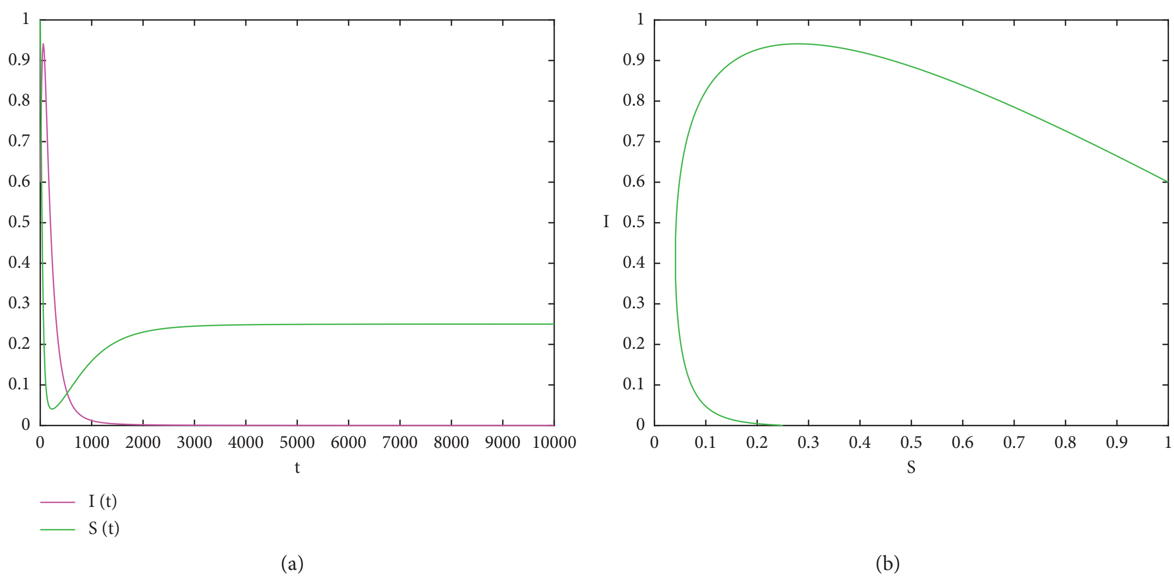


Figure 1: Simulation of solutions for system (1) with initial value $(S(0), I(0)) = (1, 0.6)$, where $R = 0.8929 < 1$. (a) The series graph of S and I for system (1). (b) The phase diagrams of S and I.

According to (78), we can get that $\mathscr{L}V \leq -1$ for all $(S, I) \in U_4$.

Thus, from the discussion of the above four cases, for a sufficiently small ε, we can get

$$\mathscr{L}V \leq -1 \text{ forall } (S, I) \in \mathbb{R}_+^2 \backslash U, \tag{89}$$

which means condition (A.2) in Lemma 3 is satisfied. Hence, all the conditions in Lemma 3 are satisfied; then, according to Lemma 3, system (2) has a unique ergodic stationary distribution $\pi(\cdot)$. This completes the proof of Theorem 3. □

5. Numerical Simulation

In this section, we employ numerical simulation to verify the main results.

For the deterministic system, we first consider the case $\phi = 1$, and the parameters for system (1) are chosen as shown in Table 1. If we let $\sigma = 0.0005$, direct calculations show that system (1) has a disease-free equilibrium $E_0(0.25, 0)$ which is globally asymptotically stable (see Figure 1), where $R = 0.8929 < 1$.

If we increase planting rate σ to 0.0015, direct calculations show that system (1) has a disease equilibrium $E_1(S^1, I^1) = (0.28, 0.1343)$ which is globally asymptotically stable (see Figure 2), where $R = 2.6786 > 1$.

For the case $\phi < 1$, the parameters are chosen as $\phi = 0.8, \beta = 0.025, \eta = 0.002$, and $\omega = 0.005$, and simple calculations show that system (1) has a disease equilibrium $E(S^*, I^*) = (0.2032, 0.1562)$ which is globally asymptotically stable (see the blue curve in Figures 3(a), 3(c), and 3(e)).

For the stochastic system, we get the following discrete system:

$$\begin{cases} S_{k+1} = S_k + [\sigma\phi - \beta S_k I_k - \eta S_k]\Delta t + \delta_1 S_k \Delta B_{1,k}, \\ I_{k+1} = I_k + [\sigma(1-\phi) + \beta S_k I_k - (\eta + \omega) I_k]\Delta t + \delta_2 I_k \Delta B_{2,k}, \end{cases} \tag{90}$$

where $\Delta B_{i,k} \triangleq B(t_{k+1}) - B(t_k)\,(i = 1, 2)$ obeys the Gaussian distribution $N(0, \Delta t)$.

Let $\delta_1 = 0.04, \delta_2 = 0.03$, and direct calculation shows $R_1 = 2.5168 > 1, R_2 = 1.4382 > 1$. According to Theorems 2

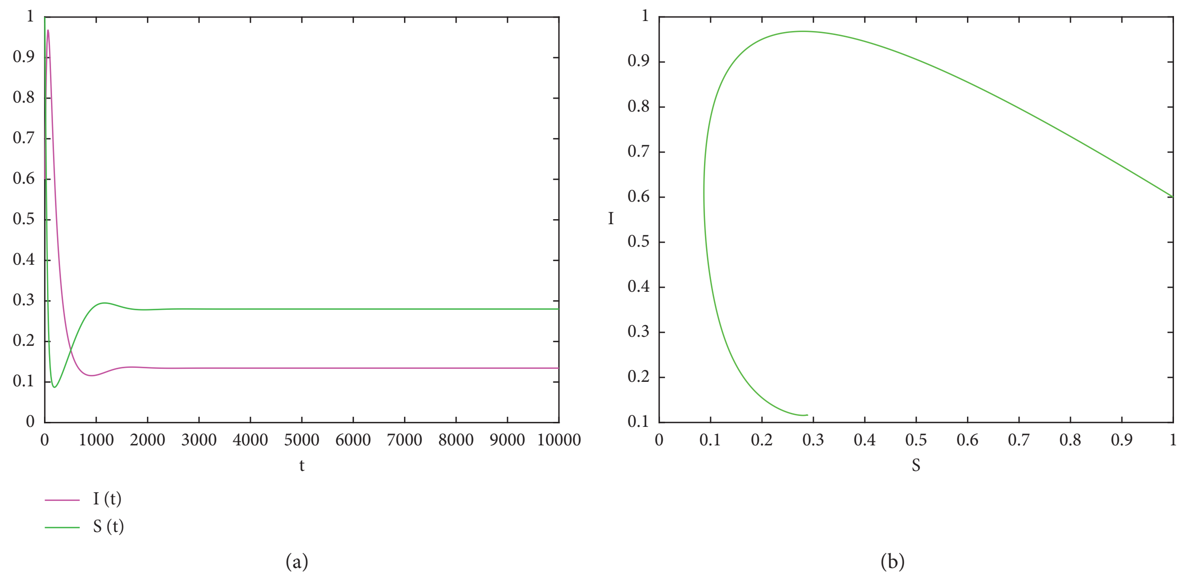


Figure 2: Simulation of solutions for system (1) with initial value $(S(0), I(0)) = (1, 0.6)$, where $R = 2.6786 > 1$. (a) The series graph of S and I for system (1). (b) The phase diagrams of S and I.

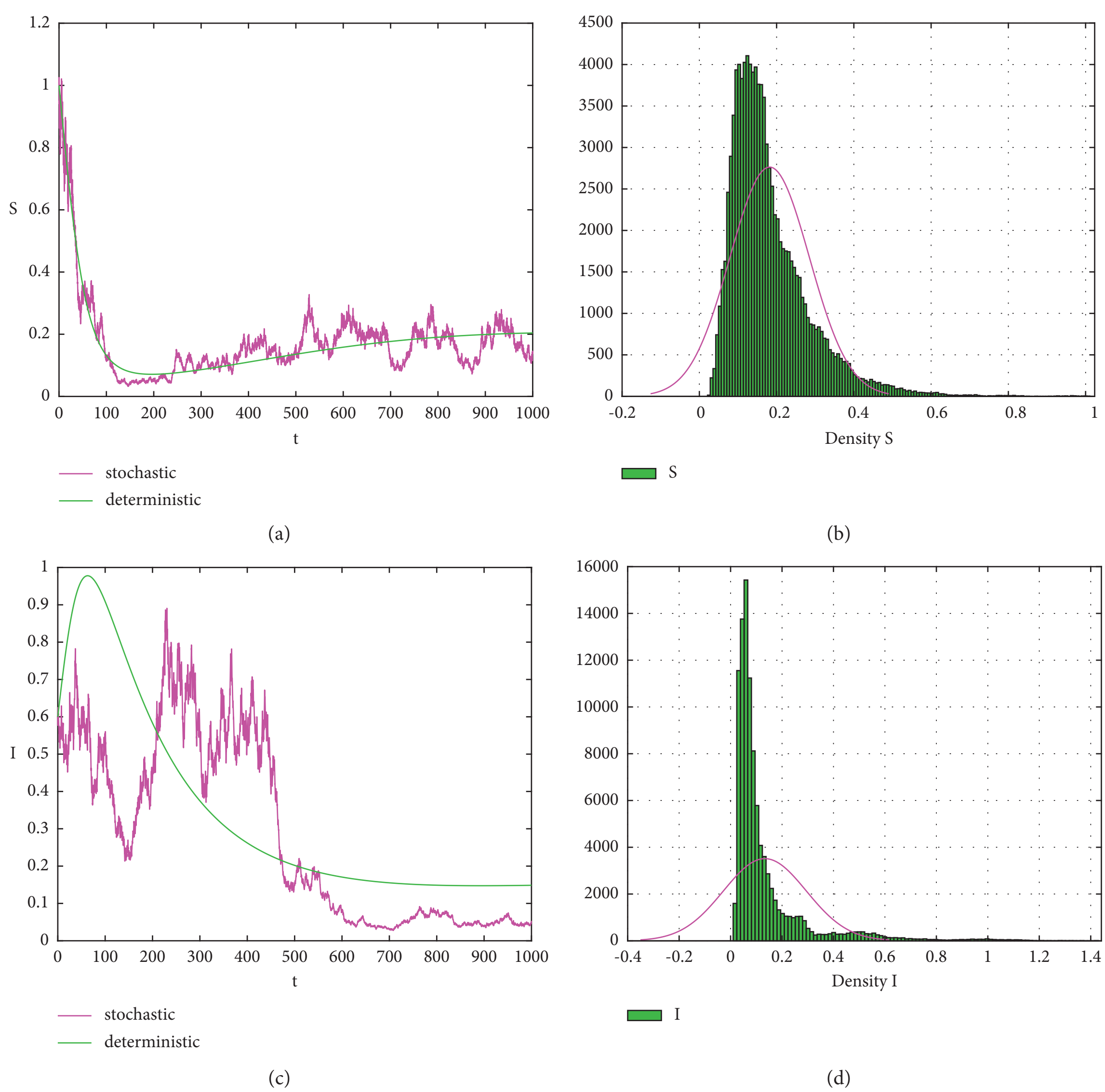
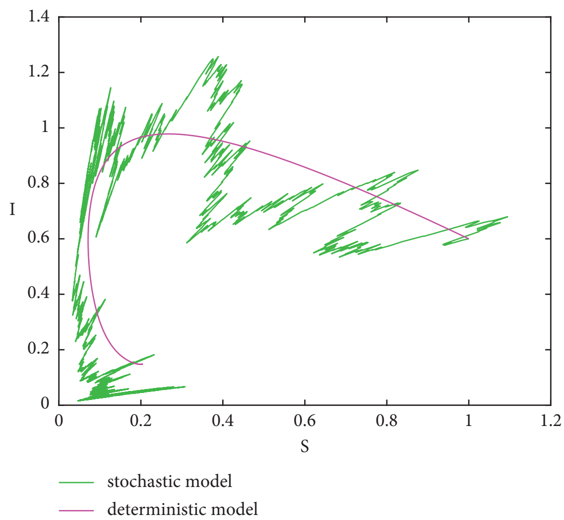
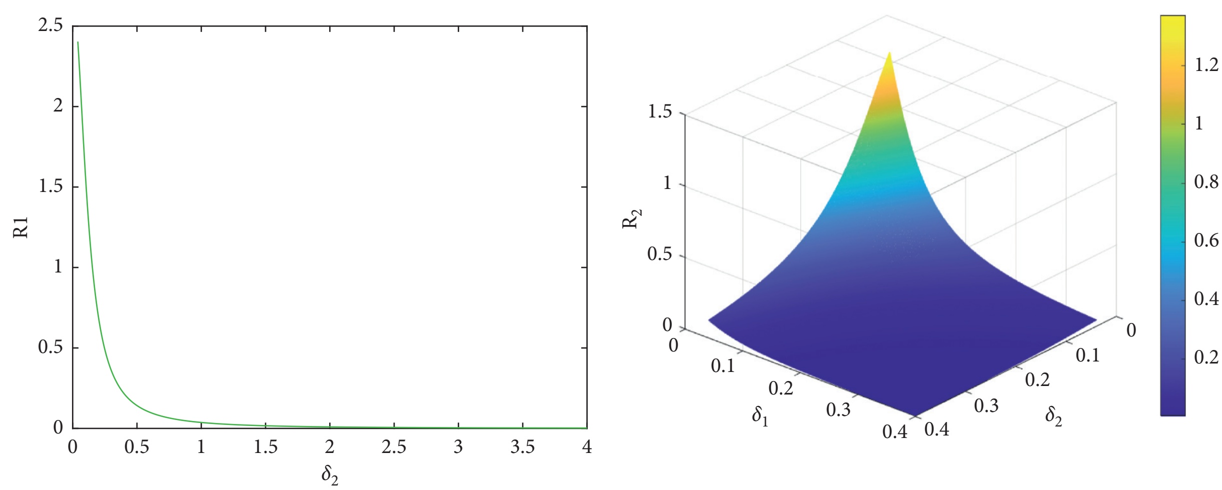


Figure 3: Continued.



(e)

Figure 3: Comparisons of solutions of system (1) and system (2), where $\delta_1 = 0.04, \delta_2 = 0.03$. (a) The sample paths of S. (b) The density functions of S in system (2). (c) The sample paths of I. (d) The density functions of I in system (2). (e) The phase diagrams of S and I.



(a) (b)

Figure 4: The relationship of the key values and intensity of white noise for stochastic system (2), where $\sigma = 0.0015, \beta = 0.025, \phi = 0.8, \eta = 0.002$, and $\omega = 0.005$. (a) The relationship of $R1$ and $\delta 2$ for stochastic system (2). (b) The relationship of $R2$ and $\delta 1, \delta 2$ for stochastic system (2).

and 3, the disease will persist eventually, and moreover system (2) has a unique stationary distribution. The red curves in Figures 3(a) and 3(c) and the blue curves in Figures 3(e) show that the solutions of the stochastic system will fluctuate around the solution of the deterministic system. Figures 3(b) and 3(d) show that system (2) has a unique stationary distribution.

From Theorems 2 and 3, we get the key values

$$R_1 = \frac{\sigma\beta}{\eta\left(\eta + \omega + \left(\delta_2^2/2\right)\right)},$$
$$R_2 = \frac{\sigma\phi\beta}{\left(\left(\delta_1^2/2\right) + \eta\right)\left(\eta + \omega + \left(\delta_2^2/2\right)\right)} \tag{91}$$

easily, and we can find that R_1 is monotonically decreasing with respect to the intensity of white noise δ_2 (see Figure 4(a)), and R_2 is monotonically decreasing with respect to the intensity of white noises δ_1 and δ_2 (see Figure 4(b)).

6. Discussion and Conclusion

In this paper, a simple stochastic model has been developed to model the spread of plant diseases. By using the theory and methods of stochastic differential equations, the dynamics of the system, such as the stochastic persistence of the disease and the existence of a stationary distribution, are discussed, and relevant sufficient conditions are obtained.

Our results show that the system is deeply affected by stochastic interference.

However, in the real world, the spread of plant diseases is very complicated. Plant diseases can be spread through air, rain, soil, insects and other organisms, grafting, sowing and transplanting, fertilizing, and irrigation [40, 41]. Considering two common transmission modes of plant disease transmission: insect and plant cuttings, Van den Bosch et al. [12] established the following model on the basis of model (1) as follows:

$$\begin{cases} \frac{dS(t)}{dt} = \sigma\phi + \sigma(1-\phi)\frac{r(1-p)I(t)+S(t)}{(1-p)I(t)+S(t)} - \eta S(t) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} = \sigma(1-\phi)\frac{(1-r)(1-p)I(t)}{(1-p)I(t)+S(t)} - \eta I(t) - \omega I(t) + \beta S(t)I(t). \end{cases} \tag{92}$$

In the model, σ represents the replanting rate, which is divided into two parts in the replanted plants, ϕ is the proportion of disease-free plants propagated in vitro, and $1-\phi$ is the proportion of plants cultivated from cuttings of infected plants. r represents the probability that some infected plants may become disease-free plants due to reversal. Vision or other diagnostic methods are used to choose whether the plant is retained. If it is identified as an infected plant, it will be discarded with a probability of p. $1-r$ indicates the probability that the infected plant has not been reversed and is still an infected plant. $1-p$ indicates the probability that the infected plant is not detected by the diagnostic method and is retained. $\sigma(1-\phi)(((1-r)(1-p)I(t))/((1-p)I(t)+S(t)))$ indicates the number of infected plants cultivated from cuttings of infected plants among all replanted plants. $\sigma(1-\phi)((r(1-p)I(t)+S(t))/((1-p)I(t)+S(t)))$ represents the number of disease-free plants cultivated from cuttings of infected plants among all replanted plants. It is an interesting question to consider the model (92) from a stochastic perspective to explore the extinction or persistence of plant diseases, which helps us understand the impact of randomness on the spread of plant diseases. We will leave this question for future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Haisu Zhang contributed to original draft preparation. Tongqian Zhang was responsible for review and editing.

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